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Replacement Decisions with Maintenance Under Uncertainty: An Imbedded Optimal Control Model

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How should a manager make replacement decisions for a chain of machines over time if each is maintained by an optimal control model addressing uncertainty of machine breakdowns? A network representation of the problem involves arcs with interdependent costs. A solution algorithm is presented and replacement considerations under technological change are incorporated into a well-known optimal control model for maintenance under uncertainty (that of Kamien and Schwartz 1971). The method is illustrated by an example.

Subject classifications: dynamic programming/optimal control: models; facilities/equipment planning: maintenance/replacement; inventory/production policies: maintenance/replacement; reliability: replacement/renewal.
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1. Introduction

In an optimal control framework, this paper addresses the question of how a machine should be maintained and when it should be replaced by another (possibly of a different technology) if deterioration and breakdowns follow a continuous probability distribution. The next section provides a background for some of the related literature. Section 3 describes some application areas for a well-known optimal control model of Kamien and Schwartz (1971) for the maintenance of a single machine and outlines a numerical solution procedure. A stochastic dynamic programming formulation is provided to simultaneously address maintenance-replacement decisions. Section 4 presents a network formulation with probabilistic routes and decision nodes, for more general models. Optimal control models are imbedded into each other, and then into a larger dynamic programming mode, and a solution method is proposed. The implications of the framework are illustrated for the model of Kamien and Schwartz in §5. The paper concludes with a numerical illustration for machines with Weibull failure rate and a discussion of avenues for future research.

2. Background

There exists a large body of literature on maintenance and replacement policies under Markovian deterioration. The paper of Derman (1962) on sequential decisions and Markov chains opened the door for a stream of research, beginning with that of Klein (1962) and leading to works such as those of Hopp and Wu (1990) and Hopp and Nair (1994), which addressed Markovian deterioration and technological change.

In a different setting, Kamien and Schwartz (1971) (in short, K-S) developed an optimal control model for the maintenance and sale date of a single machine. Though limited to the narrower scope of Pontryagin's principle, the K-S model could address a wide range of continuous probability distributions. In the extensive review of Pierskalla and Voelker (1976), the K-S model stood as the main optimal control formulation that addressed uncertainty.

If coverage of a method in textbooks is an indicator of popularity, then two such optimal control models for maintenance decisions are Thompson's (1968) deterministic model and Kamien and Schwartz's (1971) probabilistic model. (See, for example, Rapp 1974, Tu 1991, Kamien and Schwartz 1991, Sethi and Thompson 2000.) Both models addressed the maintenance and sale date of a single machine. Deterministic maintenance models have been extended to a multitude of replacement decisions over time. Building upon Thompson's (1968) model, Sethi and Morton (1972), Tapiero (1973), Sethi and Chand (1979), and Chand and Sethi (1982) addressed deterministic maintenance models integrated into a chain of machine replacements allowing probabilistic technological breakthroughs. The computational burden limited the applicability of modeling probabilistic technological change (Sethi and Thompson 1977, 2000, p. 259). More recently, building on Kamien and Schwartz's ideas, Mehrez and Berman (1994) and Mehrez et al. (2000) developed deterministic maintenance models allowing for as much as three replacements over time. Their approach allowed for the introduction time of the new machine with new technology to be Markovian. A common feature of all these models was deterministic maintenance. A key characteristic of these deterministic maintenance models is the following: The machine does not fail to get scrapped during the period for which a given maintenance policy is established; it can deteriorate, but still continues to produce at some level. Therefore, when a machine is installed, we know exactly when it will retire under the given maintenance policy.

Probabilistic maintenance, on the other hand, addresses the possibility of the cessation of production due to breakdown. Recent control models for maintenance under uncertainty include the works of Boukas and his colleagues (Boukas and Haurie 1990, Boukas et al. 1995, Boukas and Liu 2001 and references therein), who made use of Davis's (1984, 1993) piecewise deterministic Markov process and Sethi and Zhang (1994). In the earlier models of Boukas and his colleagues, transition probabilities (to failure) of their continuous-time, finite-state Markov chains depended directly on the age of the machine as a continuous variable. More recently, Boukas and Liu (2001, p. 1455) stated for these models "...the age variable...greatly increases the computational burden and may lead to the curse of dimensionality." Removing the continuous age variable, they approximated the model by four states of a continuous-time Markov chain: good, average, bad, and failure. In general, their models encompass a rich spectrum of variables, including varying production and inventory levels to meet stochastic demand for different products produced on a number of machines. On the other hand, replacement of present or failed machines by those of newer technology is not considered. In the optimal control literature, we have not been aware of probabilistic maintenance models that, in addition to maintenance, also simultaneously take into account possibilities of a chain of replacements under given scenarios of technological change.

The purpose of this paper is to extend the probabilistic single-machine K-S model (which can have a continuous-time variable as input for the hazard rate, for aging) into a wider setting, allowing maintenance decisions to take into account the implications of the possibilities of a multitude of replacements over time. The way this problem differs from replacement models that use deterministic (optimal control) maintenance segments such as, say, those of Sethi and Morton (1972) or Mehrez et al. (2000), can be summarized as follows. In contrast to deterministic models that explicitly lend themselves to dynamic programming with clear-cut regeneration nodes, the stochastic maintenance model presents additional challenges. Due to uncertainty of breakdowns, the planned (targeted) regeneration node for the replacement of a machine may be different than the actual regeneration node. The implication to the maintenance (optimal control) model is that the objective function terms change in the span of the optimization horizon for each individual machine. Put differently, the optimal control model has numerous discontinuities in its objective function integrand. When the problem is broken into smaller pieces, the costs of each remain interdependent. Local optimal control models' maintenance policies affect the breakdown probabilities of downstream ones. Treating the problem in smaller pieces with uniform objective function expressions over the "local" optimization period leads to numerous maintenance (optimal control) problems in tandem that would need to be optimized together with the dynamic programming calculations for the replacement decisions. The next section prepares the setting to address these issues.

3. The Problem

The single-machine optimal control model of Kamien and Schwartz (1971) begins with the cumulative distribution function of lifetime. Let $F_j(t)$ denote the probability that a machine of vintage j (bought when there were j periods to go until the end of the planning horizon) fails at or before t units of time from its purchase date. The term failure, or breakdown, is limited in this paper to those dysfunctions that require the ceasing of production and replacement of the machine. The effective hazard rate of this machine equals the natural hazard rate $h_j(t) = [dF_j(t)/dt]/[1 - F_j(t)]$ multiplied by $(1 - u(t))$, the latter indicating additional maintenance efforts to reduce the probability of failure at time t. In other words, the natural hazard rate h_j embodies the basic minimum maintenance requirements of the machine, while u represents what else can be done to reduce the probability of breakdown that leads to scrapping. Cost of this additional maintenance effort is $M_j(u(t))h_j(t)$.

3.1. Some Areas of Application

The control variable $u(t)$ not only may include improvements within the machine itself, but also around its external environment. It includes preventive as well as predictive measures that may prolong the life of the machine. Optimal control models are especially suited for handling continuous-time-varying decisions on temperature and moisture. Durability and strength of typical continuous-fiber composites can be significantly affected by heat as well as by even a minute presence of moisture, as demonstrated in Reifsnider and Case (2002, pp. 242–243). Other applications include monitoring of electrical closets in high-voltage distribution, monitoring of buildings via infrared thermography, vibration monitoring, and process parameter monitoring, all with the objective of favorably altering the probability distribution of lifetime of the system (Levitt 2003). Another consideration is usage of more electric power: better illumination of the location (or for longer periods), if it may reduce the probability of accidents at the expense of more kilowatt hours used.

Extending the natural hazard rate $h_j(t)$ by the control variable $u(t)$ may also, for some pay-scale structures, encompass usage of operators or supporting-services personnel with higher pay, and lower probabilities of accidents. This also reduces "defects due to improper use" (in the terminology of Gertsbakh and Kordonskiy 1969, p. 6).

For a machine operated, say, seven hours a day, for five days a week, the time parameter t may indicate "in-business" time. Maintenance operations are run after the in-business day, or do not show as a downtime of the machine that reduces the production day. Other choices of time scale may also be appropriate, as noted in Kordonsky and Gertsbakh (1993), Gertsbakh (2000, Chapter 6), and Lawless (2002, p. 241), depending on the source of failures. For example, if corrosion is the major culprit, then calendar time may be preferable. The applicability of the K-S model is limited to problems in which increase in the maintenance effort u does not reduce the standard production time of the machine.

3.2. The Kamien and Schwartz Optimal Control for Maintenance

The K-S model addresses the expected value of net cash flow. This can be viewed as the average net present value per machine if the experiment is independently repeated for a large number of times. In this context, $F_j(t)$ may also be viewed as the fraction of vintage j machines up and operating at time t of the experiments. R_j is the revenue net of all costs except maintenance $u(t)$. L_j denotes the junk value of a failed machine. In contrast to planned retirement, if the unexpected scrapping causes certain extra costs, these can be included in the L_j term as well. Breakdown leading to scrapping does not necessarily mean the physical obliteration of the machine. It can simply mean that production is terminated in such a way that a new replacement is in order. Maintenance costs are continuously differentiable with respect to u, with $M_j(0)=0$, $dM_j(u(t))/d(u(t))>0$, and $d^2M_j(u(t))/d(u(t))^2>0$. Cash generated at time t involves $R_j - M_j h_j$ if the machine is up and L_j if down. Its net expected present value is

$$w = e^{-rt}\left\{[R_j - M_j(u(t))h_j(t)](1-F_j(t)) + L_j\frac{dF_j(t)}{dt}\right\}$$

at the interest rate r. Letting $S_j(T)$ denote the resale value of a working machine at time T, the optimal control model of Kamien and Schwartz (1971) chooses $u(t)$ for $t\in[0,T]$ so as to maximize

$$J^* = \max_{u(t)} \int_{t=0}^{T} w\,dt + A(F_j(T),T)$$
$$\text{with } A(\,) = e^{-rT}S_j(T)(1-F_j(T)) \quad (1)$$

subject to

$$\frac{dF_j(t)}{dt} = (1-u(t))h_j(t)(1-F_j(t))$$
$$\text{with } 0\leqslant u(t)\leqslant 1 \text{ and initial condition } F_j(0)=0. \quad (2)$$

At time t, by the original design of the machine, $h_j(t)$ is fixed: It is a given formula. Therefore, in (2), varying $u(t)$ only affects $dF_j(t)/dt$, and thus future values of $F_j(t)$. On the other hand, setting $u(t)=0$ for all t would let the machine proceed according to the original $h_j(t)$.

In optimal control theory, a solution is achieved by choosing $u(t)$ for each point in time, so as to maximize the Hamiltonian $H = w + \lambda(t)[(1-u(t))h_j(t)(1-F_j(t))]$, where $\lambda(t)$ denotes the adjoint variable (shadow price) such that $\lambda(T) = \partial A(F_j(T),T)/\partial F_j(T) = -e^{-rT}S_j(T)$ and

$$\begin{aligned}\frac{d\lambda(t)}{dt} &= -\frac{\partial H}{\partial F_j(t)}\\ &= e^{-rt}[R_j - M_j(u(t))h_j(t) + L_j(1-u(t))h_j(t)]\\ &\quad + \lambda(t)(1-u(t))h_j(t). \qquad (3)\end{aligned}$$

Let $c(u,t)$ denote the terms in the Hamiltonian H that contain u. Expanding w in H, we get $c(u(t),t) \equiv -M_j(u(t)) - (L_j + \lambda(t)e^{rt})u(t)$. Here, $M_j(u(t))$ is a nonlinear function of $u(t)$ and optimal $u(t)$ is a continuous function of time as shown by K-S. Thus, in the K-S approach, for each t the optimal control is the value of $u(t)$ that maximizes the expression for the optimal c below:

$$c^*(t) = \max_{0\leqslant u(t)\leqslant 1}\left\{-M_j[u(t)] - (L_j + \lambda(t)e^{rt})u(t)\right\}. \quad (4)$$

3.3. A Numerical Procedure

Because optimal $u(t)$ is continuous and because $\lambda(t)$ is continuous (Pontryagin et al. 1962), the right side of Equation (3) is continuous. Therefore, the adjoint variable $\lambda(t)$ must be smooth (continuously differentiable). Furthermore, due to the special structure of the model, the state variable appears neither in the adjoint Equation (3) nor in the optimality condition (4). Taking advantage of all these properties, we use numerical methods. In a backward sweep starting from time $t=T$, with values of $\lambda(t)$ and $u(t)$ on hand and obtaining $\lambda'(t)$ from (3), one can compute numerically $\lambda(t-\Delta t)$ using a method such as Runge-Kutta, and then obtain $u(t-\Delta t)$ from (4), and decrement the clock by Δt again. After successive calculations, $t=0$ shall be reached with all values of $u(t)$ on hand. With these values on hand, a forward sweep beginning from $F_j(0)$ at increments of Δt using (1) and (2) can yield the numerical solution.

3.4. Dynamic Programming Formulation for Replacements

If during a given planning horizon one is allowed to replace a machine with a newer and more modern one, what would then be the optimal maintenance policy for each individual one, and when should the replacements take place? In stochastic models the overall planning horizon is often divided into T equal-size periods, allowing replacements only at the nodes indicating the end of the period (see, for example, Wagner 1975, pp. 715–718; Hopp and Nair 1991, p. 205; or Bylka et al. 1992, p. 490).

Define $f_{(0)}=0$, and for $n\geqslant 1$ let $f_{(n)}=$ net present value of an optimal regeneration and maintenance policy when there are n periods to go until the end of the planning horizon. Thus, subscripts in parentheses indicate the stage number of dynamic programming calculations rather than equipment vintage. Suppose that at stage n (time $=T-n$) the values of $f_{(n-1)}, f_{(n-2)}, \ldots, f_{(1)}$ are already at hand. Let $V(n,K)$ denote the optimal expected net present value for a vintage "n" machine obtained at time $T-n$, at cost of D_n dollars with the intention of keeping it for K periods ($K\leqslant n$) and subsequent replacements (if any). L_n will now cover not only the junk value of a failed machine, but also any special switching costs from a "failed-and-scrapped" machine to the new replacement. This is because, as noted in Jorgenson et al. (1967, p. 71), cost of in-service failure may exceed the cost of replacement planned well ahead as a preventive action against future failures. If there exist tighter bounds on $u(t)$ due to technology used, these will be denoted by $\underline{U_n}$ and $\overline{U_n}$. In such a setting, the following forms of optimal control problems need to be solved:

$$V(n,K)=\max_{u(t)}\sum_{\tau=0}^{K-1}\int_{\tau}^{\tau+1}\Big\{e^{-rt}\big\{[R_n-M_n[u(t)]h_n(t)][1-F_n(t)]+L_n[1-u(t)]h_n(t)[1-F_n(t)]\big\}+e^{-r(\tau+1)}f_{(n-\tau-1)}[1-u(t)]h_n(t)\cdot[1-F_n(t)]\Big\}\,dt+e^{-rK}[1-F_n(K)][S_n(K)+f_{(n-K)}]-D_n \qquad (5)$$

subject to

$$\frac{dF_n(t)}{dt}=[1-u(t)]h_n(t)[1-F_n(t)] \quad \text{with } 0\leqslant \underline{U_n}\leqslant u(t)\leqslant \overline{U_n}\leqslant 1 \qquad (6)$$

and initial condition $F_n(0)=0$. If there are alternative technologies available at time $T-n$, then (5)–(6) can be solved for each, and the alternative with the largest expected net present value may be chosen.

An economic interpretation of (5) can be observed by rearranging its terms into

$$V(n,K)=\max\left\{\begin{aligned}&\sum_{\tau=0}^{K-1}\int_{t=\tau}^{\tau+1}w(t)\,dt\\&+\sum_{\tau=0}^{K-1}\int_{t=\tau}^{\tau+1}e^{-r(\tau+1)}f_{n-\tau-1}\frac{dF_n(t)}{dt}\,dt\\&+e^{-rK}[1-F_n(K)][S_n(K)+f_{(n-K)}]-D_n\end{aligned}\right\}=\max\left\{\begin{aligned}&\int_{t=0}^{K}w(t)\,dt\\&+\sum_{\tau=0}^{K-1}e^{-r(\tau+1)}f_{(n-\tau-1)}[F_n(\tau+1)-F_n(\tau)]\\&+e^{-rK}[1-F_n(K)][S_n(K)+f_{(n-K)}]-D_n\end{aligned}\right\}.$$

In the last equation, the first expression (the integral) describes the expected present value of direct cash flow from operating and maintaining the machine, over the time interval $[0, K]$. For $\tau=0,1,\ldots,K-1$, the second expression describes the sum of present values of optimal maintenance/replacement policy when there are $n-(\tau+1)$ periods to go, multiplied by the probability of breakdown of a vintage n machine, in the just preceding period. The last expression before the D_n term describes the present value of the salvage revenue from the machine that was sold in operating condition and subsequent optimal policy, multiplied by the probability that the machine of vintage n did not break down during the K periods it was intended for use.

Into how many periods (T) should the overall planning horizon be divided? In answering this question—in other words, in choosing the length of a unit period—the following consideration needs to be taken into account. In the above model, when a machine breaks down in the middle of a period, purchase of a new machine will have to wait until the next regeneration point. If the firm replaces its machines rather quickly, then T needs to be chosen appropriately large, i.e., length of a unit period $=1/T$ needs to be reduced.

After the above values of $V(n,K)$ are obtained for each K, then $f_{(n)}$ can be obtained from

$$f_{(n)}=\max_{K=1,\ldots,n_K}[V(n,K)], \qquad n=1,2,\ldots,T,\ n_K\leqslant n. \qquad (7)$$

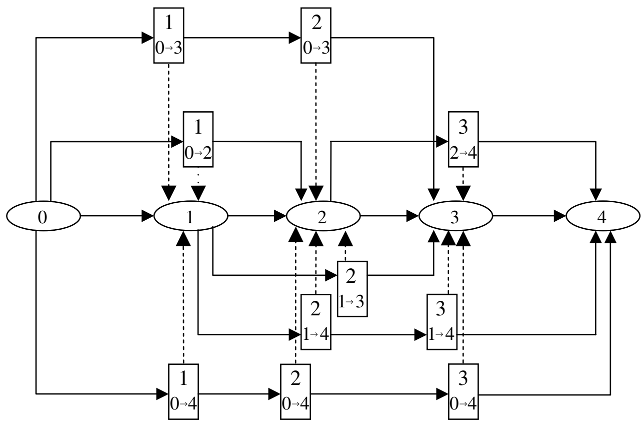
n_K is the upper bound on intended machine life for vintage n, as dictated by technical, safety, and managerial considerations. If there is no such limit, then one can set $n_K=n$. To obtain the values of $V(n,K)$, one has to consider the objective function in (5), which has discontinuities from $t=0$ to K due to different values of $f_{(n-\tau-1)}$. The next section of the paper addresses this issue.

4. A Network Representation for Imbedded Optimal Control Models

In a network representation of Equation (5), one way to cope with the changing integrands over the span of the optimization time from $t=0$ to $t=K$ is to break the targeted life span of the machine into arcs that are each a single period long. Here, an arc indicates the operation of a machine for the duration between the times represented by its starting and ending nodes. This network can conveniently allow dynamic programming if costs of individual arcs between the nodes are independent of others. Independence of arc costs is not the case, however. The maintenance decisions are intertwined, and lead to the existence of a series of arcs with sequence-dependent costs.

The network addressed here involves optimal control over the arcs and dynamic programming decisions at certain nodes. As an example, a four-period dynamic programming network is illustrated in Figure 1.

Figure 1. Replacement options for a four-period problem.



Notes. The dotted arc from node $\boxed{\substack{2\\ 0\to 4}}$ to node 2 indicates that the machine which had been intended for use between nodes 0 to 4 has broken down and been scrapped during the second period, hence, the purchase of a new machine at $t=2$. Replacements take place only at nodes that do not have dotted arrows emanating out from them.

The path over the three nodes 2, $\boxed{\substack{3\\ 2\to 4}}$, and 4 indicates purchase of a machine at $t=2$, intended for use for two periods, and to be salvaged at $t=4$. If an unplanned breakdown and scrapping occur during its first period of usage, then the vertical (dotted) arc $\boxed{\substack{3\\ 2\to 4}}$, 3 leads us to the purchase of a new machine at time $t=3$. The intensity of maintenance during the first period (arc 2, $\boxed{\substack{3\\ 2\to 4}}$) will influence the condition of the machine in its second period (arc $\boxed{\substack{3\\ 2\to 4}}$, 4), as well as the probability of taking the dotted arc (indicating breakage and scrapping) out of node $\boxed{\substack{3\\ 2\to 4}}$. Any solution approach needs to handle such interdependencies between maintenance and replacement costs in this probabilistic environment.

A backward-sweep dynamic programming solution of the problem in Figure 1 begins by relabelling the nodes to indicate time left until the end of the planning horizon. The node $t=4$ becomes $n=0$, node $t=3$ becomes node $n=1,\ldots,$ until the beginning node $n=4$, indicating that there are four periods to go. Nodes that have dotted lines emanating out labeled $\boxed{\substack{t\\ t_1\to t_2}}$ (with $t_1<t_2$) will now be denoted as $\boxed{\substack{n\\ n_1\to n_2}}$ (with $n_1>n_2$). For example, the old node $\boxed{\substack{3\\ 0\to 4}}$ will be relabeled as $\boxed{\substack{1\\ 4\to 0}}$, indicating that it is located at a time when there is one period to go, and that it is on the path from $n=4$ to $n=0$. Single-index nodes such $n=1,2,\ldots,$ are where dynamic programming decisions are taken for purchase of a new machine. Multi-index nodes such as $\boxed{\substack{n\\ n_1\to n_2}}$ serve to indicate the possibility of breakage and enter dynamic programming indirectly, through the optimal control calculations.

In the above context, the objective function (5), subject to constraint (6), relates to paths that begin and end with single-index nodes, and have solely multi-index nodes in between. Optimal control for a path between two such single-index nodes n and $n+K$ can proceed recursively by imbedding the immediate downstream arc's value of the objective function in the salvage value term of the model being calculated. Consider an arc on this path representing the life segment of a machine from age τ to $\tau+1$. For the maintenance model related to this arc, τ denotes the starting time of the local optimal control problem. K represents the number of periods the machine was intended to be used when it was bought. $F_n(\tau)$ is the initial value of the state variable, which is a given number between zero and one. The imbedded recursive optimal control problem is of the form

$$\begin{aligned} &J^*(n,\tau,K,F_n(\tau),f_{(n-\tau-1)},\ldots,f_{(0)})\\ &\quad=\max_{u(t)} J[n,\tau,K,F_n(\tau),f_{(n-\tau-1)},\ldots,f_{(0)},u(t)]\\ &\quad=\max_{u(t)}\int_{t=\tau}^{\tau+1} W_n(n,u(t),F_n(t),f_{(n-\tau-1)},t)\,dt\\ &\qquad+\bar{A}[n,\tau+1,K,F_n(\tau+1)] \end{aligned} \tag{8}$$

with $\tau<K$ and

$$\bar{A}[n,\tau+1,K,F_n(\tau+1)] =\begin{cases} e^{-r(\tau+1)}[S_n(K)+f_{(n-K)}][1-F_n(K)] \\ \qquad \text{for } \tau=K-1,\\ J^*[n,(\tau+1),K,F_n(\tau+1),f_{(n-\tau-2)},\ldots,f_{(0)}]\\ \qquad \text{for } \tau=K-2,\ldots,0\end{cases}$$

subject to

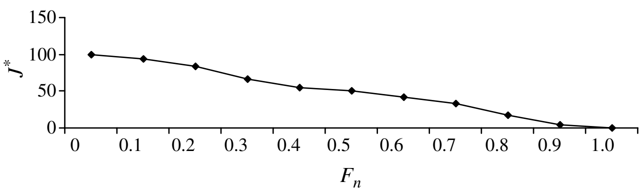
$$\frac{dF_n(t)}{dt}=g[u(t),F_n(t),t,h_n(t)] \tag{9}$$

with $0\leqslant \underline{U_n}\leqslant u(t)\leqslant \overline{U}_n\leqslant 1$, initial value of the state variable $F_n(\tau)$ given, and $F_n(\tau+1)$ free.

$W_n(n,u(t),F_n(t),f_{(n-\tau-1)},t)$ and $g[u(t),F_n(t),t,h_n(t)]$ are continuously differentiable with respect to u, F_n, and t. These two functions are not completely specified, and therefore (8)–(9) cover a general family of problems in which Kamien and Schwartz (1971) is a special case. This general problem, if solved recursively for $\tau=K-1,\ldots,0$, should eventually yield $J^*(n,0,K,0,f_{(n-1)})$. Now $V(n,K)$ can be obtained from $V(n,K)=-D_n+J^*(n,0,K,0,f_{(n-1)})$.

For $K>1$ and $\tau\leqslant K-2$, each time one attempts to solve problem (8)–(9), in the salvage value term $\bar{A}[n,\tau+1,K,F_n(\tau+1)]$ for $\tau<K-1$, $J^*(n,\tau+1,K,F_n(\tau+1),f_{(n-\tau-2)},\ldots,f_{(0)})$ needs to be represented as an analytical function of $F_n(\tau+1)$. When a closed-form expression is not available, one may express it approximately by a regression equation $\hat{J}(F_n(\tau+1))$ from the results of the previous step of the recursion as follows. Because in a given recursion, n, τ, K, and $f_{(n-\tau-1)}$ are given and fixed, problem (8)–(9) can be solved several times for different

Figure 2. Example of a relation between the starting value of F_n and the resulting J^* for a "general" cost function W_n.



values of $F_n(\tau)$ ranging from zero to one. The corresponding values of $J^*(n, \tau, K, F_n(\tau), f_{(n-\tau-1)}, \ldots, f_{(0)})$ can be regressed against $F_n(\tau)$. The estimated function will be called $\hat{J}(F_n(\tau+1))$ because it is to be used for the next step of the recursion, after τ gets decremented by one. Alternatively, a more flexible function may be fitted to all the calculated points, implying interpolation for the ranges between the available data.

A possible relation between $J^*(n, \tau, K, F_n(\tau), f_{(n-\tau-1)}, \ldots, f_{(0)})$ and $F_n(\tau)$ for a general cost function is illustrated as an example in Figure 2.

5. A Property of the Imbedded Kamien-Schwartz Model for Replacement Decisions

If an unknown nonlinear relation is approximated by some function, then as the number of stages in a recursion increases, the errors of approximation shall build up. One way to harness the size of the error is to increase the number of data points at the expense of more computational efforts. On the other hand, if a model yields a perfect fit to the functional form chosen, then the solution of imbedded optimal control and dynamic programming will yield exact results without incurring extra computational efforts, making it more attractive, as shown in the following theorem.

Theorem. *Imbedded recursions of* (8) *and* (9) *with the Kamien and Schwartz* (1971) *model yield a perfect fit in the regression equation for* $\hat{J}(F_n(\tau+1))$.

Proof. Using $g[u(t), F_n(t), t, h_n(t)] = [1-u(t)]h_n(t)[1-F_n(t)]$ and

$$W_n(\,) = e^{-rt}\{[R_n - M_n(u(t))h_n(t)][1-F_n(t)] + L_n[1-u(t)]h_n(t)[1-F_n(t)]\} + e^{-r(\tau+1)}f_{(n-\tau-1)}\{[1-u(t)]h_n(t)[1-F_n(t)]\}$$

for problem (8)–(9), the terminal value of the adjoint variable $\lambda(t)$ associated with (9) for $\tau = K-1$ at terminal value of t $(t = \tau+1 = K)$ is $\lambda(K) = -e^{-rK}[S_n(K) + f_{(n-K)}]$ and

$$\frac{d\lambda(t)}{dt} = e^{-rt}\{R_n - M_n(u(t))h_n(t) + L_n[1-u(t)]h_n(t)\} + e^{-r(\tau+1)}f_{(n-\tau-1)}[1-u(t)]h_n(t) + \lambda(t)[1-u(t)]h_n(t).$$

At time $t = \tau+1$, none of these terms are a function of $F_n(\tau)$. Therefore, neither is $u(\tau+1)$ which at $t = \tau+1$ is obtained by choosing $u(t)$ that maximizes the following:

$$\max_{\underline{U}_n \leqslant u(t) \leqslant \overline{U}_n} \{-M_n(u(t)) - (L_n + e^{-r(\tau+1-t)}f_{(n-\tau-1)} + \lambda(t)e^{rt})u(t)\}. \qquad (10)$$

This means that values of $\lambda(t-\Delta t)$ numerically computed as a function of $\lambda(t), d\lambda(t)/dt, u(t)$ do not have $F_n(\tau)$ as an argument for $t = \tau+1$, $\tau+1-\Delta t$, and $\tau+1-2\Delta t, \ldots, \Delta t+\tau$. When $u(t-\Delta t)$ has been computed for all these values of t, the next phase is a forward sweep of differential Equation (9) and the integral (8) using a fixed value of $F_n(\tau)$ as the given initial condition. Differential Equation (9) is linear and nonhomogenous of the form $dF_n(t)/dt = a(t) - a(t)F_n(t)$, and therefore its solution is of the form $F_n(t) = F_n(\tau)\cdot p(t) + q(t)$, where $F_n(\tau)$ is a constant. The integral in (8) for $J^*(n, \tau, K, F_n(\tau), f_{(n-\tau-1)}, \ldots, f_{(0)})$ is of the form $\int_{t=\tau}^{\tau+1}[1-F_n(\tau)p(t) - q(t)]z(t)\,dt$, and therefore is a linear function of $F_n(\tau)$. The same is true for the salvage value in (8) for $\tau = K-1$. The essential point here is that $p(t)$ and $q(t)$ do not depend on the initial condition $F_n(\tau)$. This means that the data for the regression come from a linear function and have to yield a perfect fit.

The maximum and minimum values of $F_n(\,)$ are 1 and 0, respectively, and when $F_n(\tau) = 1$, $J^* = 0$. Therefore, in the salvage value term of $\bar{A}[n, \tau+1, K, F_n(\tau+1)]$ in Equation (8), $J^*(n, \tau+1, K, F_n(\tau+1), f_{(n-\tau-2)}, \ldots, f_{(0)})$ can be replaced by $J^*(n, \tau+1, K, 0, f_{(n-\tau-2)}, \ldots, f_{(0)}) \cdot [1 - F_n(\tau+1)]$ and is now an analytical function of $F_n(\tau+1)$ and does not need to be estimated as a regression equation.

To begin the case for $\tau < K-1$, we now have the terminal value of λ:

$$\lambda(\tau+1) = \frac{\partial \bar{A}}{\partial F_n(\tau+1)} = -J^*(n, \tau+1, K, 0, f_{(n-\tau-2)}, \ldots, f_{(0)}). \qquad (11)$$

The arguments used above for the case of $\tau = K-1$ now apply in a similar fashion and lead again to a linear function and, therefore, a perfect fit.

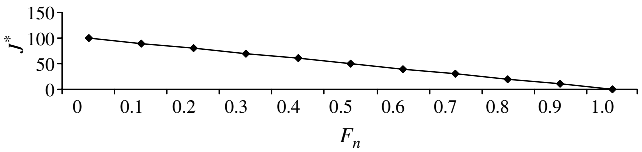
These results mean that the solution of the recursion (7) yields an exact solution for the Kamien-Schwartz maintenance model when $J^*(n, 0, K, 0, f_{(n-1)}, \ldots, f_{(0)}) - D_n$ is used for the term $V(n, K)$. Q.E.D

In contrast to the nonlinear relation in Figure 2, the just-proven single block of a straight line relation for the K-S model is illustrated in Figure 3.

6. An Illustrative Example

Because future technologies and their maintenance requirements are usually not known with certainty, one way to prepare is to consider alternative scenarios and

Figure 3. An illustration of the straight line relation between F_n and J^* for Kamien-Schwartz model.



determine the maintenance-replacement plans required by each.

As an example for one such scenario, consider a six-period problem. The vintage index j, identifying when the machine was purchased, will be measured by the number of periods from the time of purchase to the end of the planning horizon. Suppose that $r = 0.05$, $\underline{U}_j = 0$, $\overline{U}_j = 0.9$, $M_j(u(t)) = m_j(e^{c_j u(t)} - 1)$, and $S_j(T) = s_j e^{-0.5T}$; the underlying probability distribution function behind the natural hazard rate is Weibull such that $h_j(t) = b_j t^{b_j - 1}$, $L = 0.1$, and that the technologies of the different vintages yield the cash-flow parameters given in Table 1.

Initial resale value is 12% less than original cost D_j, namely, $s_j = 0.88D_j$. The difference $D_j - S_j(T)|_{T=0} = D_j - s_j$ includes ordering cost and installation cost, as well as the difference between the price of a brand-new machine versus that of a new but pre-owned one.

The primary algorithm is the dynamic programming recursion of (7) for stages $j = 1, \ldots, 6$. In each stage j, the standard K-S model is used for $K = 1$; for $K > 1$, the imbedded optimal control model is used according to (8) and (9) (in these equations n is replaced here with j).

The numerical procedure in §3.3 applies to this problem with the following modifications: $0 \leqslant \underline{U}_n \leqslant u(t) \leqslant \overline{U}_n \leqslant 1$ and Equation (10) are used for choosing the optimal value of $u(t)$. For salvage value, $\bar{A}[n, \tau+1, K, F_n(\tau+1)]$ is used as defined in Equation (8). For cases with $\tau < K - 1$, the expression obtained in §5 for the value of $\bar{A}[n, \tau+1, K, F_n(\tau+1)]$ is used in Equation (11) to compute the value of $\lambda(\tau+1)$ at the right end of the "local" optimal control problem from τ to $\tau+1$.

Stage 1 of dynamic programming consists of the numerical solution of a standard Kamien-Schwartz maintenance

Table 1. Revenue and cost parameters for a machine purchased j periods before end of the planning horizon.

t	5	4	3	2	1	0
j	1	2	3	4	5	6
R_j	71	70	64	40	37	35
m_j	1.2	1.3	1.9	1.9	2.2	2.5
c_j	4	4	4	3	2	1.5
D_j	45	35	30	28	28	20
b_j	1.3	1.28	1.26	1.22	1.2	1.15

problem for $j = 1$, with $K = 1$ and $\tau = 0$, below:

$$\max_{u(t)} \int_{t=0}^{1} e^{-0.05t}\big\{[71 - 1.2(e^{4u(t)} - 1)(1.3t^{0.3})][1 - F_1(t)] + 0.1(1 - u(t))(1.3t^{0.3})[1 - F_1(t)]\big\}\,dt + e^{-0.05}(0.88)(45e^{-0.5})[1 - F_1(1)]$$

subject to

$$\frac{dF_1(t)}{dt} = (1 - u(t))(1.3t^{0.3})[1 - F_1(t)],$$
$$0 \leqslant u(t) \leqslant 0.9, \quad \text{and} \quad F_1(0) = 0.$$

Beginning with the terminal value of $\lambda(1) = -e^{-0.05}(0.88)(45e^{-0.5}) = -22.8472$ and substituting it to (10), the optimal value of $u(1) = 0.402$ is obtained. Applying the numerical method of §3.3 for a backward sweep followed by a forward pass and subtracting D_1 yields the following result: $f_{(1)} = V(1, 1) = J^*(j = 1, \tau = 0, K = 1, F_1(0) = 0, f_{(0)} = 0) - D_1 = 19.879$.

Stage 2 solves for $f_{(2)} = \max[V(2, 1), V(2, 2)]$. For the case of $K = 1$, using standard Kamien-Schwartz optimal control yields $V(2, 1) = J^*(j = 2, \tau = 0, K = 1, F_2(0) = 0, f_{(1)} = 19.879) - D_2 = 44.218$. As for the other alternative (two periods of planned usage), i.e., $K = 2$, $V(2, 2)$ is obtained by the imbedded model of Equations (8) and (9). The first step solves for

$$\max_{u(t)} \int_{t=1}^{2} e^{-0.05t}\big\{[70 - 1.3(e^{4u(t)} - 1)(1.28t^{0.28})][1 - F_2(t)] + 0.1(1 - u(t))(1.28t^{0.28})[1 - F_2(t)]\big\}\,dt + e^{-0.1}(0.88)(35e^{-1})[1 - F_2(2)]$$

subject to

$$\frac{dF_2(t)}{dt} = (1 - u(t))(1.28t^{0.28})[1 - F_2(t)],$$
$$0 \leqslant u(t) \leqslant 0.9, \quad \text{and} \quad F_2(1) = 0$$

and yields a maximum value of 44.7523. This, as a salvage value, is imbedded into the next step as $J^*(j = 2, \tau = 1, K = 2, F_2(1), 0) = 44.7523[1 - F_2(1)]$ and requires the solution of

$$\max_{u(t)} \int_{t=0}^{1} \big\{e^{-0.05t}\big\{[70 - 1.3(e^{4u(t)} - 1)(1.28t^{0.28})][1 - F_2(t)] + 0.1(1 - u(t))(1.28t^{0.28})[1 - F_2(t)]\big\} + e^{-0.05}(19.879)(1 - u(t))(1.28t^{0.28})[1 - F_2(t)]\big\}\,dt + 44.7523[1 - F_2(1)]$$

subject to

$$\frac{dF_2(t)}{dt} = (1 - u(t))(1.28t^{0.28})[1 - F_2(t)],$$
$$0 \leqslant u(t) \leqslant 0.9, \quad \text{and} \quad F_2(0) = 0$$

and yields a maximum value of 84.1246. Subtracting from it the purchase price of \$35, $J^*(j=2, \tau=0, K=2, F_2(0)=0, f_{(1)}=19.879)-35=V(2,2)=\$49.125$. Thus, $f_{(2)}=\max(44.218, 49.125)=\$49.125$ with stage 2's optimal $K=2$.

Similarly, for the remaining stages one obtains: $f_{(3)}=\max_{K=1,2,3}[V(3,K)]=\$68.66$ with optimal $K=1$. The maximum value of $f_{(4)}=\$72.62$ is obtained from $K=1$. Continuing in the same fashion, $f_{(5)}=\$84.36$ for $K=2$ and $f_{(6)}=\$108.348$ for $K=3$ completes the solution.

These values imply the following. The machine on hand, planned for replacement at $t=3$ with a vintage 3 machine to be kept for one period, and then replaced at the fourth period with a new machine that would be planned for use for two periods, would yield the maximum value: an expected net present value of \$108.348. The corresponding maintenance effort (obtained from the imbedded Kamien-Schwartz model of stage 6 dynamic programming computations for $K=3$) in the first period begins at $t=0$ with $u(0)=0.9$, and remains the same for first two periods. In the middle of the third period, the value of u begins to decline, and at $t=3$ reaches $u(3)=0.014$. The course of action under failure is prescribed in the dynamic programming results of the previous two paragraphs. For example, if the first machine fails at, say, $t=0.8$, then there will be no production (and no revenues) for $1.0-0.8=0.2$ time units. At $t=1.0$, results of stage 5 calculations apply—namely, a new (replacement) machine begins production with intended (planned) use time of two periods, i.e., $K=2$.

The computational effort involves the following components: (1) The recursion of the dynamic programming Equation (7) is a polynomial function of T (Dreyfus and Law 1977, Chapter 2). If the total length of the planning horizon is kept constant while the number of replacement opportunities is increased, i.e., as the number of nodes T is increased, the increase in the computational effort does not rapidly become prohibitive. (2) Computational effort also depends on the method used for numerical integration. In the above example, fourth-order Runge-Kutta was used. Here, the choice of Δt determines the number of calculations and the precision of the results. The numbers given above are for $\Delta t=0.001$. To compare, other alternatives were tested. Using $\Delta t=0.01$ yielded an $f_{(6)}$ value that was less than 0.05% off from its corresponding one for $\Delta t=0.001$. Throughout the six dynamic programming stages based on the control model with $\Delta t=0.01$, none of the objective function values were more than 0.2% off from their counterparts of the case $\Delta t=0.001$. Pushing the approximation more crudely by setting $\Delta t=0.1$ yielded differences of less than 2% in each of the six stages of dynamic programming. The final value of $f_{(6)}$ obtained using $\Delta t=0.1$ was less than 0.5% off from the $f_{(6)}$ obtained using $\Delta t=0.001$. As Δt was being varied, the sensitivity of the values of $f_{(n)}$ tended to be less for larger values of n.

Programming the method in True BASIC and choosing $\Delta t=0.001$, the computation times observed on a Pentium 4 processor for problem sizes ranging from $T=10$ to $T=50$ replacement opportunities were fitted into two different models: *runtime* (*in seconds*) $=0.0375925\,T^{2.89915}$ and *runtime* (*in seconds*) $=3.7211-0.7983T+0.1334T^2+0.02298T^3$. Both models provided fit with deviations of no more than four seconds to any observation. Choosing $\Delta t=0.01$ reduced the computation time by a factor of 10. A problem with $T=50$ required 317 seconds. The same problem solved using $\Delta t=0.001$ required 3,169 seconds. The respective objective function values were different by less than 0.01%.

7. Summary and Avenues for Future Research

For a given scenario of technological change over time, what is the optimal control solution for replacement and maintenance of equipment with known natural hazard rates? While deterministic maintenance versions of this problem have been addressed during the last three decades, the probabilistic one lingered unsolved. The method presented above broke the time horizon into T replacement opportunities. For a duration between replacement opportunities at times, say, τ and $\tau+1$, the objective function of the control model (8), encompassed the expected net present value of the cash flow. In the differential equation constraint (9), the control variable $u(t)$ simply modified the original probability distribution function of the lifetime of the machine through extra maintenance effort. For any rectangular node in Figure 1, the difference $F_j(\tau+1)-F_j(\tau)$ of the incoming arrow determined the probability of which downstream arrow to take. We addressed how, via imbedding, the individual optimal control segments between times τ and $\tau+1$ should be pasted together through $\tau=0, 1, \ldots, K-1$. The overall dynamic programming effort, polynomial in T, wrapped the whole package. By choosing an appropriate value of T, the granularity of the problem may be adjusted to the specific application in industry.

The method proposed here solves the integrated replacement-maintenance problem and opens avenues for future research. The problem addressed in this paper was limited to a fixed planning horizon. The length of the planning horizon and its effects on the maintenance and replacement decisions, especially for the early periods, have practical implications to management. Because forecasting technologies can involve larger error margins as one looks further into the future, the minimum forecast horizons for robust decisions regarding the early periods of the planning horizon can be of importance and stand out as an avenue for future research in the spirit of the studies of Bylka et al. (1992), Hopp and Nair (1991, 1994), and some of the studies listed in the bibliography of Chand et al. (2002). As new periods approach, newly available information may require recomputation of the optimal policy with the updated data.

Decision making over such rolling horizons needs to be studied, and the present maintenance-replacement model may serve as one of the building blocks in such research.

Terms such as R_j were assumed to be constant over time, and S_j depended only on the age of the machine. One may wish to relax such assumptions and take into account, for example, fluctuations of the business cycle. Modification of the basic K-S model may also be explored for cases where R_j is a function of $u(t)$. An example to study may be Elsayed's (2003) data for oil refineries, where reduced operating temperatures of the industrial furnace prolong the residual lifetime of major production units, at the expense of reduced output.

The natural hazard rate may be modified to take seasonality into account. For example, for some products, winter conditions may be associated with more accidents than summer. In locations without climate control, items that are vulnerable to dampness may have a higher deterioration risk in winter. For other products it may be vice versa: Summer conditions may be more hazardous and may need (climate) control.

Another area to explore is alternative cost functions for the control variable u. If rapid variations in the control variable over a short period of time may cause additional costs, this may need to be included in the cost function in forms such as $M\{u(t),\ [u(t)-u(t-\Delta t)]^2\}$. More versatile models may incorporate control variables subject to state variable constraints using nonsmooth analysis and differential inclusions (Clarke et al. 1998, Vinter 2000).

In conclusion, alternative functional forms, cost structures, and constraints may expand the application areas of the model of Kamien and Schwartz (1971), and incorporating replacement decisions to optimal-control-maintenance models expands the time horizon of the problem. These in turn provide numerous avenues for future research.

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