Available online at www.sciencedirect.com
SCIENCE DIRECT®

ELSEVIER

Physics Letters A 336 (2005) 37–45

PHYSICS LETTERS A

www.elsevier.com/locate/pla

Schlesinger transformations for discrete second Painlevé equation: d-P_{II}

Uğurhan Muğan [a,*], Ayman Sakka [b], Paolo M. Santini [c]

[a] *Bilkent University, Department of Mathematics, 06800 Bilkent, Ankara, Turkey*
[b] *Islamic University of Gaza, Department of Mathematics, PO Box 108 Rimal, Gaza, Palestine*
[c] *Università di Roma La Sapienza, Dipartimento di Fisica, Istituto Nazionale di Fisica Nucleare, I-00185 Rome, Italy*

Received 13 March 2004; received in revised form 21 November 2004; accepted 5 January 2005

Available online 7 January 2005

Communicated by A.R. Bishop

Abstract

A method to obtain the Schlesinger transformations for the standard discrete second Painlevé equation, d–P_{II}, is given. The procedure involves formulating a Riemann–Hilbert problem for a transformation matrix which transforms the solution of the linear problem but leaves the associated monodromy data the same.

PACS: 02.30.Hq; 02.30.Ik; 02.30.Gp

Keywords: Discrete Painlevé equations; Schlesinger transformations; Inverse monodromy method

1. Introduction

A powerful method for studying the initial value problem for certain nonlinear ODEs was introduced in [1] and [2]. This method which is extension of the inverse spectral method (ISM) to ODEs, is called inverse monodromy method (IMM). It can be thought of as a nonlinear analogous of the Laplace's method. A rigorous investigation of the six Painlevé transcendents, P_I–P_{VI}, using this method has been carried out [3–5]. In particular, in these articles, it is shown that certain Riemann–Hilbert (RH) problems, occurring in the process of implementing the IMM, can be rigorously investigated. Furthermore, for special relations among the monodromy data, and for certain restrictions

* Corresponding author.
E-mail addresses: mugan@fen.bilkent.edu.tr (U. Muğan), asakka@mail.iugaza.edu (A. Sakka), paolo.santini@roma1.infn.it (P.M. Santini).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2005.01.003

of the constant parameters appearing in P_{II}–P_{VI}, these solutions have no poles. This provides the motivation for studying how the solutions of a Painlevé equations depend on their associated constant parameters.

Recently, nonlinear integrable discrete equations among which the discrete Painlevé (dP) equations are the most fundamental ones have attracted much attention. The dP equation was first obtained by Jimbo and Miwa [6]. The systematic derivation of the dP equations by using the Bäcklund transformations of the continuous Painlevé equations was given by Fokas, Grammaticos and Ramani [7]. Besides the rich mathematical structures of dP equations, such as the existence of Lax pairs, Bäcklund transformations, singularity confinement properties [8], the relation of dP equations to the continuous ones has been extensively investigated in the literature.

By exploiting the relation between the continuous and discrete Painlevé equations, in this Letter we present a method to obtain the Schlesinger transformations for the standard discrete second Painlevé equation, d-P_{II}. The same method was used to obtain the Schlesinger transformations for P_{II}–P_V [9], and for P_{VI} in [10]. These transformations lead to a class of relations between the solutions of d-P_{II} when its parameters are changed. In the case of the d-P_{II}, the singularity structure of the monodromy problem is more complicated (regular singular points at $\lambda = \pm 1$ and irregular singular points at $\lambda = 0, \infty$ of rank $r = 2$) with respect to monodromy problem of P_{II}.

Let x_n be the solution of d-P_{II} with the parameters c_0, c_2. The associated monodromy problem for d-P_{II} is $\frac{\partial Y_n}{\partial \lambda} = A_n Y_n$ where λ plays the role of spectral parameter. The implementation of the isomonodromy method necessitates the investigation of the analytic properties of $Y_n(\lambda)$ in complex λ-plane. It turns out that there exist a sectionally meromorphic function $Y_n(\lambda)$, with certain jumps across the certain contours of the complex λ-plane; these jumps are specified by the so-called monodromy data, denoted by MD. We denoted by x'_n and by Y'_n, x_n and Y_n when $(c_0, c_2) \to (c'_0, c'_2)$. It turns out that it is possible to find appropriate transformations of (c_0, c_2) such that the MD are invariant. Then $Y'_n(\lambda) = R_n(\lambda)Y_n(\lambda)$, and the Schlesinger transformation matrix $R_n(\lambda)$, can be found in *closed form*, by solving a certain simple RH-problems. The transformation matrix $R_n(\lambda)$ leads to a class of the transformations among the solutions x_n of d-P_{II}.

The standard discrete second Painlevé equation, d-P_{II}

$$2c_3(x_{n+1} + x_{n-1})\big(1 - x_n^2\big) = -x_n(2c_2 + 2n + 1) + c_0, \quad c_3 \neq 0, \tag{1}$$

can be obtained as the compatibility condition of the following linear system of equations [11],

$$\frac{\partial Y_n}{\partial \lambda} = A_n(\lambda)Y_n(\lambda), \tag{2.a}$$

$$Y_{n+1} = B_n Y_n(\lambda), \tag{2.b}$$

where

$$A_n(\lambda) = A_1\lambda + A_2 + A_3\lambda^{-1} + A_4\lambda^{-2} + A_5\lambda^{-3} + A_6\big(\lambda^2 - 1\big)^{-1}, \tag{3.a}$$

$$B_n = B_1\lambda^{-1} + B_2 + B_3\lambda, \tag{3.b}$$

and

$$A_1 = A_5 = c_3\sigma_3, \qquad A_2 = \begin{pmatrix} 0 & 2c_3x_n \\ 2c_3x_{n-1} & 0 \end{pmatrix}, \qquad A_3 = (c_2 + n - 2c_3x_nx_{n-1})\sigma_3,$$
$$A_4 = \begin{pmatrix} 0 & -2c_3x_{n-1} \\ -2c_3x_n & 0 \end{pmatrix} = -\sigma_1A_2\sigma_1, \qquad A_6 = c_0\sigma_1,$$
$$B_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & x_n \\ x_n & 0 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4}$$

σ_i, $i = 1, 2, 3$ are Pauli spin matrices and defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

The entries $(1,1)$ and $(2,2)$ of the compatibility condition $\frac{\partial B_n}{\partial\lambda} + B_n A_n = A_{n+1} B_n$ are identically satisfied and the entries $(1,2)$ and $(2,1)$ give the d-P_{II}.

2. Direct problem

The essence of the direct problem is to establish the analytic structure of Y_n in the entire complex λ-plane. Since, (2.a) is a linear ODE in λ, the analytic structure of Y_n is completely determined by its singularities. (2.a) has regular singular points at $\lambda = \pm 1$ and irregular singular points at $\lambda = 0, \infty$ with rank $r = 2$.

2.1. *Solution about* $\lambda = 0$

The formal solution $\tilde{Y}_n^{(0)}(\lambda) = (\tilde{Y}_{n,1}^{(0)}(\lambda), \tilde{Y}_{n,2}^{(0)}(\lambda))$, of (2.a) in the neighborhood of the irregular singular point $\lambda = 0$ has the expansion

$$\tilde{Y}_n^{(0)}(\lambda) = \hat{Y}_n^{(0)}(\lambda)\left(\frac{1}{\lambda}\right)^{D_n^{(0)}} e^{Q^{(0)}(\lambda)} = \left(I + \hat{Y}_{n,1}^{(0)}\lambda + \hat{Y}_{n,2}^{(0)}\lambda^2 + \cdots\right)\left(\frac{1}{\lambda}\right)^{D_n^{(0)}} e^{Q^{(0)}(\lambda)}, \tag{6}$$

where

$$\hat{Y}_{n,1}^{(0)} = \begin{pmatrix} 0 & x_{n-1} \\ -x_n & 0 \end{pmatrix}, \qquad D_n^{(0)} = -(c_2 + n)\sigma_3, \qquad Q^{(0)}(\lambda) = -\frac{c_3}{2\lambda^2}\sigma_3. \tag{7}$$

Let $Y_{n(j)}^{(0)}$, $j = 1, \ldots, 4$, be solutions of (2.a), such that $Y_{n(j)}^{(0)}(\lambda) \sim \tilde{Y}^{(0)}(\lambda)$ as $\lambda \to 0$ in the sector $S_j^{(0)}$, where the sectors are given as follows and indicated in Fig. 1,

$$\begin{aligned} &S_1^{(0)}: \quad -\frac{\pi}{4} \leqslant \arg\lambda < \frac{\pi}{4}, \qquad && S_2^{(0)}: \quad \frac{\pi}{4} \leqslant \arg\lambda < \frac{3\pi}{4}, \\ &S_3^{(0)}: \quad \frac{3\pi}{4} \leqslant \arg\lambda < \frac{5\pi}{4}, \qquad && S_4^{(0)}: \quad \frac{5\pi}{4} \leqslant \arg\lambda < \frac{7\pi}{4}, \quad |\lambda| < 1. \end{aligned} \tag{8}$$

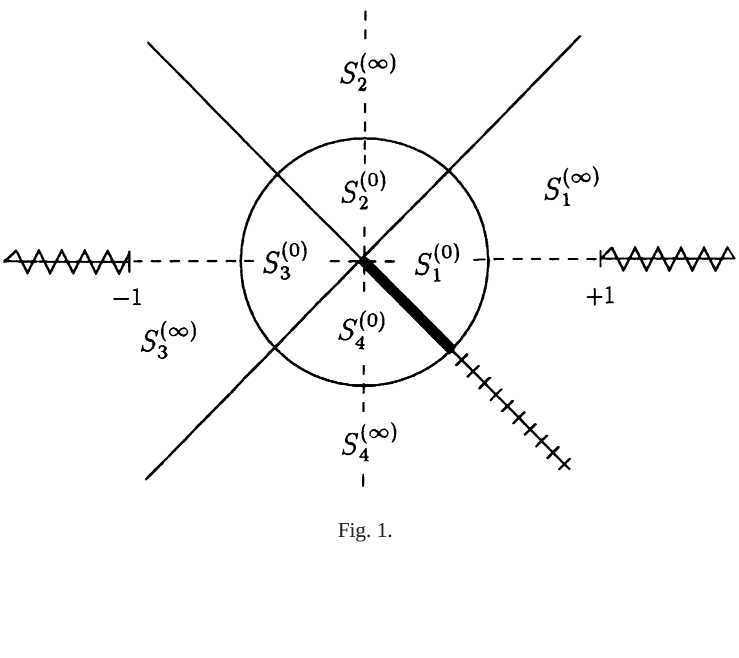

Fig. 1.

The solutions $Y^{(0)}_{n(j)}$ are related by the Stokes matrices $G^{(0)}_j$ and the monodromy matrix $M^{(0)}$ such that

$$Y^{(0)}_{n(j+1)}(\lambda) = Y^{(0)}_{n(j)}(\lambda)G^{(0)}_j, \quad \lambda \in S^{(0)}_{j+1}, \; j = 1, 2, 3,$$
$$Y^{(0)}_{n(1)}(\lambda) = Y^{(0)}_{n(4)}\bigl(\lambda e^{2i\pi}\bigr)G^{(0)}_4 M^{(0)}, \quad \lambda \in S^{(0)}_1, \tag{9}$$

where

$$G^{(0)}_1 = \begin{pmatrix} 1 & a^{(0)} \\ 0 & 1 \end{pmatrix}, \qquad G^{(0)}_2 = \begin{pmatrix} 1 & 0 \\ b^{(0)} & 1 \end{pmatrix},$$
$$G^{(0)}_3 = \begin{pmatrix} 1 & c^{(0)} \\ 0 & 1 \end{pmatrix}, \qquad G^{(0)}_4 = \begin{pmatrix} 1 & 0 \\ d^{(0)} & 1 \end{pmatrix}, \qquad M^{(0)} = e^{2i\pi D^{(0)}_n} = e^{-2i\pi c_2\sigma_3}. \tag{10}$$

2.2. Solution about $\lambda = \infty$

The formal solution $\tilde{Y}^{(\infty)}_n(\lambda) = (\tilde{Y}^{(\infty)}_{n,1}(\lambda), \tilde{Y}^{(\infty)}_{n,2}(\lambda))$, of (2.a) in the neighborhood of the irregular singular point $\lambda = \infty$ has the expansion

$$\tilde{Y}^{(\infty)}_n(\lambda) = \hat{Y}^{(\infty)}_n(\lambda)\lambda^{D^{(\infty)}_n} e^{Q^{(\infty)}(\lambda)} = \bigl(I + \hat{Y}^{(\infty)}_{n,1}\lambda^{-1} + \hat{Y}^{(\infty)}_{n,2}\lambda^{-2} + \cdots\bigr)\lambda^{D^{(\infty)}_n} e^{Q^{(\infty)}(\lambda)}, \tag{11}$$

where

$$\hat{Y}^{(\infty)}_{n,1} = \begin{pmatrix} 0 & -x_n \\ x_{n-1} & 0 \end{pmatrix}, \qquad D^{(\infty)}_n = (c_2 + n)\sigma_3, \qquad Q^{(\infty)}(\lambda) = \frac{c_3}{2}\lambda^2\sigma_3. \tag{12}$$

Let $Y^{(\infty)}_{n(j)}$, $j = 1, \ldots, 4$, be solutions of (2.a), such that $Y^{(\infty)}_{n(j)}(\lambda) \sim \tilde{Y}^{(\infty)}(\lambda)$ as $\lambda \to \infty$ in the sector $S^{(\infty)}_j$, where the sectors are given as follows and indicated in Fig. 1,

$$S^{(\infty)}_1: \quad -\frac{\pi}{4} \leqslant \arg\lambda < \frac{\pi}{4}, \qquad S^{(\infty)}_2: \quad \frac{\pi}{4} \leqslant \arg\lambda < \frac{3\pi}{4},$$
$$S^{(\infty)}_3: \quad \frac{3\pi}{4} \leqslant \arg\lambda < \frac{5\pi}{4}, \qquad S^{(\infty)}_4: \quad \frac{5\pi}{4} \leqslant \arg\lambda < \frac{7\pi}{4}, \quad |\lambda| > 1. \tag{13}$$

The solutions $Y^{(\infty)}_{n(j)}$ are related by the Stokes matrices $G^{(\infty)}_j$ and the monodromy matrix $M^{(\infty)}$ such that

$$Y^{(\infty)}_{n(j+1)}(\lambda) = Y^{(\infty)}_{n(j)}(\lambda)G^{(\infty)}_j, \quad \lambda \in S^{(\infty)}_{j+1}, \; j = 1, 2, 3,$$
$$Y^{(\infty)}_{n(1)}(\lambda) = Y^{(\infty)}_{n(4)}\bigl(\lambda e^{2i\pi}\bigr)G^{(\infty)}_4 M^{(\infty)}, \quad \lambda \in S^{(\infty)}_1, \tag{14}$$

where

$$G^{(\infty)}_1 = \begin{pmatrix} 1 & 0 \\ a^{(\infty)} & 1 \end{pmatrix}, \qquad G^{(\infty)}_2 = \begin{pmatrix} 1 & b^{(\infty)} \\ 0 & 1 \end{pmatrix},$$
$$G^{(\infty)}_3 = \begin{pmatrix} 1 & 0 \\ c^{(\infty)} & 1 \end{pmatrix}, \qquad G^{(\infty)}_4 = \begin{pmatrix} 1 & d^{(\infty)} \\ 0 & 1 \end{pmatrix}, \qquad M^{(\infty)} = e^{-2i\pi D^{(\infty)}_n} = e^{-2i\pi c_2\sigma_3}. \tag{15}$$

2.3. Solution about $\lambda = 1$

The two linearly independent solutions $Y^{(1)}_n(\lambda) = (\tilde{Y}^{(1)}_{n,1}(\lambda), \tilde{Y}^{(1)}_{n,2}(\lambda))$ of (2.a) in the neighborhood of the regular singular point $\lambda = 1$ for $c_0 \neq n$, $n \in \mathbb{Z}$, and $|\lambda - 1| < 1/2$ has the following expansion

$$Y^{(1)}_n(\lambda) = \hat{Y}^{(1)}_n(\lambda)(\lambda - 1)^{D^{(1)}} = \hat{Y}^{(1)}_{n,0}\bigl\{I + \hat{Y}^{(1)}_{n,1}(\lambda - 1) + \hat{Y}^{(1)}_{n,2}(\lambda - 1)^2 + \cdots\bigr\}(\lambda - 1)^{D^{(1)}}, \tag{16}$$

where

$$\hat{Y}^{(1)}_{n,0} = \begin{pmatrix} \mu^{(1)}_n & \nu^{(1)}_n \\ \mu^{(1)}_n & -\nu^{(1)}_n \end{pmatrix}, \qquad D^{(1)} = \frac{c_0}{2}\sigma_3, \tag{17}$$

and

$$\mu^{(1)}_n = \mu^{(1)}_0 \prod_{i=1}^{n-1}(1+x_i), \qquad \nu^{(1)}_n = \nu^{(1)}_0 \prod_{i=1}^{n-1}(1-x_i), \tag{18}$$

where $\mu^{(1)}_0$, $\nu^{(1)}_0$ are constant. (18) can be obtained by imposing the condition that $Y^{(1)}_n$ satisfies (2.b). $\hat{Y}^{(1)}_{n,1}$ satisfies

$$\hat{Y}^{(1)}_{n,1} + \left[\hat{Y}^{(1)}_{n,1}, D^{(1)}\right] = \left(\hat{Y}^{(1)}_{n,0}\right)^{-1} A^{(1)}_0 \hat{Y}^{(1)}_{n,0}, \tag{19}$$

where

$$A^{(1)}_0 = \sum_{k=1}^{5} A_k - \frac{1}{4}A_6. \tag{20}$$

Monodromy matrix about $\lambda = 1$ is defined as

$$Y^{(1)}_n\left(\lambda e^{2i\pi}\right) = Y^{(1)}_n(\lambda) M^{(1)}, \quad M^{(1)} = e^{2i\pi D^{(1)}} = e^{i\pi c_0 \sigma_3}. \tag{21}$$

2.4. *Solution about* $\lambda = -1$

The two linearly independent solutions $Y^{(-1)}_n(\lambda) = (\tilde{Y}^{(-1)}_{n,1}(\lambda), \tilde{Y}^{(-1)}_{n,2}(\lambda))$ of (2.a) in the neighborhood of the regular singular point $\lambda = -1$ for $c_0 \neq n$, $n \in \mathbb{Z}$, and $|\lambda + 1| < 1/2$ has the following expansion

$$Y^{(-1)}_n(\lambda) = \hat{Y}^{(-1)}_n(\lambda)(\lambda+1)^{D^{(-1)}} = \hat{Y}^{(-1)}_{n,0}\left\{I + \hat{Y}^{(-1)}_{n,1}(\lambda+1) + \hat{Y}^{(-1)}_{n,2}(\lambda+1)^2 + \cdots\right\}(\lambda+1)^{D^{(-1)}}, \tag{22}$$

where

$$\hat{Y}^{(-1)}_{n,0} = \begin{pmatrix} \mu^{(-1)}_n & \nu^{(-1)}_n \\ -\mu^{(-1)}_n & \nu^{(-1)}_n \end{pmatrix}, \qquad D^{(-1)} = \frac{c_0}{2}\sigma_3, \tag{23}$$

and

$$\mu^{(-1)}_n = (-1)^n \mu^{(-1)}_0 \prod_{i=1}^{n-1}(1+x_i), \qquad \nu^{(-1)}_n = (-1)^n \nu^{(-1)}_0 \prod_{i=1}^{n-1}(1-x_i), \tag{24}$$

where $\mu^{(-1)}_0$, $\nu^{(-1)}_0$ are constants. (24) can be obtained by imposing the condition that $Y^{(-1)}_n$ satisfies (2.b). $\hat{Y}^{(-1)}_{n,1}$ satisfies

$$\hat{Y}^{(-1)}_{n,1} + \left[\hat{Y}^{(-1)}_{n,1}, D^{(-1)}\right] = \left(\hat{Y}^{(-1)}_{n,0}\right)^{-1} A^{(-1)}_0 \hat{Y}^{(-1)}_{n,0}, \tag{25}$$

where

$$A^{(-1)}_0 = \sum_{k=1}^{5}(-1)^k A_k - \frac{1}{4}A_6. \tag{26}$$

Monodromy matrix about $\lambda = -1$ is defined as

$$Y^{(-1)}_n\left(\lambda e^{2i\pi}\right) = Y^{(-1)}_n(\lambda) M^{(-1)}, \qquad M^{(-1)} = e^{2i\pi D^{(-1)}} = e^{i\pi c_0 \sigma_3}. \tag{27}$$

2.5. Symmetries and monodromy data

The relation between $Y_{n(1)}^{(\infty)}$ and $Y_n^{(1)}(\lambda)$, $Y_{n(1)}^{(0)}$, and $Y_{n(3)}^{(\infty)}$ and $Y_n^{(-1)}(\lambda)$ are given by the connection matrices $E^{(k)}$, $k=-1,0,1$,

$$Y_{n(1)}^{(\infty)}(\lambda) = Y_n^{(1)}(\lambda)E^{(1)}, \tag{28.a}$$

$$Y_{n(1)}^{(\infty)}(\lambda) = Y_{n(1)}^{(0)}(\lambda)E^{(0)}, \tag{28.b}$$

$$Y_{n(3)}^{(\infty)}(\lambda) = Y_n^{(-1)}(\lambda)E^{(-1)}, \tag{28.c}$$

where

$$E^{(k)} = \begin{pmatrix} \alpha^{(k)} & \beta^{(k)} \\ \gamma^{(k)} & \delta^{(k)} \end{pmatrix}, \qquad \det E^{(k)} = 1. \tag{29}$$

Noted that, if $Y_n(\lambda)$ solve the linear differential equations (2) then $\sigma_1 Y(\frac{1}{\lambda})\sigma_1$ also solves the linear differential equations. So we have the following relation between the sectionally analytic functions $Y_{n(j)}^{(\infty)}(\lambda)$ and $Y_{n(j)}^{(0)}(\lambda)$

$$\sigma_1 Y_{n(j)}^{(\infty)}\left(\frac{1}{\lambda}\right)\sigma_1 = Y_{n(j)}^{(0)}(\lambda), \quad j=1,\ldots,4. \tag{30}$$

(30) implies the following relations

$$\sigma_1 G_j^{(\infty)}\sigma_1 = G_j^{(0)}, \quad j=1,\ldots,4, \qquad \sigma_1 E^{(0)}\sigma_1 = \left[E^{(0)}\right]^{-1}. \tag{31}$$

Similarly, both $Y_n(\lambda)$ and $\sigma_3 Y_n(\lambda e^{-i\pi})\sigma_3$ solve the linear differential equations (2). So we have the following symmetry for the sectionally analytic functions $Y_n(\lambda)$:

$$Y_{n(j+2)}^{(\infty)}(\lambda) = \sigma_3 Y_{n(j)}^{(\infty)}\left(\lambda e^{-i\pi}\right)\sigma_3, \qquad Y_{n(j+2)}^{(0)}(\lambda) = \sigma_3 Y_{n(j)}^{(0)}\left(\lambda e^{-i\pi}\right)\sigma_3, \quad j=1,2,$$
$$Y_n^{(-1)}(\lambda) = \sigma_3 Y_n^{(1)}\left(\lambda e^{-i\pi}\right)\sigma_3. \tag{32}$$

The symmetry relation (32) implies the relation

$$G_{j+2}^{(\infty)} = \sigma_3 G_j^{(\infty)}\sigma_3, \qquad G_{j+2}^{(0)} = \sigma_3 G_j^{(0)}\sigma_3, \quad j=1,2, \qquad \sigma_3 E^{(-1)}\sigma_3 = E^{(1)}. \tag{33}$$

Hence, the set of the monodromy data MD is

$$\mathrm{MD} = \left\{a^{(\infty)}, b^{(\infty)}, \alpha^{(0)}, \beta^{(0)}, \delta^{(0)}, \alpha^{(1)}, \beta^{(1)}, \gamma^{(1)}, \delta^{(1)}\right\}. \tag{34}$$

Clearly monodromy data are independent of λ. Moreover, it can be easily shown that they are also independent of n and satisfy the following consistency condition

$$G_1^{(\infty)}G_2^{(\infty)}J^{(-1)}G_3^{(\infty)}G_4^{(\infty)}M^{(\infty)}J^{(1)} = \left[E^{(0)}\right]^{-1}\prod_{j=1}^{4} G_j^{(0)}M^{(0)}E^{(0)}, \tag{35}$$

where

$$J^{(-1)} = \left[E^{(-1)}\right]^{-1}M^{(-1)}E^{(-1)}, \qquad J^{(1)} = \left[E^{(1)}\right]^{-1}M^{(1)}E^{(1)}. \tag{36}$$

In particular, the trace of the consistency conditions (35) is

$$T_1 e^{2i\pi(c_0+2c_2)} + T_2 e^{-2i\pi c_0} + T_3 e^{-2i\pi(c_0-2c_2)} + T_4 e^{2i\pi c_0} + T_5 e^{4i\pi c_2} + T_6$$
$$= e^{4i\pi c_2}\left(1 - a^{(\infty)}b^{(\infty)}\right) + a^{(\infty)}b^{(\infty)}\left(1 + a^{(\infty)}b^{(\infty)}\right) + 1, \tag{37}$$

where T_i, $i=1,\ldots,6$, can be written in terms of MD.

3. Schlesinger transformation

Let $[Y^{(\infty)}_{n(1)}(\lambda)]_-$ and $[Y^{(\infty)}_{n(1)}(\lambda)]_+$ be the limit values of $Y^{(\infty)}_{n(1)}(\lambda)$, as λ approaches to contour C_R (see Fig. 2) from above and from below, respectively, and similarly $[Y^{(\infty)}_{n(3)}(\lambda)]_+$ and $[Y^{(\infty)}_{n(3)}(\lambda)]_-$ be the limit values of $Y^{(\infty)}_{n(3)}(\lambda)$, as λ approaches to contour C_L from above and from below, respectively. Then by the definition (28.c) of the connection matrices $E^{(j)}$ and the definition (21), (27) of monodromy matrices $M^{(j)}$, $j=-1,1$, $[Y^{(\infty)}_{n(i)}(\lambda)]_\pm$, $i=1,3$, are related as follows:

$$C_R:\quad \bigl[Y^{(\infty)}_{n(1)}(\lambda)\bigr]_+ = \bigl[Y^{(\infty)}_{n(1)}(\lambda)\bigr]_- \begin{cases} J^{(1)} & \text{for } \lambda > 1, \\ I & \text{for } 1/2 < \lambda < 1, \end{cases} \tag{38}$$

$$C_L:\quad \bigl[Y^{(\infty)}_{n(3)}(\lambda)\bigr]_+ = \bigl[Y^{(\infty)}_{n(3)}(\lambda)\bigr]_- \begin{cases} J^{(-1)} & \text{for } \lambda < -1, \\ I & \text{for } -1 < \lambda < -1/2, \end{cases} \tag{39}$$

where $J^{(1)}$, $J^{(-1)}$ are given in (36).

Let $R_n(\lambda)$ be the transformation matrix which transforms the solution of the linear problem (2) as

$$Y'_n(\lambda) = R_n(\lambda) Y_n(\lambda), \tag{40}$$

but leaves the monodromy data associated with Y_n the same. Let x'_n and $c'_i = c_i + \kappa_i$ be the transformed quantities of x_n and c_i, $i=0,2$, respectively. The consistency condition of the monodromy data (35) or (37) is invariant under the transformation if $c'_0 = c_0 + p$, $c'_2 = c_2 + q/2$ where p, q are integers. Let $R_n(\lambda) = R^{(0)}_{n(j)}(\lambda)$ when λ in $S^{(0)}_j$, $j = 1,\dots,4$, $R_n(\lambda) = R^{(\infty)}_{n(i)}(\lambda)$ when λ in $S^{(\infty)}_i$, $i=2,4$, and

$$\begin{aligned} R_n(\lambda) &= \bigl[R^{(\infty)}_{n(1)}(\lambda)\bigr]_+ \quad \text{when } \lambda \in \bigl[S^{(\infty)}_1\bigr]_+, \qquad & R_n(\lambda) &= \bigl[R^{(\infty)}_{n(1)}(\lambda)\bigr]_- \quad \text{when } \lambda \in \bigl[S^{(\infty)}_1\bigr]_-, \\ R_n(\lambda) &= \bigl[R^{(\infty)}_{n(3)}(\lambda)\bigr]_+ \quad \text{when } \lambda \in \bigl[S^{(\infty)}_3\bigr]_+, \qquad & R_n(\lambda) &= \bigl[R^{(\infty)}_{n(3)}(\lambda)\bigr]_- \quad \text{when } \lambda \in \bigl[S^{(\infty)}_3\bigr]_-, \end{aligned} \tag{41}$$

where the sectors $[S^{(\infty)}_k]_\pm$, $k=1,3$, are

$$\begin{aligned} \bigl[S^{(\infty)}_1\bigr]_+&:\quad -\pi/4 \leqslant \arg\lambda < 0, \qquad & \bigl[S^{(\infty)}_1\bigr]_-&:\quad 0 \leqslant \arg\lambda < \frac{\pi}{4}, \\ \bigl[S^{(\infty)}_3\bigr]_+&:\quad 3\pi/4 \leqslant \arg\lambda < \pi, \qquad & \bigl[S^{(\infty)}_3\bigr]_-&:\quad \pi \leqslant \arg\lambda < \frac{5\pi}{4}, \end{aligned} \tag{42}$$

and $|\lambda| > 1/2$. Definition (9), (14) of the Stokes matrices, (28) of connection matrices and (38), (39) imply that the sectionally analytic transformation matrix R_n satisfies the following RH-problem on the contours indicated in

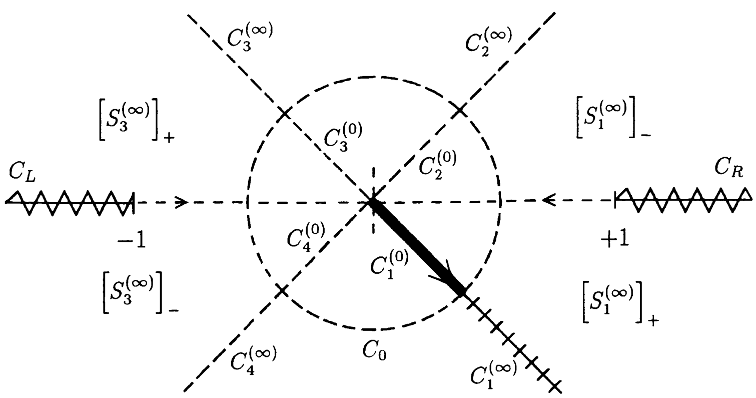

Fig. 2.

Fig. 2:

$$C_{j+1}^{(0,\infty)}:\quad R_{n(j+1)}^{(0,\infty)}(\lambda)=R_{n(j)}^{(0,\infty)}(\lambda),\qquad j=1,2,3, \tag{43}$$

$$C_{1}^{(0,\infty)}:\quad R_{n(1)}^{(0,\infty)}(\lambda)=(-1)^q R_{n(4)}^{(0,\infty)}\big(\lambda e^{2i\pi}\big), \tag{44}$$

$$C_R:\quad \big[R_{n(1)}^{(\infty)}\big]_+=\big[R_{n(1)}^{(\infty)}\big]_-\begin{cases}(-1)^p & \text{for } \lambda>1,\\ I & \text{for } 1/2<\lambda<1,\end{cases} \tag{45}$$

$$C_L:\quad \big[R_{n(3)}^{(\infty)}\big]_+=\big[R_{n(3)}^{(\infty)}\big]_-\begin{cases}(-1)^p & \text{for } \lambda<-1,\\ I & \text{for } -1<\lambda<-1/2,\end{cases} \tag{46}$$

$$C_0:\quad [R_n]_+=[R_n]_- \tag{47}$$

with the following boundary conditions

$$\begin{aligned}
&R_{n(1)}^{(0)}(\lambda)\sim\big[\hat{Y}_n^{(0)}(\lambda)\big]'\left(\frac{1}{\lambda}\right)^{\frac12 q\sigma_3}\big[\hat{Y}_n^{(0)}(\lambda)\big]^{-1},\quad \text{as } \lambda\to 0,\ \lambda\in S_1^{(0)},\\
&\big[R_{n(1)}^{(\infty)}(\lambda)\big]_+\sim\big[\hat{Y}_n^{(\infty)}(\lambda)\big]'\lambda^{\frac12 q\sigma_3}\big[\hat{Y}_n^{(\infty)}(\lambda)\big]^{-1},\quad \text{as } \lambda\to \infty,\ \lambda\in \big[S_1^{(\infty)}\big]_+,\\
&\big[R_{n(1)}^{(\infty)}(\lambda)\big]_+\sim\big[\hat{Y}_n^{(1)}(\lambda)\big]'(\lambda-1)^{\frac12 p\sigma_3}\big[\hat{Y}_n^{(1)}(\lambda)\big]^{-1},\quad \text{as } \lambda\to 1,\ \lambda\in \big[S_1^{(\infty)}\big]_+,\\
&\big[R_{n(3)}^{(\infty)}(\lambda)\big]_+\sim\big[\hat{Y}_n^{(-1)}(\lambda)\big]'(\lambda+1)^{\frac12 p\sigma_3}\big[\hat{Y}_n^{(-1)}(\lambda)\big]^{-1},\quad \text{as } \lambda\to -1,\ \lambda\in \big[S_3^{(\infty)}\big]_+.
\end{aligned} \tag{48}$$

From Eqs. (44)–(46) and the boundary conditions (48), the continuity of the RH-problem at $\lambda=0$ and consistency at $\lambda=\infty$ imply that p and q are even integers. Hence, the shifts in (c_0,c_2) are

$$(c_0',c_2')=(c_0+2k,c_2+r),\qquad k,r\in\mathbb{Z}, \tag{49}$$

and the transformation matrix R_n is analytic everywhere in λ-plane. R_n can be determined explicitly from the boundary conditions (48). It is enough to consider the particular cases $(k,r)=(\pm1,0)$ and $(k,r)=(0,\pm1)$.

For $(c_0',c_2')=(c_0+2,c_2)$, the transformation matrix is as follows:

$$R_{n,1}=\frac{r_1}{\lambda^2-1}\begin{pmatrix}(1-2\rho_1)(\lambda^2-1)+2 & -2\lambda\\ -2\lambda & (1+2\rho_1)(\lambda^2-1)+2\end{pmatrix}, \tag{50}$$

where

$$\rho_1=\frac{1}{c_0+1}\big[2c_3(x_n+1)(1-x_{n-1})+c_2+n\big],\qquad r_1^2=\frac{1}{1-2\rho_1^2}. \tag{51}$$

By using Eqs. (2.a) and (50) we can obtain the following Bäcklund transformation for x_n [12]

$$x_n'=\frac{1}{1+2\rho_1}\big[(1-2\rho_1)x_n+2\big]. \tag{52}$$

The transformation (52) breaks down if $\rho_1=-1/2$. But then $(1-2\rho_1)x_n+2$ must be zero or $c_0=-1$. Hence, d-P_{II} admits one-parameter family of solutions characterized by the following discrete Riccati equation if $c_0=-1$:

$$x_n=-1+\frac{c_2+n}{2c_3(x_{n-1}-1)}. \tag{53}$$

For $(c_0',c_2')=(c_0-2,c_2)$, the transformation matrix $R_{n,2}$ is

$$R_{n,2}=\frac{r_2}{\lambda^2-1}\begin{pmatrix}(1+2\rho_2)(\lambda^2-1)+2 & 2\lambda\\ 2\lambda & (1-2\rho_2)(\lambda^2-1)+2\end{pmatrix}, \tag{54}$$

where

$$\rho_2 = \frac{1}{c_0 - 1}\big[2c_3(x_n + 1)(1 - x_{n-1}) + c_2 + n\big], \qquad r_2^2 = \frac{1}{1 - 2\rho_2^2}. \tag{55}$$

$R_{n,2}$ yields the following Bäcklund transformation for x_n,

$$x_n' = \frac{1}{1 - 2\rho_2}\big[(1 + 2\rho_2)x_n - 2\big]. \tag{56}$$

It should be noted that, the transformation (56) can be obtain by combining (52) with $x_n'' = -x_n'$, $c_0'' = -c_0'$. Similarly, (56) breaks down if $\rho_2 = 1/2$. But then $(1 + 2\rho_2)x_n - 2$ must be zero or $c_0 = 1$. Hence, one-parameter family of solutions of d-P$_{\mathrm{II}}$ satisfy the following discrete Riccati equation if $c_0 = 1$:

$$x_n = 1 + \frac{c_2 + n}{2c_3(x_{n-1} + 1)}. \tag{57}$$

For $(c_0', c_2') = (c_0, c_2 + 1)$, the transformation matrix is $R_{n,3} = B_n$ where B_n is given in (3.b). The transformation matrix $R_{n,3}$ leads to $x_n' = x_{n+1}$. For $(c_0', c_2') = (c_0, c_2 - 1)$, the transformation matrix $R_{n,4}$ is

$$R_{n,4} = \begin{pmatrix} \frac{1}{\lambda} & -x_{n-1} \\ -x_{n-1} & \lambda \end{pmatrix} \tag{58}$$

and the corresponding transformation is $x_n' = x_{n-1}$.

Successive applications of $R_{n,i}$, $i = 1, \ldots, 4$, map $c_0' = c_0 + 2k$ and $c_2' = c_2 + r$, $k, r \in \mathbb{Z}$. Also, it should be noticed that $R_{n,1}R_{n,2} = I$.

Acknowledgements

We would like to thank Nalini Joshi for making the linear equations (2) available to us. U.M. would also like to thank Department of Physics at University of Rome La Sapienza for their support during his stay in Rome. This work was partially supported by the Scientific and Technical Research Council of Turkey (TÜBİTAK).

References

[1] H. Flaschka, A. Newell, Commun. Math. Phys. 76 (1982) 76.
[2] M. Jimbo, T. Miwa, K. Ueno, Physica D 2 (1981) 306;
M. Jimbo, T. Miwa, Physica D 2 (1981) 407;
M. Jimbo, T. Miwa, Physica D 4 (1981) 47.
[3] A.S. Fokas, X. Zhou, Commun. Math. Phys. 144 (1992) 601.
[4] A.S. Fokas, U. Muğan, X. Zhou, Inverse Problems 8 (1992) 757.
[5] U. Muğan, A. Sakka, J. Phys. A: Math. Gen. 28 (1995) 4109.
[6] M. Jimbo, T. Miwa, Physica D 2 (1981) 407.
[7] A.S. Fokas, B. Grammaticos, A. Ramani, J. Math. Anal. Appl. 180 (1993) 342.
[8] B. Grammaticos, A. Ramani, Application of Analytic and Geometric Methods to Nonlinear Differential Equations, in: P.A. Clarkson (Ed.), NATO ASI Ser. C, vol. 413, Kluwer, Dordrecht, 1993.
[9] U. Muğan, A.S. Fokas, J. Math. Phys. 33 (1992) 2031.
[10] U. Muğan, A. Sakka, J. Math. Phys. 36 (1995) 1284.
[11] N. Joshi, private communication.
[12] A.S. Carstea, A. Ramani, R. Willox, B. Grammaticos, J. Phys. A: Math. Gen. 36 (2003) 8419.