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ÜLKÜ GÜRLER & BANU YÜKSEL ÖZKAYA

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TECHNICAL NOTE

A note on "Continuous review perishable inventory systems: models and heuristics"

ÜLKÜ GÜRLER and BANU YÜKSEL ÖZKAYA

Department of Industrial Engineering, Bilkent University, 06533, Ankara, Turkey
E-mail: ulku@bilkent.edu.tr or ybanu@bilkent.edu.tr

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In a recent paper, Lian and Liu (2001) consider a continuous review perishable inventory model with renewal arrivals, batch demands and zero lead times. However, the main analytical result they provide holds only for some special cases such as Poisson arrivals with exponential interarrival times. In this note we generalize Theorem 1 of Lian and Liu (2001) for the case where the arrivals follow an arbitrary renewal process.

1. Preliminaries

In a recent paper, Lian and Liu (2001) consider a continuous review inventory model with perishable items and renewal batch demands. Using embedded Markov chain methods, they provide an approach for the solution of the problem under consideration. However as detailed below, some expressions they provide for the expected sojourn times are valid only for special cases such as Poisson arrivals. The aim of this note is to provide a modified expression for the expected sojourn times when the interarrival times follow a general distribution.

In Lian and Liu (2001), the interdemand times are assumed to have a general distribution with distribution function G and mean μ^{-1}. The n-fold convolution of G with itself is denoted by $G^{(n)}$. The batch size of the nth demand, Y_n, has an arbitrary distribution with probability mass function $b(i) = P(Y_n = i)$, cumulative distribution function $B(i)$ and the n-fold convolution with itself is given by $B^{(n)}$. The items are perishable with a constant lifetime of T time units.

Under the (s, S) replenishment policy, where $s \leq -1$, the inventory level process $\{I(t), t > 0\}$ is analyzed using an embedded Markov chain approach and Laplace transforms. Letting $J = \{s+1, s+2, \ldots, -1, S\}$, X_n is defined as the state that $I(t)$ enters when it makes the nth transition in J and Z_n is the corresponding transition epoch. The time period between two consecutive reorder points is called a *reorder cycle*. The random reorder cycle is denoted by τ and τ_i refers to the time that the inventory level changes from state i to state S. Defining an embedded Markov chain with transition probabilities $Q_{ij}(t) = P\{X_1 = j, Z_1 \leq t | X_0 = i\}$, limiting probabilities $Q_{ij} = \lim_{t\to+\infty} Q_{ij} = P\{X_1 = j | X_0 = i\}$, and using Laplace transforms, Lian and Liu (2001) provide recursive relations from which the expected cycle length can be obtained.

In order to calculate the expected cycle cost, $v_j, j \in J$ is defined as the probability that in the steady-state the inventory level process visits state j in a cycle, given as:

$$v_j = Q_{sj} + \sum_{i=j+1}^{0} v_i Q_{ij}, \quad j = -2, \ldots, s+1. \tag{1}$$

Also, for $i = 0, \ldots, S-1$, the following quantities are defined

$$U_i(t) = P(I(t) = i, Z_1 > t | X_0 = S), \tag{2}$$

$$U_i = \int_0^\infty U_i(t) \mathrm{d}t, \tag{3}$$

where U_i is the conditional expected sojourn time at inventory level i when the process starts from state S. Similarly V_i is defined as the expected sojourn time in state i in one replenishment cycle. Note that for $i = S, S-1, \ldots, 1$, $V_i = v_s U_i$ and for $i = -1, \ldots, s+1$, $V_i = v_i U_i$. Lian and Liu (2001) then present Theorem 1 to provide expressions for V_i, from which the expected total inventory holding cost in a reorder cycle is obtained.

We would like to point out here that the expressions for V_i, given in Equations (27) and (28) of Theorem 1 in Lian and Liu are not valid for general G, they hold only for special cases such as Poisson arrivals with exponential interdemand times.

2. Generalization to renewal demand intervals

We provide below the corrected version of Theorem 1 of Lian and Liu when the interdemand times have a general distribution and present an illustrative example. We should also note here that the rest of the analysis provided by Lian and Liu is valid with the modified expressions for V_i given below. Retaining the notation and the approach of Lian and Liu (2001), we have:

Theorem 1. *Let V_i be the expected sojourn time in state i in one replenishment cycle. Then for $S > 0$,*

$$V_S = \int_0^T [1 - G(t)]\mathrm{d}t, \tag{27*}$$

$$V_i = \sum_{j=1}^{S-i} b^{(j)}(S-i) \int_0^T [G^{(j)}(t) - G^{(j+1)}(t)]\mathrm{d}t, \quad i = 1, \ldots, S-1 \tag{28*}$$

$$V_i = \mu^{-1} v_i, \quad i = -1, -2, \ldots, s+1, \tag{29}$$

$$V_0 = E\tau_S - \sum_{i=1}^{S} V_i - \mu^{-1} \sum_{j=s+1}^{-1} v_j, \tag{30}$$

where v_i are as given in (1). *When $S = 0$, only* (29) *is needed and $V_0 = \mu^{-1}$.*

Proof. Only the proofs of ($27^*, 28^*$) will be given. We have

$$U_S(t) = P(I(t) = S, Z_1 > t | X_0 = S) = [1 - G(t)]\chi(t < T),$$

where $\chi(\cdot)$ is the indicator function of its argument. Then

$$U_S = \int_0^\infty U_S(t)\mathrm{d}t = \int_0^T [1 - G(t)](t)\mathrm{d}t,$$

which proves (27^*). When $i = 1, \ldots, S-1$,

$$U_i(t) = P(I(t) = i, Z_1 > t | X_0 = S),$$
$$= \sum_{j=1}^{S-i} P\left\{ \sum_{l=1}^{j} D_l \le t < \sum_{l=1}^{j+1} D_l, \sum_{l=1}^{j} Y_l = S - i \right\} \chi(t < T),$$
$$= \sum_{j=1}^{S-i} b^{(j)}(S-i)[G^{(j)}(t) - G^{(j+1)}(t)]\chi(t < T).$$

Hence

$$U_i = \sum_{j=1}^{S-i} b^{(j)}(S-i) \int_0^T [G^{(j)}(t) - G^{(j+1)}(t)]\mathrm{d}t$$

as given in (28^*) above. ■

We observe from the result above that explicit expressions for ($27^*, 28^*$) can only be obtained for special cases such as gamma distribution and in other cases numerical integration methods should be employed.

Example. Consider the case where the interarrival times have a gamma distribution with scale parameter λ, shape parameter 2, and mean $1/\mu = 2/\lambda$. Then $g(x) = \lambda^2 x\mathrm{e}^{-\lambda x}$ and $G(x) = 1 - \mathrm{e}^{-\lambda x} - \lambda x\mathrm{e}^{-\lambda x}$. Referring to (2) above, we write

$$U_S(t) = P(I(t) = S, Z_1 > t | X_0 = S),$$
$$= [1 - G(t)]\chi(t < T) = [\mathrm{e}^{-\lambda t} + \lambda t\mathrm{e}^{-\lambda t}]\chi(t < T).$$

Using (3), we obtain

$$U_S = \int_0^\infty U_S(t)\mathrm{d}t,$$
$$= \frac{2}{\lambda}\left[1 - \mathrm{e}^{-\lambda T} - \frac{\lambda}{2} T\mathrm{e}^{-\lambda T}\right] \ne \frac{1}{\mu} G(T).$$

Since for $i = S, V_i = U_i$, (27) of Lian and Liu for V_S is not valid when G is a 2-Erlang distribution. Now, considering V_i, for $i \ne S$, we note that the n-fold convolution of G is a 2n-Erlang distribution with scale parameter λ and $G^{(n)}(x)$ can be written as

$$G^{(n)}(x) = 1 - \sum_{k=0}^{2n-1} \frac{\mathrm{e}^{-\lambda x}(\lambda x)^k}{k!}.$$

In view of (28^*), V_i with $i = 1, 2, \ldots, S-1$ can be written as

$$V_i = \sum_{j=1}^{S-i} b^{(j)}(S-i) \int_{t=0}^{T} \left[G^{(j)}(t) - G^{(j+1)}(t) \right] \mathrm{d}t,$$
$$= \sum_{j=1}^{S-i} b^{(j)}(S-i) \int_{t=0}^{T} \left[\frac{\mathrm{e}^{-\lambda t}(\lambda t)^{2j}}{(2j)!} + \frac{\mathrm{e}^{-\lambda t}(\lambda t)^{2j+1}}{(2j+1)!} \right] \mathrm{d}t. \tag{4}$$

Using integration by parts for the second term in the integral in (4), we have

$$V_i = \sum_{j=1}^{S-i} b^{(j)}(S-i) \left\{ \int_{t=0}^{T} \frac{\mathrm{e}^{-\lambda t}(\lambda t)^{2j}}{(2j)!}\mathrm{d}t - \frac{t(\lambda t)^{2j}\mathrm{e}^{-\lambda t}}{(2j+1)!}\Big|_0^T + \int_0^T \frac{\mathrm{e}^{-\lambda t}(\lambda t)^{2j}}{(2j)!}\mathrm{d}t \right\},$$
$$= \sum_{j=1}^{S-i} b^{(j)}(S-i) \left\{ \frac{1}{\mu}\left[1 - \sum_{k=0}^{2j} \frac{\mathrm{e}^{-\lambda T}(\lambda T)^k}{k!}\right] - \frac{T(\lambda T)^{2j}\mathrm{e}^{-\lambda T}}{(2j+1)!} \right\},$$
$$= \sum_{j=1}^{S-i} b^{(j)}(S-i) \left\{ \frac{1}{\mu} F^{(2j+1)}(T) - \frac{T(\lambda T)^{2j}\mathrm{e}^{-\lambda T}}{(2j+1)!} \right\},$$

where F is the distribution function of an exponential random variable and $F^{(k)}$ is k-fold convolution of it.

Reference

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Biographies

Ülkü Gürler is an Associate Professor in the Industrial Engineering Department of Bilkent University, Ankara, Turkey. She has Ph.D. and M.A. degrees in Statistics, from the Wharton School of University of Pennsylvania and a B.S. degree in Statistics from the Middle East Technical University, Ankara, Turkey. She has research interests in stochastic inventory and maintenance models, Bayesian applications in inventory and maintenance problems, and nonparametric estimation with censored and truncated data.

Banu Yüksel Özkaya has B.S. and M.Sc degrees from the Industrial Engineering Department of Bilkent University. Ankara, Turkey. She is currently a Ph.D. student in the same department. Her research focuses on the analysis of perishable inventory models and multi-echelon inventory systems.

Contributed by the Engineering Statistics and Applied Probability Department