Journal of Convex Analysis
Volume 11 (2004), No. 1, 245–250

A Characterization of Polyhedral Convex Sets

Farhad Husseinov
Department of Economics, Bilkent University, 06533 Ankara, Turkey
farhad@bilkent.edu.tr

Received September 26, 2002
Revised manuscript received December 12, 2003

This paper describes a class of convex closed sets, S, in R^n for which the following property holds: for every correspondence defined on a probability space with relative open values in S its integral is a relative open subset of S. It turns out, that the only closed convex sets in R^n having this property are generalized polyhedral convex sets. In particular, the only compact convex sets in R^n having this property are polytopes.

Keywords: Polyhedral convex sets, correspondence, locally convex sets

Mathematics Subject Classification: 52B99

1. Introduction

This note describes a class of closed convex sets, S, in R^n for which the following property holds: for every correspondence defined on a probability space with relative open values in S its integral is a relative open subset of S. It turns out, that the only convex closed sets in R^n having this property (named in sequel *the relative openness of integral* property, r.o.i. property,) are generalized polyhedral convex sets (see Definition 2.1 below). In particular, the only compact convex sets in R^n having the r.o.i. property are polytopes.

This study bears on a theorem on the integral of correspondences due to Grodal [1]. Grodal used this theorem to study the closedness and continuity of the core and the set of Pareto optimal allocations.

First, we formulate here a result which drops the convexity assumption in Grodal's theorem on correspondences. Its proof can be found in Husseinov [2], where it is used to strengthen Grodal's results on the core and Pareto optimal allocations to economies with nonconvex preferences. We start with some notations. As usual, $A\triangle A' = (A\setminus A')\cup(A'\setminus A)$ is the symmetric difference of two sets A and A'. ∂X, $\operatorname{int} X$, $\operatorname{ri} X$, and $\operatorname{co} X$ will denote the boundary, interior, relative interior and convex hull of a set X in R^n, respectively. The set of all positive integers is denoted by N. For a correspondence $F : T \to R^n$, where (T, Σ, μ) is a measure space, and a $\mu-$measurable set $A \subset T$ we use a short notation $\int_A F$ for $\int_A F(t)d\mu(t)$. Instead of $\int_T F$ we write $\int F$. We denote as $\mathcal{L}_F$ the set of all integrable selections of correspondence F.

Theorem 1.1. *Let (T, Σ, μ) be a measure space and let $X : T \to R^n$ be a measurable convex-valued correspondence. Let furthemore, $\varphi : T \to R^n$ be a measurable correspondence such that $\varphi(t)$ is a relative open subset of $X(t)$ almost everywhere on T. Then*

$$int\,(\int X d\mu) \cap (\int \varphi d\mu) = int\,(\int \varphi d\mu).$$

ISSN 0944-6532 / \$ 2.50

This theorem allows to strengthen Grodal's results on the continuity of the core and the Pareto optimal allocations of economies with nonconvex preferences.

A natural question concerning Theorem 1.1 is the following. Is it true that under the assumptions of Theorem 1.1, $\int \varphi d\mu$ is a relative open subset in $\int X d\mu$? The following simple example shows that the answer is in negative.

Example 1.2. Let D be a closed circle in R^2 of radius 1 and with the center at point $(0,1)$. Define $\varphi : (0,1] \to D$ by $\varphi(t) = \{x \in D : \ ||x|| < t\}$ for $t \in (0,1]$. Clearly, $0 \in \int_0^1 \varphi(t)dt$, but 0 is not a relative interior point of $\int_0^1 \varphi(t)dt$ in D. In fact, no point of ∂D, except 0, belongs to $\int_0^1 \varphi(t)dt$. Indeed, take $a \in \partial D,\ a \neq 0$, and assume, on the contrary, $a \in \int_0^1 \varphi(t)dt$. Then there exists $f \in \mathcal{L}_\varphi$ such that $a = \int_0^1 f(t)dt$. Denote by L the line tangent to D at a. Then if $f(t) \notin L$ on a set of positive measure, we would have $a = \int_0^1 f(t)dt \notin L$. So $f(t) \in L$ for almost all $t \in (0,1]$. Since $L \cap D = \{a\}$, and $\varphi(t) \subset D$ for all $t \in (0,1]$, it follows that $f(t) = a$ almost everywhere on $(0,1]$. So, we obtain $a \in \varphi(t)$ almost everywhere on $(0,1]$. But from the definition of $\varphi(t)$ we have $a \notin \varphi(t)$ for $t \in (0, ||a||)$. This contradiction proves the assertion.

In mathematical economics correspondences with values in a convex (polyhedral) cone, frequently arise. For example, in the classical model of economy involving finitely many (n) different commodities the commodity space is assumed to be the nonnegative orthant R^n_+. So, the above question is of particular interest, from the viewpoint of mathematical economics, in the case, where $X(t) = X$ is a convex (polyhedral) cone. The idea of Example 1.2 can be extended to show that the answer, in general, is still in negative.

Example 1.3. Put $D_1 = \{x \in R^3 : \ x_1^2 + x_2^2 \leq 1 \text{ and } x_3 = 1\}$, and let C be a cone generated by D_1. Define $\varphi : (0,1] \to C$ in the following way

$$\varphi(t) = C \cap H(t)$$

where $H(t)$ is that of the two open half-spaces in R^3 defined by the plane through point $a(t) = (1,0,1+t)$ and coordinate axis $0x_2$, which contains the point $(1,0,0)$. Clearly, $a(0) = (1,0,1) \in \varphi(t)$ for every $t \in (0,1]$. Hence $(1,0,1) \in \int_0^1 \varphi(t)dt$. But obviously, no point from the relative boundary of D_1 except $(1,0,1)$ belongs to $\int_0^1 \varphi(t)dt$. Hence, $(1,0,1)$ is not a relative interior point of $\int_0^1 \varphi(t)dt$ in C.

2. Characterization of polyhedral convex sets

We will show that the answer to the above question is in positive in the case of a polyhedral convex cone. Moreover, it will be shown here that for every polyhedral convex set P in R^n the following property holds: for an arbitrary probability space (T, Σ, μ), and for an arbitrary correspondence $\varphi : T \to P$ with relative open values in P, its integral $\int \varphi$ is a relative open subset of P. It turns out, that polyhedral convex sets form the maximal class of sets in R^n possessing this property. To formulate this result we need the following definition.

Definition 2.1. A set P in R^n is said to be a *generalized polyhedral convex set* if for each $a > 0$, the intersection $C_a \cap S$, where $C_a = [-a,a]^n$, is a polytope.

Now we are ready to formulate a theorem which characterizes sets with the r.o.i. property.

Before we introduce two notions that are used in a proof of this theorem.

Definition 2.2. A *local cone* with the vertex x is an intersection of a convex cone with the vertex at x and an open ball with the center at x.

Definition 2.3. A set S in R^n is said to be *locally conical* if for each $x \in S$ there exsits an open ball $B_r(x)$ with center at x such that $B_r(x) \cap S$ is a local cone with vertex at x.

Theorem 2.4. *A convex closed set P in R^n possesses the relative openness of integral property, if and only if it is a generalized polyhedral convex set.*

Proof. Without loss of generality, we assume that P has the full dimension n. First show that if a set P in R^n is a generalized polyhedral set, then it possesses the r.o.i. property. This will be done in five steps. Proofs of steps 1,3 and 4 are carried by induction on the dimension n. In all three proofs the case $n = 1$ is simple.

Step 1. For every two relative open subsets A, B in P and $\alpha, \beta \geq 0,\ \alpha + \beta = 1$, the set $\alpha A + \beta B$ is relative open in P.

Indeed, let $z \in \alpha A + \beta B$. Then $z = \alpha x + \beta y$ for some $x \in A,\ y \in B$. If either x or y is an interior point of P, then obviously, z is an interior point of $\alpha A + \beta B$. Assume $x, y \in \partial P$. Then two cases are possible: $x = y$ and $x \neq y$. Consider the case $x = y$. Then there exists $r > 0$ such that $B_r(x) \cap P \subset A \cap B$. Since $A \cap B \subset \alpha A + \beta B$ it follows that $B_r(x) \cap P \subset \alpha A + \beta B$. That is x is a relative interior point of $\alpha A + \beta B$. Let now $x \neq y$. We will consider two subcases (a) $z \in$ int P and (b) $z \in \partial P$.

(a) Denote $(x, y) = \{(1-t)x + ty | 0 < t < 1\}$. If $z \in$ int P, then there exists $x', y' \in (x, y) \subset$ int P such that $x' \in A,\ y' \in B$ and $z = \alpha x' + \beta y'$. Then there exists $r > 0$ such that $B_r(x') \subset A$ and $B_r(y') \subset B$. Clearly, $B_r(z') = \alpha B_r(x') + \beta B_r(y') \subset \alpha A + \beta B$. So, z is an interior point of $\alpha A + \beta B$.

(b) Let $z \in \mathrm{ri}\, F$, where F is a maximal proper face of P. Let $A_0 \subset A$ and $B_0 \subset B$ be two convex relative open subsets in P containing x and y, respectively. Then by the induction assumption $\bar{B}_r(z) = B_r(z) \cap F \subset \alpha A_0 + \beta B_0$ for some $r > 0$. Since A_0 and B_0 are relative open, there are $x_0 \in A_0 \setminus F$ and $y_0 \in B_0 \setminus F$. Then $\alpha x_0 + \beta y_0 \in (\alpha A_0 + \beta B_0) \setminus F$. Clearly, $co\,(\{\alpha x_0 + \beta y_0\} \cup \bar{B}_r(z)) \subset \alpha A_0 + \beta B_0$ is a neighborhood of z in P which is contained in $\alpha A_0 + \beta B_0$. So, z is a relative interior point of $\alpha A + \beta B$. Let $z \in \alpha A + \beta B$ belong to the relative interior of a face F of dimension smaller than $n - 1$. Let F_j $(j = 1, ..., m)$ be the collection of all maximal proper faces of P containing F. By the induction assumption there exists a convex relative open set $U_j \subset F_j,\ z \in U_j$, such that $U_j \subset \alpha A + \beta B$ $(j = 1, ..., m)$. Put $U = co\,(\cup_{j=1}^m U_j)$, and show that $U \subset \alpha A + \beta B$ $(j = 1, ..., m)$. This will finish the proof, because , since U_j are relative open in F_j $(j = 1, ..., m)$, we have that U is relative open in P. Let $u \in U$. Then $u = \sum_{j=1}^m \gamma_j u_j$, for some $u_j \in U_j$ $(j = 1, ..., m),\ \gamma_j \geq 0$ and $\sum_{j=1}^m \gamma_j = 1$. Then $u_j = \alpha x_j + \beta y_j$ for some $x_j \in A,\ y_j \in B\,(j = 1, ..., m)$. It follows that

$$u = \sum_{j=1}^m \gamma_j u_j = \alpha \sum_{j=1}^m \gamma_j x_j + \beta \sum_{j=1}^m \gamma_j y_j = \alpha x + \beta y,$$

where $x = \sum_{j=1}^m \gamma_j u_j \in A_0$ and $y = \sum_{j=1}^m \gamma_j y_j \in B_0$. So, $x \in A,\ y \in B$, and hence $u \in \alpha A + \beta B$. So, Step 1 is proved.

Step 2. It follows easily from Step 1 that for an arbitrary finitely many open sets $A_1, ..., A_m \subset P$ and $\alpha_1, ..., \alpha_m \geq 0$, $\sum_{j=1}^m \alpha_j = 1$, $\sum_{j=1}^m \alpha_j A_j$ is relative open in P.

Indeed, assume that the assertion is correct for less than m sets. If some of α_j is zero, then by the induction assumption $\sum_{j=1}^m \alpha_j A_j$ is relative open in P. Assume $\alpha_j > 0,\ j = 1, ..., m$. Then

$$\sum_{j=1}^{m} \alpha_j A_j = \alpha_m A_m + \beta \sum_{j=1}^{m-1} \frac{\alpha_j}{\beta} A_j, \text{ where } \beta = \sum_{j=1}^{m-1} \alpha_j.$$

By the induction assumption $B = \sum_{j=1}^{m-1} \frac{\alpha_j}{\beta} A_j$ is relative open in P. Then by Step 1, $\sum_{j=1}^m \alpha_j A_j = \alpha_m A_m + (1 - \alpha_m) B$ is relative open in P.

Step 3. Let $A_1, A_2, ...$ be a sequence of relative open sets in P and $\sum_{j=1}^\infty \alpha_j$ a nonnegative series with sum 1. Then $\sum_{j=1}^\infty \alpha_j A_j$ is a relative open subset of P.

Without loss of generality, we can assume that A_j $(j \in N)$ are convex. Let $x = \sum_{j=1}^\infty \alpha_j x_j$, where $x_j \in A_j$ $(j \in N)$, be an arbitrary point in $\sum_{j=1}^\infty \alpha_j A_j$. If x is an interior point of P, then by Theorem 1.1, x is an interior point of $\sum_{j=1}^\infty \alpha_j A_j$. Let now x be a relative interior point of some $(n-1)-$face F of P. Since $B_j = A_j \cap F$ $(j \in N)$ is relative open in F, by the induction assumption x is an interior point of $\sum_{j=1}^\infty \alpha_j B_j$ in F. Let $a_1, ..., a_n$ be affinely independent points in $\sum_{j=1}^\infty \alpha_j B_j$ such that $x \in \text{ri co}\,\{a_1, ..., a_n\}$. For every $j \in N$ fix a point $x'_j \in A_j \setminus F$ such that $||x'_j - x_j|| < \frac{1}{2^j}$ $(j \in N)$. Then clearly, the series $\sum_{j=1}^\infty \alpha_j x'_j$ is convergent and its sum, $x' \notin F$. Since $\sum_{j=1}^\infty \alpha_j A_j$ is convex and $a_1, ..., a_n, x' \in \sum_{j=1}^\infty \alpha_j A_j$, the simplex Σ with vertices at these points is contained in $\sum_{j=1}^\infty \alpha_j A_j$. Clearly, Σ is a neighborhood of x in P. Hence x is an interior point of $\sum_{j=1}^\infty \alpha_j A_j$ relative to P.

Let now $x \in \text{ri}\, F$, where F is a face of P of dimension smaller than $n-1$, and let F_k $(k = 1, ..., m)$ be the collection of all $(n-1)-$ dimensional faces of P containing x. Then by the induction assumption x is an interior point of $\sum_{j=1}^\infty \alpha_j (A_j \cap F_k)$ $(k = 1, ..., m)$ relative to F_k, that is there exists U_k $(k = 1, ..., m)$ a convex neighborhood of x in F_k such that $U_k \subset \sum_{j=1}^\infty \alpha_j A_j$. Since A_j $(j \in N)$ are convex, $\sum_{j=1}^\infty \alpha_j A_j$ is convex. Then $co\,(\cup_{k=1}^m U_k)$, which is a neighborhood of x in $\sum_{j=1}^\infty \alpha_j A_j$, is contained in $\sum_{j=1}^\infty \alpha_j A_j$.

Step 4. In this step we show that for a generalized polyhedral set P, an atomless probability space (T, Σ, μ) and a correspondence $\varphi : T \to P$ with relative open values, $\int \varphi$ is relative open in P.

Take $z \in \int \varphi$. Let $x \in \mathcal{L}_\varphi$ be such that $z = \int x$. If z is an interior point of P, then by Theorem 1.1, $z \in \text{int}\,(\int \varphi)$. Let $z \in \partial P$, and let F_j $(j = 1, ..., m)$ be the collection of all maximal proper faces of P containing z. Since $z \in F_j$ $(j = 1, ..., m)$, it follows that for some measurable set $T_0 \subset T$ of full measure, $x(t) \in \cap_{j=1}^m F_j$ for all $t \in T_0$. Since set $\varphi(t)$ is relative open in P, sets $\varphi_j(t) = \varphi(t) \cap F_j$ are relative open in F_j $(j = 1, ..., m)$ for all $t \in T_0$. Extend φ_j $(j = 1, ..., m)$ into T putting $\varphi_j(t) = F_j$ $(j = 1, ..., m)$ for $t \in T \setminus T_0$. Then $\varphi_j : T \to F_j$ $(j = 1, ..., m)$ are measurable correspondences with nonempty relative open values. By the induction assumption, set $\int \varphi_j$ is relative open in F_j for $j = 1, ..., m$. Since $z \in \int \varphi_j$ $(j = 1, ..., m)$, there exist relative open sets $U_j \subset F_j$ $(j = 1, ..., m)$ such that $z \in U_j \subset \int \varphi_j$ $(j = 1, ..., m)$. Clearly $U = \text{co}\,(\cup_{j=1}^m U_j)$ is a neighborhood of z in P. Since set $\varphi_j(t) \subset \varphi(t)$ almost everywhere on T, we have $U_j \subset \int \varphi$ $(j = 1, ..., m)$. By Lyapunov Theorem [3], $\int \varphi$ is a convex set. Hence it contains U. So z is a relative interior point of $\int \varphi$.

Step 5. This step concludes the proof of the fact that every generalized polyhedral convex set possesses the r.o.i. property.

Let $A_k\,(k \in M)$, where $M \subset N$, be the set of all atoms in T and let $T_0 = T \setminus (\cup_{k\in M} A_k)$. Then $\int \varphi = \int_{T_0} \varphi + \sum_{k\in M} \alpha_k \varphi_k$, where $\varphi_k = \varphi(A_k)$ for $k \in M$. Denote $\alpha_0 = \mu(T_0)$. If $\alpha_0 > 0$ denote $\mu_0(E) = \frac{1}{\alpha}\mu(E)$ for sets from $\Sigma(T_0)$, where $\Sigma(T_0) = \{E \in \Sigma : E \subset T_0\}$. Then $(T_0, \Sigma(T_0), \mu_0)$ is a probability space and by Step 4, $\varphi_0 = \int_{T_0} \varphi d\mu_0$ is a relative open subset of P. Obviously, $\varphi_0 = \int_{T_0} \varphi d\mu_0 = \frac{1}{\alpha_0} \int_{T_0} \varphi d\mu$. So $\int \varphi = \alpha_0 \varphi_0 + \sum_{k\in M} \alpha_k \varphi_k$, where φ_k $(k \in M_0)$ are relative open sets in P, and $\alpha_k > 0$, $\sum_{k\in M_0} \alpha_k = 1$. By Step 3, $\int \varphi$ is a relative open subset of P.

In the next two steps we show that if a set P in R^n possesses the r.o.i. property, then it is a generalized polyhedral convex set.

Step 6. If a convex closed set P in R^n possesses the r.o.i.p. then P is locally conical.

Obviously, P is locally conical at $x \in \operatorname{int} P$. Assume P is not locally conical at $x \in \partial P$. Then for each $\varepsilon > 0$ there exists $x_\varepsilon \in B_\varepsilon(x) \cap (\partial P)$ such that $[x, x_\varepsilon] \not\subset \partial P$. Let H_ε be a supporting hyperplane of P at x_ε. Then $x \notin H_\varepsilon$. Otherwise, $[x, x_\varepsilon] \subset H_\varepsilon$, and hence $[x, x_\varepsilon] \subset \partial P$. Define $\varphi : (0, 1] \to P$, putting $\varphi(t) = B_t(x) \cap P$ for $t \in (0, 1]$. It is easily shown that $x_\varepsilon \notin \int \varphi$ for all $\varepsilon > 0$. Indeed, since $x \notin H_\varepsilon$, x belongs to the open half-space H_ε^+ defined by H_ε, closure of which contains P. Then there exists $r > 0$ such that $B_r(x) \subset H_\varepsilon^+$. Since $\varphi(t) \subset B_t(x)$ for each $t \in (0, 1]$ it follows that $\varphi(t) \subset H_\varepsilon^+$ for $t \in (0, r]$. This implies that for $y(\cdot) \in \mathcal{L}_\varphi$, $\int y \in H_\varepsilon^+$. Since $x_\varepsilon \notin H_\varepsilon^+$, we have from here $x_\varepsilon \notin \int \varphi$ for all $\varepsilon > 0$. So $\{x_\varepsilon : \varepsilon > 0\} \cap (\int \varphi) = \emptyset$, and $||x_\varepsilon - x|| < \varepsilon$ for all $\varepsilon > 0$. That is, we have points in P arbitrarily close to x, not lying in $\int \varphi$. Therefore x is not a relative interior point of $\int \varphi$. Thus, P does not possess the r.o.i. property. So, we have showed that if P possesses the r.o.i. property, then P is locally conical.

Step 7. A locally conical convex closed set is a generalized polyhedral convex set.

So, let P be a locally conical convex closed set. Then $P \cap [-a, a]^n$, as the intersection of two locally conical sets, is locally conical for every $a > 0$. Hence, it suffices to show that every locally conical convex compact set P is a polytope. Show that every extreme point x in P is isolated. Let $C = B_r(x) \cap P$ be a local cone with the vertex at x. Then for an arbitrary point $y \in C$, $y \neq x$ we have $y \in \operatorname{ri}\{(1-t)x + ty | t \in [0, b]\} \subset C$ for some number $b > 1$. So y is not an extreme point of C. We conclude that x is the only extreme point of P in $B_r(x)$. Since P is compact and every extreme point in P is isolated it follows that P has only finitely many extreme points. Indeed, assume that there are infinitely many extreme points in P. Then by compactness of P, we have that there exists a convergent sequence $\{x_k\}$ of extreme points with $x_k \neq x_l$ for $k \neq l$. Let $x_k \to x$. Since P is locally conical, there exists $r > 0$, such that $C = B_r(x) \cap P$ is a local cone. For sufficiently large index $\bar{k}$ we have $x_{\bar{k}} \in B_r(x)$. Since $x_{\bar{k}}$ is an extreme point of P, it is an extreme point of C. But we showed above that in the local cone C all points, perhaps except x, are not extreme points. The obtained contradiction proves that P has only finitely many extreme points. According to the representation theorem [4, Theorem 18.5] P is the convex hull of its extreme points. Then P is a polytope. The theorem is proved.

Theorem 2.4 contains the following characterization of polytopes.

Corollary 2.5. *A convex compact set in R^n is a polytope if and only if it possesses the relative openness of integral property.*

When P is a cone in R^n Theorem 2.4 implies the following

Corollary 2.6. *A convex closed cone in R^n is polyhedral if and only if it possesses the relative openness of integral property.*

Acknowledgements. The author is thankful to an anonymous referee for careful reading of the paper resulted in a good many improvements.

References

[1] B. Grodal: A theorem on correspondences and continuity of the core, in: Differential Games and Related Topics, H. W. Kuhn, G. P. Szego (eds.), North-Holland, Amsterdam (1971) 221–233.

[2] F. V. Hüsseinov: Theorems on correspondences and stability of the core, Economic Theory 22(4) (2003) 893–902.

[3] A. Lyapunov: Sur les fonctions-vecteurs complement additives, Bull. Acad. Sci. USSR, Ser. Math. 4 (1940) 465–478.

[4] R. T. Rockafellar: Convex Analysis, Princeton University Press, Princeton (1970).