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# Temperature dependence of the energy of a vortex in a two-dimensional Bose gas

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## Abstract

We evaluate the thermodynamic critical angular velocity  $\Omega_c(T)$  for creation of a vortex of lowest quantized angular momentum in a strictly two-dimensional Bose gas at temperature  $T$ , using a mean-field two-fluid model for the condensate and the thermal cloud. Our results show that (i) a Thomas–Fermi description of the condensate badly fails in predicting the particle density profiles and the energy of the vortex as functions of  $T$ ; and (ii) an extrapolation of a simple Thomas–Fermi formula for  $\Omega_c(0)$  is nevertheless approximately useful up to  $T \simeq 0.5T_c$ .

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## 1. Introduction

Quasi-two-dimensional (2D) Bose–Einstein-condensed gases (BEC) have been attracting increasing attention in recent years. Flatter and flatter condensates have been produced inside magneto-optical harmonic traps by squeezing the anisotropy parameter, measured as the ratio between the radial and axial trap frequencies (see Görlitz et al. [1] and references

therein). As the gas approaches the 2D limit, its collisional properties start to influence the boson–boson coupling parameter, which becomes dependent on the gas density [2,3]. Understanding the behaviour of vortices in this regime is important, since they reflect the superfluid nature of the condensate [4–6] and are expected to play a role in the transition from the superfluid to the normal state [7].

An experimental method for the creation of quantized vortices in a trapped BEC has made use of an “optical spoon” [8,9], whereby the condensate in an elongated trap is set into rotation by stirring with laser

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beams. This experiment on a confined boson gas is conceptually the analogue of the rotating bucket experiment on bulk superfluid helium, with the difference that the thermal cloud is inhomogeneously distributed and lies mostly outside the condensate cloud. A quantized vortex first appears in the condensate at a critical angular frequency of stirring which corresponds to an instability of the vortex-free state. A lower bound for the critical frequency can be assessed from the energy of a vortex, defined as the difference in the internal energy of the gas with and without a vortex. Observation of a vortex in a trapped gas is difficult since the vortex core is small in comparison to the size of the boson cloud, but the size of the core increases during free expansion and indeed vortices were first observed experimentally by releasing the trap and allowing ballistic expansion of the cloud [8].

In this Letter we consider a 2D rotating condensate at finite temperature and evaluate the particle density profiles and the energy of a vortex within a strictly 2D model for the boson–boson coupling. This is appropriate to a situation in which the  $s$ -wave scattering length starts to exceed the vertical confinement length. Previous work has established that the dimensionality of the scattering collisions strongly affects the equilibrium density profiles [10,11] and the process of free expansion of a BEC containing a vortex [12]. A similar study of the density profiles of a rotating BEC in 3D geometry at finite temperature has been carried out by Mizushima et al. [13], who also determined the location of various dynamical instabilities within the Bogoliubov–Popov theory.

We begin, therefore, by introducing our description of a 2D BEC containing a vortex at finite temperature  $T$ . This uses a mean-field two-fluid model for the condensate and the thermal cloud.

## 2. The model

The BEC is subject to an anisotropic harmonic confinement characterized by the radial trap frequency  $\omega_{\perp}$  and by the axial frequency  $\lambda\omega_{\perp}$  with  $\lambda \gg 1$ . Motions along the  $z$  direction are suppressed and the condensate wave function is determined by a 2D equation of motion in the  $\{x, y\}$  plane.

The order parameter for a 2D condensate accommodating a quantized vortex state of angular

momentum  $\hbar\kappa$  per particle is written as  $\Phi(\mathbf{r}) = \psi(r) \exp(i\kappa\phi)$ , with  $\phi$  the azimuthal angle. The wave function  $\psi(r)$  then obeys the nonlinear Schrödinger equation (NLSE)

$$\left[ -\frac{\hbar^2}{2m} \nabla^2 + \frac{\hbar^2 \kappa^2}{2mr^2} + \frac{1}{2} m \omega_{\perp}^2 r^2 + g_2 n_c(r) + 2g_1 n_T(r) \right] \psi(r) = \mu \psi(r), \quad (1)$$

where  $\mu$  is the chemical potential,  $m$  the atomic mass,  $n_c(r) = |\psi(r)|^2$  the condensate density, and  $n_T(r)$  the density distribution of the thermal cloud. The 2D coupling parameters  $g_j$ , with  $j = 2$  for condensate–condensate repulsions and  $j = 1$  for condensate–noncondensate repulsions, are given by

$$g_j = \frac{4\pi \hbar^2 / m}{\ln |4\hbar^2 / (jm\mu a^2)|}, \quad (2)$$

where  $a$  is the  $s$ -wave scattering length [2,3]. In Eq. (2) we have omitted a term due to thermal excitations [11], which is negligible in the temperature range of present interest ( $T \lesssim 0.5T_c$ , with  $T_c$  the critical temperature).

In our model the atoms in the thermal cloud are not put directly into rotation, but feel the rotating condensate through the mean-field interactions. In the Hartree–Fock approximation [14] the thermal cloud is treated as an ideal gas subject to the effective potential

$$V_{\text{eff}}(r) = \frac{1}{2} m \omega_{\perp}^2 r^2 + 2g_1 n_c(r). \quad (3)$$

The density distribution of the thermal cloud is then given by

$$n_T(r) = -\frac{m}{2\pi \hbar^2 \beta} \times \ln \left\{ 1 - \exp \left[ \beta \left( \mu - V_{\text{eff}}(r) - \frac{p_0^2}{2m} \right) \right] \right\}, \quad (4)$$

with  $\beta = 1/(k_B T)$ . The momentum cut-off  $p_0$  in Eq. (4) is a simple expedient to eliminate discontinuities in the density profiles that can occur near the Thomas–Fermi radius of the condensate. We take  $p_0 = 2\sqrt{mg_1 n_T}$  (see, for instance, [15]) which is equivalent to adding the term  $2g_1 n_T$  to the effective potential in Eq. (3).

We solve self-consistently the coupled Eqs. (1)–(4) together with the condition that the areal integral of

$n_c(r) + n_T(r)$  is equal to the total number  $N$  of particles. The differential equation (1) is solved iteratively by discretization, using a two-step Crank–Nicholson scheme [16]. In Section 3 we shall also compare the results with those obtained in the Thomas–Fermi approximation by dropping the radial kinetic energy term in Eq. (1).

### 2.1. The energy of a vortex

The critical angular velocity measured in experiments where vortex nucleation occurs from a dynamical instability [8] depends strongly on the shape of the perturbation. However, a lower bound for the angular velocity required to produce a single-vortex state can be estimated once the energies of the states with and without the vortex are known. Since the angular momentum per particle is  $\hbar\kappa$ , the critical (thermodynamic) angular velocity is given by

$$\Omega_c = \frac{E_\kappa - E_{\kappa=0}}{N\hbar\kappa}. \quad (5)$$

This expression follows by equating the energy of the vortex state in the rotating frame, that is  $E_\kappa - \Omega_c L_z$ , to the energy  $E_{\kappa=0}$  of the vortex-free state.

In the noninteracting case at zero temperature, the energy difference per particle is simply  $\hbar\kappa\omega_\perp$ , so that  $\Omega_c$  is just the trap frequency in the  $\{x, y\}$  plane. For the interacting gas at zero temperature in the Thomas–Fermi approximation, Eq. (5) reduces to the expression [4,17]

$$\Omega_c^{\text{TF}}(0) = \frac{2\hbar}{mR^2} \ln\left(\frac{0.888R}{\xi}\right). \quad (6)$$

Here  $R = (2\mu/m\omega_\perp^2)^{1/2}$  and  $\xi = R\hbar\omega_\perp/2\mu$  are the Thomas–Fermi radius and the healing length, respectively.

In the general case of an interacting gas at finite temperature, we have to evaluate numerically the total energy as the sum of four terms [18],

$$E_\kappa = E_{\text{kin},c} + E_{\text{trap}} + E_{\text{int}} + E_{\text{kin},T}. \quad (7)$$

These terms are the kinetic energy of the condensate

$$E_{\text{kin},c} = \int d^2r \psi^*(r) \left( -\frac{\hbar^2}{2m} \nabla^2 + \frac{\hbar^2 \kappa^2}{2mr^2} \right) \psi(r), \quad (8)$$

the energy of confinement

$$E_{\text{trap}} = \frac{1}{2} m \omega_\perp^2 \int d^2r r^2 [n_c(r) + n_T(r)], \quad (9)$$

the interaction energy

$$E_{\text{int}} = \frac{1}{2} \int d^2r [g_2 n_c^2(r) + 4g_1 n_c(r) n_T(r) + 2g_1 n_T^2(r)], \quad (10)$$

and the kinetic energy of the thermal cloud

$$E_{\text{kin},T} = \int d^2r \int \frac{dp}{2\pi\hbar^2} \frac{p^3}{2m} \left\{ \exp\left[ \beta \left( \frac{p^2}{2m} + V_{\text{eff}}(r) - \mu \right) \right] - 1 \right\}^{-1}. \quad (11)$$

In Section 3 we compare the angular frequency obtained by the calculation of the total energy of the gas with and without a vortex with those obtained from a Thomas–Fermi calculation and from extrapolating Eq. (6) at finite temperature through the temperature dependence of the chemical potential.

## 3. Results and discussion

For a numerical illustration we have taken  $\kappa = 1$  and chosen values of the system parameters as appropriate to the gas of  $^{23}\text{Na}$  atoms studied in the experiments of Görlitz et al. [1], namely  $\omega_\perp = 2 \times 188.4$  Hz,  $a = 2.8$  nm, and  $N = 5 \times 10^3$ . We are implicitly assuming, however, that the trap has been axially squeezed to reach the strictly 2D scattering regime.

The density profiles obtained from Eqs. (1)–(4) for the gas at three different values of the temperature (in units of the critical temperature  $T_c = (\sqrt{6N}/\pi k_B)\hbar\omega_\perp$  of the ideal Bose gas) are shown in Figs. 1 and 2. In the absence of a vortex (Fig. 1), the main point to notice is that the growth of the thermal cloud exerts an increasing repulsion on the outer parts of the condensate, constricting it towards the central region of the trap. This effect is seen only when the condensate is treated by the NLSE and persists in the presence of a vortex (Fig. 2) but, as we shall see below, is missed in the Thomas–Fermi approximation where the radial kinetic energy of the condensate is neglected. In addition, the thermal cloud penetrates the core of the vortex and enhances the expulsion of the condensate from the core region (inset of Fig. 2), in a manner which again is governed in its details by the radial kinetic energy term in the NLSE.

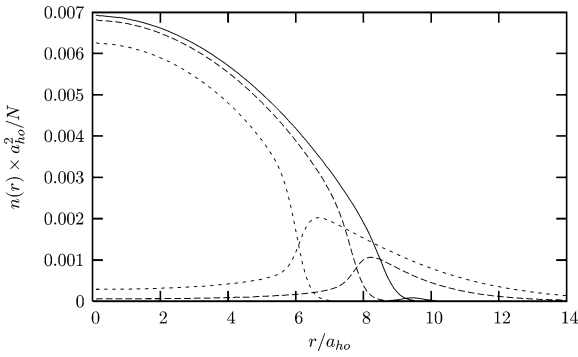


Fig. 1. Density profiles  $n(r)$  for the condensate and the thermal cloud (in units of  $a_{ho}^2/N$ , with  $a_{ho} = \sqrt{\hbar/m\omega_{\perp}}$ ) versus radial distance  $r$  (in units of  $a_{ho}$ ) at temperature  $T/T_c = 0.05$  (full line),  $0.25$  (long-dashed line), and  $0.50$  (short-dashed line).

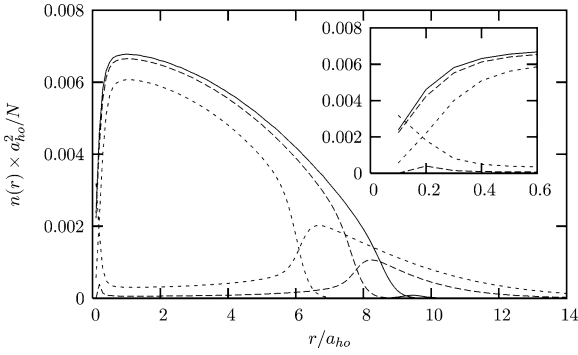


Fig. 2. Density profiles for the condensate and for the thermal cloud in the presence of a vortex. Units and symbols are as in Fig. 1. The inset shows an enlarged view of the profiles near the center of the trap.

The profiles obtained for a BEC containing a vortex at  $T = 0.5T_c$  by the full numerical calculation using the NLSE are compared in Fig. 3 with those obtained in the Thomas–Fermi approximation. The constriction of the condensate in its outer parts and its expulsion from the core region by the thermal cloud are clearly underestimated in the Thomas–Fermi theory.

Finally, Fig. 4 reports our results for the energy of the vortex as a function of  $T/T_c$ . The inaccuracies arising in the density profiles from the Thomas–Fermi treatment of the interplay between the condensate and the thermal cloud clearly lead to large errors in the estimation of  $\Omega_c(T)$ .

On the other hand the simple expression given in Eq. (6), with the two alternatives of using in it the chemical potential  $\mu(T)$  from the Thomas–Fermi ap-

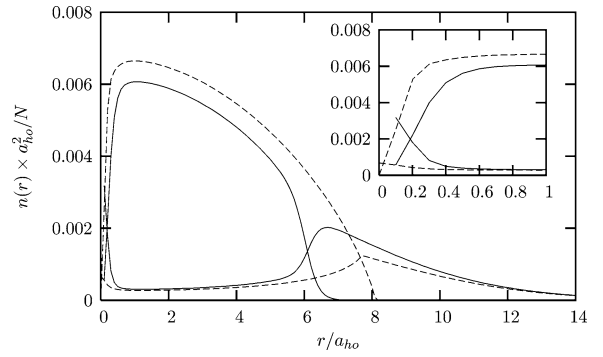


Fig. 3. Density profiles for a BEC containing a vortex at  $T/T_c = 0.50$  in the full calculation using the NLSE (full lines) and in the Thomas–Fermi approximation (dashed lines). The units are as in Fig. 1. The inset shows an enlarged view of the profiles near the center of the trap.

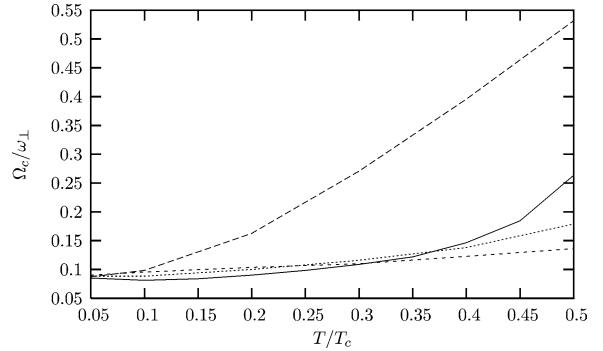


Fig. 4. The energy of a vortex, expressed as the thermodynamic critical frequency  $\Omega_c$  in units of the radial trap frequency  $\omega_{\perp}$ , as a function of  $T/T_c$ . The results from the full calculation using the NLSE (full line) are compared with those obtained from Eq. (6) with the corresponding values of  $\mu(T)$  (dotted line) and with the Thomas–Fermi values of  $\mu(T)$  (short-dashed line). The long-dashed line shows the results obtained in a calculation using the Thomas–Fermi theory.

proximation or from the full calculation using the NLSE, gives a reasonable account of the vortex energy up to  $T \simeq 0.5T_c$ .

#### 4. Summary and future directions

In summary, we have calculated the density profiles and the thermodynamic critical frequency for vortex nucleation in a strictly 2D Bose–Einstein-condensed gas at various temperatures. Our calculations have demonstrated the interplay between the thermal cloud

and the structure of the condensate in the regions where the radial kinetic energy of the condensate is playing an important role (namely, in the outer parts of the condensate and in the vortex core) and validated the use of a simple expression of the vortex energy for temperatures up to about  $0.5T_c$ .

As to future directions, a practical realization of the 2D regime that we have investigated would require a combination of axial squeezing of the trap and an enhancement of the scattering length. In relation to the latter, a treatment of a Feshbach resonance in a 2D gas will be reported shortly.

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### References

- [1] A. Görlitz, J.M. Vogels, A.E. Leanhardt, C. Raman, T.L. Gustavson, J.R. Abo-Shaer, A.P. Chittakur, S. Gupta, S. Inouye, T. Rosenband, W. Ketterle, *Phys. Rev. Lett.* 87 (2001) 130402.
- [2] S.A. Morgan, M.D. Lee, K. Burnett, *Phys. Rev. A* 65 (2002) 022706.
- [3] U. Al Khawaja, J.O. Andersen, N.P. Proukakis, H.T.C. Stoof, *Phys. Rev. A* 66 (2002) 013615.
- [4] E. Lundh, C.J. Pethick, H. Smith, *Phys. Rev. A* 58 (1998) 4816.
- [5] Y. Castin, R. Dum, *Eur. Phys. J. D* 7 (1999) 399.
- [6] F. Dalfovo, M. Modugno, *Phys. Rev. A* 61 (2000) 023605.
- [7] J.M. Kosterlitz, D.J. Thouless, *J. Phys. C* 6 (1973) 1181.
- [8] K.W. Madison, F. Chevy, W. Wohlleben, J. Dalibard, *Phys. Rev. Lett.* 84 (2000) 806.
- [9] P. Rosenbusch, V. Bretin, J. Dalibard, *Phys. Rev. Lett.* 89 (2002) 200403.
- [10] B. Tanatar, A. Minguzzi, P. Vignolo, M.P. Tosi, *Phys. Lett. A* 302 (2002) 131.
- [11] K.K. Rajagopal, P. Vignolo, M.P. Tosi, *Physica B* 344 (2004) 157.
- [12] O. Hosten, P. Vignolo, A. Minguzzi, B. Tanatar, M.P. Tosi, *J. Phys. B* 36 (2003) 2455.
- [13] T. Mizushima, T. Isoshima, K. Machida, *Phys. Rev. A* 64 (2001) 043610.
- [14] A. Minguzzi, S. Conti, M.P. Tosi, *J. Phys.: Condens. Matter* 9 (1997) L33.
- [15] N.V. Prokof'ev, O. Ruebenacker, B.V. Svistunov, *Phys. Rev. Lett.* 87 (2001) 270402.
- [16] W.H. Press, S.A. Teukolsky, W.T. Vetterling, P.B. Flannery, *Numerical Recipes*, Cambridge Univ. Press, Cambridge, 1992.
- [17] G. Baym, C.J. Pethick, *Phys. Rev. Lett.* 76 (1996) 6.
- [18] S. Giorgini, L.P. Pitaevskii, S. Stringari, *J. Low Temp. Phys.* 109 (1997) 309.