Gen. Relativ. Gravit. (2005) 37(12): 2015–2022
DOI 10.1007/s10714-005-0176-y

RESEARCH ARTICLE

Metin Gürses · Özgür Sarıoğlu

Accelerated Levi-Civita-Bertotti-Robinson metric in D dimensions

Received: 14 February 2005 / Published online: 22 November 2005

Abstract A conformally flat accelerated charge metric is found in an arbitrary dimension D. It is a solution of the Einstein-Maxwell-null fluid equations with a cosmological constant in $D \geq 4$ dimensions. When the acceleration is zero, our solution reduces to the Levi-Civita-Bertotti-Robinson metric. We show that the charge loses its energy, for all dimensions, due to the acceleration.

Keywords Accelerated charge · Einstein-Maxwell-null fluid · Levi-Civita-Bertotti-Robinson metric

1 Introduction

The Levi-Civita-Bertotti-Robinson (LBR) metric is one of the classical metrics in general relativity. It is the only Einstein-Maxwell solution which is homogeneous and has homogeneous non-null Maxwell field [1–4]. LBR space-time is a product of two spaces of constant curvature, namely it is $AdS_2 \times S^2$. Due to this property LBR metric arises also in supergravity theories [5–8]. LBR type of space-times show up also as the space-time regions closer to the horizons of extreme Reissner-Nordstrom black hole geometries [9, 10]. Its curvature invariants are all constant and the electromagnetic field tensor is covariantly constant. This property leads to the result that LBR metric is an exact solution of any theory of gravitation coupled with a $U(1)$ gauge field [9, 11].

M. Gürses (✉)
Department of Mathematics, Faculty of Sciences, Bilkent University, 06800 Ankara-Turkey
E-mail: gurses@fen.bilkent.edu.tr

Ö. Sarıoğlu
Department of Physics, Faculty of Arts and Sciences, Middle East Technical University, 06531 Ankara-Turkey
E-mail: sarioglu@metu.edu.tr

In this work we generalize the LBR metric in D-dimensions by introducing acceleration. A similar generalization was done long ago for the Reissner-Nordstrom metric in four dimensions [12]. Recently we extended the Bonnor-Vaidya formalism to Reissner-Nordstrom metric in D-dimensions [13, 14]. We used an arbitrary curve C in D-dimensional Minkowski space-time M_D and examined in detail the Einstein-Maxwell-null dust [13] and Einstein-Born-Infeld-null dust field equations [14], Yang-Mills equations [15], and Liénard-Wiechert potentials in even dimensions [16]. In the first three works [13–15], we found some new solutions generalizing the Tangherlini [17], Plebański [18], and Trautman [19] solutions, respectively. The last one, [16], indicates that the accelerated scalar or vector charged particles in even dimensions lose energy.

Our conventions are similar to the conventions of our earlier works [13, 14], [16]. In a D-dimensional Minkowski space-time M_D, we use a parametrized curve $C = \{x^\mu \in M_D |\; x^\mu = z^\mu(\tau)\,, \mu = 0, 1, 2, \ldots, D-1\,, \tau \in I\}$ such that τ is a parameter of the curve and I is an interval on the real line $\mathbb{R}$. We define the world function Ω as

$$\Omega = \eta_{\mu\nu}\,(x^\mu - z^\mu(\tau))\,(x^\nu - z^\nu(\tau)), \tag{1}$$

where x^μ is a point not on the curve C. There exists a point $z^\mu(\tau_0)$ on the non-space-like curve C which is also on the light cone with the vertex located at the point x^μ, so that $\Omega(\tau_0) = 0$. Here τ_0 is the retarded time. By using this property we find that

$$\lambda_\mu \equiv \partial_\mu\,\tau_0 = \frac{x^\mu - z^\mu(\tau_0)}{R}, \tag{2}$$

where $R \equiv \dot{z}^\mu(\tau_0)\,(x_\mu - z_\mu(\tau_0))$ is the retarded distance. Here a dot over a letter denotes differentiation with respect to τ_0. It is easy to show that λ_μ is null and satisfies

$$\lambda_{\mu,\nu} = \frac{1}{R}\,[\eta_{\mu\nu} - \dot{z}_\mu\,\lambda_\nu - \dot{z}_\nu\,\lambda_\mu - (A-\epsilon)\,\lambda_\mu\,\lambda_\nu], \tag{3}$$

$$R_{,\mu} = \dot{z}_\mu + (A-\epsilon)\lambda_\mu, \tag{4}$$

where $A \equiv \ddot{z}^\mu\,(x_\mu - z_\mu(\tau_0))$ and $\dot{z}^\mu\,\dot{z}_\mu = \epsilon = -1, 0$. Here $\epsilon = -1$ and $\epsilon = 0$ correspond to the time-like and null velocity vectors, respectively. In this work we shall consider only the case where the velocity vector is time-like ($\epsilon = -1$). One can also show explicitly that $\lambda^\mu\,\dot{z}_\mu = 1$ and $\lambda^\mu\,R_{,\,\mu} = 1$. If one defines $a \equiv A/R = \lambda^\mu \ddot{z}_\mu$, then

$$\lambda^\mu\,a_{,\,\mu} = 0. \tag{5}$$

Furthermore defining (letting $a_0 \equiv a$)

$$a_k \equiv \lambda_\mu\,\frac{d^{k+2}\,z^\mu(\tau_0)}{d\tau_0^{k+2}}, \quad k = 0, 1, 2, \ldots \tag{6}$$

one can show that

$$\lambda^\mu\,a_{k,\,\mu} = 0, \quad \forall k = 0, 1, 2, \ldots \tag{7}$$

In the next section, using [9], we present the LBR metric in arbitrary D-dimensions. In Sect. 3 we give the accelerated LBR metrics in D-dimensions. Here we find the rate of energy loss for all D. In Sect. 4 we consider a special curve C where the null fluid disappears. In particular, taking a curve C which corresponds to a constant acceleration, we find a solution of the Einstein-Maxwell field equations in D-dimensions. We show that this is also the LBR metric in an accelerated reference frame.

2 Levi-Civita-Bertotti-Robinson metrics

We assume a spherically symmetric, static space-time in D-dimensions. Among such a class of space-times, LBR geometry is a product space-time $AdS_2 \times S^{D-2}$. It has been studied for several purposes [9, 11], (see also [10]). We have the following theorem.

Theorem 1 *Let $g_{\mu\nu}$ be the metric and $F_{\mu\nu}$ be the Maxwell field given by*

$$g_{\mu\nu} = \frac{q^2}{r^2}\left[-t_\mu t_\nu + c_0^2 k_\mu k_\nu + r^2 h_{\mu\nu}\right], \tag{8}$$

$$F_{\mu\nu} = \frac{c_0}{r}(t_\mu k_\nu - t_\nu k_\mu), \tag{9}$$

where $h_{\mu\nu}$ is the metric of the $(D-2)$-dimensional sphere S^{D-2}, c_0, q are constants, $t_\mu = \delta_\mu^0$, $k_\mu = \delta_\mu^r$. Then they solve the Einstein-Maxwell field equations with a cosmological constant

$$G_{\mu\nu} = \frac{Q^2}{c_0^2}\left[F_\mu{}^\alpha F_{\nu\alpha} - \frac{1}{4}(F^{\alpha\beta} F_{\alpha\beta}) g_{\mu\nu}\right] + \Lambda g_{\mu\nu}, \tag{10}$$

where

$$Q^2 = q^2\left[(D-3)\, c_0^2 + 1\right], \quad \Lambda = \frac{1}{2q^2}\left[\frac{1}{c_0^2} - (D-3)^2\right]. \tag{11}$$

Here Q is the electric charge. This metric describes the near horizon region of the charged Tangherlini metric with a cosmological constant [9, 10] and moreover is conformally flat if $c_0 = 1$. In this case the cosmological constant vanishes only when $D = 4$. All the curvature invariants are constants, being functions of the constants, c_0, Q and D. These metrics describe the geometry of black holes in the neighborhood of their outer horizons [9–11]. Our purpose in this work is to generalize this solution when the charge moves on a curve C described in the introduction.

3 Accelerated Bertotti-Robinson metric in D-dimensions

In this section, we shall generalize the LBR metric by introducing acceleration using the curve kinematics given in the introduction. We shall consider the case when the space-time is conformally flat. We assume that the metric and the electromagnetic vector potential are given by

$$g_{\mu\nu} = e^{\psi}\, \eta_{\mu\nu}, \quad A_\mu = S\, \dot{z}_\mu, \tag{12}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, ψ and S are functions of R only. Then we have

Theorem 2 *In an arbitrary dimension D*

$$ds^2 = \frac{R_0^2}{R^2}\, \eta_{\mu\nu}\, dx^\mu\, dx^\nu, \tag{13}$$

$$A_\mu = e\, \frac{\dot{z}_\mu}{R}, \tag{14}$$

are the solutions of the Einstein-Maxwell-null (pressureless) fluid equations with cosmological constant Λ*,*

$$G_{\mu\nu} = \kappa \left[F_\mu{}^\alpha\, F_{\nu\alpha} - \frac{1}{4}\, (F^{\alpha\beta}\, F_{\alpha\beta})\; g_{\mu\nu} \right] + \kappa\, \rho \lambda_\mu\, \lambda_\nu + \Lambda g_{\mu\nu}. \tag{15}$$

The energy density of the fluid and the cosmological constant are given by

$$\kappa\, \rho = (D-2)\, (a_1 - \ddot{z}^\alpha\, \ddot{z}_\alpha), \tag{16}$$

$$\Lambda = -\frac{(D-2)(D-4)}{2\, R_0^2}, \quad e^2 = \frac{D-2}{\kappa} R_0^2, \tag{17}$$

where R_0 *is a constant,* R *is the retarded distance described in the introduction,* λ_μ *is the null vector defined in (2) and* a_1 *is defined in (6). Furthermore, the current vector* $J^\mu = \nabla_\nu F^{\mu\nu}$ *vanishes for all D (except on the curve C, see Remark 1 below).*

First of all, when the curve C is a straight line, the solution given in Theorem 2 reduces to the one in Theorem 1 with $c_0 = 1$. Moreover, we observe that the metric and the electromagnetic fields for all dimensions have the same form. This is interesting because in the case of accelerated charges in the Kerr-Schild geometry, the metric and the electromagnetic fields take different forms in different dimensions [13, 14]. Here the only D dependent quantity is the cosmological constant. On the other hand, the energy density of the null fluid depends on the acceleration parameter of the curve C. It is worthwhile to look at the energy loss formula in this case. Energy flux formula, in general, is given by (see [13]),

$$N = -\lim_{R\to\infty} \int_{S^{D-2}} \dot{z}^\alpha\, T_{\alpha\beta}\, n^\beta\, R^{-1}\, R_0^{D-2}\, d\Omega, \tag{18}$$

where $T_{\mu\nu}$ is the corresponding energy momentum tensor (of the fluid or the Maxwell field) and n^μ is a space-like vector defined through $\lambda_\mu = -\dot{z}_\mu - n_\mu/R$ (see [13] for more details). We find that $N_F = 0$ for the null fluid distribution. Its energy is conserved. The rate of change of the energy of the electromagnetic field N_E is found as

$$N_E = e^2\, R_0^{D-4}\, \kappa_1^2 \left\{ -\Omega_{D-3}\, \Gamma\left(\frac{D-2}{2}\right) \left[\frac{-\sqrt{\pi}}{\Gamma\left(\frac{D-1}{2}\right)} + 2^{D-1}\, \frac{\Gamma\left(\frac{D+2}{2}\right)}{\Gamma(D)} \right] + \Omega_{D-2} \right\}, \tag{19}$$

for $D \geq 4$, where Ω_D is the solid angle in D dimensions. Here κ_1 is the first curvature of the curve C. In calculating (19), we used the integration technique developed in our earlier works [13]. The quantity N_E given above is positive for all dimensions D and hence there is energy loss due to acceleration in all dimensions. In particular, when $D = 4$, $N_E = \frac{8\pi}{3} e^2 \kappa_1^2$ is exactly the energy loss formula due to the acceleration of a charged particle in flat space-time [20].

Remark 1 We obtain the covariant derivative of the Maxwell tensor $F_{\alpha\beta}$ as

$$\nabla_\mu F_{\alpha\beta} = \lambda_\mu (\lambda_\alpha \zeta_\beta - \lambda_\beta \zeta_\alpha) \tag{20}$$

where

$$\zeta^\mu = \frac{1}{R}\left[\frac{d^3 z^\mu}{d\tau_0^3} - a_1 \frac{d z^\mu}{d\tau_0}\right].$$

This vector vanishes if C is a straight line or has constant acceleration, in which case $F_{\alpha\beta}$ becomes a covariantly constant tensor field. Since $\lambda_\mu \zeta^\mu = 0$, then the current vector J^μ mentioned in Theorem 2 is zero everywhere except on the curve C. In fact it takes the form

$$J^\mu(x) = \frac{1}{\Omega_{D-2}} \int_C \dot{z}^\mu(\tau)\, \delta(x - z(\tau))\, d\tau,$$

where $\nabla_\mu J^\mu = 0$ identically.

Remark 2 We note that the Ricci scalar, the Ricci invariant $R^{\alpha\beta} R_{\alpha\beta}$, and the Maxwell invariant $F^{\alpha\beta} F_{\alpha\beta}$ are all constants.

$$g^{\alpha\beta} R_{\alpha\beta} = \frac{(D-1)(D-4)}{R_0^2}, \tag{21}$$

$$R^{\alpha\beta} R_{\alpha\beta} = \frac{D^3 - 8D^2 + 21D - 16}{R_0^4}, \tag{22}$$

$$F^{\alpha\beta} F_{\alpha\beta} = \frac{2}{\kappa R_0^2}(2 - D). \tag{23}$$

Similarly the curvature invariant $R^{\alpha\beta\gamma\sigma} R_{\alpha\beta\gamma\sigma}$ is also constant due to the conformal flatness. All of these invariants are equal to the corresponding invariants of the LBR metric (static case).

4 Charged particle with constant acceleration

In this section we consider only the Einstein-Maxwell field equations with a cosmological constant. To achieve this, we look for special curves C such that the null fluid energy density ρ vanishes.

Using the Serret-Frenet frame [13] in M_D, we obtain

$$\kappa\rho = (D-2)\,[-\dot{\kappa_1} \cos\theta + \kappa_1 \kappa_2 \sin\theta \cos\phi],$$

where κ_1 and κ_2 are the first two curvatures of the curve C and (θ, ϕ) are the first two angular coordinates on S^{D-2}. It is clear that the energy density ρ of the null fluid does not have a fixed sign. It changes its sign at different points. There is a non-trivial choice where κ_1 is a non-zero constant and $\kappa_2 = 0$ which leads to the vanishing of the energy density ($\rho = 0$). In this case C describes a particle moving with a constant acceleration and the Maxwell field tensor $F_{\alpha\beta}$ is covariantly constant. The total energy measured by the observer moving along the curve C is

$$E = G_{\mu\nu}\, \dot{z}^{\mu}\, \dot{z}^{\nu} = (D-2)\left[\kappa_1^2 \sin^2\theta + \frac{D-3}{2R^2}\right].$$

This vanishes asymptotically ($R \to \infty$) in the case of LBR space-time, but asymptotically it is proportional to the square of the first curvature in the space-time corresponding to our solution.

As an example of such a curve, let (using the notation $x^{\mu} = (t, x, x^2, \ldots, x^{D-1})$)

$$z^{\mu} = B\,(\sinh(w\tau_0),\ \cosh(w\tau_0) - 1,\ 0, \ldots,\ 0),$$

where B and w are constants with $\kappa_1 = w = 1/B$, and τ_0 is defined through

$$\cosh(w\tau_0) = \frac{(x+B)[t^2 - (x+B)^2 - r^2 - B^2] + 2B\,t\,R}{2B[t^2 - (x+B)^2]},$$

where $r^2 = (x^2)^2 + (x^3)^2 + \cdots + (x^{D-1})^2$. Here R is the retarded distance. It is given by

$$R = \pm\frac{1}{2B}\sqrt{(t^2 - (x+B)^2 - r^2 - B^2)^2 + 4B^2\,(t^2 - (x+B)^2)}.$$

This curve has non-zero constant first curvature κ_1 and all other curvatures ($\kappa_i = 0,\ \ i \geq 2$) are zero. The charged particle has constant acceleration $\sqrt{\ddot{z}^{\mu}\,\ddot{z}_{\mu}} = \kappa_1$ along the x-direction.

In four dimensions since the cosmological constant is also zero, we have a solution of the Einstein-Maxwell field equations representing an (constant) accelerated charged particle. The LBR metric and our solution given above are both conformally flat and solutions of the Einstein-Maxwell field equations. Furthermore, as we mentioned in Remark 2, both have the same curvature invariants. On the other hand LBR metric and our metric correspond to two distinct curves, namely, straight line and (non-zero) constant curvature cases, respectively. In spite of this difference, these two metrics are transformable to each other. Let

$$x'^{\mu} = \frac{x^{\mu} + s k^{\mu}}{1 + 2u + k^2 s}, \tag{24}$$

where $s \equiv \eta_{\mu\nu}\, x^{\mu} x^{\nu},\ u \equiv k_{\mu}\, x^{\mu}$ and k_{μ} is a constant vector with $k^2 \equiv \eta_{\mu\nu} k^{\nu} k^{\mu}$. Here we choose $k_0 = 0$, $Bk_1 = -2$ and $k^{\mu} = 0$ for all $\mu > 1$. Then it is straightforward to show that

$$ds^2 = \frac{R_0^2}{R'^2}\,\eta_{\mu\nu}\, dx'^{\mu} dx'^{\nu} = \frac{R_0^2}{x^2 + r^2}\,\eta_{\mu\nu}\, dx^{\mu} dx^{\nu}, \tag{25}$$

where

$$R' = \pm \frac{1}{2B} \sqrt{[(t')^2 - (x' + B)^2 - (r')^2 - B^2]^2 + 4B^2 \left((t')^2 - (x' + B)^2\right)}.$$

This means that our solution is expressed in an (constant) accelerated frame. This is the reason why we observe radiation.

Remark 3 The infinitesimal version of the conformal transformation (24) (conformal Killing vector in flat Minkowski space-time) is given in Penrose and Rindler [21] as

$$\xi^\mu = -2ux^\mu + sk^\mu. \tag{26}$$

They remark that "$\cdots$ *the special conformal transformations (four parameters sometimes misleadingly called uniform acceleration transformation* $\cdots$*)*." Here we observe that these special transformations really correspond to a constant acceleration.

5 Conclusion

We found accelerated LBR metrics which solve Einstein-Maxwell-null fluid field equations with a dimension dependent cosmological constant. We have obtained the energy loss formula due to the acceleration. In four dimensions it coincides with the standard energy loss formula in flat space-time. We obtain the LBR metric when the curve C is a straight line in M_D. We also showed that there is another LBR limit. When the curve C has constant acceleration our solution is transformable (by a special conformal mapping) to the LBR metric.

Acknowledgements This work is partially supported by the Scientific and Technical Research Council of Turkey and by the Turkish Academy of Sciences.

References

1. Levi-Civita, T.: R. C. Acad. Lincei **26**, 519 (1917)
2. Bertotti, B.: Phys. Rev. **116**, 1331 (1959)
3. Robinson, R.: Bull. Acad. Polon. **7**, 351 (1959)
4. Stephani, H., Kramer, D., MacCallum, M., Hoenselaers, C., Herlt, E.: Exact Solutions of Einstein's Field Equations, Cambridge Monographs on Mathematical Physics. (Cambridge: Cambridge University Press), Second Edition (2003)
5. Gibbons, G.W., Hull, C.M.: Phys. Lett B **109**, 190 (1982); Gibbons, G.W.: in Supersymmetry, Supergravity and Related Topics, eds. del Auglia, F., de Azcaraga, J., Ibanez, L.: World Scientific, Singapore, p 147, (1985)
6. Ferrara, S.: Bertotti-Robinson Geometry and Supersymmetry, Talk given at 12th Italian Conference on General Relativity and Gravitational Physics, Rome, September, (1996) (hep-th/9701163)
7. Lowe, D.A., Strominger, A.: Phys. Rev. Lett. **73**, 1468 (1994)
8. Kallosh, R., Peet, A.: Phys. Rev D **46**, 5223 (1992); Kallosh, R.: Phys. Lett B **282**, 80 (1992)
9. Gürses, M.: Phys. Rev. D **46**, 2522 (1992)
10. Cardoso, V., Dias, O.J.C., Lemos, J.P.S.: Phys. Rev. D **70**, 024002 (2004)
11. Gürses, M., Sermutlu, E.: Class. Quantum Grav. **12**, 2799 (1995)
12. Bonnor, W.B., Vaidya, P.C.: General Relativity papers in honor of Synge, J.L., edited by L. O' Raifeartaigh (Dublin: Dublin Institute for Advanced Studies) pp 119 (1972)
13. Gürses, M., Sarıoğlu, Ö.: Class. Quant Grav. **19**, 4249 (2002); Erratum, ibid **20**, 1413 (2003)

14. Gürses, M., Sarıoğlu, Ö.: Class. Quant Grav. **20**, 351 (2003)
15. Sarıoğlu, Ö.: Phys. Rev. D **66**, 085005 (2002)
16. Gürses, M., Sarıoğlu, Ö.: J. Math. Phys. **44**, 4672 (2003)
17. Tangherlini, F.R.: Nuovo Cimento **77**, 636 (1963)
18. García, A., Salazar, I.H., Plebański, J.F.: Nuovo Cimento B **84**, 65 (1984)
19. Trautman, A.: Phys. Rev. Lett. **46**, 875 (1981)
20. Jackson, J.D.: Classical Electrodynamics, New York: Wiley (1975)
21. Penrose, R., Rindler, W.: Spinors and Space-Time: Spinor and twistor methods in space-time geometry Vol. 2 Cambridge University Press, Cambridge, pp 83–84, (1986)