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Univariate $\overline{X}$ control charts for individual characteristics in a multinormal model

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The early work on multivariate statistical process control was built upon Hotelling's T^2 control chart which was developed to simultaneously monitor the means of correlated quality variables. This chart, however, has a drawback, namely, the problem of identifying the responsible variable(s) when an out-of-control signal occurs. One alternative is to use a separate $\overline{X}$ control chart for each individual characteristic with equal risks, based on Bonferroni inequality. In this study, we show that, from an economic perspective, it may be desirable to have unequal type I risks for the individual charts, because of different inspection and restoration costs associated with each variable. We obtain their risk ratios, which are measures of relative importance of the variables monitored. Then, based on these risk ratios, we develop computer algorithms for finding the exact control limits for individual variables from a multinormal distribution, in the sense that the overall type I risk of the charts is equal to the desired value. Numerical studies show that the proposed methods give optimal or near-optimal results from an economic as well as statistical point of view.

1. Introduction

Statistical process control charts are used widely in industry to control process (and therefore product) quality. Usually, measurements of process or product characteristics are considered as independent and identically distributed normal random variables, and are monitored by means of univariate control charts, which, based on prespecified control limits, signal the change in the mean and/or variance of the characteristic monitored. Sometimes, however, it is necessary to monitor more than one characteristic simultaneously in order to meet the required quality standards. When these characteristics are mutually correlated, multivariate control charts are employed to avoid poor operating performance, which may arise if the control limits of the univariate charts are not adjusted accordingly. The majority of the multivariate control schemes discussed in the literature [1], as in our study, assume that the joint probability distribution of the monitored quality characteristics is, or approximates a multivariate normal, and measurements are serially uncorrelated.

Similar to a univariate Shewhart X-bar scheme, the χ^2 (chi-squared) statistic for a p-variate normal characteristic X to set up a multivariate control chart is defined as [2]:

$$\chi^2 = n(\overline{X} - \boldsymbol{\mu}_0)' \boldsymbol{\Sigma}^{-1} (\overline{X} - \boldsymbol{\mu}_0),$$

where n is the sample size, $\overline{X}$ is the mean of the sample observations, $\boldsymbol{\mu}_0$ is the known in-control population mean vector, and $\boldsymbol{\Sigma}$ is the known in-control population variance-covariance matrix. The χ^2 chart is closely related to the Hotelling's T^2 chart [3] which uses an estimate of $\boldsymbol{\Sigma}$ instead of the true population value. According to the χ^2 chart, a sample signals a possible out-of-control condition for the process when the χ^2 statistic exceeds the cut-off value $\chi^2_{p,\alpha}$ which is the $100(1-\alpha)$th percentile of the chi-squared distribution with p degrees of freedom; α is the specified type I risk (i.e., false alarm rate). When $\boldsymbol{\Sigma}$ and $\boldsymbol{\mu}_0$ are unknown, they can be estimated from a sample drawn from or representing the in-control process. Although the χ^2 chart can be designed in a fairly straightforward manner, it is difficult to relate an out-of-control signal to a particular variable since χ^2 is an aggregate measure of the process.

To overcome the above-mentioned difficulty, one of the approaches appearing in the literature is the simultaneous use of p univariate Shewhart charts for means. For example, the well-known Bonferroni inequality gives a system of p individual charts, each with an identical type I risk (α_i) equal to the desired overall type I risk (α) for the system divided by the number of variables (p) monitored,

i.e., $\alpha_i = \alpha/p$ for all i. Since statistical process control is basically a hypothesis testing problem, the idea of using individual univariate control charts is the same as that of testing about the individual parameters in a multiparameter model. The control region obtained is rectangular, and an out-of-control signal occurs when the sample mean of any variable falls outside its control interval. The rectangular region is suitable for the case in which the user should take separate corrective action for each variable to remedy the out-of-control condition [4,5]. The system based on the Bonferroni inequality is conservative in the sense that it does not use the dependence structure of the variables monitored that the actual overall type I risk is normally smaller than the desired value α. (For more information on the accuracy of the Bonferroni and other approximate methods, see Nickerson [6] and Hayter and Tsui [7]). To improve, Hayter and Tsui [7], henceforth referred to as HT, proposed a method for determining the exact two-sided percentage points for each of the variable means of a multivariate normal distribution, given the overall type I risk (α), number of variables (p), and the population correlation matrix (**R**). In other words, the p individual univariate Shewhart charts designed based on these percentage points will have an exact overall type I risk α. Their approach involves utilizing the existing tables of a multivariate normal distribution which give the desired percentage points corresponding to some selected correlation structures and α values. For those combinations of correlations and α that are not in the table, the authors suggest using simulation instead. Like the system based on the Bonferroni inequality, HT's method assumed the individual control charts have identical type I risks.

When each individual chart has an identical type I risk and when the shifts (in units of their standard deviations) in the means of variables are equal, the *individual* out-of-control, as well as in-control, Average Run Lengths (ARLs) will be the same for every chart. The out-of-control ARL is the expected number of samples that fall within the control interval after a shift occurs. The in-control ARL is the expected number of samples falling within the control interval while the process is in control, and essentially is the inverse of the type I risk provided that the observations are independent and identically distributed. When we have identical ARLs for all charts under shifts of the same magnitude, all the variables are basically treated as equally important. This assumption has been made, in part, because no method has been heretofore developed for the more general case of unequal type I risks. In practice, however, users may desire the mean time to detect the change in the mean of one variable to be smaller than that of other variables since, for example, the cost of restoring the process to an in-control state may jointly depend on the delay incurred before a true alarm occurs as well as which variable(s) went out of control. On the other hand, it may be desirable to have the in-control ARL of some variables be as large as possible because of a high false alarm cost. These types of considerations suggest that it may not be optimal to have the same ARL for every variable.

In this study we propose an approach for obtaining an economic-statistical design of a system of p individual control charts for p correlated variables. The system will have an overall type I risk near or equal to the desired value of α, while the individual type I risks reflect the relative importance of the variables monitored based on economics. We proceed as follows. In Section 2, we propose a method, based on the idea of the economic univariate control chart originated by Duncan [8], for determining the relative importance of the variables in terms of their risk ratios, namely $r_i = \alpha_i/\alpha_1$, using the first variable as the basis for comparison. The result shows why the equal risk procedures (with $r_i = 1$), such as that of Bonferroni and HT, yield nonoptimal results when the economics is considered. Based on the relative importance obtained in Section 2, we then develop numerical methods in Section 3 to obtain the two-sided control limits for the individual charts such that the overall type I risk of the system is near or equal to the desired value of α. Our algorithms in Section 3 are different from that of HT. Comparisons with some existing results for equiprobable cases are also made. An extensive study was conducted to evaluate the accuracy of the proposed methods for unequal risk cases. Conclusions and some possible research directions are given in Section 4.

2. Relative importance of variables based on economic considerations

We now consider a simple economic model to determine the optimal relative importance of the univariate variables in a multinormal model. Assume there are p correlated variables, $X_1, X_2, \ldots, X_p$. We define two states, state 1 and state 2, which represent the in-control and out-of-control conditions for a process, respectively. When the process is in control, their means, without loss of generality, are assumed to be zero. The mean vector of the process in state 2 is equal to the shift vector $\Delta = (\delta_1, \ldots, \delta_p)$. The covariance structure of the system is assumed to be known and unchanged over time, and the individual variances are assumed to be one. Thus, there is only one assignable cause, which creates a deterministic shift in mean with a known value. We assume that the time until the special cause occurs is exponentially distributed with known mean $1/\lambda$. The implied time unit in our discussion is an hour. We assume the process is not stopped after an alarm, but is halted throughout restoration.

In developing our economic model for relative importance, we shall mainly follow Duncan [8]. However, for simplicity, we assume the sample size is fixed at one, and

the time between two consecutive sampling actions (i.e., sampling interval), denoted by N, is fixed and prespecified. These assumptions differentiate our model from the traditional economic models in which the control limits, sample size, and sampling interval are jointly determined by minimizing the total expected cost as is the case in Lorenzen and Vance [9]. In our model, the only variables to be determined are the optimal control limits, or equivalently, the optimal individual type I risks. When the sample size and sampling interval are also treated as decision variables, finding an optimal solution would be very complex [8]. Note however that, when we show nonoptimality of equal control limits for our more restrictive model, this result will also be expected for these more general models. The production volume can take on any value as long as the probability of a change in the process occurring during taking a sample is zero or negligible [8].

When α_i is the type I risk of the Shewhart chart for variable X_i, its upper and lower control limits, $\pm h_i$ ($h_i > 0$), satisfy

$$\alpha_i = 2[1 - \Phi(h_i)], \quad \text{for } i = 1, \ldots, p, \tag{1}$$

where $\Phi(\cdot)$ denotes the standard normal cumulative distribution function. The out-of-control ARL for the specified shift Δ is given by

$$ARL(\Delta) = [1 - P(-h_1 - \delta_1 < X_1 \le h_1 - \delta_1, \ldots, -h_p - \delta_p < X_p \le h_p - \delta_p)]^{-1}. \tag{2}$$

The joint probability in the right-hand side of (2) is to be calculated with X_i's being jointly distributed as a multivariate normal with a zero mean vector and the known covariance structure. Note that the ARL given by (2) is not for a particular chart, but is the *joint* ARL of p charts.

2.1. *Notation*

Let

- T_i = average false alarm inspection cost for variable X_i;
- M = average lost income per hour due to nonconforming product produced by the process operating in state 2 instead of state 1;
- W = average cost of searching the assignable cause after a true alarm;
- G = cost of obtaining and charting a sample;
- D = average time needed to discover the assignable cause;
- E = time to obtain and chart a sample of observations;
- γ = actual type I risk of the system (of p individual charts).

Also let

$$B = N\{ARL(\Delta) - 0.5 + \lambda N/12\} + E + D. \tag{3}$$

Duncan [8] considered one variable only, and derived the following approximate hourly cost function L, which is to be minimized to obtain the most economic univariate X-bar chart:

$$L = \{\lambda M B + (\gamma T/N) + \lambda W\}/(1 + \lambda B) + G/N,$$

where $\gamma T/N$ is the average false alarm inspection cost per hour in Duncan's univariate scheme, with T denoting the average false alarm inspection cost for the variable under consideration. The other terms that make up L represent the remaining three cost components (refer to notation), each averaged over a long horizon. Assuming every out-of-control signal on every chart is investigated regardless of the number of charts signaling such a condition at a particular sampling instant, we have $\gamma T = \alpha_1 T_1 + \cdots + \alpha_p T_p$ in our cost function. Hence, by using L, our overall hourly cost function is

$$TC = \{\lambda M B + (\alpha_1 T_1 + \cdots + \alpha_p T_p)/N + \lambda W\}/(1 + \lambda B) + G/N. \tag{4}$$

Similar to Saniga [10], we take an "economic statistical" approach to the problem rather than a purely economic one. This approach involves incorporating some constraints on statistical performance into the model. To this end, we set an upper bound on the overall type I risk. Let

$$Q = P(X_1 \le -h_1 - \delta_1) + P(X_1 > h_1 - \delta_1) + \cdots + P(X_p \le -h_p - \delta_p) + P(X_p > h_p - \delta_p). \tag{5}$$

The Bonferroni inequality gives

$$ARL(\Delta) \ge 1/Q, \quad \text{and} \quad (\alpha_1 + \cdots + \alpha_p) \ge \gamma.$$

To facilitate a closed form solution to our optimization problem, we will substitute $1/Q$ for $ARL(\Delta)$ in (3). Now we have the following nonlinear optimization model:

$$\underset{h_i}{\text{Minimize}} \; TC \tag{6}$$

subject to

$$(\alpha_1 + \cdots + \alpha_p) \le \alpha.$$

The relationship between α_i and h_i is given in (1). The constraint ensures that the actual overall type I risk, γ, computed from the resulting optimal h_i values will not exceed the required (prespecified) overall type I risk, α.

When the input values of such parameters as δ_i's and T_i's are different, it is extremely unlikely that the resulting optimum α_i's of (6) will be equal. By using a numerical example, we will compare the resulting costs from these optimal α_i values versus that from equal α_i values, and demonstrate that equal α_i values cannot yield a lower total cost.

The optimal solutions for the α_i's of (6) must satisfy the Karush–Kuhn–Tucker (KKT) necessary conditions for the optimum points [11]. For our nonlinear program, the conditions are

$$v_1 \alpha_i'(h_i) = TC'(h_i), \quad i = 1, \ldots, p, \tag{7}$$

$$v_1\{\alpha - (\alpha_1 + \cdots + \alpha_p)\} = 0, \quad (8)$$

$$v_i \leq 0, \quad (9)$$

where v_1 is the Lagrange multiplier, and "$'$" denotes the first partial derivative of the function with respect to its argument.

It is rather difficult to solve for α_i's and h_i's from (7)–(9) explicitly and exactly; numerical methods are normally used. However, under certain reasonable assumptions given in Appendix A, approximate closed form solutions for the risk ratios are possible and are given by:

$$\alpha_j/\alpha_1 = a[\exp(\delta_j/\delta_1)]^b\{d_{j1} - [J_{j1}/\exp(u_j)]\}^{b/u_1}, \quad \text{for } j \neq 1, \quad (10)$$

where $a = 0.00296$ and $b = 5.897$, and the formulas for computing the values of d_{j1}, J_{j1}, and u_j are given in Appendix A. Equation (10) or the result of a numerical method describes the estimated relationship between the optimal individual type I risks. The ratios of these type I risks can be regarded as measures of relative importance of the variables.

Using this information, we now can define and solve a new probability problem to determine the set of optimal or near-optimal individual type I risks such that the originally specified overall type I risk, α, is maintained. We focus on this in the next section.

3. Obtaining the control limits

Given the desired type I risk ratios from a numerical search or from (10), we now propose algorithms to obtain the control limits for a multiple univariate X-bar chart system with an overall actual (achieved) type I risk equal or close to α. Note that the exact values of α_i's are not known yet, although their ratios are known from (10).

We re-index the variables such that the one with the smallest type I risk is denoted by X_1. Then the ratio of type I risk of X_i to that of X_1 is denoted by:

$$r_i = \alpha_i/\alpha_1, \quad \text{for } i = 2, \ldots, p.$$

Let

μ_{i0} = in-control population mean of variable X_i;
σ_i = in-control population standard deviation of variable X_i.

The control procedure composed of p univariate X-bar charts is based on equal or unequal percentage points of a multivariate normal distribution. As before, we assume we have standardized variables, i.e., $\mu_{10} = \mu_{20} = \cdots = \mu_{p0} = 0$, and $\sigma_1 = \sigma_2 = \cdots = \sigma_p = 1$. The known population correlation matrix is denoted by $\mathbf{R}$. Now, the problem can be stated as follows:

Given $\alpha, r_2, \ldots, r_p$ and $\mathbf{R}$,

Find $h_1, h_2, \ldots, h_p$ such that

$$P(-h_1 < X_1 \leq h_1, \ldots, -h_p < X_p \leq h_p) = 1 - \alpha. \quad (11)$$

The values $-h_i$ and h_i are the lower and upper control limits of the individual Shewhart control chart for the standardized variable X_i. Then, the control limits for the original non-standardized variables are

$$UCL_i = \mu_{i0} + h_i\sigma_i, \quad \text{and} \quad LCL_i = \mu_{i0} - h_i\sigma_i.$$

If sample size $n \neq 1$, the constant shifts δ_i's should be replaced by $n^{-1/2}\delta_i$'s, the plotting statistics are sample means $\overline{X}_i$'s, and σ_i should be replaced by $n^{-1/2}\sigma_i$ in the above expressions. Note that $n^{-1/2}\sigma_i$ is the standard deviation of the sample mean $\overline{X}_i$.

Note that there is only one solution for (11) which satisfies the prespecified values of r_i, $i = 2, \ldots, p$. Heuristically, this can be explained as follows. First we note that α is a monotonically increasing function of $\alpha_1, \ldots, \alpha_p$. Also each α_i is a monotonically decreasing function of h_i. Let $A = \alpha_1 + \cdots + \alpha_p$. When α_i changes, A changes in the same direction as α. Note also that there is a one-to-one relationship between A and α. Once A and r_i, $i = 2, \ldots, p$, are fixed, we have a system of p linear equations in p unknowns, implying the solution $(\alpha_1, \ldots, \alpha_p)$ is unique. When $r_2 = \cdots = r_p = 1$, the values of $h_1, h_2, \ldots, h_p$ can be obtained from published tables of the percentage points of a multivariate normal (or multivariate Student t with infinite degrees of freedom) distribution [12].

If the variables are independent (i.e., $\mathbf{R} = \mathbf{I}$, the identity matrix), the control limits can be found by solving α_1 (and in turn, α_i, $i = 2, \ldots, p$) from $(1 - \alpha_1)(1 - r_2\alpha_1) \cdots (1 - r_p\alpha_1) = 1 - \alpha$. However, an analytical solution is not straightforward when variables are dependent with arbitrary correlations.

In the following three subsections, we propose algorithms to determine the percentage points of a multivariate normal distribution with two, three, and four dependent variables. The general idea of the proposed algorithms is as follows. We start with an initial set of risks, α_i's, satisfying the given risk ratios obtained from Section 2. In general, there are infinitely many such α_i's, hence an additional constraint such as $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_p$ is added (c.f. Bonferroni inequality). We then iteratively update these type I risks until the control limits, h_i's, from (1) satisfy the probability constraint in (11). Note that, at each iteration, α_i's always satisfy the given fixed risk ratios. The iterative algorithm is based on the idea of finding the zero of an equation (i.e., (11)) by successive substitutions. To our knowledge, these algorithms are the first attempts in the literature to solve the problem described in (11) for unequal α_i's. The Fortran programs based on these algorithms are available from the authors upon request.

3.1. *Bivariate case*

Let ρ be the known correlation coefficient between X_1 and X_2. Denote the cumulative distribution function of a bivariate normal with zero mean vector and correlation coefficient ρ by

$$B(x_1, x_2; \rho) = P(X_1 \le x_1, X_2 \le x_2; \rho).$$

It can be shown that

$$P(l_1 < X_1 \le u_1, l_2 < X_2 \le u_2) = B(l_1, l_2; \rho) - B(l_1, u_2; \rho) - B(u_1, l_2; \rho) + B(u_1, u_2; \rho). \tag{12}$$

The goal is to find the values h_1 and h_2 such that (see (11))

$$P(-h_1 < X_1 \le h_1, -h_2 < X_2 \le h_2) = 1 - \alpha.$$

Using (12), we have

$$1 - \alpha = B(-h_1, -h_2; \rho) - B(-h_1, h_2; \rho) - B(h_1, -h_2; \rho) + B(h_1, h_2; \rho). \tag{13}$$

Note the following equalities:

$$B(h_1, -h_2; \rho) = B(+\infty, -h_2; \rho) - B(-h_1, -h_2; -\rho), \tag{14}$$

$$B(h_1, h_2; \rho) = B(+\infty, h_2; \rho) - B(-h_1, h_2; -\rho). \tag{15}$$

By substituting (14) and (15) into (13) and rearranging, we obtain

$$\alpha_2 = \alpha + B(-h_1, -h_2; \rho) - B(-h_1, h_2; \rho) + B(-h_1, -h_2; -\rho) - B(-h_1, h_2; -\rho). \tag{16}$$

Our proposed Algorithm 1, given in Appendix B, recursively finds the exact values of h_1 and h_2 satisfying (16), given α, ρ, and r. Again, the underlying idea is to find the zero of an equation by successive substitutions. Subroutine DBNRDF from the International Mathematical and Statistical Libraries (IMSL) [13], which is accurate up to seven digits [14], can be used to evaluate $B(x_1, x_2; \rho)$.

To determine the accuracy of the limits obtained from Algorithm 1, we ran the algorithm using a full factorial experiment with the following three factors:

$$\alpha = 0.0025(0.0025)0.10; \quad r = 1(1)3; \quad \rho = 0(0.1)0.9.$$

The first value to the right of the equal sign is the lowest value (level) assigned to the corresponding factor, the number inside the parenthesis shows the constant increment of factor level, and the third number is the highest value of the corresponding factor in the experiment. The total number of level combinations was $40 \times 3 \times 10 = 1200$. At each level combination, the effective (achieved) overall type I risk, α_e, was computed based on the values of h_1 and h_2 obtained from the algorithm, namely,

$$\alpha_e = 1 - B(-h_1, -h_2; \rho) + B(-h_1, h_2; \rho) + B(h_1, -h_2; \rho) - B(h_1, h_2; \rho).$$

The difference between α and α_e can be regarded as the error due to the algorithm in obtaining the overall type I risk. Since we are mainly interested in the error for the in-control ARL of the system, we defined the following relative error in the in-control ARL for each combination:

$$e = |\alpha^{-1} - \alpha_e^{-1}|/\alpha^{-1}.$$

Because e depends on the particular factor levels, we calculated its average over all experimental runs. Based on 1200 level combinations, the average relative error was found to be 0.0006. The observed maximum e value was 0.0063, which indicates that our proposed algorithm is very accurate. Note that the relative error, e, can be reduced if we use a tighter error tolerance for h_i's in Step 4 of Algorithm 1 given in Appendix B.

3.2. *Trivariate case*

When the number of variables, p, is greater than 2, we shall use a different algorithm to find the control limits since the implementation of the approach used in the bivariate case requires a considerably longer computation time for $p > 2$. We first consider the $p = 3$ case. Let

$$G(x_1, x_2, x_3; \mathbf{R}) = P(X_1 \le x_1, X_2 \le x_2, X_3 \le x_3; \mathbf{R}),$$

which is the trivariate normal probability distribution function with zero mean vector and correlation matrix $\mathbf{R}$. To simplify the notation, henceforth, we will not explicitly include the correlation matrix $\mathbf{R}$ as an argument of $G(\cdot)$ unless there may be ambiguity, and the dimension of $G(\cdot)$ is recognized through the number of arguments inside the parenthesis.

It can be shown that

$$P(l_1 < X_1 \le u_1, l_2 < X_2 \le u_2, l_3 < X_3 \le u_3) = -G(l_1, l_2, l_3) - G(l_1, u_2, u_3) - G(u_1, l_2, u_3) - G(u_1, u_2, l_3) + G(l_1, l_2, u_3) + G(l_1, u_2, l_3) + G(u_1, l_2, l_3) + G(u_1, u_2, u_3). \tag{17}$$

Since the IMSL does not have a subroutine for computing $G(\cdot)$ for $p > 2$, we will provide a method for that purpose, following the notation of Rice *et al.* [15], who investigated a method to compute the one-sided probability $G(-x_1, -x_2, \ldots, -x_p)$ approximately. Our method is based on the same approach although it is adapted to our particular objective.

It is known that the conditional mean and variance of a standard normal variable truncated from above (i.e., $X_1 \le x_1$) are:

$$a_1 \equiv E(X_1 \mid X_1 \le x_1) = -\phi(x_1)/\Phi(x_1), \tag{18}$$

$$s_1^2 \equiv \mathrm{Var}(X_1 \mid X_1 \le x_1) = 1 - \frac{\phi(x_1)}{\Phi(x_1)}\left[\frac{\phi(x_1)}{\Phi(x_1)} + x_1\right], \tag{19}$$

where $\phi(\cdot)$ is the standard normal probability density function. Let ρ_{ij} be the correlation coefficient between X_i and X_j. Then

$$a_{2|1} \equiv E(X_2 \mid X_1 \le x_1) = \rho_{12} a_1, \tag{20}$$

$$s^2_{2|1} \equiv \mathrm{Var}(X_2 \mid X_1 \le x_1) = 1 - \rho^2_{12}(1 - s_1^2) = 1 + \rho_{12} x_1 a_{2|1} - a^2_{2|1}. \tag{21}$$

The conditional pdf, $\phi_{2|1}$, of X_2 given $X_1 \le x_1$ is normal when $\rho_{12} = 0$, and increasingly deviates from normality as ρ_{12} increases. Assuming $\phi_{2|1}$ is normal [16], we approximate

$$G(x_1, x_2) = P(X_1 \le x_1, X_2 \le x_2) = P(X_2 \le x_2 \mid X_1 \le x_1) P(X_1 \le x_1),$$

by $F(x_1, x_2)$, where

$$F(x_1, x_2) = \Phi(z_{2|1})\Phi(x_1), \tag{22}$$

and where the standardized threshold, $z_{2|1}$, is given by

$$z_{2|1} = (x_2 - a_{2|1})/s_{2|1}. \tag{23}$$

To calculate the trivariate probability in (17), we first note that

$$G(x_1, x_2, x_3) = P(X_3 \le x_3 \mid X_2 \le x_2, X_1 \le x_1) \times P(X_2 \le x_2 \mid X_1 \le x_1) P(X_1 \le x_1). \tag{24}$$

The mean and variance, $a_{3|1}$ and $s^2_{3|1}$, of X_3 conditional on $X_1 \le x_1$ can be calculated similar to $a_{2|1}$ and $s^2_{2|1}$. Using the bivariate approximation (22) in (24), we can approximate $G(x_1, x_2, x_3)$ by

$$F(x_1, x_2, x_3) = \Phi(z_{3|12})\Phi(z_{2|1})\Phi(x_1), \tag{25}$$

where

$$\begin{aligned} z_{3|12} &= (z_{3|1} - a_{3|12})/s_{3|12}, \\ z_{3|1} &= (x_3 - a_{3|1})/s_{3|1}, \\ a_{3|12} &= -\rho_{23|1}\phi(z_{2|1})/\Phi(z_{2|1}), \end{aligned} \tag{26}$$

$$\begin{aligned} s_{23|1} &= \mathrm{Cov}(X_2, X_3 \mid X_1 \le x_1) = \rho_{23} - \rho_{12}\rho_{13}(1 - s_1^2), \\ \rho_{23|1} &= \mathrm{Corr}(X_2, X_3 \mid X_1 \le x_1) = s_{23|1}/(s_{2|1} s_{3|1}), \\ s^2_{3|12} &= 1 + \rho_{23|1} z_{2|1} a_{3|12} - a^2_{3|12}. \end{aligned} \tag{27}$$

Thus, from (11) and (17) we have:

$$\begin{aligned} 1 - \alpha \approx\ & -F(-h_1, -h_2, -h_3) - F(-h_1, h_2, h_3) \\ & - F(h_1, -h_2, h_3) - F(h_1, h_2, -h_3) \\ & + F(-h_1, -h_2, h_3) + F(-h_1, h_2, -h_3) \\ & + F(h_1, -h_2, -h_3) + F(h_1, h_2, h_3). \end{aligned} \tag{28}$$

We propose Algorithm 2 in Appendix C, based on successive substitutions for finding the values of h_1, h_2, and h_3 (and α_1, α_2, and α_3) satisfying (28). The idea is the following. First, for given ρ, α, and the individual risk ratios, (28) can be rewritten as a quadratic function of α_3, for example. By solving α_3 and by using the method of successive substitutions, we find the optimal or near-optimal value of α_3 and hence that of α_1 and α_2, satisfying (28).

In order to assess the accuracy of Algorithm 2, we compared our results for the special case where $r_2 = r_3 = 1$ and $\rho_{12} = \rho_{13} = \rho_{23} = \rho$ with the "exact" values h_1, h_2, and h_3 tabulated by Bechhofer and Dunnett [12]. Rice *et al.* [15] have pointed out that this special case comparison sufficiently indicates the degree of the accuracy obtained by their approximation method for the general case, since their method, as with Algorithm 2, does not depend on the assumption of any particular correlation structure or type I risk ratios. The sensitivity of the results of the underlying method to changes in the parameters α, ρ and p are discussed in more detail in Rice *et al.* [15]. We again utilized IMSL subroutines for computing univariate probabilities in the intermediate steps. Our computer implementation showed that, as the value of ρ increases, the algorithm yields more conservative control intervals; i.e., the obtained h_i exceeds the "exact" h_i. To alleviate this effect, we incorporated the procedure described below into Algorithm 2.

Hohenbichler and Rackwitz [17] have presented a method, henceforth referred to as HR, to obtain a lower bound for $G(x_1, x_2, \ldots, x_p; \mathbf{R})$. Their method is similar to that of Rice *et al.* [15], in the sense that the one-sided probability is approximated by using conditional expectations. The manner in which we incorporated the HR procedure into Algorithm 2 can be outlined as follows. In Step 1, for the last two positive terms in (28) we find $z_{2|1}$, $z_{3|1}$, and $\rho_{23|1}$ by using the HR procedure instead of using (23), (26), and (27). This change reduces the values of these two terms, and consequently, the h_i values obtained at the end of the algorithm become smaller, a result which helps to correct the bias observed in Algorithm 2. To distinguish the resulting procedure from Algorithm 2, we call it the "improved" version which is, in essence, a combination of Algorithm 2 and the HR procedure [18].

3.3. *Quadrivariate and higher-order cases*

We also developed an algorithm similar to Algorithm 2 for the quadrivariate case, in which $1 - \alpha$ can be expressed as (with $p = 4$)

$$1 - \alpha = \sum_{a_1, \ldots, a_p = +1 \text{ or } -1} \left[\left(\prod_{j=1}^{p} a_j \right) G(a_1 h_1, \ldots, a_p h_p) \right], \tag{29}$$

where the sum is taken over all combinations of $a_1, a_2, \ldots, a_p$, with $a_i = -1$ or $+1$ for $i = 1, 2, \ldots, p$. To obtain an approximation, F, for G in (29) for $p \ge 4$, we

follow an approach similar to that in Section 3.2 to obtain (cf. Rice *et al.* [15]):

$$F(x_1, x_2, \ldots, x_p) = \Phi(z_{p|0,\ldots,p-1})\Phi(z_{p-1|0,\ldots,p-2}) \cdots \times \Phi(z_{2|0,1})\Phi(z_{1|0}),$$

where, for $j = 1, 2, \ldots, i-1$, and $\ell = 1, 2, \ldots, k-1$,

$$z_{i|0,\ldots,j} = (z_{i|0,\ldots,j-1} - a_{i|0,\ldots,j})/s_{i|0,\ldots,j},$$

$$a_{j|0,\ldots,j-1}\rho_{ji|0,\ldots,j-1} = a_{i|0,\ldots,j},$$

$$a_{j|0,\ldots,j-1} = -\phi(z_{j|0,\ldots,j-1})/\Phi(z_{j|0,\ldots,j-1}),$$

$$s^2_{k|0,\ldots,\ell} = 1 + \rho_{k\ell|0,\ldots,\ell-1}a_{k|0,\ldots,\ell}z_{\ell|0,\ldots,\ell-1} - a^2_{k|0,\ldots,\ell},$$

and, for $m, n > j$,

$$\rho_{mn|0,\ldots,j} = \{\rho_{mn|0,\ldots,j-1} - \rho_{jm|0,\ldots,j-1}\rho_{jn|0,\ldots,j-1}a_{j|0,\ldots,j-1} \times (a_{j|0,\ldots,j-1} - z_{j|0,\ldots,j-1})\}/(s_{m|0,\ldots,j}s_{n|0,\ldots,j}),$$

with initial values $z_{i|0} = x_i$, $\rho_{ji|0} = \rho_{ji}$, $a_{i|0} = a_i$, and $s^2_{i|0} = s^2_i$ given in (18) and (19). Special cases for $p = 2$ and 3 were given in Sections 3.1 and 3.2, respectively.

To solve for $\pm h_i$, we can make the following modifications to Algorithm 2. In Step 3 of Algorithm 2 we substitute terms containing α_4 in place of $\Phi(z_{4|123})$ and $\Phi(x_1)$, analogous to the three-variable case. Thus, Step 4 now involves finding the smaller root of a quadratic equation in α_4. The remaining steps are modified in a similar manner; for instance, in Step 6 we compare $|h_{4n} - h_4|$ versus a prespecified small constant.

Once again, we checked the accuracy via comparisons with the values available for the equicorrelated, equal percentage points case given by Bechhofer and Dunnett [12]. It was observed that, for four variables, the h_i values found from our algorithm were lower than the "exact" h_i values, creating a bias of the opposite nature to the error observed in the three variable case. Therefore, to obtain an "improved" procedure, we again made suitable adjustments using the HR procedure. This time, we used the values of $z_{2|1}$, $z_{3|1}$, and $\rho_{23|1}$ obtained from the HR procedure in calculating the four terms with only one negative h_i on the right side of (29).

The two-sided percentage points obtained from the aforementioned algorithms together with the "exact" values given by Bechhofer and Dunnett [12] for $p = 2, 3$, and 4 are presented in Table 1. Each approximate value shown in Table 1 was calculated in at most 40 iterations, using the improved versions in the three and four variable cases. Clearly, the algorithms given for the three and four variable cases can be extended readily to the cases where $p > 4$. However, from a practical point of view, the increase in computation time may be a limiting factor for these extensions.

3.4. *Numerical examples*

We now present a numerical example to illustrate the use of the algorithms developed. Consider two quality characteristics with a correlation coefficient of 0.6. The problem is to find the optimal control limits for the given numerical values of the following parameters: $n = 1$, $N = 1$, $\lambda = 0.01$, $M = \$800$, $W = \$200$, $G = \$6$, $T_1 = \$800$, $T_2 = \$100$, $D = 0.7$, and $E = 0.05$. Algorithm 1 is used to find the h_1 and h_2 values, and then compute *ARL* (Δ) from (2) and *TC* from (4). Thus, for example, for $\alpha = 0.05$ and $\delta_1 = \delta_2 = 3$, we obtain $r = 32.38$, which gives (from (10) and (1)) $h_1 = 3.17$ and $h_2 = 1.97$. There is a 42.3% (=[\$43.59 − \$25.15]/\$43.59) reduction in expected cost per hour when we use the proposed method rather than HT to determine the control limits, indicating equiprobable limits of HT are not always optimal under an economic consideration (see Serel [18] for more examples).

It is now possible to make some general observations. The difference between the expected total costs for equal

Table 1. Values of h_i from the proposed algorithms versus exact values (exact values are given in bold)

α	p	ρ									
		0	*0.1*	*0.2*	*0.3*	*0.4*	*0.5*	*0.6*	*0.7*	*0.8*	*0.9*
0.01	2	**2.806 23**	**2.805 91**	**2.804 89**	**2.802 93**	**2.799 60**	**2.794 27**	**2.784 95**	**2.772 98**	**2.752 18**	**2.715 39**
		2.806 92	2.806 81	2.806 24	2.804 74	2.801 46	2.795 59	2.786 52	2.773 10	2.752 22	2.715 44
	3	**2.934 16**	**2.933 68**	**2.932 11**	**2.929 01**	**2.923 66**	**2.915 00**	**2.901 43**	**2.880 40**	**2.847 04**	**2.788 99**
		2.934 10	2.933 78	2.932 48	2.929 86	2.925 34	2.917 70	2.905 65	2.886 35	2.853 79	2.792 36
	4	**3.022 20**	**3.021 62**	**3.019 66**	**3.015 74**	**3.008 89**	**2.997 74**	**2.980 28**	**2.953 33**	**2.910 95**	**2.838 05**
		3.022 00	3.021 32	3.019 36	3.015 58	3.009 52	3.000 41	2.987 17	2.968 08	2.940 65	2.898 74
0.05	2	**2.236 48**	**2.235 63**	**2.233 04**	**2.228 53**	**2.221 75**	**2.212 13**	**2.198 72**	**2.179 88**	**2.152 44**	**2.108 14**
		2.238 21	2.237 42	2.234 81	2.229 93	2.222 56	2.212 44	2.198 79	2.179 93	2.152 47	2.108 19
	3	**2.387 74**	**2.386 40**	**2.382 30**	**2.375 14**	**2.364 34**	**2.348 97**	**2.327 56**	**2.297 63**	**2.254 35**	**2.185 35**
		2.387 69	2.386 91	2.383 65	2.377 21	2.367 22	2.352 21	2.330 63	2.298 71	2.250 04	2.166 52
	4	**2.490 92**	**2.489 23**	**2.484 06**	**2.475 00**	**2.461 29**	**2.441 77**	**2.414 62**	**2.376 79**	**2.322 43**	**2.236 49**
		2.489 91	2.487 96	2.482 05	2.472 26	2.457 72	2.438 03	2.411 65	2.376 68	2.330 10	2.265 06

and unequal type I risks is larger for the case where the shifts in the means are of the same magnitude. Also, as α increases, the expected savings from setting r based on (10) instead of using $r = 1$ appear to increase. We also note that, because it is derived heuristically, in some cases (10) may lead to an expected total cost slightly higher than that given by the HT scheme. This is due to the fact that some of the assumptions in Appendix A were not completely satisfied. But in this case we can always depend on numerical search methods to provide exact results.

We also studied the changes in the value of r given by (10) as the cost parameters T_1, T_2, M, and W changed. Note that (10) and Appendix A imply a larger value for r as α increases when everything else is held fixed. We also observed that r is insensitive to changes in W, and inversely related to M. Another result which is reported in Serel [18] is that, to a large extent, just the difference between T_1 and T_2, and not their specific values, determines the value of r in (10).

4. Conclusion

Multivariate quality control can be expected to gain more attention in manufacturing environments in the near future as the demand for more specialized statistical tools by quality practitioners increases, due to increasingly complex products, rising competition, and advances in information technology. Although it has long been recognized that the consideration of the cost outcomes of statistical decision rules is important for effective implementation of these tools in practice, the multivariate statistical process control methods developed in recent years have not been studied within an economic framework, partly because of analytical intractability. The T^2 control chart, while it is the most known multivariate control scheme, needs to be supported by other procedures, since it does not *per se* identify the variable that caused the problem in the process. The multiple univariate X-bar control charts method can be regarded as a good alternative for the T^2 chart because of its better diagnostic feature. By developing a model that takes into account the economic impact of the decision rules associated with a multiple univariate X-bar control scheme, our study considerably enhances the success potential of this control scheme in practice. The computer algorithms for finding unequal percentage points of a multivariate normal distribution, which are the first of their kind in the literature, are not only suitable in the current context of statistical process control, but it is possible to utilize these algorithms for improving the statistical decision rules developed for other applications where the decision model is based on rectangular probability regions. Some references which use this type of probability region for applications different from the process control can be found in Joe [19].

With respect to future research, a possible direction to pursue is to investigate the effect of simultaneous changes in both the means and variances of the variables on the performance of a multiple univariate chart system. It is well known that changes in the variance will also impact the control chart for the mean. Therefore, if it is desired to assign an overall type I risk to the system consisting of both mean and variance control charts, we would try to allocate the total probability of type I risk among the system components as efficiently as possible. It would be useful to have some guiding answers to this problem in both the univariate and multivariate cases. Previously, Crowder [20] has addressed the issue of adjusting control limits of range and mean charts for a given overall type I risk. He suggested that if the shifts in, say deviation, are of principal concern, then the control interval for the range chart should be narrowed accordingly. However, he did not elaborate on how to make this adjustment optimally. Saniga [10] has investigated the problem in the context of a joint economic statistical design of mean and deviation charts.

Other future research might focus on a more rigorous and detailed treatment of the economic model of Section 2, and further refinements of the approximations developed in this study. Finally, it would be useful to develop a multiple univariate bootstrapping procedure, which would eliminate the need to make distributional assumptions, such as multinormality, as well as needing large amounts of data to derive the control limits.

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Appendices

Appendix A

Derivation of equation (10)

Let $\phi(\cdot)$ be the standard normal probability density function, then we have (from (1)):

$$\alpha_i'(h_i) = -2\phi(h_i) < 0. \tag{A1}$$

Equation (3) implies

$$B'(h_i) = -NQ^{-2}Q'(h_i) > 0,$$

and (5) implies

$$Q'(h_i) = -\phi(h_i + \delta_i) - \phi(h_i - \delta_i) < 0.$$

Note that (7), (9), and (A1) imply that for a point h_i^* to be an optimum point, $TC'(h_i)$ evaluated at $h_i = h_i^*$ should be nonnegative.

Points satisfying (7)

Let

$$b_1 = M - [\alpha_1 T_1 + \cdots + \alpha_p T_p]/N - \lambda W,$$
$$b_2 = 1 + \lambda B,$$
$$b_3 = \lambda B'(h_i)b_1 + (T_i/N)\alpha_i'(h_i)(1 + \lambda B). \tag{A2}$$

Then from (4),

$$TC'(h_i) = b_3 b_2^{-2}.$$

It should be noted that b_2 is always positive, and so is b_1 for realistic values of parameters, which implies the first term on the right side of (A2) will generally be positive. On the other hand, the second term on the right side of (A2) is always negative. $\alpha_i'(h_i)$ becomes less negative as h_i increases. Since from (3) B is proportional to $ARL(\Delta)$, B will decrease as δ_i increases. In light of these observations, for large h_i^* and δ_i values, $TC'(h_i)$ evaluated at $h_i = h_i^*$ is likely to be nonnegative. For example, $TC'(h_i) \geq 0$ when $\delta_i \geq 2.5$ and $h_i \geq 2.5$. $\delta_i \geq 2.5$ is a reasonable assumption in practice; otherwise, one should not use Shewhart's X-bar charts because they are not effective for detecting small shifts (e.g., shifts with $\delta_i \leq 2.5$). The assumption $h_i \geq 2.5$ is true when α_i's or α is small, see (1) and the constraint in (6).

Derivation of (10)

From (7), we have

$$\alpha_i'(h_i)/\alpha_j'(h_j) = TC'(h_i)/TC'(h_j).$$

Define $w_i = \{\phi(h_i + \delta_i) + \phi(h_i - \delta_i)\}\phi(h_i)$, for $i = 1, \ldots, p$. After some manipulations, we obtain

$$(w_j - w_i)(\lambda b_1 Q^{-2} N) = (T_j - T_i)[2(1 + \lambda B)/N]. \tag{A3}$$

It can be shown that

$$w_i = \exp(-0.5\delta_i^2)[\exp(-h_i\delta_i) + \exp(h_i\delta_i)]. \tag{A4}$$

We assume that δ_i and h_i are relatively large so that $TC'(h_i)$ evaluated at h_i is nonnegative, and the points satisfying (7) constitute a minima. This assumption about the values of δ_i and h_i also implies that removal of the term $\exp(-h_i\delta_i)$ will not change the value of w_i significantly. After eliminating this term and some more algebra, we obtain from (A3) and (A4)

$$\exp(h_i/h_j) = \exp(\delta_j/\delta_i)\{d_{ji} - [J_{ji}/\exp(h_j\delta_j)]\}^{1/h_j\delta_i}, \tag{A5}$$

where

$$d_{ji} = \exp\{0.5(\delta_i^2 - \delta_j^2)\},$$
$$J_{ji} = (T_j - T_i)\exp(0.5\delta_i^2)[2(1 + \lambda B)/(\lambda b_1 Q^{-2} N^2)]. \tag{A6}$$

We now develop an approximation method to write (A5) in terms of α_i and α_j. To express the left-hand side of (A5) in terms of α_i and α_j, we used least squares regression to

find the values of a and b to minimize the total squared error of the following hypothesized relationship over the discretized interval $2.00 \leq h_1 \leq h_j \leq 3.00$ with an increment of 0.05:

$$\alpha_j/\alpha_1 = a[\exp(h_1/h_j)]^b. \quad \text{(A7)}$$

SAS results were: $a = 0.00296$ and $b = 5.897$, with $R^2 = 0.9654$. This high value of the coefficient of multiple determination, R^2, indicates the approximation in (A7) is extremely good. Combining (A5) and (A7), we have

$$\alpha_j/\alpha_1 = a[\exp(\delta_j/\delta_1)]^b\{d_{j1} - [J_{j1}/\exp(h_j\delta_j)]\}^{b/h_j\delta_1}.$$

We still need to make more approximations since the values of h_j, which should be known to evaluate the right side of the expression, are not known yet. We approximate h_i, for $i = 1, \ldots, p$ by

$$h_b = \Phi^{-1}(1 - 0.5\alpha/p),$$

which is the Bonferroni based control limit given equal type I risks for all variables. In order to evaluate J_{j1}, we use the following approximations:

$$ARL(\Delta) \cong 1/Q, \text{ and}$$
$$b_1 \cong M - [\alpha(T_1 + \cdots + T_p)/(pN)] - \lambda W,$$

where all the control limits needed for computing Q are assumed to be $\pm h_b$. Hence we obtain (10) with $u_j = h_b\delta_j$, for $j = 1, 2, \ldots, p$.

Appendix B

Algorithm 1 for bivariate case

Step 0. Set the initial values of h_1 and h_2 based on a Bonferroni region (i.e., $\alpha = \alpha_1 + \alpha_2$):

$$\alpha_1 = \alpha/(1 + r) \quad \text{and} \quad \alpha_2 = r\alpha_1,$$

such that $h_1 = \Phi^{-1}(1 - \alpha_1/2)$ and $h_2 = \Phi^{-1}(1 - \alpha_2/2)$.

Step 1. Update α_2 from (16).

Step 2. Update α_1 from $\alpha_1 = \alpha_2/r$.

Step 3. Compute new values of h_1 and h_2 (from (1)):

$$h_{in} = \Phi^{-1}(1 - 0.5\alpha_i), \quad i = 1, 2.$$

Step 4. If $|h_{2n} - h_2| < 0.0001$ go to Step 7.

Step 5. $h_2 = h_{2n}$, $h_1 = h_{1n}$.

Step 6. Go to Step 1.

Step 7. Stop. Current values of h_1 and h_2 are the solutions.

Appendix C

Algorithm 2 for trivariate case

We first rearrange the variables so that $h_1 \geq h_2 \geq h_3$ (or $\alpha_1 \leq \alpha_2 \leq \alpha_3$).

Step 0. Set the initial values of h_1, h_2, and h_3 based on a Bonferroni region; that is

$$\alpha_1 = \alpha/(1 + r_2 + r_3), \ \alpha_2 = r_2\alpha_1, \ \alpha_3 = r_3\alpha_1, \text{ and}$$
$$h_i = \Phi^{-1}(1 - \alpha_i/2).$$

Step 1. Compute the values of $z_{3|12}$ and $z_{2|1}$ for each of the eight terms on the right side of (28).

Step 2. Define a multiplier term k_i, $i = 1, \ldots, 8$, for each of the eight terms on the right side of (28) as follows:

$$\text{for } x_3 = h_3, \text{ then } k_i = [1 - \Phi(z_{3|12})]/[1 - \Phi(x_3)];$$
$$\text{for } x_3 = -h_3, \text{ then } k_i = \Phi(z_{3|12})/\Phi(x_3).$$

Step 3. Make the following substitutions in each of the eight terms on the right side of (28): substitute $(1 - 0.5k_i\alpha_3)$ for $\Phi(z_{3|12})$ if $x_3 = h_3$,

$$0.5k_i\alpha_3 \quad \text{for } \Phi(z_{3|12}) \quad \text{if } x_3 = -h_3,$$
$$(1 - 0.5\alpha_3/r_3) \quad \text{for } \Phi(x_1) \quad \text{if } x_1 = h_1,$$
$$0.5\alpha_3/r_3 \quad \text{for } \Phi(x_1) \quad \text{if } x_1 = -h_1.$$

Thus, by using the multipliers k_i, we can replace $\Phi(z_{3|12})$ by a term that explicitly contains α_3.

Step 4. Now, we can write (28) as a quadratic equation in α_3. Let the smaller root of this equation be y. Then, let

$$\alpha_3 = y, \ \alpha_2 = r_2y/r_3, \ \alpha_1 = y/r_3.$$

Step 5. Compute new values of h_1, h_2, and h_3

$$h_{in} = \Phi^{-1}(1 - 0.5\alpha_i/2)$$

Step 6. If $|h_{3n} - h_3| < 0.0001$, go to Step 9.

Step 7. Let $h_1 = h_{1n}$, $h_2 = h_{2n}$, $h_3 = h_{3n}$.

Step 8. Go to Step 1.

Step 9. Stop. Current values of h_1, h_2, and h_3 are the approximate solution to (28).

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