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2-Killing vector fields on warped product manifolds

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This paper provides a study of 2-Killing vector fields on warped product manifolds as well as characterization of this structure on standard static and generalized Robertson–Walker space-times. Some conditions for a 2-Killing vector field on a warped product manifold to be parallel are obtained. Moreover, some results on the curvature of a warped product manifolds in terms of 2-Killing vector fields are derived. Finally, we apply our results to describe 2-Killing vector fields of some well-known warped product space-time models.

Keywords: Warped product manifold; 2-Killing vector field; parallel vector fields; standard static space-time and generalized Robertson–Walker space-time.

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1. Introduction

Killing vector fields have well-known geometrical and physical interpretations and have been studied on Riemannian and pseudo-Riemannian manifolds for a long time. The number of independent Killing vector fields measures the degree of symmetry of a Riemannian manifold. Thus, the problems of existence and characterization of Killing vector fields are important and are widely discussed by both mathematicians and physicists [4, 5, 7, 8, 16, 17, 20, 26, 31, 32].

Generalization of Killing vector fields has a long history in mathematics for different scales and purposes [10–12, 21]. In [28], the concept of 2-Killing vector

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fields, as a new generalization of Killing vector fields, was first introduced and studied on Riemannian manifolds. The relations between 2-Killing vector fields, curvature and monotone vector fields are obtained. Finally, a characterization of 2-Killing vector field on $\mathbb{R}^n$ is derived.

At this point, we want to emphasize that the concept of monotone vector fields introduced by Németh (see [19, 22–24]) and since then they have been studied as a research topic in the area of (nonlinear) analysis on Riemannian manifolds (see also [6] as an additional reference to the above list). As noted above, the connections between monotone vector fields and 2-Killing vector fields have been established. In addition to that, by using space-like hypersurfaces of a Lorentzian manifold (see [25]), these topics have been received attention in Lorentzian geometry as well. Thus the notion of 2-Killing vector fields is important in different branches of mathematics from (nonlinear) analysis on Riemannian manifolds to Lorentzian geometry.

As far as we know, the concept of 2-Killing vector fields has been studied neither on warped products nor on space-time models up to this paper in which we intent to fill this gap in the literature by providing a complete study of 2-Killing vector fields on such spaces. In this way, all the results related to 2-Killing vector fields and thus monotone vector fields can be easily extended to a special class of manifolds, namely, warped product manifolds.

We organize the paper as follows. In Sec. 2, we state well-known connection related formulas of warped product manifolds and Killing vector fields. Thus some of the proofs are omitted. In Sec. 3, as the core of the paper, the relation between 2-Killing vector fields on a warped product manifold and 2-Killing vector fields on the fiber and base manifolds is discussed. Here, we now state our main results; the following theorem represents an important and helpful identity.

Theorem 3.1. *Let $\zeta = (\zeta_1, \zeta_2) \in \mathfrak{X}(M_1 \times_f M_2)$ be a vector field on a warped product manifold of the form $M_1 \times_f M_2$. Then*

$$\begin{aligned}(\mathcal{L}_\zeta \mathcal{L}_\zeta g)(X,Y) &= (\mathcal{L}^1_{\zeta_1}\mathcal{L}^1_{\zeta_1} g_1)(X_1,Y_1) + f^2(\mathcal{L}^2_{\zeta_2}\mathcal{L}^2_{\zeta_2} g_2)(X_2,Y_2)\\ &\quad + 4f\zeta_1(f)(\mathcal{L}^2_{\zeta_2} g_2)(X_2,Y_2) + 2f\zeta_1(\zeta_1(f))g_2(X_2,Y_2)\\ &\quad + 2\zeta_1(f)\zeta_1(f)g_2(X_2,Y_2)\end{aligned}$$

for any vector fields $X, Y \in \mathfrak{X}(M_1 \times_f M_2)$.

The proof of this result contains long computations that have been done using previous results on warped product manifolds (see Appendix A). As an immediate consequence, the relation between 2-Killing vector fields on a warped product manifold and those on product factors is given.

Some conditions for a 2-Killing vector field to be parallel vector field are considered in the following theorem.

Theorem 3.2. *Let $\zeta \in \mathfrak{X}(M_1 \times_f M_2)$ be a vector field on a warped product manifold of the form $M_1 \times_f M_2$. Then*

(1) *$\zeta = \zeta_1 + \zeta_2$ is parallel if ζ_i is a 2-Killing vector field,* $\mathrm{Ric}^i(\zeta_i, \zeta_i) \leq 0, i = 1, 2$ *and f is constant.*
(2) *$\zeta = \zeta_1$ is parallel if ζ_1 is a 2-Killing vector field,* $\mathrm{Ric}^1(\zeta_1, \zeta_1) \leq 0$, *and $\zeta_1(f) = 0$.*
(3) *$\zeta = \zeta_2$ is parallel if ζ_2 is a 2-Killing vector field,* $\mathrm{Ric}^2(\zeta_2, \zeta_2) \leq 0$, *and f is constant.*

The preceding theorem also provides some results on the curvature of a warped product manifold in terms of 2-Killing vector fields.

Theorem 3.3. *Suppose that $\zeta \in \mathfrak{X}(M_1 \times_f M_2)$ is a nontrivial 2-Killing vector field. If $D_\zeta \zeta$ is parallel along a curve γ, then*

$$K(\zeta, \dot{\gamma}) \geq 0.$$

Finally, in Sec. 4, we apply these results on standard static space-times and generalized Robertson–Walker space-times. For instance, the following result is obtained.

Theorem 4.1. *Let $\bar{M} = I_f \times M$ be a standard static space-time with the metric $\bar{g} = -f^2 \mathrm{d}t^2 \oplus g$. Suppose that $u : I \to \mathbb{R}$ is smooth and ζ is a vector field on F. Then $\bar{\zeta} = u\partial_t + \zeta$ is a 2-Killing vector field on $\bar{M}$ if one of the following conditions is satisfied*:

(1) *ζ is 2-Killing on $M, u = a$ and $f\zeta(f) = b$ where $a, b \in \mathbb{R}$.*
(2) *ζ is 2-Killing on $M, u = (rt + s)^{\frac{1}{3}}$ and $\zeta(f) = 0$ where $r, s \in \mathbb{R}$.*

Furthermore, the converse of this result and many others on generalized Robertson–Walker space-times are discussed.

2. Preliminaries

In this section, we will provide basic definitions and curvature formulas about warped product manifolds and Killing vector fields.

Suppose that (M_1, g_1, D_1) and (M_2, g_2, D_2) are two $\mathcal{C}^\infty$ pseudo-Riemannian manifolds equipped with Riemannian metrics g_i where D_i is the Levi-Civita connection of the metric g_i for $i = 1, 2$. Further suppose that $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$ are the natural projection maps of the Cartesian product $M_1 \times M_2$ onto M_1 and M_2, respectively. If $f : M_1 \to (0, \infty)$ is a positive real-valued smooth function, then the warped product manifold $M_1 \times_f M_2$ is the product manifold $M_1 \times M_2$ equipped with the metric tensor $g = g_1 \oplus g_2$ defined by

$$g = \pi_1^*(g_1) \oplus (f \circ \pi_1)^2 \pi_2^*(g_2),$$

where * denotes the pull-back operator on tensors [9, 27]. The function f is called the warping function of the warped product manifold $M_1 \times_f M_2$. In particular, if

$f = 1$, then $M_1 \times_1 M_2 = M_1 \times M_2$ is the usual Cartesian product manifold. It is clear that the submanifold $M_1 \times \{q\}$ is isometric to M_1 for every $q \in M_2$. Moreover, $\{p\} \times M_2$ is homothetic to M_2. Throughout this paper we use the same notation for a vector field and for its lift to the product manifold.

Let D be the Levi-Civita connection of the metric tensor g. The following proposition is well known [9].

Proposition 2.1. *Let $(M_1 \times_f M_2, g)$ be a Riemannian warped product manifold with warping function $f > 0$ on M_1. Then*

(1) $D_{X_1} Y = D^1_{X_1} Y_1 \in \mathfrak{X}(M_1)$,
(2) $D_{X_1} Y_2 = D_{Y_2} X_1 = \frac{X_1(f)}{f} Y_2$,
(3) $D_{X_2} Y_2 = -f g_2(X_2, Y_2) \nabla^1 f + D^2_{X_2} Y_2$

for all $X_i, Y_i \in \mathfrak{X}(M_i)$, with $i = 1, 2$ where $\nabla^1 f$ is the gradient of f.

A vector field $\zeta \in \mathfrak{X}(M)$ on a pseudo-Riemannian manifold (M, g) with metric g is called a Killing vector field if

$$\mathcal{L}_\zeta g = 0,$$

where $\mathcal{L}_\zeta$ is the Lie derivative on M with respect to ζ. One can redefine Killing vector fields using the following identity. Let ζ be a vector field, then

$$(\mathcal{L}_\zeta g)(X, Y) = g(D_X \zeta, Y) + g(X, D_Y \zeta) \tag{2.1}$$

for any vector fields $X, Y \in \mathfrak{X}(M)$. A simple yet useful characterization of Killing vector fields is given in the following proposition. The proof is straightforward by using the symmetry in the above identity.

Lemma 2.1. *If (M, g, D) is a pseudo-Riemannian manifold with Riemannian connection D. A vector field $\zeta \in \mathfrak{X}(M)$ is a Killing vector field if and only if*

$$g(D_X \zeta, X) = 0 \tag{2.2}$$

for any vector field $X \in \mathfrak{X}(M)$.

Now we consider Killing vector fields on Riemannian warped product manifolds. The following simple result will help us to present a characterization of Killing vector fields on warped product manifolds.

Lemma 2.2. *Let $\zeta \in \mathfrak{X}(M_1 \times_f M_2)$ be a vector field on the pseudo-Riemannian warped product manifold $M_1 \times_f M_2$ with warping function f. Then for any vector field $X \in \mathfrak{X}(M_1 \times_f M_2)$ we have*

$$g(D_X \zeta, X) = g_1(D^1_{X_1} \zeta_1, X_1) + f^2 g_2(D^2_{X_2} \zeta_2, X_2) + f \zeta_1(f) \|X_2\|_2^2. \tag{2.3}$$

Proof. Using Proposition 2.1, we get

$$\begin{aligned} g(D_X\zeta, X) &= g_1(D^1_{X_1}\zeta_1 - fg_2(X_2,\zeta_2)\nabla f, X_1) + f^2 g_2(D^2_{X_2}\zeta_2 + \zeta_1(\ln f)X_2 \\ &\quad + X_1(\ln f)\zeta_2, X_2) \\ &= g_1(D^1_{X_1}\zeta_1, X_1) - fg_2(X_2,\zeta_2)X_1(f) + f^2 g_2(D^2_{X_2}\zeta_2, X_2) \\ &\quad + f\zeta_1(f)g_2(X_2,X_2) + fX_1(f)g_2(\zeta_2,X_2) \\ &= g_1(D^1_{X_1}\zeta_1, X_1) + f^2 g_2(D^2_{X_2}\zeta_2, X_2) + f\zeta_1(f)\|X_2\|_2^2. \end{aligned}$$

□

The preceding two theorems give us a characterization of Killing vector fields on warped product manifolds. They are immediate consequence of the previous result.

Theorem 2.1. *Let $\zeta = (\zeta_1,\zeta_2) \in \mathfrak{X}(M_1 \times_f M_2)$ be a vector field on the pseudo-Riemannian warped product manifold $M_1 \times_f M_2$ with warping function f. Then ζ is a Killing vector field if one of the following conditions holds*:

(1) $\zeta = (\zeta_1, 0)$ *and ζ_1 is a killing vector field on M_1.*
(2) $\zeta = (0, \zeta_2)$ *and ζ_2 is a killing vector field on M_2.*
(3) ζ_i *is a Killing vector field on M_i, for $i = 1, 2$ and $\zeta_1(f) = 0$.*

The converse of the above result is considered in the following result.

Theorem 2.2. *Let $\zeta = (\zeta_1,\zeta_2) \in \mathfrak{X}(M_1 \times_f M_2)$ be a killing vector field on the warped product manifold $M_1 \times_f M_2$ with warping function f. Then*

(1) ζ_1 *is a Killing vector field on M_1.*
(2) ζ_2 *is a Killing vector field on M_2 if $\zeta_1(f) = 0$.*

In [16], the authors proved similar results on standard static space-times using the following proposition.

Proposition 2.2. *Let $\zeta = (\zeta_1,\zeta_2) \in \mathfrak{X}(M_1 \times_f M_2)$ be a vector field on the warped product manifold $M_1 \times_f M_2$ with warping function f. Then*

$$(\mathcal{L}_\zeta g)(X,Y) = (\mathcal{L}^1_{\zeta_1} g_1)(X_1,Y_1) + f^2(\mathcal{L}^2_{\zeta_2} g_2)(X_2,Y_2) + 2f\zeta_1(f)g_2(X_2,Y_2), \tag{2.4}$$

where $\mathcal{L}^i_{\zeta_i}$ is the Lie derivative on M_i with respect to ζ_i, for $i = 1, 2$.

3. 2-Killing Vector Fields

In this section after we define and state fundamental results about 2-Killing vector fields, we obtain the main results of the paper.

A vector field $\zeta \in \mathfrak{X}(M)$ is called a 2-Killing vector field on a pseudo-Riemannian manifold (M, g) if

$$\mathcal{L}_\zeta \mathcal{L}_\zeta g = 0, \tag{3.1}$$

where $\mathcal{L}_\zeta$ is the Lie derivative in the direction of ζ on M [28].

The following two results [28] are needed to exploit the above definition.

Proposition 3.1. *Let $\zeta \in \mathfrak{X}(M)$ be a vector field on a pseudo-Riemannian manifold M. Then*

$$(\mathcal{L}_\zeta \mathcal{L}_\zeta g)(X,Y) = g(D_\zeta D_X \zeta - D_{[\zeta,X]}\zeta, Y) + g(X, D_\zeta D_Y \zeta - D_{[\zeta,Y]}\zeta) + 2g(D_X\zeta, D_Y\zeta) \tag{3.2}$$

for any vector fields $X, Y \in \mathfrak{X}(M)$.

The following result is quite direct and helpful.

Corollary 3.1. *A vector field ζ is 2-Killing if and only if*

$$R(\zeta, X, \zeta, X) = g(D_X\zeta, D_X\zeta) + g(D_X D_\zeta \zeta, X) \tag{3.3}$$

for any vector field $X \in \mathfrak{X}(M)$.

The symmetry of Eq. (3.2) shows that ζ is 2-Killing if and only if

$$g(D_\zeta D_X \zeta - D_{[\zeta,X]}\zeta, X) + g(D_X\zeta, D_X\zeta) = 0.$$

Example 3.1. Let M be the two-dimensional Euclidean space, i.e. $(\mathbb{R}^2, \mathrm{d}s^2)$ where $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$. A vector field $\zeta = u\partial_x + v\partial_y \in \mathfrak{X}(M)$ is 2-Killing if

$$(\mathcal{L}^I_\zeta \mathcal{L}^I_\zeta g_I)(X,Y) = 0$$

for any vector fields X, Y, where $\mathcal{L}_\zeta$ is the Lie derivative on $\mathbb{R}^2$ with respect to ζ. Now it is easy to show that ζ is 2-Killing vector field on M if and only if

$$uu_{xx} + 2u_x^2 = 0,$$
$$vv_{yy} + 2v_y^2 = 0.$$

By making use of the above proposition one can get sufficient and necessary conditions for a vector field $\zeta = (\zeta_1, \zeta_2) \in \mathfrak{X}(M_1 \times_f M_2)$ to be a 2-Killing on the pseudo-Riemannian warped product manifold $M_1 \times_f M_2$. The following theorem represents a similar such.

Theorem 3.1. *Let $\zeta = (\zeta_1, \zeta_2) \in \mathfrak{X}(M_1 \times_f M_2)$ be a vector field on the warped product manifold $M_1 \times_f M_2$. Then*

$$\begin{aligned}(\mathcal{L}_\zeta \mathcal{L}_\zeta g)(X,Y) &= (\mathcal{L}^1_{\zeta_1}\mathcal{L}^1_{\zeta_1} g_1)(X_1,Y_1) + f^2(\mathcal{L}^2_{\zeta_2}\mathcal{L}^2_{\zeta_2} g_2)(X_2,Y_2)\\ &\quad + 4f\zeta_1(f)(\mathcal{L}^2_{\zeta_2} g_2)(X_2,Y_2) + 2f\zeta_1(\zeta_1(f))g_2(X_2,Y_2)\\ &\quad + 2\zeta_1(f)\zeta_1(f)g_2(X_2,Y_2)\end{aligned}$$

for any vector fields $X, Y \in \mathfrak{X}(M_1 \times_f M_2)$.

Proof. See Appendix A. □

The following results are direct consequences of the above theorem.

Corollary 3.2. *Let $\zeta = \zeta_1 + \zeta_2 \in \mathfrak{X}(M_1 \times_f M_2)$ be a vector field on the warped product manifold of the form $M_1 \times_f M_2$. If $\zeta_1 + \zeta_2$ is a 2-Killing vector field on $M_1 \times_f M_2$, then ζ_1 is a 2-Killing vector field on M_1.*

Corollary 3.3. *Let $\zeta \in \mathfrak{X}(M_1 \times_f M_2)$ be a vector field on the warped product manifold of the form $M_1 \times_f M_2$. Suppose that ζ_1 and ζ_2 are 2-Killing vector fields on M_1 and M_2, respectively. Then $\zeta_1 + \zeta_2$ is a 2-Killing vector field on $M_1 \times_f M_2$ if and only if*

(1) $\zeta_1(f) = 0$, *or*
(2) ζ_2 *is a homothetic vector field on M_2 with homothetic factor c (i.e. $\mathfrak{L}^2_{\zeta_2} g_2 = cg_2$) such that*

$$f\zeta_1(\zeta_1(f)) + \zeta_1(f)\zeta_1(f) = -2cf\zeta_1(f).$$

Corollary 3.4. *Let $\zeta = \zeta_1 + \zeta_2 \in \mathfrak{X}(M_1 \times_f M_2)$ be a vector field on the warped product manifold $M_1 \times_f M_2$. Then ζ is a 2-Killing vector field on $M_1 \times_f M_2$ if one of the following conditions holds:*

(1) *The vector field ζ_i is a 2-Killing vector field on $M_i, i = 1, 2$, and $\zeta_1(f) = 0$.*
(2) *$\zeta = \zeta_2$ and ζ_2 is a 2-Killing vector field on M_2.*

Theorem 3.2. *Let $\zeta \in \mathfrak{X}(M_1 \times_f M_2)$ be a vector field on the warped product manifold $M_1 \times_f M_2$. Then*

(1) *$\zeta = \zeta_1 + \zeta_2$ is parallel if ζ_i is a 2-Killing vector field, and $\mathrm{Ric}^i(\zeta_i, \zeta_i) \le 0, i = 1, 2$ and also f is constant.*
(2) *$\zeta = \zeta_1$ is parallel if ζ_1 is a 2-Killing vector field, and $\mathrm{Ric}^1(\zeta_1, \zeta_1) \le 0$, and also $\zeta_1(f) = 0$.*
(3) *$\zeta = \zeta_2$ is parallel if ζ_2 is a 2-Killing vector field, and $\mathrm{Ric}^2(\zeta_2, \zeta_2) \le 0$, and also f is constant.*

Proof. Suppose that

$$\{e_1, e_2, \ldots, e_m\}$$

is an orthonormal frame in T_pM_1 and

$$\{e_{m+1}, e_{m+2}, \ldots, e_{m+n}\}$$

is an orthonormal frame in T_qM_2 for some point $(p, q) \in M_1 \times M_2$. Then

$$\{\overline{e}_1, \overline{e}_2, \ldots, \overline{e}_{m+n}\}$$

is an orthonormal frame in $T_{(p.q)}(M_1 \times M_2)$ where

$$\overline{e}_i = \begin{cases} e_i, & 1 \le i \le m, \\ \dfrac{1}{f} e_i, & m+1 \le i \le m+n. \end{cases}$$

Thus for any vector field $\zeta \in \mathfrak{X}(M_1 \times_f M_2)$ we have

$$\begin{aligned}\mathrm{Tr}(g(D\zeta, D\zeta)) &= \sum_{i=1}^{m+n} g(D_{\overline{e}_i}\zeta, D_{\overline{e}_i}\zeta) \\ &= \sum_{i=1}^{m} g(D_{e_i}\zeta, D_{e_i}\zeta) + \frac{1}{f^2}\sum_{i=m+1}^{m+n} g(D_{e_i}\zeta, D_{e_i}\zeta). \end{aligned} \tag{3.4}$$

Using Proposition 2.1, the first term is given by

$$\begin{aligned}\sum_{i=1}^{m} g(D_{e_i}\zeta, D_{e_i}\zeta) &= \sum_{i=1}^{m} g(D^1_{e_i}\zeta_1 + e_i(\ln f)\zeta_2, D^1_{e_i}\zeta_1 + e_i(\ln f)\zeta_2) \\ &= \sum_{i=1}^{m} g(D^1_{e_i}\zeta_1, D^1_{e_i}\zeta_1) + \sum_{i=1}^{m} g(e_i(\ln f)\zeta_2, e_i(\ln f)\zeta_2) \\ &= \mathrm{Tr}(g_1(D^1\zeta_1, D^1\zeta_1)) + \|\zeta_2\|_2^2 \sum_{i=1}^{m}(e_i(\ln f))^2 \\ &= \mathrm{Tr}(g_1(D^1\zeta_1, D^1\zeta_1)) + \|\zeta_2\|_2^2\|\nabla f\|_1^2 \end{aligned} \tag{3.5}$$

and the second term is given by

$$\begin{aligned}\frac{1}{f^2}\sum_{i=m+1}^{m+n} g(D_{e_i}\zeta, D_{e_i}\zeta) &= \frac{1}{f^2}\sum_{i=m+1}^{m+n} g(\zeta_1(\ln f)e_i + D^2_{e_i}\zeta_2 \\ &\quad - fg_2(e_i,\zeta_2)\nabla f, \zeta_1(\ln f)e_i + D^2_{e_i}\zeta_2 - fg_2(e_i,\zeta_2)\nabla f) \\ &= n(\zeta_1(\ln f))^2 + \sum_{i=m+1}^{m+n} g_2(D^2_{e_i}\zeta_2, D^2_{e_i}\zeta_2) \\ &\quad + \|\nabla f\|_1^2 \sum_{i=m+1}^{m+n}(g_2(e_i,\zeta_2))^2, \end{aligned} \tag{3.6}$$

$$\frac{1}{f^2}\sum_{i=m+1}^{m+n} g(D_{e_i}\zeta, D_{e_i}\zeta) = \frac{n}{f^2}(\zeta_1(f))^2 + \mathrm{Tr}(g_2(D^2\zeta_2, D^2\zeta_2)) + \|\nabla f\|_1^2\|\zeta_2\|_2^2. \tag{3.7}$$

By using Eqs. (3.5) and (3.7), Eq. (3.4) becomes

$$\begin{aligned}\mathrm{Tr}(g(D\zeta, D\zeta)) &= \mathrm{Tr}(g_1(D^1\zeta_1, D^1\zeta_1)) + \mathrm{Tr}(g_2(D^2\zeta_2, D^2\zeta_2)) \\ &\quad + 2\|\zeta_2\|_2^2\|\nabla f\|_1^2 + \frac{n}{f^2}(\zeta_1(f))^2. \end{aligned} \tag{3.8}$$

Now suppose that ζ_i is a 2-Killing vector field and $\mathrm{Ric}^i(\zeta_i,\zeta_i) \le 0$, then ζ_i is a parallel vector field with respect to the metric g_i and hence

$$\mathrm{Tr}(g_1(D^1\zeta_1, D^1\zeta_1)) = \mathrm{Tr}(g_2(D^2\zeta_2, D^2\zeta_2)) = 0.$$

Then for a constant function f, we have

$$\operatorname{Tr}(g(D\zeta, D\zeta)) = 0.$$

Thus ζ is a parallel vector field with respect to the metric g. One can easily prove the last two assertions using Eq. (3.8). $\square$

Corollary 3.5. *Let $\zeta \in \mathfrak{X}(M_1 \times_f M_2)$ be a vector field on a warped product manifold of the form $M_1 \times_f M_2$. Then*

$$\begin{aligned}\operatorname{Tr}(g(D\zeta, D\zeta)) &= \operatorname{Tr}(g_1(D^1\zeta_1, D^1\zeta_1)) + \operatorname{Tr}(g_2(D^2\zeta_2, D^2\zeta_2)) \\ &\quad + 2\|\zeta_2\|_2^2\|\nabla f\|_1^2 + \frac{n}{f^2}\zeta_1(f)\zeta_1(f).\end{aligned}$$

Theorem 3.3. *Assume that $\zeta \in \mathfrak{X}(M_1 \times_f M_2)$ is a nontrivial 2-Killing vector field on the warped product manifold $M_1 \times_f M_2$. If $D_\zeta\zeta$ is parallel along a curve γ, then*

$$K(\zeta, \dot{\gamma}) \geq 0.$$

Proof. Let $\zeta \in \mathfrak{X}(M_1 \times_f M_2)$ be a nontrivial 2-Killing vector field, then

$$\begin{aligned}0 &= g(D_\zeta D_X\zeta, Y) - g(D_{[\zeta,X]}\zeta, Y) + 2g(D_X\zeta, D_Y\zeta) \\ &\quad + g(X, D_\zeta D_Y\zeta) - g(X, D_{[\zeta,Y]}\zeta)\end{aligned}$$

for any vector fields $X, Y \in \mathfrak{X}(M_1 \times_f M_2)$. Take $X = Y = T = \dot{\gamma}$, then

$$\begin{aligned}g(D_\zeta D_T\zeta, T) - g(D_{[\zeta,T]}\zeta, T) + g(D_T\zeta, D_T\zeta) &= 0, \\ g(D_\zeta D_T\zeta - D_{[\zeta,T]}\zeta, T) &= -g(D_T\zeta, D_T\zeta).\end{aligned}$$

Since $D_\zeta\zeta$ is parallel along a curve γ, $D_T D_\zeta\zeta = 0$ and hence

$$\begin{aligned}g(R(\zeta, T)\zeta, T) &= -g(D_T\zeta, D_T\zeta), \\ R(\zeta, T, T, \zeta) &= -g(D_T\zeta, D_T\zeta), \\ K(\zeta, \dot{\gamma}) &= \|D_T\zeta\|^2 * A(\zeta, \dot{\gamma}) \geq 0,\end{aligned}$$

where $A(\zeta, \dot{\gamma})$ is area of the parallelogram generated by ζ and $\dot{\gamma}$. $\square$

The above result can be proved by using Corollary 3.1 as follows.

Let $\zeta \in \mathfrak{X}(M_1 \times_f M_2)$ be a nontrivial 2-Killing vector field, then

$$\begin{aligned}R(\zeta, T, \zeta, T) &= g(D_T\zeta, D_T\zeta) + g(D_T D_\zeta\zeta, T) \\ &= \|D_T\zeta\|^2 + 0 \\ &= \|D_T\zeta\|^2 \geq 0.\end{aligned}$$

Moreover, if $D_\zeta\zeta = 0$, then $K(\zeta, X) \geq 0$ for any vector field $X \in \mathfrak{X}(M_1 \times_f M_2)$.

Now, we will state yet another condition for a vector field on warped product manifolds to be 2-Killing.

Let (M, g) be an n-dimensional pseudo-Riemannian manifold. Suppose that X and Y are vector fields on M. Then denote:

$$\mathfrak{F}(X, Y) = g(\nabla_X \nabla_Y X, Y) + g(\nabla_Y X, \nabla_Y X) - g(\nabla_{[X,Y]} X, Y).$$

Note that X is a 2-Killing vector field if $\mathfrak{F}(X, Y) = 0$ for any vector field Y on M. We can prove many of the above results using the following theorem.

Theorem 3.4. *Let $\zeta \in \mathfrak{X}(M_1 \times_f M_2)$ be a vector field on the warped product manifold of the form $M_1 \times_f M_2$. Then*

$$\begin{aligned}\mathfrak{F}(\zeta_1 + \zeta_2, X_1 + X_2) = {} & \mathfrak{F}_1(\zeta_1, X_1) + f^2 \mathfrak{F}_2(\zeta_2, X_2) \\ & + (f\zeta_1(f) + \zeta_1(f)\zeta_1(f)) g_2(X_2, X_2) \\ & + 2f\zeta_1(f) g_2(\nabla_{X_2} \zeta_2, X_2).\end{aligned}$$

4. 2-Killing Vector Fields of Warped Product Space-Times

We will apply our main results to some well-known warped product space-time models to characterize their 2-Killing vector fields.

4.1. 2-*Killing vector fields of standard static space-times*

We begin by defining standard static space-times.

Let (M, g) be an n-dimensional Riemannian manifold and $f : M \to (0, \infty)$ be a smooth function. Then $(n+1)$-dimensional product manifold $I \times M$ furnished with the metric tensor

$$\bar{g} = -f^2 \mathrm{d}t^2 \oplus g$$

is called a standard static space-time and is denoted by $\bar{M} = I_f \times M$ where I is an open, connected subinterval of $\mathbb{R}$ and $\mathrm{d}t^2$ is the Euclidean metric tensor on I.

Note that standard static space-times can be considered as a generalization of the Einstein static universe [1–3, 8, 13–16].

Theorem 4.1. *Let $\bar{M} = I_f \times M$ be a standard static space-time with the metric $\bar{g} = -f^2 \mathrm{d}t^2 \oplus g$. Suppose that $u : I \to \mathbb{R}$ is smooth on I. Then $\bar{\zeta} = u\partial_t + \zeta$ with $\zeta \in \mathfrak{X}(M)$ is a 2-Killing vector field on $\bar{M}$ if one of the following conditions is satisfied*:

(1) *ζ is 2-Killing on $M, u = a$ and $f\zeta(f) = b$ where $a, b \in \mathbb{R}$.*
(2) *ζ is 2-Killing on $M, u = (rt + s)^{\frac{1}{3}}$ and $\zeta(f) = 0$ where $r, s \in \mathbb{R}$.*

Proof. Let $\bar{X} = x\partial_t + X \in \mathfrak{X}(\bar{M})$ and $\bar{Y} = y\partial_t + Y \in \mathfrak{X}(\bar{M})$ be any vector fields on $\bar{M}$ where $X, Y \in \mathfrak{X}(M)$ and x, y are smooth real-valued functions on I. Using

Theorem 3.1, we have

$$
\begin{aligned}
&(\bar{\mathcal{L}}_{\bar{\zeta}}\bar{\mathcal{L}}_{\bar{\zeta}}\bar{g})(\bar{X},\bar{Y})\\
&\quad= (\mathcal{L}_{\zeta}\mathcal{L}_{\zeta}g)(X,Y) + f^2(\mathcal{L}^I_{u\partial_t}\mathcal{L}^I_{u\partial_t}g_I)(x\partial_t,y\partial_t) + 4f\zeta(f)(\mathcal{L}^2_{\zeta_2}g_2)(x\partial_t,y\partial_t)\\
&\qquad + 2f\zeta(\zeta(f))g_I(x\partial_t,y\partial_t) + 2\zeta(f)\zeta(f)g_I(x\partial_t,y\partial_t).
\end{aligned}
$$

Note that for a vector $u\partial_t$ field on I, we have

$$
\begin{aligned}
\mathcal{L}_{\zeta}g_I(x\partial_t,y\partial_t) &= 2\dot{u}g_I(x\partial_t,y\partial_t),\\
\mathcal{L}_{\zeta}\mathcal{L}_{\zeta}g_I(x\partial_t,y\partial_t) &= (2u\ddot{u}+4\dot{u}^2)g_I(x\partial_t,y\partial_t).
\end{aligned}
$$

Then

$$
\begin{aligned}
&(\bar{\mathcal{L}}_{\bar{\zeta}}\bar{\mathcal{L}}_{\bar{\zeta}}\bar{g})(\bar{X},\bar{Y})\\
&\quad= (\mathcal{L}_{\zeta}\mathcal{L}_{\zeta}g)(X,Y) + f^2(2u\ddot{u}+4\dot{u}^2)g_I(x\partial_t,y\partial_t) + 8\dot{u}f\zeta(f)g_I(x\partial_t,y\partial_t)\\
&\qquad + 2\zeta(f\zeta(f))g_I(x\partial_t,y\partial_t).
\end{aligned}
\tag{4.1}
$$

The vector field ζ is 2-Killing on M and the function u in both conditions (1) and (2) is a solution of

$$(2u\ddot{u}+4\dot{u}^2) = 0.$$

Thus Eq. (4.1) becomes

$$(\bar{\mathcal{L}}_{\bar{\zeta}}\bar{\mathcal{L}}_{\bar{\zeta}}\bar{g})(\bar{X},\bar{Y}) = 2[4f\zeta(f)\dot{u} + \zeta(f\zeta(f))]g_I(x\partial_t,y\partial_t). \tag{4.2}$$

Finally, condition (1) implies that $\dot{u} = \zeta(f\zeta(f)) = 0$ and condition (2) implies that $\zeta(f) = 0$. Consequently, condition (1) or condition (2) implies that

$$(\bar{\mathcal{L}}_{\bar{\zeta}}\bar{\mathcal{L}}_{\bar{\zeta}}\bar{g})(\bar{X},\bar{Y}) = 0$$

and so $\bar{\zeta}$ is 2-Killing on $\bar{M}$. □

The converse of the above theorem is considered in the following corollary. The proof is straightforward.

Corollary 4.1. *Assume that $\bar{M}$ is a standard static space-time of the form $I_f \times M$ and $\bar{\zeta} = u\partial_t + \zeta$ is a 2-Killing vector field on $\bar{M}$. Then ζ is a 2-Killing vector field on M. Moreover, the vector field $u\partial_t$ is a 2-Killing vector field on I if $\zeta(f) = 0$.*

Example 4.1. Let $\zeta = u(t)\partial_t + v(x)\partial_x$ be a vector field on the warped product manifold $\bar{M} = I_f \times \mathbb{R}$ with warping function f and the metric tensor $\mathrm{d}s^2 = -f^2\mathrm{d}t^2 + \mathrm{d}x^2$. To prove that ζ is a 2-Killing vector field, we can use Eq. (4.1). If $\bar{X} = x\partial_t + X$ and $\bar{Y} = y\partial_t + Y$ are two vector fields on $\bar{M}$, then

$$
\begin{aligned}
(\bar{\mathcal{L}}_{\bar{\zeta}}\bar{\mathcal{L}}_{\bar{\zeta}}\bar{g})(\bar{X},\bar{Y}) &= (\mathcal{L}_{\zeta}\mathcal{L}_{\zeta}g)(X,Y) + f^2(2u\ddot{u}+4\dot{u}^2)g_I(x\partial_t,y\partial_t)\\
&\quad + 8\dot{u}f\zeta(f)g_I(x\partial_t,y\partial_t) + 2\zeta(f\zeta(f))g_I(x\partial_t,y\partial_t),
\end{aligned}
\tag{4.3}
$$

where $\zeta = v(x)\partial_x$ and $g = \mathrm{d}x^2$. It is now easy to show that

$$\zeta(f) = vf', \quad \zeta(f\zeta(f)) = v^2 ff'' + v^2 f'^2 + vv'ff',$$

$$(\mathcal{L}_\zeta \mathcal{L}_\zeta g)(\partial_x, \partial_x) = 2vv'' + 4v'^2.$$

Also, an orthogonal basis of $\mathfrak{X}(M)$ is $\{\partial_t, \partial_x\}$. Thus Eq. (4.3) becomes

$$(\bar{\mathcal{L}}_{\bar{\zeta}} \bar{\mathcal{L}}_{\bar{\zeta}} \bar{g})(\partial_x, \partial_x) = 2vv'' + 4v'^2,$$

$$(\bar{\mathcal{L}}_{\bar{\zeta}} \bar{\mathcal{L}}_{\bar{\zeta}} \bar{g})(\partial_x, \partial_t) = 0,$$

$$\left(\bar{\mathcal{L}}_{\bar{\zeta}} \bar{\mathcal{L}}_{\bar{\zeta}} \bar{g}\right)(\partial_t, \partial_x) = 0,$$

$$\left(\bar{\mathcal{L}}_{\bar{\zeta}} \bar{\mathcal{L}}_{\bar{\zeta}} \bar{g}\right)(\partial_t, \partial_t) = -f^2(2u\ddot{u} + 4\dot{u}^2) - 8\dot{u}vff' - 2v^2ff'' - 2v^2f'^2 - 2vv'ff'.$$

Now if $u\partial_t$ and $v\partial_t$ are 2-Killing vector fields on I and $\mathbb{R}$, respectively, then

$$2u\ddot{u} + 4\dot{u}^2 = 2vv'' + 4v'^2 = 0.$$

Consequently, ζ is 2-Killing if $f' = 0$. One can obtain the same result by using the definition of 2-Killing vector fields (see Appendix B).

4.2. *2-Killing vector fields of generalized Robertson–Walker space-times*

We first define generalized Robertson–Walker space-times.

Let (M, g) be an n-dimensional Riemannian manifold and $f : I \to (0, \infty)$ be a smooth function. Then $(n+1)$-dimensional product manifold $I \times M$ furnished with the metric tensor

$$\bar{g} = -\mathrm{d}t^2 \oplus f^2 g$$

is called a generalized Robertson–Walker space-time and is denoted by $\bar{M} = I \times {}_f M$ where I is an open, connected subinterval of $\mathbb{R}$ and $\mathrm{d}t^2$ is the Euclidean metric tensor on I.

This structure was introduced to the literature to extend Robertson–Walker space-times [18, 30, 29]

Due to Corollary 3.2, we need to determine 2-Killing vector fields on I. Suppose that $\zeta_1 = h\partial_t$ is a vector field on I where h is a smooth function on I. Then

$$\begin{aligned}(\mathfrak{L}^I_{h\partial_t} \mathfrak{L}^I_{h\partial_t} g_I)(\partial_t, \partial_t) &= -2hh'' - 4(h')^2 \\ &= -2(hh'' + 2(h')^2).\end{aligned}$$

Therefore, $\zeta_1 = h\partial_t$ is a 2-Killing vector field on I if and only if $hh'' = -2(h')^2$.

In this case, one can solve the last differential equation and obtain that $h(t) = (at - b)^{\frac{1}{3}}$ for some $a, b \in \mathbb{R}$ where $t \in I$ and $t \neq \frac{b}{a}$.

Thus to characterize 2-Killing vector fields on the generalized Robertson–Walker space-time of the form $\bar{M} = I \times_f M$, one can focus on vector fields of the form $(at - b)^{\frac{1}{3}}\partial_t + V$.

An easy application of Corollary 3.3 leads us to the following result.

Proposition 4.1. *Let $\bar{M} = I \times_f M$ be a generalized Robertson–Walker space-time with the metric tensor $\bar{g} = -\mathrm{d}t^2 \oplus f^2 g$. Suppose that V is a 2-Killing vector field on (M, g). Then a vector field $(at-b)^{\frac{1}{3}}\partial_t + V$ is a 2-Killing vector field on $(\bar{M}, \bar{g})$ if V is a homothetic vector field on (M, g) with c satisfying*

$$\frac{a}{3} f\dot{f} + (f\ddot{f} + \dot{f}^2)(at-b) = -2cf\dot{f}(at-b)^{\frac{2}{3}}.$$

Remark 4.1. At this point, we want to emphasize that we prefer not to apply Corollary 3.4 since condition (1) implies that the warping function f of a generalized Robertson–Walker space-time of the form $\bar{M} = I \times_f M$ is constant and hence the underlying warped product turns out to be just a trivial product.

Appendix A. Proof of Theorem 3.1

Using Propositions 2.1 and 3.1, we get

$$\begin{aligned}(\mathcal{L}_\zeta \mathcal{L}_\zeta g)(X, Y) &= g(D_\zeta D_X \zeta, Y) + g(X, D_\zeta D_Y \zeta) - g(D_{[\zeta,X]}\zeta, Y) - g(X, D_{[\zeta,Y]}\zeta) \\ &\quad + 2g(D_X\zeta, D_Y\zeta).\end{aligned}$$

The first term T_1 is given by

$$\begin{aligned}T_1 &= g(D_\zeta D_X \zeta, Y) \\ &= g\left(D_\zeta\left(D^1_{X_1}\zeta_1 + \frac{1}{f}\zeta_1(f)X_2 + \frac{1}{f}X_1(f)\zeta_2 + D^2_{X_2}\zeta_2 - fg_2(X_2,\zeta_2)\triangledown f\right), Y\right) \\ &= g\left(D^1_{\zeta_1}D^1_{X_1}\zeta_1 + \frac{1}{f}\zeta_1(\zeta_1(f))X_2 + \frac{1}{f}\zeta_1(X_1(f))\zeta_2 + \frac{1}{f}\zeta_1(f)D^2_{X_2}\zeta_2\right. \\ &\quad - \zeta_1(f)g_2(X_2,\zeta_2)\triangledown f - fg_2(X_2,\zeta_2)D^1_{\zeta_1}\triangledown f + \frac{1}{f}(D^1_{X_1}\zeta_1)(f)\zeta_2 \\ &\quad + \frac{1}{f}\zeta_1(f)D^2_{\zeta_2}X_2 - \zeta_1(f)g_2(X_2,\zeta_2)\triangledown f + \frac{1}{f}X_1(f)D^2_{\zeta_2}\zeta_2 \\ &\quad - X_1(f)g_2(\zeta_2,\zeta_2)\triangledown f + D^2_{\zeta_2}D^2_{X_2}\zeta_2 - fg_2(D^2_{X_2}\zeta_2,\zeta_2)\triangledown f \\ &\quad \left. - fg_2(D^2_{\zeta_2}X_2,\zeta_2)\triangledown f - fg_2(X_2, D^2_{\zeta_2}\zeta_2)\triangledown f - g_2(X_2,\zeta_2)(\triangledown f)(f)\zeta_2, Y\right)\end{aligned}$$

and so

$$\begin{aligned}T_1 &= g_1(D^1_{\zeta_1}D^1_{X_1}\zeta_1, Y_1) + f\zeta_1(\zeta_1(f))g_2(X_2, Y_2) + f\zeta_1(X_1(f))g_2(\zeta_2, Y_2) \\ &\quad + f\zeta_1(f)g_2(D^2_{X_2}\zeta_2, Y_2) - \zeta_1(f)Y_1(f)g_2(X_2,\zeta_2) - fg_2(X_2,\zeta_2)g_1(D^1_{\zeta_1}\triangledown f, Y_1) \\ &\quad + f(D^1_{X_1}\zeta_1)(f)g_2(\zeta_2, Y_2) + f\zeta_1(f)g_2(D^2_{\zeta_2}X_2, Y_2) - \zeta_1(f)Y_1(f)g_2(X_2,\zeta_2) \\ &\quad + fX_1(f)g_2(D^2_{\zeta_2}\zeta_2, Y_2) - X_1(f)Y_1(f)g_2(\zeta_2,\zeta_2) + f^2g_2(D^2_{\zeta_2}D^2_{X_2}\zeta_2, Y_2) \\ &\quad - fY_1(f)g_2(D^2_{X_2}\zeta_2,\zeta_2) - fY_1(f)g_2(D^2_{\zeta_2}X_2,\zeta_2) - fY_1(f)g_2(X_2, D^2_{\zeta_2}\zeta_2) \\ &\quad - f^2g_2(X_2,\zeta_2)(\triangledown f)(f)g_2(\zeta_2, Y_2)\end{aligned}$$

$$\begin{aligned}
&= g_1(D^1_{\zeta_1} D^1_{X_1}\zeta_1, Y_1) + f^2 g_2(D^2_{\zeta_2} D^2_{X_2}\zeta_2, Y_2)\\
&\quad + f\zeta_1(\zeta_1(f))g_2(X_2, Y_2) + f\zeta_1(X_1(f))g_2(\zeta_2, Y_2) + f\zeta_1(f)g_2(D^2_{X_2}\zeta_2, Y_2)\\
&\quad - f\zeta_1(Y_1(f))g_2(X_2, \zeta_2) + fg_2(X_2, \zeta_2)(D^1_{\zeta_1} Y_1)(f)\\
&\quad + fg_2(\zeta_2, Y_2)(D^1_{X_1}\zeta_1)(f) + f\zeta_1(f)g_2(D^2_{\zeta_2} X_2, Y_2) - 2\zeta_1(f)Y_1(f)g_2(X_2, \zeta_2)\\
&\quad + fX_1(f)g_2(D^2_{\zeta_2}\zeta_2, Y_2) - X_1(f)Y_1(f)g_2(\zeta_2, \zeta_2)\\
&\quad - fY_1(f)g_2(D^2_{X_2}\zeta_2, \zeta_2) - fY_1(f)g_2(D^2_{\zeta_2} X_2, \zeta_2) - fY_1(f)g_2(X_2, D^2_{\zeta_2}\zeta_2)\\
&\quad - f^2 g_2(X_2, \zeta_2)g_2(\zeta_2, Y_2)(\bigtriangledown f)(f).
\end{aligned}$$

Exchanging X and Y we get the second term T_2 and so

$$\begin{aligned}
T_1 + T_2 &= g(D_\zeta D_X \zeta, Y) + g(D_\zeta D_Y \zeta, X)\\
&= g_1(D^1_{\zeta_1} D^1_{X_1}\zeta_1, Y_1) + f^2 g_2(D^2_{\zeta_2} D^2_{X_2}\zeta_2, Y_2) + g_1(D^1_{\zeta_1} D^1_{Y_1}\zeta_1, X_1)\\
&\quad + f^2 g_2(D^2_{\zeta_2} D^2_{Y_2}\zeta_2, X_2) + 2f\zeta_1(\zeta_1(f))g_2(X_2, Y_2)\\
&\quad - 2X_1(f)Y_1(f)g_2(\zeta_2, \zeta_2) - 2f^2 g_2(X_2, \zeta_2)g_2(\zeta_2, Y_2)(\bigtriangledown f)(f)\\
&\quad + f\zeta_1(f)g_2(D^2_{X_2}\zeta_2, Y_2) + fg_2(X_2, \zeta_2)(D^1_{\zeta_1} Y_1)(f)\\
&\quad + fg_2(\zeta_2, X_2)(D^1_{Y_1}\zeta_1)(f) + f\zeta_1(f)g_2(D^2_{Y_2}\zeta_2, X_2)\\
&\quad + fg_2(Y_2, \zeta_2)(D^1_{\zeta_1} X_1)(f) + fg_2(\zeta_2, Y_2)(D^1_{X_1}\zeta_1)(f)\\
&\quad + f\zeta_1(f)g_2(D^2_{\zeta_2} X_2, Y_2) - 2\zeta_1(f)Y_1(f)g_2(X_2, \zeta_2) - fY_1(f)g_2(D^2_{X_2}\zeta_2, \zeta_2)\\
&\quad + f\zeta_1(f)g_2(D^2_{\zeta_2} Y_2, X_2) - 2\zeta_1(f)X_1(f)g_2(Y_2, \zeta_2) - fX_1(f)g_2(D^2_{Y_2}\zeta_2, \zeta_2)\\
&\quad - fY_1(f)g_2(D^2_{\zeta_2} X_2, \zeta_2) - fX_1(f)g_2(D^2_{\zeta_2} Y_2, \zeta_2).
\end{aligned}$$

The third term is given by

$$\begin{aligned}
T_3 &= g(D_{[\zeta,X]}\zeta, Y)\\
&= g(D_{[\zeta_1,X_1]}\zeta_1 + D_{[\zeta_2,X_2]}\zeta_1 + D_{[\zeta_1,X_1]}\zeta_2 + D_{[\zeta_2,X_2]}\zeta_2, Y)\\
&= g\left(D^1_{[\zeta_1,X_1]}\zeta_1 + \frac{1}{f}\zeta_1(f)[\zeta_2, X_2] + \frac{1}{f}[\zeta_1, X_1](f)\zeta_2 + D^2_{[\zeta_2,X_2]}\zeta_2\right.\\
&\quad \left. - fg_2([\zeta_2, X_2], \zeta_2)\bigtriangledown f, Y\right)\\
&= g_1(D^1_{[\zeta_1,X_1]}\zeta_1, Y_1) + f\zeta_1(f)g_2([\zeta_2, X_2], Y_2) + f[\zeta_1, X_1](f)g_2(\zeta_2, Y_2)\\
&\quad + f^2 g_2(D^2_{[\zeta_2,X_2]}\zeta_2, Y_2) - fg_2([\zeta_2, X_2], \zeta_2)Y_1(f)\\
&= g_1(D^1_{[\zeta_1,X_1]}\zeta_1, Y_1) + f^2 g_2(D^2_{[\zeta_2,X_2]}\zeta_2, Y_2) + f\zeta_1(f)g_2([\zeta_2, X_2], Y_2)\\
&\quad + fg_2(\zeta_2, Y_2)[\zeta_1, X_1](f) - fg_2([\zeta_2, X_2], \zeta_2)Y_1(f).
\end{aligned}$$

Exchanging X and Y we get the fourth term T_4 and so

$$\begin{aligned}T_3 + T_4 = {} & g_1(D^1_{[\zeta_1,X_1]}\zeta_1, Y_1) + g_1(D^1_{[\zeta_1,Y_1]}\zeta_1, X_1) + f^2 g_2(D^2_{[\zeta_2,X_2]}\zeta_2, Y_2)\\ & + f^2 g_2(D^2_{[\zeta_2,Y_2]}\zeta_2, X_2) + f\zeta_1(f)g_2([\zeta_2, X_2], Y_2) + fg_2(\zeta_2, Y_2)[\zeta_1, X_1](f)\\ & - fY_1(f)g_2([\zeta_2, X_2], \zeta_2) + f\zeta_1(f)g_2([\zeta_2, Y_2], X_2) + fg_2(\zeta_2, X_2)[\zeta_1, Y_1](f)\\ & - fX_1(f)g_2([\zeta_2, Y_2], \zeta_2).\end{aligned}$$

The last term T_5 is given by

$$\begin{aligned}(1/2)T_5 &= g(D_X\zeta, D_Y\zeta)\\ &= g\left(D^1_{X_1}\zeta_1 + \frac{1}{f}\zeta_1(f)X_2 + \frac{1}{f}X_1(f)\zeta_2 + D^2_{X_2}\zeta_2 - fg_2(X_2,\zeta_2)\triangledown f,\right.\\ &\qquad \left.\times D^1_{Y_1}\zeta_1 + \frac{1}{f}\zeta_1(f)Y_2 + \frac{1}{f}Y_1(f)\zeta_2 + D^2_{Y_2}\zeta_2 - fg_2(Y_2,\zeta_2)\triangledown f\right)\\ &= g_1(D^1_{X_1}\zeta_1, D^1_{Y_1}\zeta_1) - fg_2(Y_2,\zeta_2)(D^1_{X_1}\zeta_1)(f) + \zeta_1(f)\zeta_1(f)g_2(X_2, Y_2)\\ &\quad + \zeta_1(f)Y_1(f)g_2(X_2,\zeta_2) + f\zeta_1(f)g_2(X_2, D^2_{Y_2}\zeta_2)\\ &\quad + \zeta_1(f)X_1(f)g_2(\zeta_2, Y_2) + X_1(f)Y_1(f)g_2(\zeta_2,\zeta_2) + fX_1(f)g_2(\zeta_2, D^2_{Y_2}\zeta_2)\\ &\quad + f\zeta_1(f)g_2(D^2_{X_2}\zeta_2, Y_2) + fY_1(f)g_2(D^2_{X_2}\zeta_2,\zeta_2) + f^2g_2(D^2_{X_2}\zeta_2, D^2_{Y_2}\zeta_2)\\ &\quad - fg_2(X_2,\zeta_2)(D^1_{Y_1}\zeta_1)(f) + f^2g_2(X_2,\zeta_2)g_2(Y_2,\zeta_2)g_1(\triangledown f,\triangledown f).\end{aligned}$$

Then

$$\begin{aligned}(\mathcal{L}_\zeta\mathcal{L}_\zeta g)(X,Y) = {} & (\mathcal{L}^1_{\zeta_1}\mathcal{L}^1_{\zeta_1}g_1)(X_1,Y_1) + f^2(\mathcal{L}^2_{\zeta_2}\mathcal{L}^2_{\zeta_2}g_2)(X_2,Y_2)\\ & + 4f\zeta_1(f)(\mathcal{L}^2_{\zeta_2}g_2)(X_2,Y_2) + 2f\zeta_1(\zeta_1(f))g_2(X_2,Y_2)\\ & + 2\zeta_1(f)\zeta_1(f)g_2(X_2,Y_2).\end{aligned}$$

Appendix B. Space-Time Example

In this section we deal with a standard static space-time of the form $I_f \times \mathbb{R}$. Using Proposition 2.1, one can establish the following

(1) $\nabla_{\partial_x}\partial_x = 0$,
(2) $\nabla_{\partial_t}\partial_x = \nabla_{\partial_x}\partial_t = \partial_x(\ln f)\partial_t = \frac{f'}{f}\partial_t$ and
(3) $\nabla_{\partial_t}\partial_t = ff'\partial_x$

on the warped product manifold $I_f \times \mathbb{R}$. It is clear that

$$[\bar\zeta, \partial_t] = -\dot u\partial_t, \quad [\bar\zeta, \partial_x] = -v'\partial_x.$$

Also, we have

$$\nabla_{\partial_t}\bar\zeta = uff'\partial_x + \frac{1}{f}(\dot u f + vf')\partial_t, \quad \nabla_{\partial_x}\bar\zeta = v'\partial_x + \frac{1}{f}(uf')\partial_t$$

and

$$\nabla_{\bar{\zeta}}\nabla_{\partial_t}\bar{\zeta} = [uvff'' + 2uvf'^2 + 2u\dot{u}ff']\partial_x + \frac{1}{f}[v^2f'' + vv'f' + v\dot{u}f' - u^2ff'^2 + u\ddot{u}f]\partial_t,$$

$$\nabla_{\bar{\zeta}}\nabla_{\partial_x}\bar{\zeta} = (vv'' + u^2f'^2)\partial_x + \frac{1}{f}(u\dot{u}f' + uv'f' + uvf'')\partial_t.$$

Finally,

$$\nabla_{[\bar{\zeta},\partial_t]}\bar{\zeta} = -u\dot{u}ff'\partial_x - \frac{1}{f}(\dot{u}vf' + \dot{u}^2f)\partial_t,$$

$$\nabla_{[\bar{\zeta},\partial_x]}\bar{\zeta} = -v'^2\partial_x - \frac{1}{f}(uv'f')\partial_t.$$

Now we can evaluate 2-Killing forms on $I_f \times \mathbb{R}$ as follows

$$(\bar{\mathcal{L}}_{\bar{\zeta}}\bar{\mathcal{L}}_{\bar{\zeta}}g)(\partial_x,\partial_x) = 2[vv'' + 2v'^2],$$

$$(\mathcal{L}_{\bar{\zeta}}\mathcal{L}_{\bar{\zeta}}g)(\partial_t,\partial_x) = 0,$$

$$(\mathcal{L}_{\bar{\zeta}}\mathcal{L}_{\bar{\zeta}}g)(\partial_x,\partial_t) = 0,$$

$$(\mathcal{L}_{\bar{\zeta}}\mathcal{L}_{\bar{\zeta}}g)(\partial_t,\partial_t) = -2f^2[u\ddot{u} + 2\dot{u}^2] - 2[v^2ff'' + vv'ff'] - 8\dot{u}vff' - 2v^2f'^2.$$

which is what we have done before.

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