FINITENESS AND QUASI-SIMPLICITY FOR SYMMETRIC $K3$-SURFACES

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Abstract

We compare the smooth and deformation equivalence of actions of finite groups on $K3$-surfaces by holomorphic and antiholomorphic transformations. We prove that the number of deformation classes is finite and, in a number of cases, establish the expected coincidence of the two equivalence relations. More precisely, in these cases we show that an action is determined by the induced action in the homology. On the other hand, we construct two examples to show first that, in general, the homological type of an action does not even determine its topological type, and second that $K3$-surfaces X and $\bar{X}$ with the same Klein action do not need to be equivariantly deformation equivalent even if the induced action on $H^{2,0}(X)$ is real, that is, reduces to multiplication by ± 1.

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1. Introduction

1.1. Questions

In this paper, we study equivariant deformations of complex $K3$-surfaces with symmetry groups, where by a symmetry we mean either a holomorphic or an antiholomorphic transformation of the surface. Although the automorphism group of a particular $K3$-surface may be infinite, we confine ourselves to finite group actions and address the following two questions (see Sections 1.4 – 1.6 for precise definitions):

finiteness: whether the number of actions, counted up to equivariant deformation and isomorphism, is finite, and

quasi-simplicity: whether the differential topology of an action determines it up to the above equivalence.

The response to the second question, as it is posed, is obviously in the negative. For example, given an action on a surface X, the same action on the complex conjugate surface $\bar{X}$ is diffeomorphic to the original one but often not deformation equivalent to it. Thus, we pose this question in a somewhat weaker form:

weak quasi-simplicity: does the differential topology of an action determine it up to equivariant deformation and (anti-) isomorphism?

To our knowledge, these questions have never been posed explicitly, and, moreover, despite numerous related partial results, they both have remained open.

One may notice a certain ambiguity in the statements of the above questions, especially in what concerns quasi-simplicity: we do not specify whether we consider diffeomorphic actions on true $K3$-surfaces or, more generally, diffeomorphic actions on surfaces diffeomorphic to a $K3$-surface. Fortunately, a surface diffeomorphic to a $K3$-surface is a $K3$-surface (see [FM2]), and the two versions turn out to be equivalent. Thus, we confine ourselves to true $K3$-surfaces and respond to both the finiteness and (to a great extent) weak quasi-simplicity questions (Section 1.7).

1.2. A brief retrospective of the method

Following the founding work by I. Piatetski-Shapiro and I. Shafarevich [PS], we base our study on the global Torelli theorem. When combined with Vik. Kulikov's theorem on surjectivity of the period map [Ku], this fundamental result essentially reduces the finiteness and quasi-simplicity questions to certain arithmetic problems. It is this approach that was used by V. Nikulin in [Ni2] and [Ni3], where he established (partially implicitly) the finiteness and quasi-simplicity results for polarized $K3$-surfaces with symplectic actions of finite abelian groups and for those with real structures. (Partial preliminary results, based on the injectivity of the period map, are found in [Ni1] for symplectic actions and in [K] for real structures.) In [DIK], we extended these results to real Enriques surfaces (which can be regarded as $K3$-surfaces with certain actions of $\mathbb{Z}_2 \times \mathbb{Z}_2$). In fact, [Ni2], [Ni3], and [DIK] give a complete deformation classification of the respective surfaces. It was while studying real Enriques surfaces that we got interested in the above questions and obtained our first results in this direction.

In all cases above, one starts by using the global Torelli theorem to show that the deformation class of a surface is determined by the induced action in its 2-homology and thus to reduce the problem to a (sometimes quite elaborate) study of the induced action. One of our principal results (Theorem 1.7.2) extends this statement to a wide class of actions, thus making it possible to complete the classification in many cases. (For example, G. Xiao's paper [X] seems very promising in classifying $K3$-surfaces with symplectic finite group actions; eventually it reduces the study of the induced actions to the study of certain definite sublattices in the homology of the orbit space, which is also a $K3$-surface in this case.) On the other hand, in Proposition 6.1.1 we construct an example of an action of a relatively simple group (the dihedral group of order 6) whose deformation and topological types cannot be read from the homology. The study of such actions would require new tools that would let one enumerate the walls in the period space that do matter.

1.3. Related results

One can find a certain similarity between our finiteness results and the finiteness in the theory of moduli of complex structures on 4-manifolds, which states (see [FM1] and [F]) that the moduli space of Kählerian complex structures on a given underlying differentiable compact 4-manifold has finitely many components. (By Kählerian we mean a complex structure admitting a Kähler metric. In the case of surfaces, this is a purely topological restriction: the complex structures on a given compact 4-manifold X are Kählerian if and only if the first Betti number $b_1(X; \mathbb{Q})$ is even.) Moreover, the moduli space is connected as soon as there is a Kählerian representative of Kodaira dimension at most zero (as is the case for $K3$-surfaces and complex 2-tori); for Kodaira dimension one, there are at most two deformation classes, which are represented by X and $\bar{X}$ (see [FM1]). Examples of general type surfaces X not deformation equivalent to $\bar{X}$ are found in [KK] and [C].

The principal result of our paper can be regarded as an equivariant version of the above statements for $K3$-surfaces. The finiteness theorem (Theorem 1.7.1) is closely related to a series of results from the theory of algebraic groups that go back to C. Jordan [J]. The original Jordan theorem states that $\mathrm{SL}(n, \mathbb{Z})$ contains but a finite number of conjugacy classes of finite subgroups. A. Borel and Harish-Chandra (see [BH] and [Bo]) generalized this statement to any arithmetic subgroup of an algebraic group; further recent generalizations are due to V. Platonov [P]. Note that, together with the global Torelli theorem, these Jordan-type theorems (applied to the 2-cohomology lattice of a $K3$-surface) imply that the number of different finite groups acting faithfully on $K3$-surfaces is finite. A complete classification of finite groups acting symplectically (i.e., identically on holomorphic forms) on $K3$-surfaces is found in Sh. Mukai [M] (see also Sh. Kondō [Ko1] and G. Xiao [X]; the abelian groups were first classified by Nikulin [Ni3]; unlike Mukai, who listed only the groups, Nikulin gave a description of the homological actions (cf. Section 1.2) and their moduli spaces and showed that the latter are connected). A sharp bound on the order of a group acting holomorphically on a $K3$-surface is given by Kondō [Ko2]; it is based on Nikulin's bound on the restriction of the induced action to the group of transcendental cycles. Here, as in the study of the components of the moduli space, the crucial starting point is a thorough analysis of the transcendental part of the action over $\mathbb{Q}$ (cf. almost geometric actions in Section 2.6); it was originated in [Ni3].

Among other related finiteness results found in the literature, we would like to mention a theorem by Piatetski-Shapiro and Shafarevich [PS] stating that the automorphism group of an algebraic $K3$-surface is finitely generated, our [DIK] generalization of this theorem to all $K3$-surfaces, and H. Sterk's [St] finiteness results on the classes of irreducible curves on an algebraic $K3$-surface. Note that all these results deal with individual surfaces rather than with their deformation classes. They are related to the finiteness of the number of conjugacy classes of finite subgroups in the group of Klein automorphisms of a given variety. As a special case, one can ask whether the number of conjugacy classes of real structures on a given variety is finite. For the latter question, the key tool is the Borel-Serre [BS] finiteness theorem for Galois cohomology of finite groups; as an immediate consequence, it implies finiteness of the number of conjugacy classes of real structures on an abelian variety. In [DIK] we extended this statement to all surfaces of Kodaira dimension at least 1 and to all minimal Kähler surfaces. Remarkably, finiteness of the number of conjugacy classes of real structures on a given rational surface is still an open question.

Unlike finiteness, the quasi-simplicity question does not make much sense for individual varieties. In the past it was mainly studied for deformation equivalence of real structures: Given a deformation family of complex varieties, is a real variety within this family determined up to equivariant deformation by the topology of the real structure? The first nontrivial result in this direction, concerning real cubic surfaces in $\mathbb{P}^3$, was discovered by F. Klein and L. Schläfli (see, e.g., the survey [DK1]). At present, the answer is known for curves (essentially due to F. Klein and G. Weichold; see, e.g., the survey [N]), complex tori (essentially due to A. Comessatti [Co1], [Co2]), rational surfaces (A. Degtyarev and V. Kharlamov [DK2]), ruled surfaces (J.-Y. Welschinger [W]), $K3$-surfaces (essentially due to Nikulin [Ni2]), Enriques surfaces (see [DIK]), hyperelliptic surfaces (F. Catanese and P. Frediani [CF]), and some sporadic surfaces of general type (e.g., so-called Bogomolov-Miayoka-Yau surfaces; see Kharlamov and Kulikov [KK]).

Note that, for the above classes of special surfaces, topological invariants that determine the deformation class are known. Together with quasi-simplicity, this implies finiteness (as the invariants take values in finite sets). Finiteness also holds for varieties of general type (in any dimension); as for such varieties, the Hilbert scheme is quasi-projective.

1.4. Terminology conventions

Unless stated otherwise, all complex varieties are supposed to be nonsingular, and differentiable manifolds are C^∞. A *real variety* (X, conj) is a complex variety X equipped with an antiholomorphic involution conj. In spite of the fact that we work with antiholomorphic transformations as well, we reserve the term *isomorphism* for biholomorphic maps, using *anti-isomorphism* for bi-antiholomorphic ones.

When working with period spaces, it is convenient to equip a $K3$-surface X with the fundamental class γ_X of a Kähler structure on X. We call γ_X a *polarization* of X. Strictly speaking, since we do not assume that γ_X is ample (nor even that X is algebraic or γ_X is an integral class), it would probably be more appropriate to invent a different term (quasi-polarization, K-polarization, Kählerization, ...). However, as in this paper it does not lead to confusion, we decided to avoid awkward language and use a familiar term in a slightly more general sense.

1.5. Augmented groups and Klein actions

An *augmented group* is a finite group G supplied with a homomorphism $\kappa\colon G \to \{\pm 1\}$. (We do not exclude the case when κ is trivial.) Denote the kernel of κ by G^0. A *Klein action* of a group G on a complex variety X is a group action of G on X by both holomorphic and antiholomorphic maps. Assigning $+1$ (resp., -1) to an element of G acting holomorphically (resp., antiholomorphically), one obtains a natural augmentation $\kappa\colon G \to \{\pm 1\}$. An action is called *holomorphic* (resp., *properly Klein*) if $\kappa = 1$ (resp., if $\kappa \neq 1$).

Replacing the complex structure J on a complex variety X with its conjugate $(-J)$, one obtains another complex variety, commonly denoted by $\bar{X}$, with the same underlying differentiable manifold. An automorphism of X is as well an automorphism of $\bar{X}$; it can also be regarded as an antiholomorphic bijection between X and $\bar{X}$. Thus, a Klein G-action on X can as well be regarded as a Klein action on $\bar{X}$, with the same augmentation $\kappa\colon G \to \{\pm 1\}$ and the same subgroup G^0. These two actions are obviously diffeomorphic, but they do not need to be isomorphic.

A Klein action of a group G on a complex variety X gives rise to the induced action $G \to \operatorname{Aut} H^*(X)$, $g \mapsto g^*$, in the cohomology ring of X. Since we deal with $K3$-surfaces, which are simply connected, and since all elements of G are orientation-preserving in this dimension, the induced action reduces essentially to the action on the group $H^2(X)$, regarded as a lattice via the intersection index form. For our purpose, it is more convenient to work with the *twisted induced action* $\theta_X\colon G \to \operatorname{Aut} H^2(X)$, $g \mapsto \kappa(g)g^*$. The latter, considered up to conjugation by lattice automorphisms, is called the *homological type* of the original Klein action on X. Clearly, it is a topological invariant.

1.6. Smooth deformations

A (*smooth*) *family*, or *deformation*, of complex varieties is a proper submersion $p\colon X \to S$ with differentiable, not necessarily compact or complex, manifolds X, S supplied with a fiberwise integrable complex structure on the bundle $\operatorname{Ker} dp$. The varieties $X_s = p^{-1}(s)$, $s \in S$, are called *members* of the family. Given a group G, a family $p\colon X \to S$ is called *G-equivariant* if it is supplied with a smooth fiberwise G-action that restricts to a Klein action on each fiber.

Two complex varieties X, Y supplied with Klein actions of a group G are called *equivariantly deformation equivalent* if there is a chain $X = X_0, X_1, \ldots, X_k = Y$ of complex varieties X_i with Klein actions of G such that for each $i = 0, \ldots, k-1$ the varieties X_i and X_{i+1} are G-isomorphic to members of a G-equivariant smooth family. (By a G-isomorphism we mean a biholomorphic map ϕ such that $\phi g = g\phi$ for any $g \in G$.)

Clearly, the equivariant deformation equivalence is an equivalence relation, G-equivariantly deformation equivalent varieties are G-diffeomorphic, and the homological type of a G-action is a deformation invariant.

1.7. The principal results

Let X be a $K3$-surface with a Klein action of a finite group G. Then G^0 acts on the subspace $H^{2,0}(X) \cong \mathbb{C}$, which gives rise to a natural representation $\rho\colon G^0 \to \mathbb{C}^*$. If G is finite, the image of ρ belongs to the unit circle $S^1 \subset \mathbb{C}^*$. We refer to ρ as the *fundamental representation* associated with the original Klein action. It is a deformation but, in general, not a topological invariant of the action. A typical example is the same Klein action on $\bar{X}$; its associated fundamental representation is the conjugate $\bar{\rho}\colon g \mapsto \overline{\rho g} \in \mathbb{C}^*$.

As shown below (Proposition 4.3.1), in the case of finite group actions on a $K3$-surface X, the twisted induced action θ_X determines the subgroup G^0 and "almost" determines the fundamental representation $\rho\colon G^0 \to S^1$: from θ_X, one can recover a pair ρ, $\bar{\rho}$ of complex conjugate fundamental representations.

THEOREM 1.7.1 (Finiteness theorem)
The number of equivariant deformation classes of $K3$-surfaces with faithful Klein actions of finite groups is finite.

THEOREM 1.7.2 (Quasi-simplicity theorem)
Let X and Y be two $K3$-surfaces with finite group G Klein actions of the same homological type. Assume that either
(1) *the action is holomorphic, or*
(2) *the associated fundamental representation ρ is real; that is, $\rho = \bar{\rho}$.*
Then either X or $\bar{X}$ is G-equivariantly deformation equivalent to Y. If the associate fundamental representation is trivial, then X and $\bar{X}$ are G-equivariantly deformation equivalent.

Remark. If ρ is nonreal, the deformation classes of X and $\bar{X}$ are distinguished by the fundamental representation (ρ and $\bar{\rho}$). In Proposition 6.4.1 we give an example when X and $\bar{X}$ are not deformation equivalent even though ρ is real.

Remark. In Proposition 6.1.1 we discuss another example, that of a properly Klein action of the dihedral group $\mathbb{D}_3$ whose deformation class is not determined by its homological type and associated fundamental representation. However, the actions constructed differ by their topology. Thus, they do not constitute a counterexample to quasi-simplicity of $K3$-surfaces (in its weaker form), and the problem still remains open.

Note that this phenomenon is somewhat unusual and unexpected for $K3$-surfaces since in all examples known before, such as (real) $K3$-surfaces, (real) Enriques surfaces, and $K3$-surfaces with an involution, the deformation class (and hence the topological type of the action) can be read from the induced action on the homology. However, all these examples are covered by Theorem 1.7.2.

A real variety (X, conj) with a real (i.e., commuting with conj) holomorphic G^0-action can be regarded as a complex variety with a Klein action of the extended group $G = G^0 \times \mathbb{Z}_2$, the $\mathbb{Z}_2$-factor being generated by conj. Note that if X is a $K3$-surface with a real holomorphic G^0-action, the associated fundamental representation $\rho\colon G^0 \to \mathbb{C}^*$ is real.

COROLLARY 1.7.3
Let X and Y be two real $K3$-surfaces with real holomorphic G^0-actions, so that the extended Klein actions of $G = G^0 \times \mathbb{Z}_2$ have the same homological type. Then X and Y are G-equivariantly deformation equivalent.

The methods used in the paper can as well be applied to the study of finite group Klein actions on 2-dimensional complex tori. (The corresponding version of the global Torelli theorem was first discovered by Piatetski-Shapiro and Shafarevich [PS] and then corrected by T. Shioda [S].) The analogs of Theorem 1.7.1 and 1.7.2 for 2-tori are Theorems A.1.1 (finiteness) and A.1.2 (quasi-simplicity) proved in Appendix A. For holomorphic actions preserving a point, this is a known result; it is contained in the classification of finite group actions on 2-tori by A. Fujiki [Fu], where a complete description of the moduli spaces is also given. (The results for holomorphic actions on Jacobians go back to F. Enriques and F. Severi [ES1] and, on general abelian surfaces, back to G. Bagnera and M. de Franchis [BF].) We give a short proof not using the classification, extend the results to nonlinear Klein actions, and compare the complex conjugated actions. As a straightforward consequence, we obtain analogous results for hyperelliptic surfaces. A number of tools used in Appendix A are close to those used by Fujuki in his study of the relation between symplectic actions and root systems.

Note that Theorem A.1.2 is stronger than its counterpart Theorem 1.7.2 for $K3$-surfaces; one does not need any additional assumption on the action. On the other hand, we show that, in quite a number of cases, a 2-torus X is not equivariantly deformation equivalent to $\bar{X}$ (see Section A.4).

Together, Theorems 1.7.1, 1.7.2, A.1.1, and A.1.2 give finiteness and quasi-simplicity results for $K3$-surfaces, Enriques surfaces, 2-tori, and hyperelliptic surfaces, that is, for all Kähler surfaces of Kodaira dimension zero.

Among other results not directly related to the proofs, worth mentioning is Theorem 5.2.1, which compares the homological types of Klein actions on a singular $K3$-surface and on close nonsingular ones. There is also a generalization that applies to any surface provided that the singularities are simple.

1.8. Idea of the proof

As already mentioned, our study is based on the global Torelli theorem. As is known, in order to obtain a good period space, one should *mark* the $K3$-surfaces, that is, fix isomorphisms $H^2(X) \to L = 2E_8 \oplus 3U$ (see Section 1.10 for the notation). Technically, it is more convenient to deal with the period space $K\Omega_0$ of marked polarized $K3$-surfaces, which, in turn, is a sphere bundle over the period space Per_0 of marked Einstein $K3$-surfaces (see Section 4.1 for details). According to Kulikov [Ku], one has $\operatorname{Per}_0 = \operatorname{Per} \smallsetminus \Delta$, where Per is a contractible homogeneous space (the space of positive definite 3-subspaces in $L \otimes \mathbb{R}$) and Δ is the set of the subspaces orthogonal to roots of L.

Now we fix a finite group G and an action $\theta\colon G \to \operatorname{Aut} L$. This gives rise to the equivariant period spaces $K\Omega_0^G$ and $\operatorname{Per}_0^G = \operatorname{Per}^G \smallsetminus \Delta$ of marked $K3$-surfaces with the given homological type of Klein G-action. Note that we are interested only in *geometric* actions, that is, those for which the spaces Per_0^G or $K\Omega_0^G$ are nonempty. Given a $K3$-surface, its markings compatible with θ differ by elements of the group $\operatorname{Aut}_G L$

of the automorphisms of L commuting with G. Thus, the finiteness and the (weak) quasi-simplicity problems reduce essentially to the study of the set of connected components of the orbit space $\mathfrak{M}^G = \operatorname{Per}_0^G / \operatorname{Aut}_G L$. In fact, the desired result (connectedness or finiteness of the number of connected components) can be obtained with a smaller group $A \subset \operatorname{Aut}_G L$, depending on the nature of the action. (A description of such "underfactorized" moduli spaces is given in Sections 4.4.2 – 4.4.7.) Furthermore, the quotient space $\operatorname{Per}_0^G / A$ can be replaced with a subspace $\operatorname{Int}\Gamma \smallsetminus \Delta$, where Γ is an appropriate convex (hence connected) fundamental domain of the action of A on Per^G, and it remains to enumerate the *walls* in $\operatorname{Int}\Gamma$, that is, the strata of $\Delta \cap \operatorname{Int}\Gamma$ of codimension 1.

1.9. Contents of the paper

In Section 2 we give the basic definitions and cite some known results on lattices and group actions on them. In Section 2.6 we introduce the notion of *almost geometric* actions. This notion can be regarded as the "$\mathbb{Z}$-independent" (i.e., defined over $\mathbb{R}$) part of the necessary condition for an action to be realizable by a $K3$-surface. We study the invariant subspaces of an almost-geometric action and show, in particular, that such an action determines the augmentation of the group and, up to complex conjugation, the associated fundamental representation.

In Section 3 we introduce and study *geometric* actions, which we define in arithmetical terms. The main goals of this section are Theorems 3.1.2 and 3.1.3, which establish certain connectedness and finiteness properties of appropriate fundamental domains of groups of automorphisms of the lattice preserving a given geometric action.

In Section 4 we introduce the equivariant period and moduli spaces and show that an action on the lattice is geometric (in the sense of Section 3) if and only if it is realizable by a $K3$-surface. We give a detailed description of certain "underfactorized" moduli spaces and use it to prove the main results.

Section 5 deals with equivariant degenerations of $K3$-surfaces: we discuss the behavior of the twisted induced action along the walls of the period space.

In Section 6 we discuss two examples to show that, in general, the deformation type of a Klein action is not determined by its homological type and associated fundamental representation.

In Appendix A we treat the case of 2-tori.

1.10. Common notation

We freely use the notation $\mathbb{Z}_n$ and $\mathbb{D}_n$ for the cyclic group of order n and dihedral group of order $2n$, respectively. We use A_n, D_n, E_6, E_7, and E_8 for the even *negative* definite lattices generated by the root systems of the same name, and U for the hyperbolic plane (indefinite unimodular even lattice of rank 2). All other nonstandard symbols are explained in the text.

2. Actions on lattices

2.1. Lattices

An (*integral*) *lattice* is a free abelian group L of finite rank supplied with a symmetric bilinear form $b\colon L\otimes L\to\mathbb{Z}$. We usually abbreviate $b(v,w)=v\cdot w$ and $b(v,v)=v^2$. For any ring $\Lambda\supset\mathbb{Z}$ we use the same notation b (as well as $v\cdot w$ and v^2) for the linear extension $(v\otimes\lambda)\otimes(w\otimes\mu)\mapsto(v\cdot w)\lambda\mu$ of b to $L\otimes\Lambda$. A lattice L is called *even* if $v^2=0 \bmod 2$ for all $v\in L$; otherwise, L is called *odd*. Let $L^\vee=\operatorname{Hom}(L,\mathbb{Z})$ be the dual abelian group. The lattice L is called *nondegenerate* (*unimodular*) if the *correlation homomorphism* $L\to L^\vee$, $v\mapsto b(v,\,\cdot\,)$, is a monomorphism (resp., isomorphism). The cokernel of the correlation homomorphism is called the *discriminant group* of L and denoted by $\operatorname{discr}L$. The group $\operatorname{discr}L$ is finite (trivial) if and only if L is nondegenerate (resp., unimodular).

The assignment $(x \bmod L, y \bmod L)\mapsto(x\cdot y)\bmod\mathbb{Z}$, $x,y\in L^\vee$, is a well-defined bilinear form $b\colon\operatorname{discr}L\otimes\operatorname{discr}L\to\mathbb{Q}/\mathbb{Z}$. If L is even, there is also a quadratic extension $q\colon\operatorname{discr}L\to\mathbb{Q}/2\mathbb{Z}$ of b given by $x \bmod L\mapsto x^2 \bmod 2\mathbb{Z}$.

Given a lattice L, we denote by σ_+L and σ_-L its inertia indexes and by $\sigma L=\sigma_+L-\sigma_-L$ its signature. We call a nondegenerate lattice L *elliptic* (resp., *hyperbolic*) if $\sigma_+L=0$ (resp., if $\sigma_+L=1$). The terminology is not quite standard; we change the sign of the forms, and we treat a positive definite lattice of rank 1 as hyperbolic. This is caused by the fact that our lattices are related (explicitly or implicitly) to the Neron-Severi groups of complex surfaces.

A sublattice $M\subset L$ is called *primitive* if the quotient L/M is torsion-free. Given a sublattice $M\subset L$, we denote by $M\widehat{\ }$ its *primitive hull* in L, that is, the minimal primitive sublattice containing M: $M\widehat{\ }=\{v\in L\mid kv\in M \text{ for some } k\in\mathbb{Z},\, k\neq 0\}$.

An element $v\in L$ of square (-2) is called a *root*.* A *root system* is a lattice generated (over $\mathbb{Z}$) by roots. Recall that any elliptic root system decomposes, uniquely up to the order of the summands, into an orthogonal sum of irreducible elliptic root systems, that is, those of type A_n, D_n, E_6, E_7, or E_8.

2.2. Automorphisms

An *isometry* (*dilation*) of a lattice L is an automorphism $a\colon L\to L$ preserving the form (resp., multiplying the form by a fixed number $\neq 0$). All isometries of L constitute a group; we denote it by $\operatorname{Aut}L$. If L is nondegenerate, there is a natural repre-

*Traditionally, the roots are the elements of square (-2) or (-1). We exclude the case of square (-1) as we consider only even lattices.

sentation $\operatorname{Aut} L \to \operatorname{Aut}\operatorname{discr} L$. Denote its kernel $\operatorname{Aut}^0 L$. It is a finite-index normal subgroup of $\operatorname{Aut} L$ consisting of the "universally extensible" automorphisms. More precisely, an automorphism a of L belongs to $\operatorname{Aut}^0 L$ if and only if a extends to any suplattice $L' \supset L$ identically on $L^\perp$.

Given a vector $v \in L$, $v^2 \neq 0$, denote by $\mathfrak{s}_v$ the reflection against the hyperplane orthogonal to v, that is, the isometry of $L \otimes \mathbb{R}$ defined by $x \mapsto x - ((x \cdot v)/v^2)v$. If $\mathfrak{s}_v(L) \subset L$ (which is always the case when $v^2 = \pm 1$ or ± 2), we use the same notation for the induced automorphism of L. The subgroup $W(L) \subset \operatorname{Aut} L$ generated by the reflections against the hyperplanes orthogonal to roots of L is called the *Weil group* of L. Clearly, $W(L)$ is a normal subgroup of $\operatorname{Aut} L$ and $W(L) \subset \operatorname{Aut}^0 L$.

We recall a few facts on automorphisms of root systems; details can be found, for example, in [Bou]. Let R be an elliptic root system. The hyperplanes orthogonal to roots in R divide the space $R \otimes \mathbb{R}$ into several connected components called *cameras* of R, and the Weil group $W(R)$ acts transitively on the set of cameras. For each camera C of R, there is a canonical semidirect product decomposition $\operatorname{Aut} R = W(R) \rtimes S_C$, where $S_C \subset \mathrm{O}(R \otimes \mathbb{R})$ is the group of symmetries of C. (As an abstract group, S_C can be identified with the group of symmetries of the Dynkin diagram of R.) In particular, if an element $g \in \operatorname{Aut} R$ preserves C, one has $g \in S_C$. More generally, if g preserves a face $C' \subset C$, then in the decomposition $g = sw$, $s \in S_C$, the element w belongs to the Weil group of the root system generated by the roots of R orthogonal to C'.

2.3. Actions

Let G be a group. A G-action on a lattice L is a representation $\theta\colon G \to \operatorname{Aut} L$. In what follows we always assume that G is finite. Given a ring $\Lambda \supset \mathbb{Z}$, we use the same notation θ for the extension $g \mapsto \theta g \otimes \mathrm{id}_\Lambda$ of the action to $L \otimes \Lambda$. Denote by $\operatorname{Aut}_G(L \otimes \Lambda)$ the group of G-equivariant Λ-isometries of $L \otimes \Lambda$, that is, the centralizer of θG in $\operatorname{Aut}(L \otimes \Lambda)$, and let $W_G(L) = W(L) \cap \operatorname{Aut}_G L$ and $\operatorname{Aut}^0_G L = \operatorname{Aut}^0 L \cap \operatorname{Aut}_G L$.

A submodule $M \subset L \otimes \Lambda$ is called *G-invariant* if $\theta g(M) \subset M$ for any $g \in G$; it is galled *G-characteristic* if $a(M) \subset M$ for any $a \in \operatorname{Aut}_G(L \otimes \Lambda)$.

Let $\mathbb{K} \subset \mathbb{C}$ be a field. For an irreducible $\mathbb{K}$-linear representation ξ of G, we denote by $L_\xi(\mathbb{K})$ the ξ-isotypic subspace of $L \otimes \mathbb{K}$, that is, the maximal invariant subspace of $L \otimes \mathbb{K}$ that is a sum of irreducible representations isomorphic to ξ. Given a subfield $\Bbbk \subset \mathbb{K}$, denote by $L_\xi(\Bbbk)$ the minimal $\Bbbk$-subspace of $L \otimes \Bbbk$ such that $L_\xi(\Bbbk) \otimes_\Bbbk \mathbb{K} \supset L_\xi(\mathbb{K})$, and for a subring $\mathfrak{O} \subset \Bbbk$, $\mathfrak{O} \ni 1$, let $L_\xi(\mathfrak{O}) = L_\xi(\Bbbk) \cap (L \otimes \mathfrak{O})$. Clearly, $L_\xi(\Bbbk)$ is the space of an isotypic $\Bbbk$-representation of G, and $L_\xi(\mathfrak{O})$ is G-invariant and G-characteristic. If $\Bbbk$ is an algebraic number field and $\mathfrak{O}$ is an order in $\Bbbk$, then $L_\xi(\mathfrak{O})$ is a finitely generated abelian group and $L_\xi(\Bbbk) = L_\xi(\mathfrak{O}) \otimes_\mathfrak{O} \Bbbk$.

We use the shortcut L^G for $L_1(\mathbb{Z}) = \{x \in L \mid gx = x \text{ for all } g \in G\}$.

2.4. Extending automorphisms

Below we recall a few simple facts on extending automorphisms of lattices. All the results still hold if the lattices involved are supplied with an action of a finite group G and the automorphisms are G-equivariant. One can also consider lattices defined over an order in an algebraic number field.

LEMMA 2.4.1
Let M be a nondegenerate lattice, and let $M' \subset M$ be a sublattice of finite index. Then the groups $\operatorname{Aut} M$ and $\operatorname{Aut} M'$ have a common finite-index subgroup.

LEMMA 2.4.2
Let M be a lattice, and let $M' \subset M$ be a nondegenerate sublattice. Then the group of automorphisms of M' extending to M has finite index in $\operatorname{Aut} M'$.

LEMMA 2.4.3
Let M be a nondegenerate lattice, and let A be a group acting by isometries on $M \otimes \mathbb{Q}$. Assume that there is a finite-index sublattice $M' \subset M$ such that $a(M') \subset M$ for any $a \in A$. Then A has a finite-index subgroup acting on M.

Proof
It suffices to apply Lemma 2.4.1 to the A-invariant sublattice $\sum_{a \in A} a(M') \subset M$. □

COROLLARY 2.4.4
Let M^+ and M^- be two nondegenerate lattices, and let $J\colon M^- \to M^+$ be a dilation invertible over $\mathbb{Q}$. Then there exists a finite-index subgroup $A^+ \subset \operatorname{Aut} M^+$ such that the correspondence $a \mapsto a \oplus J^{-1} a J$ restricts to a well-defined homomorphism $A^+ \to \operatorname{Aut}(M^+ \oplus M^-)$.

2.5. Fundamental polyhedra

Given a real vector space V with a nondegenerate quadratic form, we denote by $\mathscr{H}(V)$ the space of maximal positive definite subspaces of V. Note that $\mathscr{H}(V)$ is a contractible space of nonpositive curvature. If $\sigma_+ V = 1$ (i.e., if V is hyperbolic), one can define $\mathscr{H}(V)$ as the projectivization $\mathscr{C}(V)/\mathbb{R}^*$ of the positive cone $\mathscr{C}(V) = \{x \in V \mid x^2 > 0\}$.

Fix an algebraic number field $\Bbbk \subset \mathbb{R}$, and let $\mathfrak{O}$ be the ring of integers of $\Bbbk$. Consider a hyperbolic integral lattice M and a hyperbolic sublattice $M' \subset M \otimes \Bbbk$ defined over $\mathfrak{O}$, that is, such that $\mathfrak{O} M' \subset M'$. Let $\mathscr{H}' = \mathscr{H}(M' \otimes_{\mathfrak{O}} \mathbb{R})$. Then any group A acting by isometries on M and preserving M' acts on $\mathscr{H}'$. Since M is a hyperbolic integral lattice and $(M')^\perp \subset M$ is elliptic, the induced action is discrete, and the Dirichlet domain with center at a generic $\Bbbk$-rational point of $\mathscr{H}'$ is a $\Bbbk$-rational polyhedral fundamental domain of the action. Any such domain is called a *rational Dirichlet polyhedron* of A (in $\mathscr{H}'$).

The following theorem treats the classical case where $M = M'$ is an integral lattice and $A = \operatorname{Aut} M$. It is due to C. L. Siegel [Si], H. Garland, M. S. Raghunathan [GR], and N. J. Wielenberg [Wi].

THEOREM 2.5.1
Let M be a hyperbolic integral lattice. Then the rational Dirichlet polyhedra of the full automorphism group $\operatorname{Aut} M$ in $\mathscr{H}(M)$ are finite. Unless M has rank 2 and represents zero, the polyhedra have finite volume.

COROLLARY 2.5.2
Let M be a hyperbolic integral lattice. Then the closure in $\mathscr{H}(M) \cup \partial\mathscr{H}(M)$ of any rational Dirichlet polyhedron of $\operatorname{Aut} M$ in $\mathscr{H}(M)$ is the convex hull of a finite collection of rational points.

2.6. The fundamental representations
Let $\theta\colon G \to \operatorname{Aut} L$ be a finite group action on a nondegenerate lattice L with $\sigma_+ L = 3$. We say that θ is *almost geometric* if there is a G-invariant flag $\ell \subset \mathfrak{w}$, where $\mathfrak{w} \subset L \otimes \mathbb{R}$ is a positive definite 3-subspace and ℓ is a 1-subspace with trivial G-action.

LEMMA 2.6.1
Let $\theta\colon G \to \operatorname{Aut} L$ be a finite group action on a lattice L with $d = \sigma_+ L > 0$. Then for any positive definite G-invariant d-subspace $\mathfrak{w} \subset L \otimes \mathbb{R}$, the induced action $\theta_{\mathfrak{w}}\colon G \to \mathrm{O}(\mathfrak{w}) = \mathrm{O}(d)$ is determined by θ up to conjugation in $\mathrm{O}(d)$. In particular, the augmentation $\kappa\colon G \to \mathrm{O}(\mathfrak{w}) \xrightarrow{\det} \{\pm 1\}$ is uniquely determined by θ.

Proof
Given another subspace $\mathfrak{w}'$ as in the statement, the orthogonal projection $\mathfrak{w}' \to \mathfrak{w}$ is nondegenerate and G-equivariant. Hence, the induced representations $\theta_{\mathfrak{w}}, \theta_{\mathfrak{w}'}\colon G \to \mathrm{O}(d)$ are conjugated by an element of $\mathrm{GL}(d)$. Since G is finite, they are also conjugated by an element of $\mathrm{O}(d)$. Indeed, it is sufficient to treat the case of irreducible representation, where the result follows from the uniqueness of a G-invariant scalar product up to a constant factor. $\square$

Given an almost geometric action $\theta\colon G \to \operatorname{Aut} L$, we always assume G augmented via κ above, so that an element $c \in G$ does not belong to $G^0 = \operatorname{Ker}\kappa$ if and only if it reverses the orientation of $\mathfrak{w}$. From Lemma 2.6.1 it follows that the existence of a 1-subspace ℓ with trivial G-action does not depend on the choice of a G-invariant positive definite 3-subspace $\mathfrak{w}$. Furthermore, the induced action on $\mathfrak{w}_0 = \ell^\perp \subset \mathfrak{w}$ is also independent of $\mathfrak{w}$. Choosing an orientation of $\mathfrak{w}_0$, one obtains a 2-dimensional representation $\rho\colon G^0 \to \mathrm{SO}(\mathfrak{w}_0) = S^1$. In what follows, we identify S^1 with the unit circle in $\mathbb{C}$ and often regard representations in S^1 as 1-dimensional complex representations. In particular, we consider the spaces (lattices) $L_\rho(\Lambda)$ (Section 2.3) associated with θ. Note that $L_\rho(\mathbb{C})$ is the ρ-eigenspace of G^0. Changing the orientation of $\mathfrak{w}_0$ replaces ρ with its conjugate $\bar\rho$. In view of Lemma 2.6.1, the unordered pair $(\rho, \bar\rho)$ is determined by θ; we call ρ and $\bar\rho$ the *fundamental representations* associated with θ. The order of the image $\rho(G^0)$ is called the *order* of θ and is denoted $\operatorname{ord}\theta$.

LEMMA 2.6.2

Let $\xi\colon G^0 \to S^1$ be a nonreal representation (i.e., $\bar\xi \ne \xi$). Then the map $L_\xi(\mathbb{C}) \to L_\xi(\mathbb{R})$, $\omega \mapsto (\omega + \bar\omega)/2$, is an isomorphism of $\mathbb{R}$-vector spaces. In particular, the space $L_\xi(\mathbb{R})$ inherits a natural complex structure J_ξ (induced from the multiplication by i in $L_\xi(\mathbb{C})$), which is an antiselfadjoint isometry. One has $J_{\bar\xi} = -J_\xi$.

Proof

The proof is straightforward. The metric properties of J_ξ follow from the fact that $\omega^2 = 0$ for any eigenvector ω (of any isometry) corresponding to an eigenvalue α with $\alpha^2 \ne 1$. □

LEMMA 2.6.3

Let θ be an almost geometric action, and let ρ be an associated fundamental representation. Assume that $\kappa \ne 1$. Then any element $c \in G \smallsetminus G^0$ restricts to an involution $c_\rho\colon L_\rho(\mathbb{R}) \to L_\rho(\mathbb{R})$. If ρ is not real, then c_ρ is J_ρ-antilinear; in particular, the (±1)-eigenspaces $V_\rho^\pm$ of c_ρ are interchanged by J_ρ.

Proof

Clearly, c takes ρ-eigenvectors of G^0 to ρ^c-eigenvectors, where ρ^c is the representation $g \mapsto \rho(c^{-1}gc)$. Since, by the definition of fundamental representations, there is a ρ-eigenvector ω taken to a $\bar\rho$-eigenvector, one has $\rho^c = \bar\rho$ and the space $L_\rho(\mathbb{R})$ is c-invariant. Furthermore, the vector $\operatorname{Re}\omega$ is invariant under c_ρ^2. Since $c^2 \in G^0$, one has $c_\rho^2 = \mathrm{id}$.

If ρ is nonreal, then c interchanges $L_\rho(\mathbb{C})$ and $L_{\bar\rho}(\mathbb{C})$. Since c commutes with the complex conjugation, the isomorphism $\omega \mapsto (\omega + \bar\omega)/2$ (Lemma 2.6.2) conjugates

c_ρ with the antilinear involution $\omega \mapsto c(\bar\omega)$ on $L_\rho(\mathbb{C})$. $\square$

LEMMA 2.6.4
Let θ be an almost geometric action, let ρ be an associated fundamental representation, and let $\Bbbk \subset \mathbb{R}$ be a field. Then the space $L_\rho(\Bbbk)$ is G-invariant and the induced G-action on $L_\rho(\Bbbk)$ factors through an action of the cyclic group $\mathbb{Z}_n$ (if $\kappa = 1$) or the dihedral group $\mathbb{D}_n$ (if $\kappa \neq 1$), where $n = \operatorname{ord}\theta$. The induced $\mathbb{Z}_n$-action is $\Bbbk$-isotypic; the $\mathbb{D}_n$-action is $\Bbbk$-isotypic unless $n \leq 2$.

Proof
All statements are obvious if $\kappa = 1$. Assume that $\kappa \neq 1$, and pick an element $c \in G \smallsetminus G^0$. The intersection $Q = L_\rho(\Bbbk) \cap c(L_\rho(\Bbbk))$ is defined over $\Bbbk$, and $Q \otimes_\Bbbk \mathbb{R}$ contains $L_\rho(\mathbb{R})$ (Lemma 2.6.3). Hence, $Q \supset L_\rho(\Bbbk)$ and $L_\rho(\Bbbk)$ is G-invariant. Further, the endomorphisms c^2 and $g - c^{-1}gc$ of $L_\rho(\Bbbk) \otimes_\Bbbk \mathbb{R}$ (where $g \in G^0$) are defined over $\Bbbk$ and annihilate $L_\rho(\mathbb{R})$ (Lemma 2.6.3 again); due to the minimality of $L_\rho(\Bbbk)$, they are trivial. $\square$

3. Folding the walls

3.1. Geometric actions
A finite group action $\theta\colon G \to \operatorname{Aut} L$ on an even nondegenerate lattice L with $\sigma_+ L = 3$ is called *geometric* if it is almost geometric and the sublattice $L^\bullet = (L^G + L_\rho(\mathbb{Z}))^\perp$ contains no roots, where ρ is a fundamental representation of θ.

Consider a geometric action θ, and fix an associated fundamental representation ρ. If $\kappa \neq 1$, fix an element $c \in G \smallsetminus G^0$, and denote by $V_\rho^\pm$ and $V^\pm$ its (± 1)-eigenspaces in $L_\rho(\mathbb{R})$ and $L_\rho(\mathbb{Q})$, respectively (Lemmas 2.6.3 and 2.6.4). Let $M^\pm = V^\pm \cap L$ be the (± 1)-eigenlattices of c in $L_\rho(\mathbb{Z})$. If $\rho \neq 1$, the spaces $V_\rho^\pm$ and $V^\pm$ are hyperbolic. The following lemma is a straightforward consequence of Lemmas 2.6.3 and 2.6.4.

LEMMA 3.1.1
The subspaces $V_\rho^\pm$ and $V^\pm$ and the sublattices $M^\pm$ are G-characteristic; they are G-invariant if and only if $\operatorname{ord}\theta \leq 2$. If $\rho \neq 1$, there is a well-defined action of $\operatorname{Aut}_G L$ on $\mathscr{H}(V_\rho^\pm)$; it is discrete and, up to isomorphism, independent of the choice of an element $c \in G \smallsetminus G^0$.

In view of this lemma, one can consider corresponding G-actions and introduce the following rational Dirichlet polyhedra.

- $\Gamma_1 \subset \mathscr{H}(L^G \otimes \mathbb{R})$ is a rational Dirichlet polyhedron of $W_G((L^G \oplus L^\bullet)\widehat{\ })$; it is defined whenever $\rho \neq 1$, so that $\sigma_+ L^G = 1$.

- $\Gamma_\rho^\pm \subset \mathscr{H}(V_\rho^\pm)$ are some rational Dirichlet polyhedra of $W_G((M^\pm \oplus L^\bullet)\widehat{\ })$; they are defined whenever ρ is real and $\kappa \neq 1$. (To define Γ_ρ^+, one needs to assume, in addition, that $\rho \neq 1$, so that $\sigma_+ M^+ = 1$.)
- $\Sigma_\rho^\pm \subset \mathscr{H}(V_\rho^\pm)$ are some rational Dirichlet polyhedra of $\operatorname{Aut}_G^0(L_\rho(\mathbb{Z}))$; they are defined whenever ρ is nonreal and $\kappa \neq 1$.

Given a vector $v \in L$, put $\mathrm{h}(v) = \{x \in L \otimes \mathbb{R} \mid x \cdot v = 0\}$, and introduce the following notation:

- $\mathrm{h}_1(v) = \mathrm{h}(v) \cap (L^G \otimes \mathbb{R})$;
- if ρ is real and $\kappa \neq 1$, then $\mathrm{h}_\rho^\pm(v) = \mathrm{h}(v) \cap V_\rho^\pm$;
- if ρ is nonreal, then $\mathrm{h}_\rho(v) = \{x \in L_\rho(\mathbb{R}) \mid x \cdot v = J_\rho x \cdot v = 0\}$; if, besides, $\kappa \neq 1$, then $\mathrm{h}_\rho^\pm(v) = \mathrm{h}_\rho(v) \cap V_\rho^\pm$.

We use the same notation $\mathrm{h}_1(v)$ and $\mathrm{h}_\rho^\pm(v)$ for the projectivizations of the corresponding spaces in $\mathscr{H}(L^G \otimes \mathbb{R})$ and $\mathscr{H}(V_\rho^\pm)$, respectively (whenever the space is hyperbolic).

The goal of this section is to prove the following two theorems.

THEOREM 3.1.2
Let $\theta\colon G \to \operatorname{Aut} L$ be a geometric action, and let ρ be an associated fundamental representation. If $\rho \neq 1$, then for any root $v \in L_\rho(\mathbb{Z})^\perp$ the intersection $\mathrm{h}_1(v) \cap \operatorname{Int}\Gamma_1$ is empty. If ρ is real and $\kappa \neq 1$, then for any root $v \in (L^G \oplus M^\mp)^\perp$ the intersection $\mathrm{h}_\rho^\pm(v) \cap \operatorname{Int}\Gamma_\rho^\pm$ is empty. (For Γ_ρ^+ to be well defined, one needs to assume, in addition, that $\rho \neq 1$.)

THEOREM 3.1.3
Let $\theta\colon G \to \operatorname{Aut} L$ be a geometric action with nonreal associated fundamental representation ρ and $\kappa \neq 1$. Then $\Sigma_\rho^\pm$ intersects finitely many distinct subspaces $\mathrm{h}_\rho^\pm(v)$ defined by roots $v \in (L^G)^\perp$.

Theorem 3.1.2 is proved at the end of Section 3.2. Theorem 3.1.3 is proved in Section 3.6.

3.2. Walls in the invariant sublattice

THEOREM 3.2.1
Let N be an even lattice, and let G be a finite group acting on N so that $(N^G)^\perp \subset N$ is negative definite. Let $v \in N$ be a root whose projection to $N^G \otimes \mathbb{R}$ has negative square. Then either

(1) *the orthogonal complement $(N^G)^\perp$ contains a root, or*

(2) *there is an element of* $W_G(N)$ *whose restriction to* N^G *is the reflection against the hyperplane* $\mathrm{h}(v) \cap (N^G \otimes \mathbb{R})$.

COROLLARY 3.2.2
In the above notation, assume that N *is hyperbolic and* $(N^G)^\perp$ *contains no roots. Then for any root* $v \in N$, *the intersection of* $\mathrm{h}(v)$ *with the interior of a rational Dirichlet polyhedron of* $W_G(N)$ *in* $\mathscr{H}(N^G)$ *is empty.*

To prove Theorem 3.2.1 we need a few facts on automorphisms of root systems. Let R be an even root system, and let G be a finite group acting on R. The action is called *admissible* if the orthogonal complement $(R^G)^\perp$ contains no roots, and it is called *b-transitive* if there is a root whose orbit generates R.

LEMMA 3.2.3
Given a finite group G *action on an elliptic root system* R, *the following statements are equivalent:*
(1) *the action is admissible;*
(2) *the action preserves a camera of* R;
(3) *the action factors through the action of a subgroup of the symmetry group of a camera of* R.

Proof
An action is admissible if and only if R^G does not belong to a wall $\mathrm{h}(v)$ defined by a root $v \in R$. On the other hand, R^G contains an inner point of a camera if and only if this camera is preserved by the action. □

COROLLARY 3.2.4
Up to isomorphism, there are two faithful admissible b-transitive actions on irreducible even root systems: the trivial action on A_1 *and a* $\mathbb{Z}_2$*-action on* A_2 *interchanging two roots* u, v *with* $u \cdot v = 1$.

Proof
The statement follows from Lemma 3.2.3, the classification of irreducible root systems, and the natural bijection between the symmetries of a camera and the symmetries of its Dynkin diagram. □

Proof of Theorem 3.2.1
Pick a vector v as in the statement, and consider the sublattice $R \subset N$ generated by the orbit of v. Under the assumptions, R is an even root system, and the induced G-

action on R is b-transitive. Assume that the action on R is admissible (as otherwise $(R^G)^\perp$, and thus $(N^G)^\perp$, would contain a root). Then, in view of Corollary 3.2.4, the lattice R splits into an orthogonal sum of several copies of either A_1 or A_2, and the vector $\bar{v} = \sum_{g\in G} g(v)$ has the form $\sum m_i a_i$, $m_i \in \mathbb{Z}$, where each a_i is a generator of A_1 or the sum of two generators of A_2 interchanged by the action. Since the a_i's are mutually orthogonal roots, the composition of the reflections $\mathfrak{s}_{a_i}$ is the desired automorphism of N. $\square$

Proof of Theorem 3.1.2
The statement for Γ_1 follows immediately from Theorem 3.2.1 applied to $N = L_\rho(\mathbb{Z})^\perp$. To prove the assertion for $\Gamma_\rho^\pm$, consider the induced G-action $\theta_{\mathfrak{w}}\colon G \to \mathrm{O}(\mathfrak{w})$, where $\mathfrak{w}$ is as in the definition of an almost geometric action (Section 2.6), and note that, under the hypotheses ($\rho \ne 1$ is real), $\theta_{\mathfrak{w}}$ factors through the abelian subgroup $C \subset \mathrm{O}(\mathfrak{w})$ generated by the central symmetry c and a reflection s. Thus, the statement for Γ_ρ^+ (resp., Γ_ρ^-) follows from Theorem 3.2.1 applied to the lattice $N = (L^G \oplus M^-)^\perp$ (resp., $N = (L^G \oplus M^+)^\perp$) with the twisted action $g \mapsto r(g)\theta(g)$, where $r\colon G \to \{\pm 1\}$ is the composition of $\theta_{\mathfrak{w}}$ and the homomorphism $c \mapsto -1$, $s \mapsto 1$ (resp., $c \mapsto -1$, $s \mapsto -1$). $\square$

3.3. The group $\mathrm{Aut}_G L$

As before, let $\theta\colon G \to \mathrm{Aut}\, L$ be an almost geometric action, and let ρ be a fundamental representation of θ. Recall (Lemma 2.6.4) that the induced G-action on $L_\rho(\mathbb{Z})$ factors through the group $G' = \mathbb{Z}_n$ (if $\kappa = 1$) or $\mathbb{D}_n$ (if $\kappa \ne 1$), where $n = \mathrm{ord}\,\theta > 2$. Let $\mathbb{K}$ be the cyclotomic field $\mathbb{Q}(\exp(2\pi i/n))$, and let $\Bbbk \subset \mathbb{K}$ be the real part of $\mathbb{K}$, that is, the extension of $\mathbb{Q}$ obtained by adjoining the real parts of the primitive nth roots of unity. Both $\mathbb{K}$ and $\Bbbk$ are abelian Galois extensions of $\mathbb{Q}$. Denote by $\mathfrak{O}_{\mathbb{K}}$ and $\mathfrak{O}$ the rings of integers of $\mathbb{K}$ and $\Bbbk$, respectively. Unless specified otherwise, we regard $\Bbbk$ and $\mathbb{K}$ as subfields of $\mathbb{C}$ via their standard embeddings. An isotypic $\Bbbk$-representation of G' corresponding to a pair of conjugate primitive nth roots of unity is called *primitive*.

LEMMA 3.3.1
For any primitive irreducible $\Bbbk$-representation ξ of G', the restriction homomorphism $\mathrm{Aut}_G L \to \mathrm{Aut}_G L_\xi(\mathfrak{O})$ is well defined and its image has finite index. If $L = L_\xi(\mathbb{Z})$, the restriction is a monomorphism.

Proof
In view of Lemmas 2.4.2 and 2.6.4, it suffices to consider the case when $L = L_\xi(\mathbb{Z})$ and $G = G'$. The restriction homomorphism is well defined as any G-equivariant

isometry of $L_\xi(\mathbb{Z})$, after extension to $L_\xi(\mathbb{Z}) \otimes \Bbbk$, must preserve the $\Bbbk$-isotypic subspaces. It is a monomorphism since $L_\xi(\mathbb{Q})$ is the minimal $\mathbb{Q}$-vector space such that $L_\xi(\mathbb{Q}) \otimes \Bbbk$ contains $L_\xi(\Bbbk)$. (If an element $g \in \operatorname{Aut}_G L_\xi(\mathbb{Z})$ restricts to the identity of $L_\xi(\mathfrak{O})$, then $\operatorname{Ker}(g - \mathrm{id})$ is a $\mathbb{Q}$-vector space with the above property; hence, it must contain $L_\xi(\mathbb{Q})$.)

It remains to prove that, up to finite index, any G-equivariant $\mathfrak{O}$-automorphism g of $L_\xi(\mathfrak{O})$ extends to a G-equivariant automorphism of $L_\xi(\mathbb{Z}) \otimes \mathfrak{O}$ defined over $\mathbb{Z}$. Up to finite index, one has an orthogonal decomposition $L_\xi(\mathbb{Z}) \otimes \mathfrak{O} \supset \bigoplus L_{\xi_i}(\mathfrak{O})$, the summation over all primitive irreducible representations ξ_i of G. For each such representation ξ_i there is a unique element $\mathfrak{g}_i \in \operatorname{Gal}(\Bbbk/\mathbb{Q})$ such that $\xi_i = \mathfrak{g}_i \xi$, and the automorphism $\bigoplus \mathfrak{g}_i g \mathfrak{g}_i^{-1}$ of $\bigoplus L_{\xi_i}(\mathfrak{O})$ is Galois invariant, that is, defined over $\mathbb{Z}$. □

Now let $\kappa \neq 1$ (i.e., let $G' = \mathbb{D}_n$). Put $M_\xi^\pm = V_\xi^\pm \cap (L \otimes \mathfrak{O})$, and denote by $\operatorname{Aut} M_\xi^\pm$ the group of isometries of $M_\xi^\pm$ defined over $\mathfrak{O}$. (Note that $V_\rho^\pm$ are defined over $\Bbbk$ and thus can be regarded as subspaces of $L_\rho(\Bbbk)$.)

LEMMA 3.3.2
For any primitive irreducible $\Bbbk$-representation ξ of $G' = \mathbb{D}_n$, the restriction homomorphism $\operatorname{Aut}_G L_\xi(\mathfrak{O}) \to \operatorname{Aut} M_\xi^\pm$ is a well-defined monomorphism, and its image has finite index.

Proof
Again, it suffices to consider the case $G = G'$. Obviously, any G-equivariant automorphism of $L_\xi(\mathfrak{O})$ preserves $M_\xi^\pm$. To prove the converse (say, for M_ξ^+), note that, up to a factor, the map J_ξ is defined over $\Bbbk$ (as this is obviously true for an irreducible representation, where $\dim_\Bbbk V_\xi^+ = \dim_\Bbbk V_\xi^- = 1$); that is, there is a dilation $J = kJ_\xi$ of $L_\xi(\Bbbk)$ interchanging V_ξ^+ and V_ξ^-. Furthermore, the factor can be chosen so that $J(M_\xi^-) \subset M_\xi^+$. Since any extension of an isometry $a \in \operatorname{Aut} M_\xi^+$ to $L_\xi(\mathfrak{O})$ must commute with J, on $M_\xi^+ \oplus M_\xi^-$ it must be given by $a \oplus J^{-1} a J'$. On the other hand, due to Corollary 2.4.4, the latter expression does define an extension for all a in a finite-index subgroup of $\operatorname{Aut} M_\xi^+$. □

COROLLARY 3.3.3
The polyhedron $\Sigma_\rho^\pm$ is the union of finitely many copies of a rational Dirichlet polyhedron of $\operatorname{Aut} M_\rho^\pm$ *in* $\mathscr{H}_\rho^\pm$.

3.4. Dirichlet polyhedra: The case $\varphi(\operatorname{ord} \rho) = 2$
Recall that φ is the Euler function: $\varphi(n)$ is the number of positive integers less than n prime to n. Alternatively, $\varphi(n)$ is the degree of the cyclotomic extension of $\mathbb{Q}$ of

order n. Consider a hyperbolic sublattice $M \subset L$, and denote by $\mathscr{H} = \mathscr{H}(M \otimes \mathbb{R})$ the corresponding hyperbolic space. Given a vector $v \in M$, let

$$\mathrm{h}_M(v) = \bigl(\mathrm{h}(v) \cap \mathscr{C}(M \otimes \mathbb{R})\bigr)/\mathbb{R}^* \subset \mathscr{H}.$$

LEMMA 3.4.1
Let $\ell \subset \mathscr{H}$ be a line whose closure intersects the absolute $\partial\mathscr{H}$ at rational points. Then for any integer a there are at most finitely many vectors $v \in M$ such that $v^2 = a$ and the hyperplane $\mathrm{h}_M(v)$ intersects ℓ.

Proof
Let $u_1, u_2 \in M$ be some vectors corresponding to the intersection points $\ell \cap \partial\mathscr{H}$. Then u_1, u_2 span a (scaled) hyperbolic plane $U \subset M$ and the orthogonal complement $U^\perp \subset M$ is elliptic. Therefore, $U \oplus U^\perp$ is of finite index d in M.

Let v be a vector as in the statement. Since $\mathrm{h}_M(v)$ intersects ℓ, one has $v = \lambda b u_1 + (\lambda - 1) b u_2 + v'$ for some $v' \in \frac{1}{d} U^\perp$ and $\lambda \in (0, 1)$. Thus, the equation $v^2 = a$ turns into $-b^2\lambda(1-\lambda) + (v')^2 = a$. Since dv' belongs to a negative definite lattice, $\lambda(1-\lambda) > 0$, and both λbd and $(1-\lambda)bd$ are integers, this equation has finitely many solutions. □

COROLLARY 3.4.2
Let $Q \subset \mathscr{H}$ be a polyhedron whose closure in $\mathscr{H} \cup \partial\mathscr{H}$ is a convex hull of finitely many rational points. Then for any integer a there are at most finitely many vectors $v \in M$ such that $v^2 = a$ and the hyperplane $\mathrm{h}_M(v)$ intersects Q.

Proof
Each edge of Q either is a compact subset of $\mathscr{H}$ or has a rational endpoint on the absolute. In the former case, since the hyperplanes $\mathrm{h}_M(v)$ form a discrete set, the edge intersects finitely many of them. In the latter case, both the intersection points of the absolute and the line containing the edge are rational, and the edge intersects finitely many hyperplanes $\mathrm{h}_M(v)$ due to Lemma 3.4.1. Finally, if a hyperplane does not intersect any edge of Q, it contains at least $\dim \mathscr{H}$ vertices of Q at the absolute and is determined by those vertices. Since Q has finitely many vertices, the number of such hyperplanes is also finite. □

COROLLARY 3.4.3 (of Corollaries 3.4.2 and 2.5.2)
Assume that $\kappa \neq 1$ and $\varphi(\operatorname{ord}\theta) = 2$ (so that $M_\rho^\pm$ are defined over $\mathbb{Z}$), and let $\Pi_\rho^\pm$ be some rational Dirichlet polyhedra of $\operatorname{Aut} M_\rho^\pm$ *in $\mathscr{H}_\rho^\pm$. Then for any integer a there are at most finitely many vectors $v \in M_\rho^\pm$ such that $v^2 = a$ and the subspace $\mathrm{h}_\rho^\pm(v)$ intersects $\Pi_\rho^\pm$ or $J_\rho(\Pi_\rho^\mp)$.*

3.5. Dirichlet polyhedra: The case $\varphi(\operatorname{ord}\theta) \geq 4$

Recall that an algebraic number field F has exactly $\deg(F/\mathbb{Q})$ distinct embeddings to $\mathbb{C}$. Denote by $r(F)$ the number of real embeddings (i.e., those whose image is contained in $\mathbb{R}$), and denote by $c(F)$ the number of pairs of conjugate nonreal ones. Clearly, $r(F) + 2c(F) = \deg F$. The following theorem is due to Dirichlet (see, e.g., [BSh]).

THEOREM 3.5.1

The rank of the group of units (i.e., invertible elements of the ring of integers) of an algebraic number field F is $r(F) + c(F) - 1$.

Let $n = \operatorname{ord}\theta$, and assume that $\varphi(n) \geq 4$. Let $\Bbbk$, $\mathfrak{O}$, and $M^{\pm}_{\rho}$ be as in Section 3.3. Note that $r(\Bbbk) = \deg \Bbbk = \varphi(n)/2 \geq 2$ and $c(\Bbbk) = 0$.

LEMMA 3.5.2

If $\kappa \neq 1$, $\varphi(n) \geq 4$, and $\dim_{\Bbbk} V^{\pm}_{\rho} = 2$, then the rational Dirichlet polyhedra of $\operatorname{Aut} M^{\pm}_{\rho}$ in $\mathscr{H}^{\pm}_{\rho}$ are compact.

Proof

Since $\mathscr{H}^{\pm}_{\rho}$ are hyperbolic lines, it suffices to show that the groups $\operatorname{Aut} M^{\pm}_{\rho}$ are infinite. Consider one of them, say, $\operatorname{Aut} M^{+}_{\rho}$. The lattice M^{+}_{ρ} contains a finite-index sublattice M' whose Gramm matrix (after, possibly, dividing the form by an element of $\mathfrak{O}$) is of the form

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -d \end{bmatrix} \quad \text{with } d > 0 \text{ and } \sqrt{d} \notin \Bbbk.$$

In the former case (which occurs if the form represents zero over $\Bbbk$), the automorphisms of M' are of the form

$$A_{\lambda} = \begin{bmatrix} \pm\lambda & 0 \\ 0 & \pm 1/\lambda \end{bmatrix},$$

where $\lambda \in \mathfrak{O}^{*}$ is a unit of $\Bbbk$. Thus, in this case, $\operatorname{Aut} M^{+}_{\rho}$ contains a free abelian group of rank $r(\Bbbk) - 1 > 0$.

In the latter case, the automorphisms of M' are of the form

$$\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix} \qquad \text{or} \qquad B_{\lambda} = \begin{bmatrix} \alpha & d\beta \\ \beta & \alpha \end{bmatrix},$$

where $\alpha, \beta \in \mathfrak{O}$ and $\lambda = \alpha + \beta\sqrt{d}$ is a unit of $F = \Bbbk(\sqrt{d})$ such that $\alpha^2 - \beta^2 d = 1$. We show that the group of such units is at least $\mathbb{Z}$.

The map $\mu\colon \alpha+\beta\sqrt{d}\mapsto \alpha^2-\beta^2 d$ is a homomorphism from the group of units of F to the group of units of $\Bbbk$, and its cokernel is finite. As $d>0$, the quadratic extension F of $\Bbbk$ has at least two real embeddings to $\mathbb{C}$, that is, $r(F)\ge 2$. Since $r(F)+2c(F)=2\deg\Bbbk=2r(\Bbbk)$, one has* $\operatorname{rk}\operatorname{Ker}\mu=r(F)/2\ge 1$.

The coefficients α,β of all integers $\alpha+\beta\sqrt{d}$ of F have "bounded denominators"; that is, $\alpha,\beta\in\frac{1}{N}\mathfrak{O}$ for some $N\in\mathbb{N}$ (since the abelian group generated by α's and β's has finite rank and $\mathfrak{O}$ has maximal rank). Hence, for any $\lambda\in\operatorname{Ker}\mu$, the map B_λ defines an isometry of V^+ taking $N\cdot M'$ into M', and Lemma 2.4.3 applies. □

Remark. Note that if $\varphi(n)>4$, the form cannot represent zero over $\Bbbk$. Indeed, $\operatorname{Aut}M_\rho^\pm$ would otherwise contain a free abelian group of rank at least 2, which would contradict the discreteness of the action.

The next theorem (as well as Lemma 3.5.2) can probably be deduced from the Godement criterion. We chose to give here an alternative self-contained proof.

THEOREM 3.5.3
If $\kappa\ne 1$ and $\varphi(n)\ge 4$, then the rational Dirichlet polyhedra of $\operatorname{Aut}M_\rho^\pm$ *in $\mathscr{H}_\rho^\pm$ are compact.*

Proof
Let $m=\dim_\Bbbk V_\rho^\pm$. The assertion is obvious if $m=1$, and it is the statement of Lemma 3.5.2 if $m=2$. If $m>2$ and a rational Dirichlet polyhedron $\Pi\subset\mathscr{H}_\rho^+$ is not compact, one can find a line $\mathscr{H}'=\mathscr{H}(V'\otimes_\Bbbk\mathbb{R})$, $V'\subset V_\xi^+$, such that $\Pi\cap\mathscr{H}'$ is not compact. (If $\Pi=\mathscr{H}_\rho^+$, one can take for V' any hyperbolic 2-subspace. Otherwise, one can replace Π with one of its noncompact facets and proceed by induction.) Applying Lemma 3.5.2 to $M'=V'\cap L_\rho(\mathfrak{O})$, one concludes that the polyhedron $\Pi'\subset\mathscr{H}'$ of $\operatorname{Aut}M'$ is compact. On the other hand, in view of Lemma 2.4.2, $\Pi\cap\mathscr{H}'$ must be a finite union of copies of Π'. □

COROLLARY 3.5.4
Assume that $\kappa\ne 1$ and $\varphi(\operatorname{ord}\theta)\ge 4$, and let $\Pi_\rho^\pm$ be some rational Dirichlet polyhedra of $\operatorname{Aut}M_\rho^\pm$ *in $\mathscr{H}_\rho^\pm$. Then for any integer a there are at most finitely many vectors $v\in M^\pm$ such that $v^2=a$ and the subspace $\mathrm{h}_\rho^\pm(v)$ intersects $\Pi_\rho^\pm$ or $J_\rho(\Pi_\rho^\mp)$.*

*In fact, under the assumption on the signature of the form, F has exactly two real embeddings to $\mathbb{C}$, namely, $\Bbbk(\sqrt{d})$ and $\Bbbk(-\sqrt{d})$. In particular, modulo torsion one has $\operatorname{Ker}\mu=\mathbb{Z}$. Indeed, the other embeddings are $\Bbbk(\pm\sqrt{\mathfrak{g}(d)})$, where $\mathfrak{g}\in\operatorname{Gal}(\Bbbk/\mathbb{Q})$ and $\mathfrak{g}\ne 1$, and since all spaces $L_{\mathfrak{g}\rho}(\Bbbk)$ are negative definite, one has $\mathfrak{g}(d)<0$.

3.6. Proof of Theorem 3.1.3

In view of Corollary 3.3.3, one can replace $\Sigma_\rho^\pm$ in the statement with the rational Dirichlet polyhedra $\Pi_\rho^\pm$ of $\operatorname{Aut} M^\pm$ in $\mathscr{H}_\rho^\pm$.

For a root $v \in (L^G)^\perp$, denote by $v^\pm$ its projections to $V^\pm$ (the (± 1)-eigenspaces of c on $L_\rho(\mathbb{Q}) \otimes \mathbb{R}$), and denote by $v_\rho^\pm$ its projections to $V_\rho^\pm$. The projections $v^\pm$ are rational vectors with uniformly bounded denominators; that is, there is an integer N, depending only on θ, such that $Nv^\pm \in M^\pm$. Under the assumption (ρ is nonreal and $\kappa \neq 1$), the set $\mathrm{h}_\rho^+(v)$ is not empty if and only if each of $v_\rho^\pm$ either is trivial or has negative square. In any case, $(v_\rho^\pm)^2 \le 0$ and, hence, $(v^\pm)^2 \le 0$. Thus, the squares $(Nv^\pm)^2$ take finitely many distinct integral values, and the statement of the theorem follows from Corollaries 3.4.3 and 3.5.4. □

4. The proof

4.1. Period spaces related to K3-surfaces

Let $L = 2E_8 \oplus 3U$. Consider the variety Per of positive definite 3-subspaces in $L \otimes \mathbb{R}$. It is a homogeneous symmetric space (of noncompact type)

$$\mathrm{Per} = \mathrm{SO}^+(3, 19)/\,\mathrm{SO}(3) \times \mathrm{SO}(19).$$

The orthogonal projection of a positive definite 3-subspace to another one is nondegenerate. Hence, one can orient all the subspaces in a coherent way; this gives an orientation of the canonical 3-dimensional vector bundle over Per. In what follows we assume that such an orientation is fixed; the corresponding orientation of a space $\mathfrak{w} \in \mathrm{Per}$ is referred to as its *prescribed orientation*.

Given a vector $v \in L$ with $v^2 = -2$, let $\mathfrak{h}_v \subset \mathrm{Per}$ be the set of the 3-subspaces orthogonal to v. Put

$$\mathrm{Per}_0 = \mathrm{Per} \smallsetminus \bigcup_{v \in L,\, v^2 = -2} \mathfrak{h}_v.$$

The space Per_0 is called the *period space of marked Einstein K3-surfaces*.

There is a natural S^2-bundle $K\Omega \to \mathrm{Per}$, where

$$K\Omega = \bigl\{(\mathfrak{w}, \gamma) \bigm| \mathfrak{w} \in \mathrm{Per},\ \gamma \in \mathfrak{w},\ \gamma^2 = 1\bigr\}.$$

The pullback $K\Omega_0$ of Per_0 is called the *period space of marked Kähler K3-surfaces*. Finally, let Ω be the variety of oriented positive definite 2-subspaces of $L \otimes \mathbb{R}$; it is called the *period space of marked K3-surfaces*. One can identify Ω with the projectivization

$$\bigl\{\omega \in L \otimes \mathbb{C} \bigm| \omega^2 = 0,\ \omega \cdot \bar\omega > 0\bigr\}/\mathbb{C}^*, \tag{4.1.1}$$

associating to a complex line generated by ω the plane $\{\mathrm{Re}(\lambda\omega) \mid \lambda \in \mathbb{C}\}$ with the orientation given by a basis $\mathrm{Re}\,\omega$, $\mathrm{Re}\,i\omega$. Thus, Ω is a 20-dimensional complex variety, which is an open subset of the quadric defined in the projectivization of $L \otimes \mathbb{C}$ by $\omega^2 = 0$. The spaces $K\Omega_0$ and Per_0 are (noncompact) real analytic varieties of dimensions 59 and 57, respectively.

4.2. Period maps

A *marking* of a $K3$-surface X is an isometry $\varphi\colon H^2(X) \to L$. It is called *admissible* if the orientation of the space $\mathfrak{w} = \langle \operatorname{Re}\varphi(\omega), \operatorname{Im}\varphi(\omega), \varphi(\gamma)\rangle$, where $\omega \in H^{2,0}(X)$ and γ is the fundamental class of a Kähler structure on X, coincides with its prescribed orientation. A *marked $K3$-surface* is a $K3$-surface X equipped with an admissible marking. Two marked $K3$-surfaces (X, φ) and (Y, ψ) are *isomorphic* if there exists a biholomorphism $f\colon X \to Y$ such that $\psi = \varphi \circ f^*$. Denote by $\mathscr{T}$ the set of isomorphism classes of marked $K3$-surfaces.

The *period map* $\mathrm{per}\colon \mathscr{T} \to \Omega$ sends a marked $K3$-surface (X, φ) to the 2-subspace $\{\operatorname{Re}\varphi(\omega) \mid \omega \in H^{2,0}(X)\}$, the orientation given by $(\operatorname{Re}\varphi(\omega), \operatorname{Re}\varphi(i\omega))$. (We always use the same notation φ for various extensions of the marking to other coefficient groups.) Alternatively, $\mathrm{per}(X, \varphi)$ is the line $\varphi(H^{2,0}(X))$ in the complex model (4.1.1) of Ω.

A *marked polarized $K3$-surface* (see the discussion in Section 1.4) is a $K3$-surface X equipped with the fundamental class γ_X of a Kähler structure and an admissible marking $\varphi\colon H^2(X) \to L$. Two marked polarized $K3$-surfaces (X, φ, γ_X) and (Y, ψ, γ_Y) are *isomorphic* if there exists a biholomorphism $f\colon X \to Y$ such that $\psi = \varphi \circ f^*$ and $f^*(\gamma_Y) = \gamma_X$. Denote by $K\mathscr{T}$ the set of isomorphism classes of marked polarized $K3$-surfaces.

The *period map* $\mathrm{per}^K : K\mathscr{T} \to K\Omega$ sends a triple $(X, \varphi, \gamma_X) \in K\mathscr{T}$ to the point $(\mathfrak{w}, \varphi(\gamma_X)) \in K\Omega$, where $\mathfrak{w} = \mathrm{per}(X, \varphi) \oplus \varphi(\gamma_X) \in \mathrm{Per}$ is as above. When this does not lead to confusion, we abbreviate $\mathrm{per}^K(X, \varphi, \gamma_X)$ as $\mathrm{per}^K(X)$.

As is known (see [PS] and [Ku], or [Siu]), the period map per^K is a bijection to $K\Omega_0$, and the image of per is Ω_0. Moreover, $K\Omega_0$ is a fine period space of marked polarized $K3$-surfaces; that is, the following statement holds (see [B]).

THEOREM 4.2.1
The space $K\Omega_0$ is the base of a universal smooth family of marked polarized $K3$-surfaces, that is, a family $p\colon \Phi \to K\Omega_0$ such that any other smooth family $p'\colon X \to S$ of marked polarized $K3$-surfaces is induced from p by a unique smooth map $S \to K\Omega_0$. The latter is given by $s \mapsto \mathrm{per}^K(X_s)$, where X_s is the fiber over $s \in S$.

Since the only automorphism of a $K3$-surface identical on the homology is the identity (see [PS]), Theorem 4.2.1 can be rewritten in a slightly stronger form.

THEOREM 4.2.2
For any smooth family $p'\colon X \to S$ of marked polarized $K3$-surfaces, there is a unique smooth fiberwise map $X \to \Phi$ (see Theorem 4.2.1) that covers the map $S \to K\Omega_0$, $s \mapsto \operatorname{per}^K(X_s)$, of the bases and is an isomorphism of marked polarized $K3$-surfaces in each fiber.

COROLLARY 4.2.3
Let (X, γ_X) and (Y, γ_Y) be two polarized $K3$-surfaces, and let $g\colon H^2(Y) \to H^2(X)$ be an isometry such that $g(\gamma_Y) = \gamma_X$. Then we have the following.
(1) *If $g(H^{2,0}(Y)) = H^{2,0}(X)$, then g is induced by a unique holomorphic map $X \to Y$, which is a biholomorphism.*
(2) *If $g(H^{2,0}(Y)) = H^{0,2}(X)$, then $-g$ is induced by a unique antiholomorphic map $X \to Y$, which is an antibiholomorphism.*

4.3. Equivariant period spaces
In this section we construct the period space of marked polarized $K3$-surfaces with a G-action of a given homological type. Recall that we define the homological type as the class of the twisted induced action $\theta_X\colon G \to \operatorname{Aut} H^2(X)$ modulo conjugation by elements of $\operatorname{Aut} H^2(X)$. A marking takes θ_X to an action $\theta\colon G \to \operatorname{Aut} L$. Note in this respect that, since we work with admissible markings only, it would be more natural to consider θ_X up to conjugation by elements of the subgroup $\operatorname{Aut} L \cap \mathrm{O}^+(L \otimes \mathbb{R})$. However, this stricter definition would be equivalent to the original one as the central element $-\operatorname{id} \in \operatorname{Aut} L$ belongs to $\mathrm{O}^-(L \otimes \mathbb{R})$.

PROPOSITION 4.3.1
Let X be a $K3$-surface supplied with a Klein action of a finite group G. Then the twisted induced action $\theta_X\colon G \to \operatorname{Aut} H^2(X)$ is geometric, and the augmentation $\kappa\colon G \to \{\pm 1\}$ and the pair $\rho, \bar\rho\colon G^0 \to S^1$ of complex conjugated fundamental representations introduced in Section 1.7 coincide with those determined by θ_X (see Section 2.6).

Proof
Since G is finite, X admits a Kähler metric preserved by the holomorphic elements of G and conjugated by the antiholomorphic elements. Take for γ_X the fundamental class of such a metric. Pick also a holomorphic form on X, and denote by ω its cohomology class. Let $\mathfrak{w}$ be the space spanned by γ_X, $\operatorname{Re}\omega$, and $\operatorname{Im}\omega$, and let $\ell \subset \mathfrak{w}$ be the subspace generated by γ_X. Then the flag $\ell \subset \mathfrak{w}$ attests the fact that θ_X is almost

geometric, and this flag can be used to define κ and ρ. As γ_X and ω cannot be simultaneously orthogonal to an integral vector $v \in H^2(X)$ of square (-2), the action is geometric. $\square$

Let $\theta\colon G \to \operatorname{Aut} L$ be an almost geometric action on L. The assignment $g\colon \mathfrak{w} \mapsto \kappa(g)g(\mathfrak{w})$, where $g \in G$ and $-\mathfrak{w}$ stands for $\mathfrak{w}$ with the opposite orientation, defines a G-action on the space Per. Denote by Per^G the subspace of the G-fixed points, and let $\mathrm{Per}_0^G = \mathrm{Per}^G \cap \mathrm{Per}_0$. There is a natural map $K\Omega^G \to \mathrm{Per}^G$, where

$$K\Omega^G = \left\{(\mathfrak{w}, \gamma) \,\middle|\, \mathfrak{w} \in \mathrm{Per}^G,\ \gamma \in \mathfrak{w}^G,\ \gamma^2 = 1\right\}$$

with $\mathfrak{w}^G$ standing for the G-invariant part of $\mathfrak{w}$. Put $K\Omega_0^G = \{(\mathfrak{w}, \gamma) \in K\Omega^G \mid \mathfrak{w} \in \mathrm{Per}_0^G\}$, and denote by Ω^G (resp., Ω_0^G) the image of $K\Omega^G$ (resp., $K\Omega_0^G$) under the projection $K\Omega \to \Omega$. The following statement is a paraphrase of the definitions.

PROPOSITION 4.3.2
An almost geometric action $\theta\colon G \to \operatorname{Aut} L$ is geometric if and only if the space $K\Omega_0^G$ (as well as Per_0^G and Ω_0^G) is nonempty.

Let (X, φ) be a marked $K3$-surface. We say that a Klein G-action on X and an action $\theta\colon G \to \operatorname{Aut} L$ are *compatible* if for any $g \in G$ one has $\theta_X g = \varphi^{-1} \circ \theta g \circ \varphi$, where $\theta_X\colon G \to \operatorname{Aut} H^2(X)$ is the twisted induced action. If a marking is not fixed, we say that a Klein G-action on X is compatible with θ if X admits a compatible admissible marking, that is, if θ_X is isomorphic to θ.

PROPOSITION 4.3.3
An action $\theta\colon G \to L$ is compatible with a Klein G-action on a marked $K3$-surface if and only if θ is geometric. Furthermore, $K\Omega_0^G$ is a fine period space of marked polarized $K3$-surfaces with a Klein G-action compatible with θ; that is, it is the base of a universal smooth family of marked polarized $K3$-surfaces with a Klein G-action compatible with θ.

Proof
The “only if” part follows from Proposition 4.3.1, and the “if” part from Corollary 4.2.3 and Proposition 4.3.2. The fact that $K\Omega_0^G$ is a fine period space is an immediate consequence of Theorem 4.2.2. $\square$

PROPOSITION 4.3.4
Let $\kappa\colon G \to \{\pm 1\}$ be the augmentation, and let $\rho\colon G^0 \to S^1$ be a fundamental representation associated with θ. If $\rho = 1$, then the spaces $K\Omega^G$ and Ω^G are connected. If $\rho \neq 1$, then the space $K\Omega^G$ (resp., Ω^G) consists of two components, which are

transposed by the involution $(\mathfrak{w}, \gamma) \mapsto (\mathfrak{w}, -\gamma)$ *(resp., the involution reversing the orientation of* 2*-subspaces). If, besides,* $\rho \neq \bar{\rho}$*, the two components of* $K\Omega^G$ *(or* Ω^G*) are in a one-to-one correspondence with the two fundamental representations* ρ*,* $\bar{\rho}$*.*

Proof

Since Per is a hyperbolic space and G acts on Per by isometries, the space Per^G is contractible. The projections $K\Omega^G \to \mathrm{Per}^G$ and $K\Omega_0^G \to \mathrm{Per}_0^G$ are (trivial) S^p-bundles, where $p = 0$ if $\rho \neq 1$, $p = 1$ if $\rho = 1$ and $\kappa \neq 1$, and $p = 2$ if $\rho = 1$ and $\kappa = 1$. Finally, since each space $\mathfrak{w} \in \mathrm{Per}$ has its prescribed orientation, a choice of a G-invariant vector $\gamma \in \mathfrak{w}$ determines an orientation of $\gamma^\perp \subset \mathfrak{w}$ and hence a fundamental representation. □

4.4. The moduli spaces

Fix a geometric action $\theta\colon G \to \operatorname{Aut} L$, and consider the space $K\mathfrak{M}^G = K\Omega_0^G/\operatorname{Aut}_G L$. In view of Proposition 4.3.3, it is the "moduli space" of polarized $K3$-surfaces with Klein G-actions compatible with θ. Given such a surface (X, γ_X), pick a marking $\varphi\colon H^2(X) \to L$ compatible with θ, and denote by $\mathfrak{m}^K(X, \gamma_X) = \mathfrak{m}^K(X)$ the image of $\mathrm{per}^K(X, \varphi, \gamma_X)$ in $K\mathfrak{M}^G$. Since any two compatible markings differ by an element of $\operatorname{Aut}_G L$, the point $\mathfrak{m}^K(X, \gamma_X)$ is well defined. The following statement is an immediate consequence of Proposition 4.3.3 and the local connectedness of $K\Omega_0^G$.

PROPOSITION 4.4.1

Let (X, γ_X) *and* (Y, γ_Y) *be two polarized* $K3$*-surfaces with Klein* G*-actions compatible with* θ*. Then* X *and* Y *are* G*-equivariantly deformation equivalent if and only if* $\mathfrak{m}^K(X)$ *and* $\mathfrak{m}^K(Y)$ *belong to the same connected component of* $K\mathfrak{M}^G$*.*

In Lemmas 4.4.2–4.4.7 we give a more detailed description of period and moduli spaces. We use the notation of Section 3.1.

LEMMA 4.4.2 (The case $\rho = 1$, $\kappa = 1$)

If $\rho = 1$ *and* $\kappa = 1$*, then* $K\Omega_0^G \cong (\mathscr{H}(L^G) \smallsetminus \Delta) \times S^2$*, where* $\operatorname{codim} \Delta \geq 3$*. In particular,* $K\Omega_0^G$ *and hence* $K\mathfrak{M}^G$ *are connected.*

LEMMA 4.4.3 (The case $\rho = 1$, $\kappa \neq 1$)

If $\rho = 1$ *and* $\kappa \neq 1$*, then* $K\mathfrak{M}^G$ *is a quotient of the connected space* $((\mathscr{H}(L^G) \times \operatorname{Int}\Gamma_\rho^-) \smallsetminus \Delta) \times S^1$*, where* $\operatorname{codim} \Delta \geq 2$*. In particular,* $K\mathfrak{M}^G$ *is connected.*

LEMMA 4.4.4 (The case $\rho \neq 1$ real, $\kappa = 1$)

If $\rho \neq 1$ *is real and* $\kappa = 1$*, then* $K\mathfrak{M}^G$ *is a quotient of the two-component space*

$\big((\operatorname{Int}\Gamma_1 \times \mathscr{H}(L_\rho(\mathbb{R}))) \smallsetminus \Delta\big) \times S^0$, *where* codim $\Delta \geq 2$. *In particular,* $K\mathfrak{M}^G$ *has at most two connected components, which are interchanged by the complex conjugation* $X \mapsto \bar{X}$.

LEMMA 4.4.5 (The case $\rho \neq 1$ real, $\kappa \neq 1$)
If $\rho \neq 1$ *is real and* $\kappa \neq 1$, *then* $K\mathfrak{M}^G$ *is a quotient of the two-component space* $\big((\operatorname{Int}\Gamma_1 \times \operatorname{Int}\Gamma_\rho^+ \times \operatorname{Int}\Gamma_\rho^-) \smallsetminus \Delta\big) \times S^0$, *where* codim $\Delta \geq 2$. *In particular,* $K\mathfrak{M}^G$ *has at most two connected components, which are interchanged by the complex conjugation* $X \mapsto \bar{X}$.

LEMMA 4.4.6 (The case ρ nonreal, $\kappa = 1$)
If ρ *is nonreal and* $\kappa = 1$, *then* $K\mathfrak{M}^G$ *is a quotient of the two-component space* $((\operatorname{Int}\Gamma_1 \times \mathbb{P}_J\mathscr{C}_\rho) \smallsetminus \Delta) \times S^0$, *where* $\mathbb{P}_J\mathscr{C}_\rho$ *is the space of positive definite (over* $\mathbb{R}$*)* J_ρ*-complex lines in* $L_\rho(\mathbb{R})$ *and* codim $\Delta \geq 2$. *In particular,* $K\mathfrak{M}^G$ *has at most two connected components, which are interchanged by the complex conjugation* $X \mapsto \bar{X}$.

LEMMA 4.4.7 (The case ρ nonreal, $\kappa \neq 1$)
If ρ *is nonreal and* $\kappa \neq 1$, *then* $K\mathfrak{M}^G$ *is a quotient of the space* $((\operatorname{Int}\Gamma_1 \times \Sigma_\rho^+) \smallsetminus \Delta) \times S^0$, *where* Δ *is the union of a subset of codimension at least* 2 *and finitely many hyperplanes of the form* $\operatorname{Int}\Gamma_1 \times (\mathrm{h}_\rho^\pm(v) \cap \Sigma_\rho^+)$ *defined by roots* $v \in (L^G)^\perp$. *This space has finitely many connected components; hence, so does* $K\mathfrak{M}^G$.

Proof of Lemmas 4.4.2 – 4.4.5
One has

- $\operatorname{Per}^G = \mathscr{H}(L^G \otimes \mathbb{R})$ in Lemma 4.4.2 (i.e., $\rho = 1, \kappa = 1$),
- $\operatorname{Per}^G = \mathscr{H}(L^G \otimes \mathbb{R}) \times \mathscr{H}(V_\rho^-)$ in Lemma 4.4.3 (i.e., $\rho = 1, \kappa \neq 1$),
- $\operatorname{Per}^G = \mathscr{H}(L^G \otimes \mathbb{R}) \times \mathscr{H}(L_\rho(\mathbb{R}))$ in Lemma 4.4.4, (i.e., $\rho \neq 1$ real, $\kappa = 1$), and
- $\operatorname{Per}^G = \mathscr{H}(L^G \otimes \mathbb{R}) \times \mathscr{H}(V_\rho^+) \times \mathscr{H}(V_\rho^-)$ in Lemma 4.4.5 (i.e., $\rho \neq 1$ real, $\kappa \neq 1$).

Thus, in each case, Per^G is a product $\prod \mathscr{H}(L_i \otimes \mathbb{R})$ of the hyperbolic spaces of orthogonal indefinite sublattices $L_i \subset L$ such that $\bigoplus_i L_i \oplus L^\bullet$ is a finite-index sublattice in L. Consider the quotient $\mathscr{Q}_0 = \operatorname{Per}_0^G / W$, where $W = \prod W_i$ (the product in $W_G(L)$) and $W_i = 1$ if $\sigma_+ L_i > 1$ or $W_i = W_G((L_i \oplus L^\bullet)\widehat{\ })$ if $\sigma_+ L_i = 1$. The quotient $\mathscr{Q}_0$ can be identified with a subspace of $\mathscr{Q} = \prod \operatorname{Int}\Gamma_i$, where Γ_i is a fundamental Dirichlet polyhedron of W_i in $\mathscr{H}(L_i \otimes \mathbb{R})$. (Note that $\Gamma_i = \mathscr{H}(L_i \otimes \mathbb{R})$ unless $\sigma_+ L_1 = 1$.) Put $\Delta = \mathscr{Q} \smallsetminus \mathscr{Q}_0$; it is the union of the walls $\mathfrak{h}_v \cap \mathscr{Q}$ over all roots $v \in L$.

For a root $v \in L$, one has $\operatorname{codim}(\mathfrak{h}_v \cap \mathscr{Q}) \geq \sum \sigma_+ L_i$, the summation over all i such that the projection of v to L_i is nontrivial. Thus, a wall $\mathfrak{h}_v \cap \mathcal{Q}$ may have codimension 1 only if $v \in (L_i \oplus L^\bullet)\widehat{\ }$ and $\sigma_+ L_i = 1$. However, in this case $\mathfrak{h}_v \cap \mathcal{Q} = \emptyset$ due to Theorem 3.1.2. Hence, codim $\Delta \geq 2$ and the space $\mathcal{Q}_0$ is connected. □

Proof of Lemma 4.4.6
In this lemma, $\operatorname{Per}_0^G / W_G((L^G \oplus L^\bullet)\widehat{\ })$ can be identified with a subset of $\operatorname{Int}\Gamma_1 \times \mathbb{P}_J \mathscr{C}_\rho$, and the proof follows the lines of the proof of Lemmas 4.4.2 – 4.4.5. □

Proof of Lemma 4.4.7
One has $\operatorname{Per}^G = \mathscr{H}(L^G \otimes \mathbb{R}) \times \mathscr{H}(V_\rho^+)$, and the quotient space $\mathcal{Q}_0 = \operatorname{Per}_0^G / (W_G(L_\rho(\mathbb{Z})^\perp) \cdot \operatorname{Aut}_G^0(L_\rho(\mathbb{Z}))$ can be identified with a subset of $\operatorname{Int}\Gamma_1 \times \Sigma_\rho^+$. Now the statement follows from Theorems 3.1.2 and 3.1.3. □

4.5. Proofs of Theorems 1.7.1 and 1.7.2
Theorem 1.7.2 follows from Lemmas 4.4.2 – 4.4.6. Theorem 1.7.1 consists, in fact, of two statements: finiteness of the number of equivariant deformation classes within a given homological type of G-actions (of a given group G), and finiteness of the number of homological types of faithful actions. The former is a direct consequence of Lemmas 4.4.2 – 4.4.7. The latter is a special case of the finiteness of the number of conjugacy classes of finite subgroups in an arithmetic group (see [BH] and [Bo]). □

5. Degenerations

5.1. Passing through the walls
Let $L = 2E_8 \oplus 3U$. Consider a geometric G-action $\theta\colon G \to \operatorname{Aut} L$. Pick a G-invariant elliptic root system $R \subset L$. Denote by $\bar{R}$ the sublattice of L generated by all roots in $(R + L^\bullet)\widehat{\ }$. Clearly, $\bar{R}$ is a G-invariant root system; it is called the *θ-saturation* of R. We say that R is *θ-saturated* if $R = \bar{R}$. Any θ-saturated root system R is *saturated*, that is, R contains all roots in $R\widehat{\ }$.

Fix a camera C of $\bar{R}$, and denote by S_C its group of symmetries. Then, for any $g \in G$, the restriction of θg to $\bar{R}$ admits a unique decomposition $s_g w_g$, $s_g \in S_C$, $w_g \in W(\bar{R})$. Let $\theta_R(g) = (\theta g) w_g^{-1} \in \operatorname{Aut} L$ with w_g extended to L identically on $\bar{R}^\perp$. We call the map $\theta_R\colon G \to \operatorname{Aut} L$ the *degeneration* of θ at R.

PROPOSITION 5.1.1
The map θ_R is a geometric G-action. Up to conjugation by an element of $W(\bar{R})$, it does not depend on the choice of a camera C of $\bar{R}$ and is the only action with the following properties:
(1) *the action induced by θ_R on $\bar{R}$ is admissible;*

(2) *θ and θ_R induce the same action on each of the following sets: $\bar{R}^\perp$, $\operatorname{discr}\bar{R}$, and the set of irreducible components of $\bar{R}$.*

Conversely, if $\bar{R} \subset L$ is a saturated root system and $\theta_R\colon G \to L$ is an action satisfying (1) and (2) above, then $\bar{R}$ is θ-saturated and θ_R is a degeneration of θ at $\bar{R}$.

Proof

Clearly, both θ and θ_R factor through a subgroup of $\operatorname{Aut}\bar{R} \times \operatorname{Aut}\bar{R}^\perp$. The composition of θ_R with the projection to $\operatorname{Aut}\bar{R}^\perp$ coincides with that of θ; the composition of θ_R with the projection to $\operatorname{Aut}\bar{R}$ is the composition of θ, the projection to $\operatorname{Aut}\bar{R}$, and the quotient homomorphism $\operatorname{Aut}\bar{R} \to S_C \subset \operatorname{Aut}\bar{R}$. Hence, θ_R is a homomorphism. Furthermore, another choice of a camera C' of $\bar{R}$ leads to another representation $\operatorname{Aut}\bar{R} \to S_{C'} \subset \operatorname{Aut}\bar{R}$, which is conjugated to the original one by a unique element $w_0 \in W(\bar{R})$; the latter can be regarded as an automorphism of L.

All other statements follow directly from the construction. For the uniqueness part, it suffices to notice that, for any irreducible root system R' and a camera C' of R', the natural homomorphism $S_{C'} \to \operatorname{Aut}\operatorname{discr}R'$ is a monomorphism. □

PROPOSITION 5.1.2

Let R be a θ-saturated root system, and let $R' \subset R$ be the sublattice generated by all roots in $R \cap (L^G)^\perp$. Then, up to conjugation by an element of $W(R)$, the degenerations θ_R and $\theta_{R'}$ coincide. In particular, θ_R can be chosen to coincide with θ on $(R')^\perp$.

Proof

Take for C a camera adjacent to the intersection of the mirrors defined by the roots of R'. Then C has an invariant face (possibly empty), and the decomposition $\theta g|_R = s_g w_g$ has $w_g \in W(R')$ for any $g \in G$. □

If the action is properly Klein, one can take for R the θ-saturated root system generated by all roots in $(L^G)^\perp$ orthogonal to a given wall $\mathrm{h}^+_\rho(v)$. The resulting degeneration is called the *degeneration* at the wall $\mathrm{h}^+_\rho(v)$.

Remark. The degeneration construction gives rise to a partial order on the set of homological types of geometric actions of a given finite group G.

5.2. Degenerations of $K3$-surfaces

Let (G, κ) be an augmented group. Denote by D_ε the disk $\{s \in \mathbb{C} \mid |s| < \varepsilon\}$. The composition of κ and the $\{\pm 1\}$-action via the complex conjugation $s \mapsto \bar{s}$ is a Klein G-action on D_ε. A *G-equivariant degeneration of $K3$-surfaces* is a nonsingular complex 3-manifold X supplied with a Klein G-action and a G-equivariant (with respect to the above G-action on D_ε) proper analytic map $p\colon X \to D_\varepsilon$ so that the following requirements are satisfied.

- The projection p has no critical values except $s = 0$.
- The fibers $X_s = p^{-1}(s)$ of p are normal $K3$-surfaces, nonsingular unless $s = 0$.

(By a singular $K3$-surface we mean a surface whose desingularization is $K3$.) Given a degeneration X, denote by $\pi_s\colon \tilde{X}_s \to X_s$, $s \in D_\varepsilon$, the minimal resolution of singularities of X_s (see, e.g., [L]). (Note that $\tilde{X}_s = X_s$ unless $s = 0$.) From the uniqueness of the minimal resolution, it follows that any Klein action lifts from X_s to $\tilde{X}_s$. Thus, if either $\kappa = 1$ or s is real, $\tilde{X}_s$ inherits a natural Klein action of G.

THEOREM 5.2.1
Let $p\colon X \to D_\varepsilon$ be a G-equivariant degeneration of $K3$-surfaces. Pick a regular value $t \in D_\varepsilon$ real if $\kappa \neq 1$. Denote by $R \subset H^2(X_t)$ the subgroup Poincaré dual to the kernel of the inclusion homomorphism $H_2(X_t) \to H_2(X) = H_2(X_0)$. Then R is a saturated elliptic root system and the twisted induced G-action on $H^2(\tilde{X}_0)$ is isomorphic to the degeneration at R of the twisted induced G-action on $H^2(X_t)$.

Remark. A statement analogous to Theorem 5.2.1 holds, in a more general situation, for a family of complex surfaces whose singular fiber at $s = 0$ has at worst simple singularities, that is, those of type A_n, D_n, E_6, E_7, or E_8. The only difference is the fact that one can no longer claim that the root system R is saturated, and one should consider the degeneration at R *without* passing to its saturation first. (In particular, the algebraic definition of degeneration should be changed. Our choice of the definition, incorporating the saturation operation, was motivated by our desire to assure that the result should be a geometric action.) The proof given below applies to the general case with obvious minor modifications.

Proof
It is more convenient to switch to the twisted induced actions θ_s in the homology groups $H_2(X_s)$, $s \in D_\varepsilon$; they are Poincaré dual to the twisted induced actions in the cohomology.

Let $\imath_s\colon H_2(\tilde{X}_s) \to H_2(X)$, $s \in D_\varepsilon$, be the composition of $(\pi_s)_*$ and the inclusion homomorphism. Put $R_s = \operatorname{Ker} \imath_s$. Consider sufficiently small G-invariant open balls $B_i \subset X$ about the singular points of X_0, and let $B = \bigcup B_i$. One can assume that t is real and sufficiently small, so that $M_i = X_t \cap B_i$ are Milnor fibers of the singular points. Then there is a G-equivariant diffeomorphism $d'\colon X_t \smallsetminus B \to X_0 \smallsetminus B$.

Recall that all singular points of the $K3$-surface X_0 are simple and that R_0 is a saturated elliptic root system (Lemma 5.2.2). In particular, d' extends to a diffeomorphism $d\colon X_t \to \tilde X_0$. Note that neither d nor the induced isomorphism $d_*\colon H_2(X_t) \to H_2(\tilde X_0)$ is canonical and that d_* does not need to be G-equivariant. However, d_* does preserve the G-action on the sets of irreducible components of the root systems R_t, R_0 (as it is just the G-action on the set of singular points of X_0), and in view of natural identifications $R_s^\perp = H_2(X_s \smallsetminus B)/\operatorname{Tors}$ and $\operatorname{discr} R_s = H_1(\partial(X_s \smallsetminus B))$, $s = t, 0$, and the fact that d' commutes with G, the restrictions of d_* to $R_t^\perp$ and $\operatorname{discr} R_t$ are G-equivariant. Finally, the action induced by θ_0 on R_0 is admissible; it preserves the camera defined by the exceptional divisors in $\tilde X_0$ (Lemma 3.2.3). Thus, after identifying $H_2(X_t)$ and $H_2(\tilde X_0)$ via d_*, the actions $\theta = \theta_t$ and $\theta_R = \theta_0$ have properties (1) and (2) from Proposition 5.1.1, and Proposition 5.1.1 implies that θ_0 is the degeneration of θ_t at R_t. □

For completeness, we outline the proof of the following lemma, which refines the well-known fact that a $K3$-surface can have at worst simple singular points.

LEMMA 5.2.2
Let X be a $K3$-surface. Then any negative definite sublattice $R \subset H^2(X)$ generated by classes of irreducible curves is a saturated root system.

Proof
As follows from the adjunction formula, any irreducible curve $C \subset X$ of negative self-intersection is a (-2)-curve, that is, a nonsingular rational curve of self-intersection (-2). Thus, any sublattice R as in the statement is an elliptic root system generated by classes of irreducible (-2)-curves.

From the Riemann-Roch theorem it follows that, given a root $r \in \operatorname{Pic} X$, there is a unique (-2)-curve $C \subset X$ whose cohomology class is $\pm r$. Thus, the set of all roots in $\operatorname{Pic} X$ splits into disjoint union $\Delta_+ \cup \Delta_-$, where Δ_+ is the set of *effective* roots (those realized by curves) and $\Delta_- = -\Delta_+$. Furthermore, the set Δ_+ is closed with respect to positive linear combinations, and the function $\#\colon \Delta_+ \to \mathbb{N}$ counting the number of components of the curve representing a root $r \in \Delta_+$ is a well-defined homomorphism in the sense that, whenever a root r is decomposed into $\sum a_i r_i$ for some $r_i \in \Delta_+$ and $a_i \in \mathbb{N}$, one has $r \in \Delta_+$ and $\#r = \sum a_i \#r_i$. (Note that, if X is algebraic, the roots $r \in \Delta_+$ with $\#r = 1$ define the walls of the rational Dirichlet polyhedron of $\operatorname{Aut}\operatorname{Pic} X$ in $\mathscr{H}(\operatorname{Pic} X \otimes \mathbb{R})$ containing the fundamental class of a Kähler structure; see, e.g., [PS] or [DIK]. If X is nonalgebraic, they define the walls of a distinguished camera of $\operatorname{Pic} X$.)

Now let $R \in \operatorname{Pic} X$ be a root system as in the statement, and let $\bar R \supset R$ be its saturation in $\operatorname{Pic} X$. Consider the subsets $\bar\Delta_\pm = \bar R \cap \Delta_\pm$. They form a partition of the set of roots of $\bar R$, one has $\bar\Delta_- = -\bar\Delta_+$, and $\bar\Delta_+$ is closed with respect to positive linear combinations. Hence, there is a unique camera C of $\bar{R}$ such that $\bar{\Delta}_+$ is the set of roots positive with respect to C (see, e.g., [Bou]); this means that the roots $r_1, \dots, r_k \in \bar{\Delta}_+$ defining the walls of C form a basis of $\bar{R}$ and each root $r \in \bar{\Delta}_+$ is a positive linear combination of the r_i's. Hence, any root $r \in \bar{\Delta}_+$ with $\# r = 1$ must be one of the r_i's. Since R is generated by such roots, one has $R = \bar{R}$. □

6. Are $K3$-surfaces quasi-simple?

6.1. $K\mathfrak{M}^G$ with walls

Here we construct an example of a geometric action of the group $G = \mathbb{D}_3$ (with ρ nonreal and $\kappa \neq 1$) whose associated space $K\mathfrak{M}^G$ has more than two components (i.e., the action of $\operatorname{Aut}_G L$ on the set of connected components of Per_0^G is not transitive). This shows that the assumptions on the action in Theorem 1.7.2 cannot be removed. However, the resulting Klein actions on $K3$-surfaces are not diffeomorphic (Proposition 6.2.1), that is, they do not constitute a counterexample to quasi-simplicity of $K3$-surfaces.

PROPOSITION 6.1.1

There is a homological type of $\mathbb{D}_3$-action on $L \cong 3U \oplus 2E_8$ realizable by six $\mathbb{D}_3$-equivariant deformation classes of $K3$-surfaces. More precisely, there is a geometric action of $G = \mathbb{D}_3$ on L such that the corresponding moduli space $K\mathfrak{M}^G$ consists of three pairs of complex conjugate connected components.

Proof

Fix a decomposition $L = P \oplus Q$, where $P \cong 2U$ and $Q \cong U \oplus 2E_8$. Define a $\mathbb{D}_3$-action on L as follows. On Q, the $\mathbb{Z}_3$ part of $\mathbb{D}_3$ acts trivially, and each nontrivial involution of $\mathbb{D}_3$ acts via multiplication by -1. On P, fix a basis u_1, v_1, u_2, and v_2 so that $u_i^2 = v_i^2 = 0$, $u_i \cdot v_i = 1$, and $u_i \cdot u_j = v_i \cdot v_j = u_i \cdot v_j = 0$ for $i \neq j$. Choose an order 3 element t and an order 2 element s in $\mathbb{D}_3$, and define their action on P by the matrices

$$T = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad S = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix},$$

respectively. Note that $L^\bullet$ is trivial. Hence, according to Proposition 4.3.3, the constructed $\mathbb{D}_3$-action on L is realizable by a Klein $\mathbb{D}_3$-action on a $K3$-surface.

The associated fundamental representation of the constructed action is nonreal. Hence, $K\mathfrak{M}^G \cong (\operatorname{Per}_0^G / \operatorname{Aut}_G L) \times S^0$, and it suffices to show that $\operatorname{Per}_0^G / \operatorname{Aut}_G L$ has three connected components.

One has $L^G = Q$ and $L_\rho(\mathbb{Z}) = P$. The lattice M^+ (the $(+1)$-eigenlattice of s) is generated by $w_1 = u_1 + v_1 + u_2$ and $w_2 = u_1 + u_2 - v_2$, and one has $w_1^2 = 2$, $w_2^2 = -2$, and $w_1 \cdot w_2 = 0$. We assert that the only nontrivial automorphism of M^+ that extends to an equivariant automorphism of P is the multiplication by -1; thus, $\operatorname{Aut}_G P = \{\pm 1\}$. Indeed, $\operatorname{Aut} M^+$ consists of the four automorphisms $w_1 \mapsto \varepsilon_1 w_1$, $w_2 \mapsto \varepsilon_2 w_2$, where $\varepsilon_1, \varepsilon_2 = \pm 1$, and the equivariant extension to $P \otimes \mathbb{Q}$ is uniquely given by the additional conditions $t(w_i) \mapsto \varepsilon_i t(w_i)$. If $\varepsilon_1 \neq \varepsilon_2$, the extension is not integral.

Thus, the action of $\operatorname{Aut}_G L$ on $\mathscr{H}^+$ is trivial, the fundamental domain Σ_ρ^+ coincides with $\mathscr{H}^+$, and, in view of Theorem 4.4.7, one has $\operatorname{Per}_0^G / \operatorname{Aut}_G L = (\tilde\Gamma_1 \times \mathscr{H}^+) \smallsetminus \Delta$, where $\tilde\Gamma_1 = \operatorname{Int}\Gamma_1/\operatorname{Aut} Q$ and Δ is the union of a subset of codimension at least 2 and the hyperplanes $\tilde\Gamma_1 \times \mathrm{h}_\rho^\pm(v)$ defined by roots $v \in P$. (Since $\dim \mathscr{H}^+ = 1$, each nonempty set $\mathrm{h}_\rho^\pm(v)$ is a hyperplane.) Let $v \in P$ be a root, and let $v^\pm$ be its projections to $V^\pm$. Since $2v^\pm \in M_\rho^\pm$ and M_ρ^+ has no vectors of square -4, the condition $\mathrm{h}_\rho^+(v) \neq \emptyset$ implies that either $v^+ = 0$ (and then $(v^-)^2 = -2$), or $(v^+)^2 = -2$ (and then $v^- = 0$), or $(2v^+)^2 = -8 - (2v^-)^2 = -2$ or -6. Each $M_\rho^\pm$ contains, up to sign, one vector of square (-2) and two vectors of square (-6). Comparing their images under J_ρ, one concludes that the space $\mathscr{H}^+$ is divided into three components by the two walls $\mathrm{h}_\rho^+(w_2)$ and $\mathrm{h}_\rho^+(2w_2 - w_1)$. □

6.1.2

Before discussing this example in more detail, we introduce another geometric $\mathbb{D}_3$-action on L with the same sublattice $L^G = Q = U \oplus 2E_8$. In the above notation, replace S with the matrix

$$S' = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix},$$

and keep the rest unchanged. For the new action, one has $M_\rho^\pm \cong U(2)$. The only possible wall in $\mathscr{H}^+$ is $\mathrm{h}_\rho^+(w^+)$, where $w^+ \in M_\rho^+$ is the only vector of square -4. However, $J_\rho w^+$ is not proportional to the vector $w^- \in M_\rho^-$ of square -4; hence, the action is realized by a single $\mathbb{D}_3$-equivariant deformation class of $K3$-surfaces.

In view of the following lemma, there are exactly two (up to isomorphism) geometric $\mathbb{D}_3$-actions on L with $L^G \cong U \oplus 2E_8$.

LEMMA 6.1.3

Up to automorphism, there are three nontrivial $\mathbb{Z}_3$-actions on the lattice $P \cong 2U$; their invariant sublattices are isomorphic to either A_2, or $A_2(-1)$, or zero. The last action admits two, up to isomorphism, extensions to a $\mathbb{D}_3$-action.

Proof

Let $t \in \mathbb{Z}_3$ be a generator. Pick a primitive vector u_1 of square zero, and let $u_2 = t(u_1)$. If $t(u_1) = u_1$ for any such u_1, the action is trivial. If $u_1 \cdot u_2 = a \neq 0$, then u_1, u_2, and $t^2(u_1)$ span a sublattice P' of rank 3. In this case, $a = \pm 1$, and the action is uniquely recovered using the fact that its restriction to $(P')^\perp$ (a sublattice of rank 1) is trivial. Finally, if $u_1 \cdot u_2 = 0$ and u_1, u_2 are linearly independent, then one must have $t(u_2) = -u_1 - u_2$. Completing u_1, u_2 to a basis u_1, v_1, u_2, v_2 as in the proof of Proposition 6.1.1, one can see that the system $T^3 = \mathrm{id}$, $\mathrm{Gr} = T^* \,\mathrm{Gr}\, T$ (where T is the matrix of t and Gr is the Gramm matrix) has a unique solution (the one indicated in the proof of Proposition 6.1.1).

Consider the last action and an involution $s\colon P \to P$, $ts = st^{-1}$. The invariant space M^+ of s is either U, or $U(2)$, or $\langle 2\rangle \oplus \langle -2\rangle$. The consideration above shows that the $\mathbb{Z}_3$-orbit of any primitive vector u_1 of square zero is standard and spans a sublattice of rank 2. Start from $u_1 \in M^+$, and complete it to a basis u_1, v_1, u_2, v_2, as above. The set of solutions to the system $TS = ST^{-1}$, $S^2 = \mathrm{id}$, $\mathrm{Gr} = S^* \,\mathrm{Gr}\, S$ for the matrix S of s depends on one parameter a, $s(v_2) = au_1 - v_2$, and a change of variables shows that only the values $a = 0$ or 1 produce essentially different actions (resp., with $M^+ \cong U(2)$ or $\langle 2\rangle \oplus \langle -2\rangle$). □

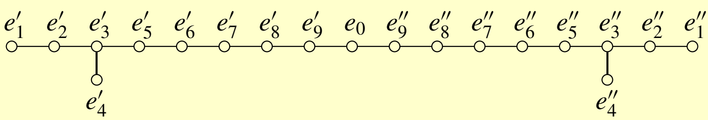
6.2. Geometric models

In this section, we give a geometric description (via elliptic pencils) of the six families constructed in Proposition 6.1.1. At a result, at the end of the section we prove the following statement.

PROPOSITION 6.2.1

All three pairs of complex conjugate deformation families constructed in Proposition 6.1.1 differ by the topological type of the $\mathbb{D}_3$-action.

Fix a decomposition $Q = \operatorname{Pic} X \cong 2E_8 \oplus U$. Let $e_1', \ldots, e_8'$, $e_1'', \ldots, e_8''$ be some standard bases for the E_8-components, and let u, v be a basis for the U component, so that $u^2 = v^2 = 0$ and $u \cdot v = 1$. Under an appropriate choice of γ (a small perturbation of $u + v$), the graph of (-2)-curves on X is the following:



Here $e_0 = u - v$, $e_9' = v - 2e_1' - 4e_2' - 6e_3' - 3e_4' - 5e_5' - 4e_6' - 3e_7' - 2e_8'$, and $e_9'' = v - 2e_1'' - 4e_2'' - 6e_3'' - 3e_4'' - 5e_5'' - 4e_6'' - 3e_7'' - 2e_8''$.

Consider the equivariant elliptic pencil $\pi\colon X \to \mathbb{P}^1$ defined by the effective class v. From the diagram above it is clear that the pencil has a section e_0 and two singular fibers of type $\tilde{E}_8$, whose components are $e'_1, \dots, e'_9$ and $e''_1, \dots, e''_9$, respectively, and has no other reducible singular fibers. (We use the same notation for a (-2)-curve and for its class in L.) Counting the Euler characteristic shows that the remaining singular fibers are either $4\tilde{A}_0^*$, or $2\tilde{A}_0^* + \tilde{A}_0^{**}$, or $2\tilde{A}_0^{**}$. (Here $\tilde{A}_0^*$ and $\tilde{A}_0^{**}$ stand for a rational curve with a node or a cusp, resp.) In any case, at least one of these singular fibers must also remain fixed under the $\mathbb{Z}_3$-action; hence, the $\mathbb{Z}_3$-action on the base of the pencil has three fixed points and thus is trivial. This implies, in particular, that the pencil has no fibers of type $\tilde{A}_0^*$; the normalization of such a fiber would have three fixed points (the two branches at the node and the point of intersection with e_0), and the $\mathbb{Z}_3$-action on it and hence on the whole surface would have to be trivial. Thus, the types of the singular fibers of the pencil are $2\tilde{A}_0^{**} + 2\tilde{E}_8$.

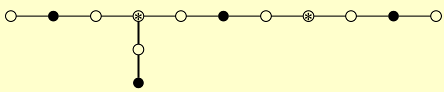
Let us study the action of $\mathbb{Z}_3$ on the fibers of the pencil. Each fiber has at least one fixed point: the point of intersection with e_0. For nonsingular fibers this implies that

(1) they all have j-invariant $j = 0$ (as there is only one elliptic curve admitting a $\mathbb{Z}_3$-action with a fixed point) and

(2) each nonsingular fiber has two more fixed points.

Denote the closure of the union of these additional fixed points by C. This is a curve fixed under the $\mathbb{Z}_3$-action. In particular, it must intersect the cuspidal fibers at the cusps. The action on the $\tilde{E}_8$ singular fibers can easily be recovered starting from the points of intersection with e_0 and using the following simple observation: in appropriate coordinates (x, y), a generator $g \in \mathbb{Z}_3$ acts via $(x, y) \mapsto (x, \varepsilon y)$ in a neighborhood of a point of a fixed curve $y = 0$, and via $(x, y) \mapsto (\varepsilon^2 x, \varepsilon^2 y)$ in a neighborhood of an isolated fixed point $(0, 0)$. (Here ε is the eigenvalue of ω: $g(\omega) = \varepsilon\omega$.) One concludes that the components e'_3, e'_7, e''_3, and e''_7 are fixed, the intersection points of pairs of other components are isolated fixed points, and C intersects the $\tilde{E}_8$ fibers at some points of e'_1 and e''_1. In particular, the restriction $\pi\colon C \to \mathbb{P}^1$ is a double covering with four branch points; hence, C is a nonsingular elliptic curve.

Let $\tilde{X}$ be X with isolated fixed points blown up, and let $\tilde{Y} = \tilde{X}/\mathbb{Z}_3$. This is a rational ruled surface with two singular fibers $\tilde{F}'$, $\tilde{F}''$ (the images of the $\tilde{E}_8$ fibers of X) whose adjacency graphs are as follows:



(Here ∘, •, and ⊛ stand for a nonsingular rational curve of self-intersection -1, -3, and -6, resp.; an edge corresponds to a simple intersection point of the curves.) The image $\tilde{R}$ of the section e_0 has self-intersection (-6) and intersects the rightmost curve in the graph; the image $\tilde{D}$ of the section C has self-intersection zero and intersects the leftmost curve in the graph. The branch divisor of the covering $\tilde{X} \to \tilde{Y}$ is $\tilde{R} + \tilde{D} +$ (the (-6)-components) $-$ (the (-3)-components).

Contract the singular fibers of $\tilde{Y}$ to obtain a geometrically ruled surface Y. Denote by R, D, F', and F'' the images of $\tilde{R}$, $\tilde{D}$, $\tilde{F}'$, and $\tilde{F}''$, respectively. The contraction can be chosen so that $R^2 = 0$, that is, $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then $D^2 = 8$ and D is a curve of bidegree $(2, 2)$. It is tangent to F' and F'', and R passes through the tangency points.

The above construction respects the $\mathbb{D}_3$-action on X, and Y inherits a canonical real structure with respect to which D, R, F', and F'', as well as the base of the pencil, are real; one has $Y_{\mathbb{R}} = S^1 \times S^1$.

Recall that, up to rigid isotopy, the embedding $D_{\mathbb{R}} \subset Y_{\mathbb{R}}$ is one of the following:

(1) $D_{\mathbb{R}}$ is empty;

(2) $D_{\mathbb{R}}$ consists of one oval (a component contractible in $Y_{\mathbb{R}}$);

(3) $D_{\mathbb{R}}$ consists of two ovals;

(4) $D_{\mathbb{R}}$ consists of two components, each realizing the class $(0, 1)$ in $H_1(Y_{\mathbb{R}})$;

(5) $D_{\mathbb{R}}$ consists of two components, each realizing the class $(1, 0)$ in $H_1(Y_{\mathbb{R}})$;

(6) $D_{\mathbb{R}}$ consists of two components, each realizing the class $(1, 1)$ in $H_1(Y_{\mathbb{R}})$.

(The basis in $H_1(Y_{\mathbb{R}})$ is chosen so that $R_{\mathbb{R}}$ realizes $(1, 0)$ and $F'_{\mathbb{R}}$ realizes $(0, 1)$.) Now one can easily indicate four topologically distinct types of action. Since p' and p'' are on the same generatrix R, the embedding $D_{\mathbb{R}} \subset Y_{\mathbb{R}}$ is either

(a) as in (2), or

(b) as in (3) (the points p', p'' are in the same component of $D_{\mathbb{R}}$), or

(c) as in (4) (the points p', p'' are in the different components of $D_{\mathbb{R}}$).

In case (c), there are two possibilities:

(c′) $F'_{\mathbb{R}}$ and $F''_{\mathbb{R}}$ belong to (the closure of) the same component of $Y_{\mathbb{R}} \setminus D_{\mathbb{R}}$;

(c″) $F'_{\mathbb{R}}$ and $F''_{\mathbb{R}}$ belong to (the closure of) distinct components of $Y_{\mathbb{R}} \setminus D_{\mathbb{R}}$.

Note that, according to Lemma 3.2.4, any model constructed does necessarily realize either the action of Proposition 6.1.1 or the action of Section 6.1.2.

The models of types (a) and (b) (resp., (a) and (c′)) can be joined through a singular elliptic $K3$-surface whose desingularization has a fiber of type $\tilde{A}_2$. In view of Proposition 5.2.1, these types realize the action of Proposition 6.1.1. Hence, the remaining type (c″) realizes the action of Section 6.1.2.

Proof of Proposition 6.2.1

The surfaces in question are represented by the above models of types (a), (b), and (c′), which differ topologically by the number of components of $C_{\mathbb{R}} \cong D_{\mathbb{R}}$ and by whether $C_{\mathbb{R}}$ has a component bounding a disk in $X_{\mathbb{R}}$. $\square$

6.3. The four families in their Weierstrass form

Since the four families constructed above are Jacobian fibrations (i.e., have sections), are isotrivial, and have singular fibers of type $2\tilde{A}_0^{**}+2\tilde{E}_8$, their Weierstrass equations are of the form

$$y^2z = x^3 + (u^2 - v^2)^5 p_2(u, v)z^3,$$

where $(u : v)$ are homogeneous real coordinates in $\mathbb{P}^1$, p_2 is a degree 2 homogeneous real polynomial with simple roots other than $u = \pm v$, and (x, y, z) are regarded as coordinates in the bundle $\mathbb{P}(\mathcal{O}(6) \oplus \mathcal{O}(4) \oplus \mathcal{O})$ over $\mathbb{P}^1(u : v)$. Isomorphisms between such elliptic fibrations are given by projective transformations in $\mathbb{P}^1(u : v)$ and coordinate changes of the form $x \mapsto k^4x$, $y \mapsto k^6y$, $z \mapsto z$, $u \mapsto ku$, $v \mapsto kv$, $k \in \mathbb{R}^*$. By means of such isomorphisms, the equation can be reduced to one of the following four families:

$$y^2z = x^3 + (u^2 - v^2)^5(u - cv)(u - \bar{c}v)z^3, \quad c \neq \bar{c},$$
$$y^2z = x^3 \pm (u^2 - v^2)^5(u - av)(u - bv)z^3, \quad -1 < a < b < 1,$$

and

$$y^2z = x^3 + (u^2 - v^2)^5(u - av)(u - bv)z^3, \quad -1 < a < 1 < b.$$

The $\tilde{E}_8$ singular fibers are those with $u^2 = v^2$. Each of the surfaces can be equipped with any of the two $\mathbb{D}_3$-actions generated by the complex conjugation and the multiplication of x by either $\exp(2\pi i/3)$ or $\exp(-2\pi i/3)$.

The exceptional family, that is, that with the action of Section 6.1.2, is the one with the last equation. To see this, one can explicitly construct two cycles in M^- with square 0 and intersection 2. For one of them, we pick a skew-invariant under the complex conjugation circle ξ in an elliptic fiber between $u = av$ and $u = v$ and drag it along a loop in $\mathbb{P}^1(u : v)$ around $u = -v$ and $u = av$. The other (singular) cycle is constructed from a circle η in the same fiber with $T\eta = \bar{\eta}$, where T is the monodromy operator about the fiber $u = v$. We drag it along a loop around $u = v$ and pull its ends together into the cusp of the fiber $u = av$.

Note that the real part of the double section of the surfaces in the first family has only one connected component, so it correspond to the series (a). One component of the double section of the surfaces given by the second equation with the sign $-$ bounds a disc in the real part of the surface, so it corresponds to series (b). The same equation with the sign $+$ gives series (c′).

Thus, one obtains another description of the six disjoint families constructed in Proposition 6.1.1. The bijection between the set of isomorphism classes of $K3$-surfaces with a $\mathbb{D}_3$-action such that $L^G = U \oplus 2E_8$ and the set of surfaces given by the above four equations (considered up to projective transformations of the base and rescalings) can be used for an alternative proof of Proposition 6.1.1.

6.4. Distinct conjugate components with the same real ρ

In this section, we construct an example of a geometric action θ of a certain group $G = \widetilde{T}_{192}$ (with $\rho \neq 1$ real and $\kappa = 1$) whose moduli space has two distinct components interchanged by the conjugation $X \mapsto \bar{X}$. Note that, since ρ is real, the components are *not* distinguished by the associated fundamental representations.

Recall that the group T_{192} can be described as follows. Consider the form $\Phi(u, v) = u^4 + v^4 - 2\sqrt{-3}\,u^2v^2$. Its group of unitary isometries is the so-called *binary tetrahedral group* $T_{24} \subset U(2)$; it can be regarded as a $\mathbb{Z}_3$-extension of the Klein group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} \subset \mathbb{H}$. (Note that the double projective line ramified at the roots of Φ is a hexagonal elliptic curve. An order 3 element of T_{24} can be given, for example, by the matrix

$$q = \frac{1}{-1 + i\sqrt{3}} \begin{bmatrix} -1 - i & 1 - i \\ -1 - i & -1 + i \end{bmatrix},$$

whose determinant is $(-1 + i\sqrt{3})/2$.) The center of T_{24} is $\{\pm 1\} \subset Q_8$. Identify two copies of $T_{24}/Q_8 \cong \mathbb{Z}_3$ via $[q] \mapsto [q]^{-1}$, and let T' be the fibered central product $(T_{24} \times_{\mathbb{Z}_3} T_{24})/\{c_1 = c_2\}$, where c_1 and c_2 are the central elements in the two factors. Then T_{192} is the semidirect product $T' \rtimes \mathbb{Z}_2$, the generator t of $\mathbb{Z}_2$ acting via transposing the factors.

Denote by $\widetilde{T}_{192}$ the extension of T_{192} by an element c subject to the relations $c^2 = c_1 = c_2$, $c^{-1}tc = c_1 t$, and $ac = ca$ for any a in either of the two copies of $T_{24} \subset T_{192}$. Augment this group via $\kappa\colon \widetilde{T}_{192} \to \widetilde{T}_{192}/T_{192} = \mathbb{Z}_2$.

PROPOSITION 6.4.1

There is a geometric action of $G = \widetilde{T}_{192}$ on $L = 3U \oplus 2E_8$ such that the associated fundamental representation ρ is real and the corresponding moduli space $K\mathfrak{M}^G$ consists of a pair of conjugate points X, $\bar{X}$.

Proof

Consider the quartic $X \subset \mathbb{P}^3$ given by the polynomial $\Phi(x_0, x_1) + \Phi(x_2, x_3)$. According to Mukai [M], there is a T_{192}-action on X with $\rho = 1$. It can be described as follows. The central product $(T_{24} \times T_{24})/\{c_1 = c_2\}$ acts via block diagonal linear automorphisms of $\Phi \oplus \Phi$, the two factors acting separately in (x_0, x_1) and (x_2, x_3). The fundamental representation of the induced action on X has order 3, and its kernel extends to a symplectic T_{192}-action via the involution $(x_0, x_1) \leftrightarrow (x_2, x_3)$.

The described T_{192}-action on X extends to a $\widetilde{T}_{192}$-action, the element $c \in \widetilde{T}_{192}$ acting via $(x_0 : x_1 : x_2 : x_3) \mapsto (ix_0 : ix_1 : x_2 : x_3)$, so that $\rho(c) = -1$. Choosing an isometry $H^2(X) \to L$, one obtains a geometric $\widetilde{T}_{192}$-action on L.

Fix a marking $H^2(X) = L$, and consider the twisted induced action on L. We assert that the corresponding period space consists of two points X and $\bar{X}$, both admitting a unique embedding into $\mathbb{P}^3$ compatible with a projective representation of $\widetilde{T}_{192}$. Indeed, as follows, for example, from the results of Xiao [X], for any action of the group $G' = T_{192}$ with $\rho = 1$, one has $\operatorname{rk} L^\bullet = 19$; hence, $\operatorname{Per}^{G'}$ is a single point $\mathfrak{w} \subset L \otimes \mathbb{R}$ and $K\Omega^{G'} = S^2$. Passing to $G = \widetilde{T}_{192}$ decomposes $\mathfrak{w}$ into $\ell = \mathfrak{w}^G$ and $\ell^\perp$ and reduces $K\Omega^G$ to a pair of points. Since the action is induced from $\mathbb{P}^3$, the line ℓ is generated by an integral vector of square 4, and this is the only (primitive) polarization of the surface compatible with the action.

It remains to show that X does not admit an antiholomorphic automorphism commuting with $\widetilde{T}_{192}$. Any such automorphism would preserve ℓ and hence would be induced from an antiholomorphic automorphism a of $\mathbb{P}^3$. Since a commutes with $\widetilde{T}_{192}$, it must fix the four intersection points of X with the line C given by $\{x_0 = x_1 = 0\}$. In particular, a must preserve C. On the other hand, the roots of Φ do not lie on a circle and thus cannot be fixed by an antihomography. □

A. Appendix. Finiteness and quasi-simplicity for 2-tori

A.1. Klein actions on 2-tori

In this section we prove analogs of Theorems 1.7.1 and 1.7.2 for complex 2-tori (or just 2-tori, for brevity). The *homological type* of a finite group G Klein action on a 2-torus X is the twisted induced action $\theta_X\colon G \to \operatorname{Aut} H^2(X)$ on the lattice $H^2(X) \cong 3U$, considered this time up to conjugation by *orientation-preserving* lattice automorphisms. As in the case of $K3$-surfaces, one has $H^{2,0}(X) \cong \mathbb{C}$, and the action of G^0 on $H^{2,0}(X)$ gives rise to a natural representation $\rho\colon G^0 \to \mathbb{C}^*$ called the *associated fundamental representation*. Both θ_X and ρ are deformation invariants of the action; θ_X is also a topological invariant.

Our principal results for 2-tori are the following two theorems.

THEOREM A.1.1 (Finiteness theorem)
The number of equivariant deformation classes of complex 2-tori with faithful Klein actions of finite groups of uniformly bounded order (for any given bound) is finite.

Remark. Note that the order of groups acting on 2-tori and not containing pure translations is bounded (cf. Theorem A.1.4). In particular, there are finitely many deformation classes of such actions.

THEOREM A.1.2 (Quasi-simplicity theorem)
Let X and Y be two complex 2-tori with diffeomorphic finite group G Klein actions. Then either X or $\bar{X}$ is G-equivariantly deformation equivalent to Y. If the associate

fundamental representation is trivial, then X and $\bar{X}$ are G-equivariantly deformation equivalent.

COROLLARY A.1.3
The number of equivariant deformation classes of hyperelliptic surfaces with faithful Klein actions of finite groups is finite. If X and Y are two hyperelliptic surfaces with diffeomorphic finite group G Klein actions, then either X or $\bar{X}$ is G-equivariantly deformation equivalent to Y.

Recall that, after fixing a point zero on a 2-torus X, one can identify X with the quotient space $T_0(X)/H_1(X;\mathbb{Z})$ and thus regard it as a group. Then with each (anti)automorphism t of X one can associate its linearization dt preserving zero, and hence, any Klein action θ on X gives rise to its linearization $d\theta$ consisting of (anti)holomorphic autohomomorphisms of X. As is known (see, e.g., [VS] or [Ch]), the original action θ is uniquely determined by $d\theta$ and a certain element

$$a(\theta) \in H^2\big(G; H_1(X)\big) = H^1\big(G; T_0(X)/H_1(X;\mathbb{Z})\big),$$

the latter depending only on the equivalence class of the extension $1 \to H_1(X) \to \mathscr{G} \to G \to 1$, where $\mathscr{G}$ is the lift of G to the group of (anti)holomorphic transformations of the universal covering T_0X of X. In particular, $a(\theta)$ is a topological invariant.

Clearly, both the homological type of a Klein action θ and its fundamental representation ρ depend only on the linearization $d\theta$. Since the group $H^2(G; H_1(X))$ is finite and $a(\theta)$ is a topological invariant, the general case of Theorems A.1.1 and A.1.2 reduces to the case of linear actions. Thus, from now on, *we consider only actions preserving zero.* All (anti)automorphisms preserving zero are group homomorphisms, and they all commute with the automorphism $-\mathrm{id}\colon X \to X$. For simplicity, *we always assume that* $-\mathrm{id} \in G$. For such actions, we prove Theorems A.1.4 and A.1.5, which imply Theorems A.1.1 and A.1.2.

THEOREM A.1.4
The number of equivariant deformation classes of complex 2-tori with faithful linear Klein actions of finite groups preserving zero is finite.

THEOREM A.1.5
Let X and Y be two complex 2-tori with linear finite group G Klein actions of the same homological type. Then either X or $\bar{X}$ is G-equivariantly deformation equivalent to Y. If the associate fundamental representation is trivial, then X and $\bar{X}$ are G-equivariantly deformation equivalent.

These theorems are proved at the end of Section A.3.

Remark. Note that, speaking about linear actions, Theorem A.1.5 is somewhat stronger than Theorem A.1.2 as it also asserts that the diffeomorphism type of a linear action is determined by its homological type.

Remark. In the case of real actions (see Section 1.7), the surfaces X and $\bar{X}$ are obviously equivariantly isomorphic. The same remark applies to Theorem A.1.3, which gives us *gratis* the following generalization of the corresponding result by F. Catanese and P. Frediani [CF] for real structures on hyperelliptic surfaces. Let X and Y be two complex 2-tori with real structures and with real holomorphic G^0-actions, so that the extended Klein actions of $G = G^0 \times \mathbb{Z}_2$ have the same homological type and the same value of $a(\theta)$. Then X and Y are G-equivariantly deformation equivalent.

A.2. Periods of marked 2-tori

Let Λ be an oriented free abelian group of rank 4. Put $L = \bigwedge^2 \Lambda^\vee$. The orientation of Λ defines an identification $\bigwedge^4 \Lambda^\vee = \mathbb{Z}$ and turns L into a lattice via $\operatorname{per}\colon L \otimes L \to \bigwedge^4 \Lambda^\vee = \mathbb{Z}$. It is isomorphic to $3U$. Denote $\operatorname{Aut}^+ L = \operatorname{Aut} L \cap \mathrm{SO}^+(L \otimes \mathbb{R})$.

Let $\mathscr{J}$ be the set of complex structures on $\Lambda \otimes \mathbb{R}$ compatible with the orientation of Λ. Further, let Ω be the set of oriented positive definite 2-subspaces in $L \otimes \mathbb{R}$. As in (4.1.1), one can identify Ω with the space $\{\omega \in L \otimes \mathbb{C} \mid \omega^2 = 0,\ \omega \cdot \bar{\omega} > 0\}/\mathbb{C}^*$. Both $\mathscr{J}$ and Ω have natural structures of smooth manifolds. Let $\operatorname{per}\colon \mathscr{J} \to \Omega$ be the map defined via $J \mapsto (x^1 + iJ^*x^1) \wedge (x^2 + iJ^*x^2)$, where $J \in \mathscr{J}$, J^* is the adjoint operator on $L^\vee$, and $x^1, x^2 \in L^\vee \otimes \mathbb{R}$ are any two vectors generating $L^\vee \otimes \mathbb{R}$ over $\mathbb{C}$ (with respect to the complex structure J^*).

The following statement is essentially contained in [PS] and [S].

PROPOSITION A.2.1

The map $\operatorname{per}\colon \mathscr{J} \to \Omega$ *is a well-defined diffeomorphism. The map* $\mathrm{SL}(\Lambda) \to \operatorname{Aut}^+ L$*,* $\varphi \mapsto \wedge^2\varphi^*$*, is an epimorphism; its kernel is the center* $\{\pm 1\} \subset \mathrm{SL}(\Lambda)$*. An element* $\varphi \in \mathrm{SL}(\Lambda)$ *commutes with a complex structure* $J \in \mathscr{J}$ *if and only if its image* $\wedge^2\varphi^*$ *preserves* $\operatorname{per} J$.

Proof

We briefly indicate the proof. A simple calculation in coordinates shows that the map $\operatorname{per}\colon \mathscr{J} \to \Omega$ is an immersion and generically one-to-one. (Remarkably, the equations involved are partially linear.) Since $\mathscr{J}$ and Ω are connected homogeneous spaces of the same dimension, per is a diffeomorphism.

The map $\mathrm{SL}(\Lambda \otimes \mathbb{R}) \to \mathrm{O}(L \otimes \mathbb{R})$, $\varphi \mapsto \wedge^2\varphi^*$, is a homomorphism of Lie groups of the same dimension. Hence, it takes the connected group $\mathrm{SL}(\Lambda \otimes \mathbb{R})$ to the component of unity $\mathrm{SO}^+(L \otimes \mathbb{R})$. The pullback of $\mathrm{Aut}^+ L \subset \mathrm{SO}^+(L \otimes \mathbb{R})$ is a discrete subgroup of $\mathrm{SL}(\Lambda \otimes \mathbb{R})$ containing $\mathrm{SL}(\Lambda)$; on the other hand, $\mathrm{SL}(\Lambda)$ is a maximal discrete subgroup (see [R]); hence, it coincides with the pullback.

The last statement follows from the naturality of the construction; one has $\operatorname{per}(\varphi J \varphi^{-1}) = \wedge^2 \varphi^*(\operatorname{per} J)$. $\square$

A 1-*marking* of a 2-torus X is a group isomorphism $\varphi_1\colon \Lambda \to H_1(X)$. We call a 1-marking *admissible* if it takes the orientation of Λ to the canonical orientation of $H_1(X)$ (induced from the complex orientation of X). A 2-*marking* of X is a lattice isomorphism $\varphi\colon H^2(X) \to L$. Since $H^2(X) = \bigwedge^2 H^1(X)$, every 1-marking φ_1 defines a 2-marking $\varphi = \wedge^2 \varphi_1^*$. A 2-marking is called *admissible* if it has the form $\wedge^2 \varphi_1^*$ for some admissible 1-marking φ_1. Any two admissible 1-markings differ by an element of $\mathrm{SL}(\Lambda)$; in view of Proposition A.2.1, any two admissible 2-markings differ by an element of $\mathrm{Aut}^+ L$ and any admissible 2-marking has the form $\wedge^2 \varphi_1^*$ for exactly two admissible 1-markings φ_1.

From now on, by a 1- (resp., 2-)marked torus we mean a 2-torus with a fixed admissible 1- (resp., 2-)marking. Isomorphisms of marked tori are defined in the obvious way (cf. Section 4.2). Clearly, 1-marked tori have no automorphisms; the group of (marked) automorphisms of a 2-marked torus is $\{\pm \mathrm{id}\}$.

Consider the space $\Phi = \mathscr{J} \times (\Lambda \otimes \mathbb{R})/\Lambda$ and the projection $p\colon \Phi \to \mathscr{J}$. The bundle $\operatorname{Ker} dp$ has a tautological complex structure, which converts $p\colon \Phi \to \mathscr{J}$ to a family of 1-marked tori. This family is obviously universal. In view of Proposition A.2.1, this implies the following statement, called the global Torelli theorem for 2-marked tori.

THEOREM A.2.2
The family $p\colon \Phi \to \Omega$ is a universal smooth family of 2-marked complex 2-tori; that is, any other smooth family $p'\colon X \to S$ of 2-marked complex 2-tori is induced from p by a unique smooth map $S \to \Omega$.

A.3. Equivariant period spaces

The following statement is similar to Proposition 4.3.1; it relies on Proposition A.2.1 and on the fact that a finite group action admits an equivariant Kähler metric.

PROPOSITION A.3.1
Given a Klein action of a finite group G on a complex 2-torus X, the twisted induced action $\theta_X\colon G \to \mathrm{Aut}\, H^2(X)$ is almost geometric (Section 2.6); its image belongs to $\mathrm{Aut}^+ H^2(X)$.

Now we proceed as in the case of $K3$-surfaces. Let $\theta\colon G \to \operatorname{Aut}^+ L$ be an almost geometric action, and denote by $\Omega^G \subset \Omega$ the fixed point set of the induced action $g\colon \mathfrak{v} \mapsto \kappa(g)g(\mathfrak{v})$, $\mathfrak{v} \in \Omega$. (As before, $-\mathfrak{v}$ stands here for $\mathfrak{v}$ with the opposite orientation.) Then the following holds.

PROPOSITION A.3.2
The space Ω^G is a fine period space of 2-marked complex 2-tori with a Klein G-action compatible with θ; that is, it is the base of a universal smooth family of 2-marked complex 2-tori with a Klein G-action compatible with θ.

PROPOSITION A.3.3
Let $\kappa\colon G \to \{\pm 1\}$ be the augmentation, and let $\rho\colon G^0 \to S^1$ be a fundamental representation associated with θ. If $\rho = 1$, then the space Ω^G is connected. If $\rho \neq 1$, then the space Ω^G consists of two components, which are transposed by the involution $\mathfrak{v} \mapsto -\mathfrak{v}$.

Proof
As in the case of $K3$-surfaces, one can consider the contractible space Per^G and sphere bundle $K\Omega^G \to \operatorname{Per}^G$ and use the fibration $K\Omega^G \to \Omega^G$ with contractible fibers. □

Proof of Theorems A.1.4 and A.1.5
Theorem A.1.5 follows from A.3.2 and A.3.3. In view of A.1.5, Theorem A.1.4 follows from the finiteness of the number of homological types of faithful actions (cf. Section 4.5). □

A.4. Comparing X and $\bar{X}$
As a refinement of Theorem A.1.2, we show that in most cases the 2-tori X and $\bar{X}$ are not equivariantly deformation equivalent.

PROPOSITION A.4.1
Consider a faithful finite group G Klein action on a complex 2-torus X. Assume that G^0 has an element of order greater than 2 acting nontrivially on holomorphic 2-forms. Then X is not G-equivariantly deformation equivalent to $\bar{X}$.

Proof
Let $g \in G$ be an element as in the statement. The assertion is obvious if the associated fundamental representation ρ is nonreal. Thus, one can assume that ρ is real and $\rho(g) = -1$. A simple calculation (using the fact that g is orientation-preserving,

ord $g > 2$, and $\wedge^2 g^*$ has eigenvalue (-1) of multiplicity at least 2) shows that in this case the eigenvalues of the action of g on Λ are of the form ξ, $\bar{\xi}$, $-\bar{\xi}$, $-\xi$ for some $\xi \notin \mathbb{R}$. Hence, there is a distinguished square root $\sqrt{g} \in \mathrm{SL}(\Lambda \otimes \mathbb{R})$. (One chooses the arguments of the eigenvalues in the interval $(-\pi, \pi)$ and divides them by 2.) The automorphism $\wedge^2(\sqrt{g})^*$ has order 4 on the (only) g-skew-invariant 2-subspace $\mathfrak{v}$; hence, it defines a distinguished orientation on $\mathfrak{v}$. $\square$

Remark. As a comment on the proof of Proposition A.4.1, we would like to emphasize a difference between $K3$-surfaces and 2-tori. Under the assumptions of Proposition A.4.1, if ρ is real, it is still possible that there is an element $a \in \mathrm{Aut}^+_G L$ interchanging the two points $\mathfrak{v}$ and $-\mathfrak{v}$ of Ω^G (representing X and $\bar{X}$). However, unlike the case of $K3$-surfaces, this does not imply that X and $\bar{X}$ are G-isomorphic; an additional requirement is that a lift of a to $\mathrm{SL}(\Lambda)$ should commute with G.

A.5. Remarks on symplectic actions

We would like to outline here a simple way to enumerate all symplectic (i.e., identical on the holomorphic 2-forms) finite group actions on 2-tori. (This result is contained in the classification by Fujiki [Fu], who calls symplectic actions special.) Our approach follows that of Kondō [Ko1] to the similar problem for $K3$-surfaces.

In view of Propositions A.3.1 and A.3.2, it suffices to consider finite group actions on $L \cong 3U$ identical on a positive definite 3-subspace in $L \otimes \mathbb{R}$. Let $\theta \colon G \to \mathrm{Aut}^+ L$ be such an action, and let $L^\bullet = (L^G)^\perp$. Then $L^\bullet$ is a negative definite lattice of rank at most 3, and the induced G-action on $L^\bullet$ is orientation-preserving and trivial on $\operatorname{discr} L^\bullet$ (as it is on $\operatorname{discr} L^G$). Standard calculations with discriminant forms (cf. [Ko1]) show that $L^\bullet$ can be embedded into E_8 (the only negative definite unimodular even lattice of rank 8), and the G-action on $L^\bullet$ extends to E_8 identically on $E_8^G = (L^\bullet)^\perp \subset E_8$. Since $\operatorname{Aut} E_8 = W(E_8)$, the lattice $L^\bullet$ is the orthogonal complement of a face of a camera of E_8. Hence, $L^\bullet$ is a root system contained in A_3, $A_2 \oplus A_1$, or $3A_1$, and $G/\operatorname{Ker}\theta$ is a subgroup of $W(L^\bullet) \cap \mathrm{SO}(L^\bullet \otimes \mathbb{R})$. It remains to observe that any such lattice admits a unique (up to isomorphism) embedding to L, and hence the pair $L^\bullet$, $G/\operatorname{Ker}\theta \subset W(L^\bullet)$ determines a G-action on L up to automorphism.

In particular, one obtains a complete list of finite groups G acting faithfully and symplectically on 2-tori. One has $\operatorname{Ker}\theta = \{\pm \mathrm{id}\}$, and the group $G/\operatorname{Ker}\theta$ is a subgroup of $W(L^\bullet) \cap \mathrm{SO}(L^\bullet \otimes \mathbb{R})$ for $L^\bullet = A_3$, $A_2 \oplus A_1$, or $3A_1$, that is, of $\mathfrak{A}_4$ (alternating group on 4 elements), $\mathfrak{S}_3$ (symmetric group on 3 elements), or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Lifting the action from L to Λ (Proposition A.2.1), one finds that G is a subgroup of the binary tetrahedral group T_{24}, the binary dihedral group Q_{12}, or the Klein (quaternion) group Q_8.

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