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Non-autonomous Svinolupov–Jordan KdV systems

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Abstract

Non-autonomous Svinolupov–Jordan KdV systems are considered. The integrability criteria of such systems are associated with the existence of recursion operators. A new non-autonomous KdV system and its recursion operator is obtained for all N. The examples for $N = 2$ and 3 are studied in detail. Some possible transformations which map some systems to autonomous ones are also discussed.

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There has recently been an increasing interest in the study of integrable nonlinear partial differential equations on associative and non-associative algebras [1] and in their recursion operators [2, 3]. It is well known that one class of integrable autonomous multi-component KdV equations (Korteweg–de Vries), associated with a Jordan algebra J (commutative and non-associative),

$$q^i_t = q^i_{xxx} + s^i_{jk} q^j q^k_x \qquad s^i_{jk} = s^i_{kj} \quad i, j, k = 1, 2, \ldots, N \tag{1}$$

has been considered by Svinolupov [4] where q^i are real and depend on the variables x and t. The constant parameters s^i_{jk} are structure constants, with respect to some basis e_i, of a Jordan algebra J defined by

$$e_i \circ e_j = s^k_{ij} e_k \tag{2}$$

and satisfy the Jordan identities

$$s^k_{pr} F^i_{ljk} + s^k_{jr} F^i_{lpk} + s^k_{jp} F^i_{lrk} = 0 \tag{3}$$

where

$$F^i_{plj} = s^i_{jk} s^k_{lp} - s^i_{lk} s^k_{jp} \tag{4}$$

is the associator of the Jordan algebra [4]. The integrability criteria of the multi-component Jordan KdV system (JKdV)(1) are associated with the existence of higher symmetries and the corresponding recursion operator.

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Theorem 1 (Svinolupov). *Let s^i_{jk} be the structure constants of a Jordan algebra, i.e., satisfy the identities (3). The system (1) possesses a recursion operator of the form*

$$\mathcal{R}^i_j = \delta^i_j D^2 + \tfrac{2}{3}\, s^i_{jk}\, q^k + \tfrac{1}{3}\, s^i_{jk}\, q^k_x\, D^{-1} + \tfrac{1}{9}\, (s^i_{jm} s^m_{kl} - s^i_{km} s^m_{jl})\, q^l\, D^{-1}\, q^k\, D^{-1}. \tag{5}$$

We only need to prove that $\mathcal{R}$ satisfies the integrability condition [5]

$$\mathcal{R}^i_{j,t} = K'^i_k\ \mathcal{R}^k_j\ -\ \mathcal{R}^i_k\ K'^k_j \tag{6}$$

with respect to (3) where K'^i_k is the Fréchet derivative of system (1). Therefore, the existence of the recursion operator ensures that system (1) possesses an infinite series of symmetries.

Svinolupov established a one-to-one correspondence between Jordan algebras and the subsystems (reducible, irreducible, completely reducible) of system (1).

Definition 1. *A system of type (1) is called reducible (triangular) if it decouples into the form*

$$U^i_t = F^i(U^k, U^k_x, U^k_{xxx}) \qquad i,k = 1,2,\ldots,K \qquad 0 < K < N \tag{7}$$

$$V^a_t = G^a(U^b, U^b_x, V^b, V^b_x, V^b_{xxx}) \qquad a,b = 1,2,\ldots,N-K \tag{8}$$

under a certain linear transformation which leaves the system (1) invariant. If not, it is irreducible. A system is called completely reducible if the second equation given above does not contain the dynamical variables U^i and U^i_x.

Example 1. For $N = 2$, the complete classification, with respect to Jordan algebra, was given by Svinolupov [6]:

$$u_t = u_{xxx} + 2c_0 u u_x \qquad v_t = v_{xxx} + c_0(uv)_x \tag{9}$$

$$u_t = u_{xxx} + c_0 u u_x \qquad v_t = v_{xxx} + c_0(uv)_x \tag{10}$$

$$u_t = u_{xxx} \qquad v_t = v_{xxx} + c_0\, u u_x \tag{11}$$

where c_0 is an arbitrary constant. The reducible systems (9) and (10) correspond to the JKdV and trivially JKdV (associator is zero) respectively. The last system is completely reducible system.

Example 2. For $N = 3$.

(i) The system

$$\begin{aligned} u_t &= u_{xxx} - c_0(u^2 - v^2 - w^2)_x \\ v_t &= v_{xxx} - c_0(uv)_x \\ w_t &= w_{xxx} - c_0(uw)_x \end{aligned} \tag{12}$$

is the only irreducible JKdV system [6, 7].

(ii) A reducible JKdV system is

$$\begin{aligned} u_t &= u_{xxx} - 2c_0 u u_x \\ v_t &= v_{xxx} - c_0(uv)_x \\ w_t &= w_{xxx} - c_0(uw)_x. \end{aligned} \tag{13}$$

In this paper we investigate the non-autonomous Svinolupov JKdV systems. For this purpose, we consider the non-autonomous form of the system (1) as

$$q^i_t = q^i_{xxx} + \tilde{s}^i_{jk}(t) q^j q^k_x \qquad \tilde{s}^i_{jk}(t) = \tilde{s}^i_{kj}(t) \qquad i,j,k = 1,2,\ldots,N \tag{14}$$

where $\tilde{s}^i_{jk}(t)$ are sufficiently differentiable functions. In particular, for $N = 1$ the system (14) is the well known cylindrical KdV (cKdV) equation [8]

$$u_t = u_{xxx} + \frac{6}{\sqrt{t}} u u_x \tag{15}$$

which possesses a recursion operator [9]

$$\mathcal{R} = tD^2 + 4\sqrt{t}u + \tfrac{1}{3}x + \tfrac{1}{6}(12\sqrt{t}u_x + 1)D^{-1}. \tag{16}$$

We are now in a position to propose a recursion operator for the integrability of system (14). Moreover, motivated by the results obtained in [4, 6] and [9–11] we may state the following theorem.

Theorem 2. *Let s^i_{jk} be the structure constants of a Jordan algebra, i.e. satisfy the identities (3). System (14) possesses a recursion operator of the form*

$$\mathcal{R}^i_j = t\delta^i_j\, D^2 + \tfrac{2}{3}\sqrt{t}\, s^i_{jk}\, q^k + \tfrac{1}{3}\delta^i_j x + (\tfrac{1}{3}\sqrt{t}\, s^i_{jk}\, q^k_x + \tfrac{1}{6}\delta^i_j)\, D^{-1} + \tfrac{1}{9} F^i_{lkj}\, q^l\, D^{-1}\, q^k\, D^{-1}. \tag{17}$$

Proof. We start with the ansatz

$$\mathcal{R}^i_j = z^i_j(t)\, D^2 + a^i_{jk}(t)\, q^k + H^i_j(x,t) + (c^i_{jk}(t)\, q^k_x + w^i_j(t))\, D^{-1} + \tilde{F}^i_{lkj}(t)\, q^l\, D^{-1}\, q^k\, D^{-1} \tag{18}$$

where z^i_j, $a^i_{jk}(t)$, $c^i_{jk}(t)$, $\tilde{F}^i_{lkj}(t)$, $w^i_j(t)$ and $H^i_j(x,t)$ are sufficiently differentiable functions. By the use of integrability condition (6) with

$$K'^i_j = \delta^i_j D^3 + \tilde{s}^i_{jk} q^k_x + \tilde{s}^i_{kj} q^k D \tag{19}$$

which is the Fréchet derivative of (14), a direct calculation gives

$$\begin{aligned}
& a^i_{jl} + c^i_{jl} - z^i_k \tilde{s}^k_{jl} = 0 \qquad c^i_{jl} = \tfrac{1}{3} z^i_k \tilde{s}^k_{jl} \\
& z^i_k \tilde{s}^k_{jl} - z^k_j \tilde{s}^i_{lk} = 0 \qquad 3a^i_{jl} + z^k_j \tilde{s}^i_{kl} - 3 z^i_k \tilde{s}^k_{lj} = 0 \\
& a^i_{jk} \tilde{s}^k_{ml} + a^i_{km} \tilde{s}^k_{jl} - a^k_{jl} \tilde{s}^i_{mk} - a^k_{jm} \tilde{s}^i_{kl} \\
& + c^i_{kl} \tilde{s}^k_{jm} - c^k_{jl} \tilde{s}^i_{mk} - 3\tilde{F}^i_{mlj} - 3\tilde{F}^i_{lmj} = 0 \\
& \tilde{s}^k_{ml} \tilde{F}^i_{kpj} - \tilde{s}^i_{mk} \tilde{F}^k_{lpj} - \tilde{s}^i_{kl} \tilde{F}^k_{mpj} = 0 \\
& [a^i_{km} \tilde{s}^k_{lj} - a^k_{jl} \tilde{s}^i_{mk}]_{(l\,m)} = 0 \qquad c^i_{jk} \tilde{s}^k_{lm} - c^k_{jl} \tilde{s}^i_{mk} - 3\tilde{F}^i_{lmj} = 0 \\
& [\tfrac{1}{2} \tilde{F}^i_{lkj} \tilde{s}^k_{mp} - \tilde{F}^k_{plj} \tilde{s}^i_{mk}]_{(l\,m\,p)} = 0 \\
& [\tfrac{1}{2} \tilde{F}^i_{pkj} \tilde{s}^k_{ml} - \tilde{F}^i_{pmk} \tilde{s}^k_{jl}]_{(l\,m)} = 0 \\
& a^i_{jl,t} - H^k_{j\,x} \tilde{s}^i_{lk} - w^k_j \tilde{s}^i_{lk} + w^i_k \tilde{s}^k_{jl} = 0 \\
& H^i_{j,2x} = 0 \qquad H^k_j \tilde{s}^i_{lk} - H^i_k \tilde{s}^k_{jl} = 0 \qquad H^i_{j,t} - H^i_{j,3x} = 0 \\
& \tilde{F}^i_{ilk,t} = 0 \qquad c^i_{jl,t} - w^k_j \tilde{s}^i_{kl} = 0 \qquad z^i_{j,t} - 3H^i_{j,x} = 0, \;\; w^i_{j,t} = 0
\end{aligned} \tag{20}$$

where the subscript round brackets denote the symmetrization. These equations can be simplified further:

$$\begin{aligned}
& a^i_{jl} = \tfrac{2}{3} z^i_k \tilde{s}^k_{jl} \qquad c^i_{jl} = \tfrac{1}{3} z^i_k \tilde{s}^k_{jl} \qquad H^i_j = x\Gamma^i_j + \beta^i_j \qquad z^i_j = 3\, t\, \Gamma^i_j \\
& \beta^i_k \tilde{s}^k_{lj} - \beta^k_j \tilde{s}^i_{lk} = 0 \qquad \Gamma^i_k \tilde{s}^k_{lj} - \Gamma^k_j \tilde{s}^i_{lk} = 0 \\
& w^i_j = w_0 \delta^i_j \qquad \Gamma^k_j \tilde{s}^i_{lk} - \tilde{s}^i_{lk} w^k_j - w^i_k \tilde{s}^k_{jl} = 0 \qquad s^i_{jl,t} = -3 M^i_k w^k_p s^p_{jl} \\
& \tilde{F}^i_{lmj} = \tfrac{1}{9} z^i_k [\tilde{s}^k_{jp} \tilde{s}^p_{ml} - \tilde{s}^k_{mp} \tilde{s}^p_{jl}] \qquad \tilde{s}^k_{pr} \tilde{F}^i_{ljk} + \tilde{s}^k_{jr} \tilde{F}^i_{lpk} + \tilde{s}^k_{jp} \tilde{F}^i_{lrk} = 0
\end{aligned} \tag{21}$$

where $M^i_k z^k_j = \delta^i_j$, β^i_j and w_0 are constants. These equations are the necessary conditions for system (14) to be integrable. Hence, without loss of generality, we can take $w_0 = \frac{1}{6}$, $\beta^i_j = 0$ and it follows that

$$\Gamma^i_j = \frac{1}{3}\delta^i_j \qquad \tilde{s}^i_{jk} = \frac{1}{\sqrt{t}} s^i_{jk} \qquad \tilde{F}^i_{lmj} = \frac{1}{9} F^i_{lmj} \tag{22}$$

where s^i_{jk} are the structure constants of (1). This completes the proof of the theorem. □

We now give some examples.

Example 3. For $N = 2$.
(i) The system

$$u_t = u_{xxx} + \frac{2c_0}{\sqrt{t}} u u_x$$
$$v_t = v_{xxx} + \frac{c_0}{\sqrt{t}} (uv)_x \tag{23}$$

is the non-autonomous JKdV where c_0 is an arbitrary constant. The recursion operator $\mathcal{R}$ for the above system is

$$\mathcal{R} = \begin{pmatrix} \mathcal{R}_0^0 & \mathcal{R}_1^0 \\ \mathcal{R}_0^1 & \mathcal{R}_1^1 \end{pmatrix} \tag{24}$$

with

$$\begin{aligned}
\mathcal{R}_0^0 &= tD^2 + \frac{1}{3}x + \frac{4c_0}{3}\sqrt{t}u + \frac{1}{6}(4c_0\sqrt{t}u_x + 1)D^{-1} \\
\mathcal{R}_1^0 &= 0 \\
\mathcal{R}_0^1 &= \frac{2c_0}{3}\sqrt{t}v + \frac{c_0}{3}\sqrt{t}v_x D^{-1} - \frac{c_0^2}{9} u D^{-1} v D^{-1} \\
\mathcal{R}_1^1 &= tD^2 + \frac{1}{3}x + \frac{2c_0}{3}\sqrt{t}u + \frac{1}{6}(2c_0\sqrt{t}u_x + 1)D^{-1} + \frac{c_0^2}{9} u D^{-1} u D^{-1}.
\end{aligned} \tag{25}$$

(ii) The non-autonomous reducible JKdV is

$$u_t = u_{xxx} + \frac{c_1}{\sqrt{t}} u u_x$$
$$v_t = v_{xxx} + \frac{c_1}{\sqrt{t}} (uv)_x \tag{26}$$

which corresponds to the perturbation system of the cKdV equation [12]. Here c_1 is an arbitrary constant. The recursion operator for this system is

$$\begin{aligned}
\mathcal{R}_0^0 &= tD^2 + \frac{1}{3}x + \frac{2c_1}{3}\sqrt{t}u + \frac{1}{6}(2c_1\sqrt{t}u_x + 1)D^{-1} \\
\mathcal{R}_1^0 &= 0 \\
\mathcal{R}_0^1 &= \frac{2c_1}{3}\sqrt{t}v + \frac{c_1}{3}\sqrt{t}v_x D^{-1} \\
\mathcal{R}_1^1 &= tD^2 + \frac{1}{3}x + \frac{2c_1}{3}\sqrt{t}u + \frac{1}{6}(2c_1\sqrt{t}u_x + 1)D^{-1}.
\end{aligned} \tag{27}$$

Example 4. For $N = 3$.
(i) The non-autonomous irreducible JKdV system is

$$\begin{aligned}
u_t &= u_{xxx} - \frac{c_0}{\sqrt{t}}(u^2 - v^2 - w^2)_x \\
v_t &= v_{xxx} - \frac{c_0}{\sqrt{t}}(uv)_x \\
w_t &= w_{xxx} - \frac{c_0}{\sqrt{t}}(uw)_x.
\end{aligned} \tag{28}$$

(ii) The non-autonomous reducible JKdV system

$$\begin{aligned}
u_t &= u_{xxx} - \frac{2c_0}{\sqrt{t}}uu_x \\
v_t &= v_{xxx} - \frac{c_0}{\sqrt{t}}(uv)_x \\
w_t &= w_{xxx} - \frac{c_0}{\sqrt{t}}(uw)_x
\end{aligned} \tag{29}$$

is the extension of (9). The recursion operators for systems (28) and (29) are too long, hence we do not give them here.

Finally, we establish linear transformations between autonomous and non-autonomous systems. In the scalar case, the KdV and cKdV equations are equivalent since their solutions are related by simple Lie-point transformation [13–17].

$$u(x,t) = t^{-1/2}u'(xt^{-1/2}, -2t^{-1/2}) - \tfrac{1}{12}xt^{-1/2}. \tag{30}$$

Here we present a generalization of this result to the case of systems of evolution equations.

Definition 2. *Two systems of equations*

$$\begin{aligned} u^i_t &= u^i_{xxx} + f(x,t,u^i,u^i_x) \\ u'^i_\sigma &= u'^i_{\xi\xi\xi} + g(\xi,\sigma,u'^i,u'^i_\xi) \end{aligned} \tag{31}$$

will be called equivalent if there exists a change of variables of the form

$$\begin{aligned} &\xi = \alpha(t)x + \beta(t) \qquad \sigma = \gamma(t) \\ &u^i(x,t) = \Gamma(t)u'^i(\xi(x,t),\sigma(x,t)) + \eta(x,t) \end{aligned} \tag{32}$$

which is invertible. The first result is given in the following statement.

Proposition 1. *The system*

$$\begin{aligned} u_t &= u_{xxx} + \frac{c_0}{\sqrt{t}}uu_x \\ v_t &= v_{xxx} + \frac{c_1}{\sqrt{t}}(uv)_x \end{aligned} \tag{33}$$

where c_0 and c_1 arbitrary constants, may be transformed into the autonomous perturbation of the KdV system

$$\begin{aligned} u'_\sigma &= u'_{\xi\xi\xi} + c_0u'u'_\xi \\ v'_\sigma &= v'_{\xi\xi\xi} + c_1(u'v')_\xi \end{aligned} \tag{34}$$

through a transformation of the form (32) if and only if $c_0 = c_1$.

The validity of this proposition allows us to state the following proposition.

Proposition 2. *The non-autonomous JKdV system (26) is transformed into the autonomous JKdV system (10) through the transformation of the form*

$$\begin{aligned} u(x,t) &= t^{-1/2}u'(xt^{-1/2}, -2t^{-1/2}) - \frac{1}{2c_1}xt^{-1/2} \\ v(x,t) &= t^{-1/2}v'(xt^{-1/2}, -2t^{-1/2}). \end{aligned} \tag{35}$$

Similar to propositions 1 and 2 we have the following statement.

Proposition 3. *The non-autonomous JKdV system (28) is transformed into the autonomous JKdV system (12) through the transformation*

$$\begin{aligned} u(x,t) &= t^{-1/2}u'(xt^{-1/2}, -2t^{-1/2}) + \frac{1}{4c_0}xt^{-1/2} \\ v(x,t) &= t^{-1/2}v'(xt^{-1/2}, -2t^{-1/2}) \\ w(x,t) &= t^{-1/2}w'(xt^{-1/2}, -2t^{-1/2}). \end{aligned} \tag{36}$$

From the above discussions we have the following result.

Proposition 4. *The non-autonomous JKdV system (23) (or its extension (29)) cannot be transformed into the JKdV system (9) (or its extension (13)) through a transformation of the form (32).*

We have observed that for some special cases of $N = 2$ and 3 time-dependent systems transform to time-independent cases. This comes indeed from the type of the Jordan algebra. For general N we have the following statement.

Proposition 5. *A Jordan system (14) is equivalent to an autonomous Jordan system (1) if there exists an element* $\boldsymbol{a}$ *of* J *such that* $\boldsymbol{a}^2 = \boldsymbol{a}$ *and* $q \circ \boldsymbol{a} = q$ *for all* $q \in J$.

Proof. We write the system of equations (14) in the form $q_t = q_{xxx} + \frac{1}{\sqrt{t}}\, q \circ q_x$, where q takes values in a Jordan algebra J. Take the point transformation

$$\begin{aligned} &q(x,t) = t^{-1/2}\, v(\xi,\tau) - \tfrac{1}{2}\, x t^{-1/2}\, \boldsymbol{a} \\ &\xi = x t^{-1/2} \qquad \tau = -2t^{-1/2}. \end{aligned} \tag{37}$$

Then equations for v become time independent. □

The transformable case in the $N = 2$ (example (2.ii)) is the case with $\boldsymbol{a} = e_1$ where $\{e_i, i = 1, 2\}$ are a basis of J. The example (4.i) in the $N = 3$ case is also transformable because the element $\boldsymbol{a} = -\frac{1}{2c_0}\, e_1$ satisfies the condition $\boldsymbol{a}^2 = \boldsymbol{a}$.

We would like to remark on the symmetries of (14). The first symmetry is the x-translational symmetry $\sigma_1^i = q_x^i$. The next one is the scale symmetry $\sigma_2^i = t\, q_t^i + \frac{1}{3}\, x q_x^i + \frac{1}{6}\, q^i$. The first generalized symmetry is given by $\sigma_3^i = \mathcal{R}^i_j\, \sigma_2^j$, where $\mathcal{R}$ is the recursion operator (17) of the system (14). This symmetry is nonlocal and contains the associator (tensor F^i_{jkl}) of the algebra J. There exists also an additional symmetry, the Galilean symmetry, $\eta_1^i = \sqrt{t}\, s^i_{jk}\, q_x^k\, \zeta^j + \frac{1}{2}\zeta^i$ for system (14) satisfying $s^i_{jk}\, \zeta^j = \delta^i_k$. Here we remark also that the element $\boldsymbol{k} = \zeta^i\, e_i$ of J satisfies $\boldsymbol{k}^2 = \boldsymbol{k}$. Hence, due to proposition 5 the corresponding systems are transformable to autonomous KdV systems (1). In the general case, $F \neq 0$, $\sigma^i = \Lambda^i_j\, \zeta^j$ is a symmetry of the non-autonomous JKdV system (14) for all $\boldsymbol{k}$, where

$$\Lambda^i_j = \tfrac{1}{3}\, \sqrt{t}\, s^i_{jk}\, q_x^k + \tfrac{1}{6}\, \delta^i_j + \tfrac{1}{9}\, F^i_{lkj}\, q^l D^{-1}\, q^k. \tag{38}$$

In the case of time-dependent recursion operators (and time-dependent evolution equations) there is an ambiguity in calculating the higher-order time-dependent symmetries. It is claimed that the recursion operators do not, in general, map symmetries to symmetries [18]. This violates the most important property of the recursion operator. We observed that not the recursion operator but the standard determination of the symmetries must be modified [19]. The time-dependent symmetries of (14) can be obtained from the following equations:

$$\sigma^i_{n+1} = \bar{\sigma}^i_{n+1} + \Lambda^i_j \int^t \mathrm{d}t'\, \Pi D^2 \sigma^j_n \qquad i, j = 1, 2, \ldots, N \tag{39}$$

where $\bar{\sigma}^i_{n+1}$ are the symmetries generated by the standard application of the operator D^{-1}. (i.e. $DD^{-1} = D^{-1}D = 1$) and Π is the projection operator defined in [18] by $\Pi\, f(t, x, q^i, q^i_x, \ldots) = f(t, 0, 0, 0, \ldots)$ where f is an arbitrary function.

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