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On a transformation between hierarchies of integrable equations

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Abstract

A transformation between a hierarchy of integrable equations arising from the standard R-matrix construction on the algebra of differential operators and a hierarchy of integrable equations arising from a deformation of the standard R-matrix is given.

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In a recent paper [1] a new hierarchy of integrable equations has been constructed through the deformation of a standard R-matrix on the algebra of pseudo-differential operators. We give a transformation between the hierarchy constructed in [1] and a hierarchy obtained through a standard R-matrix. The transformation is between corresponding vector fields (i.e. symmetries).

Let $\mathfrak{g}$ be the Lie algebra of pseudo-differential operators

$$\mathfrak{g} = \left\{ \sum_{i\in\mathbf{Z}} u_i(x) D^i \right\} \tag{1}$$

with the commutator $[L_1, L_2] = L_1L_2 - L_2L_1$. The algebra $\mathfrak{g}$ can be decomposed into Lie subalgebras $\mathfrak{g}_{\geqslant k} = \{\sum_{i\geqslant k} u_i(x)D^i\}$ and, $\mathfrak{g}_{i<k} = \{\sum_{i<k} u_i(x)D^i\}$ where $k = 0, 1, 2$ (only for such k one has Lie subalgebras). The standard R-matrix is given by $R_k = \frac{1}{2}(P_{\geqslant k} - P_{<k})$, where $P_{\geqslant k}$ and $P_{<k}$ are projection operators on $\mathfrak{g}_{\geqslant k}$ and $\mathfrak{g}_{<k}$, respectively. The Lax hierarchy is

$$L_{t_n} = \left[R\left(L^n\right), L\right] = \left[\left(L^n\right)_{\geqslant k}, L\right], \quad L \in \mathfrak{g},\ n = 1, 2, \ldots. \tag{2}$$

The above equations involves infinitely many fields. To have a consistent closed equations with a finite number of fields we restrict the Lax operators as follows

$$k=0 \quad L_0 = D^N + u_{N-2}D^{N-2} + \cdots + u_1D + u_0, \tag{3}$$

$$k=1 \quad L_1 = D^N + u_{N-1}D^{N-1} + \cdots + u_0 + D^{-1}u_{-1}, \tag{4}$$

$$k=2 \quad L_2 = u_ND^N + u_{N-1}D^{N-1} + \cdots + D^{-1}u_{-1} + D^{-2}u_{-2}. \tag{5}$$

See [2] for more details on the R-matrix formalism.

Recently in [1] the deformations of the above R-matrices were introduced. Most of the introduced deformed R-matrices do not lead to the new hierarchies. A new hierarchy is obtained through a deformation of R-matrix $R_1 = \frac{1}{2}(P_{\geqslant 1} - P_{<1})$. Let

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$P_{=i}(L)=(L)_{=i}$ denotes coefficient of D^i in the expansion of $L\in\mathfrak{g}$. Then the deformed R-matrix is

$$\tilde{R}=\frac{1}{2}(P_{\geqslant 1}-P_{<1})+\varepsilon P_{=0}(\cdot)D, \tag{6}$$

where ε is a deformation parameter. The hierarchy is

$$L_{t_n}=\big[\tilde{R}\big(L^n\big),L\big],\quad L\in\mathfrak{g},\ n=1,2,\ldots. \tag{7}$$

The above equations involves infinitely many fields, to have a consistent closed equation with finite number of fields we restrict the Lax operator as $\tilde{L}=u_N D^N+u_{N-1}D^{N-1}+\cdots+u_0+D^{-1}u_{-1}$. Then the new hierarchy is

$$\tilde{L}_{t_n}=\big[\big(\tilde{L}^n\big)_{\geqslant 1}+\epsilon\big(\tilde{L}^n\big)_{=0}D,\tilde{L}\big],\quad n=1,2,\ldots, \tag{8}$$

note that $\tilde{L}=L_2|_{u_{-2}=0}$. See [1] for more details.

In this work we shall show that the new hierarchy (8) is related to the hierarchy corresponding to R-matrix R_2 with reduced Lax operator $\tilde{L}=L_2|_{u_{-2}=0}$. So we relate hierarchy (8) to the hierarchy

$$\tilde{L}_{t_n}=\big[\big(\tilde{L}^n\big)_{\geqslant 2},\tilde{L}\big],\quad n=1,2,\ldots. \tag{9}$$

We note that both hierarchies have the same Lax operator.

The construction of the transformation is based on expressing $(\tilde{L}^n)_{=1}$ and $(\tilde{L}^n)_{=0}$ in terms of coefficients of $[(\tilde{L}^n)_{\geqslant 2},\tilde{L}]$, for $n\in\mathbf{N}$.

Proposition 1. *Let $\tilde{L}=L_2|_{u_{-2}=0}$, then*

$$\big(\big[\big(\tilde{L}^n\big)_{\geqslant 2},\tilde{L}\big]\big)_{=N}=-\big(\big[\big(\tilde{L}^n\big)_{=1}D,\tilde{L}\big]\big)_{=N}, \tag{10}$$

$$\big(\big[\big(\tilde{L}^n\big)_{\geqslant 1},\tilde{L}\big]\big)_{=N-1}=-\big(\big[\big(\tilde{L}^n\big)_{=0},\tilde{L}\big]\big)_{=N-1} \tag{11}$$

for all N (N is order of operator $\tilde{L}$).

Proof. Comparing powers of D on the right- and left-hand side of the equality

$$\big[\big(\tilde{L}^n\big)_{\geqslant 1},\tilde{L}\big]=-\big[\big(\tilde{L}^n\big)_{<1},\tilde{L}\big], \tag{12}$$

we have

$$\big(\big[\big(\tilde{L}^n\big)_{\geqslant 1},\tilde{L}\big]\big)_{=N}=0. \tag{13}$$

Then

$$\big(\big[\big(\tilde{L}^n\big)_{\geqslant 2},\tilde{L}\big]\big)_{=N}=-\big(\big[\big(\tilde{L}^n\big)_{=1}D,\tilde{L}\big]\big)_{=N}. \tag{14}$$

In the same way, comparing powers of D on the right- and left-hand side of the equality

$$\big[\big(\tilde{L}^n\big)_{\geqslant 0},\tilde{L}\big]=-\big[\big(\tilde{L}^n\big)_{<0},\tilde{L}\big] \tag{15}$$

we have

$$\big(\big[\big(\tilde{L}^n\big)_{\geqslant 0},\tilde{L}\big]\big)_{=N-1}=0. \tag{16}$$

So,

$$\big(\big[\big(\tilde{L}^n\big)_{\geqslant 1},\tilde{L}\big]\big)_{=N-1}=-\big(\big[\big(\tilde{L}^n\big)_{=0},\tilde{L}\big]\big)_{=N-1}. \tag{17}$$

The above equalities (10) and (11) allows us to express $(\tilde{L}^n)_{=1}$ and $(\tilde{L}^n)_{=0}$ in terms of coefficients of $[(\tilde{L}^n)_{\geqslant 2},\tilde{L}]$ for all N. □

Let us give an example for $N=1$.

Proposition 2. *Consider the Lax operator $\tilde{L}=uD+v+D^{-1}w$. Let*

$$\big[\big(\tilde{L}^n\big)_{\geqslant 2},\tilde{L}\big]=f_nD+g_n+D^{-1}h_n, \tag{18}$$

which gives the hierarchy (9) with the standard R-matrix and

$$\big[\big(\tilde{L}^n\big)_{\geqslant 1}+\big(\tilde{L}^n\big)_{=0}D,\tilde{L}\big]=p_nD+q_n+D^{-1}r_n, \tag{19}$$

which gives the hierarchy (8) with the deformed R-matrix, $n = 1, 2, \ldots$. The coefficients f_n, g_n, h_n, p_n, q_n, r_n are functions of u, v, w and their derivatives. Then

$$(p_n, q_n, r_n)^T = \mathcal{T}(f_n, g_n, h_n)^T$$

where

$$\mathcal{T} = \begin{pmatrix} \varepsilon u_x D^{-1} v_x D^{-1} u^{-2} - \varepsilon u v_x D^{-1} u^{-2} & \varepsilon u_x D^{-1} u^{-1} - \varepsilon & 0 \\ u v_x D^{-1} u^{-2} + \varepsilon v_x D^{-1} v_x D^{-1} u^{-2} & 1 + \varepsilon v_x D^{-1} u^{-1} & 0 \\ ((uw)_x + \varepsilon w v_x) D^{-1} u^{-2} + \varepsilon w_x D^{-1} v_x D^{-1} u^{-2} + w u^{-1} & \varepsilon w u^{-1} + \varepsilon w_x D^{-1} u^{-1} & 1 \end{pmatrix}. \tag{20}$$

Proof. Let $(\tilde{L}^n)_{=1} = A_n$ and $(\tilde{L}^n)_{=0} = B_n$. The equality (10) implies that

$$f_n = -\big([A_n D, \tilde{L}]\big)_{=1}, \tag{21}$$

hence, we can find

$$A_n = u D^{-1} u^{-2} f_n. \tag{22}$$

Using the equality (11) we have

$$g_n + \big([A_n D, \tilde{L}]\big)_{=0} = -\big([B_n, \tilde{L}]\big)_{=0}, \tag{23}$$

hence, we can find

$$B_n = D^{-1}\big(u^{-1} g_n + v_x D^{-1} u^{-2} f_n\big). \tag{24}$$

From the equality

$$\big[(\tilde{L}^n)_{\geqslant 1} + \varepsilon(\tilde{L}^n)_{=0} D, \tilde{L}\big] = \big[(\tilde{L}^n)_{\geqslant 2}, \tilde{L}\big] + \big[(A_n + \varepsilon B_n) D, \tilde{L}\big] \tag{25}$$

we can find the transformation between the vector fields

$$\begin{aligned} p_n &= u_x \varepsilon B_n - u \varepsilon B_{n,x}, \\ q_n &= g_n + v_x (A_n + \varepsilon B_n), \\ r_n &= h_n + \big(w(A_n + \varepsilon B_n)\big)_x, \end{aligned} \tag{26}$$

where A_n and B_n are given by (22) and (24), respectively. Thus we obtain the transformation operator $\mathcal{T}$ in (20).

If we apply operator $\mathcal{T}$ to the simple symmetry $(u_x, v_x, w_x)^T$ we obtain $(0, 0, 0)^T$. Applying the operator $\mathcal{T}$ to $(0, 0, 0)^T$ we get

$$\begin{pmatrix} p_1 \\ q_1 \\ r_1 \end{pmatrix} = \begin{pmatrix} \varepsilon(v u_x - u v_x + u_x) \\ u v_x + \varepsilon(v v_x + v_x) \\ (uw)_x + \varepsilon(vw)_x + \varepsilon w_x \end{pmatrix}. \tag{27}$$

This is the deformed system (8) for $n = 1$ (with the inclusion of the symmetry $(u_x, v_x, w_x)^T$), [1]. If we take symmetry of the hierarchy (9) corresponding to $n = 2$ (this is the reduced system [2,3])

$$\begin{pmatrix} f_2 \\ g_2 \\ h_2 \end{pmatrix} = \begin{pmatrix} u^2 u_{xx} + 2u^2 v_x \\ u^2 v_{xx} + 2u(uw)_x \\ -(u^2 w)_{xx} \end{pmatrix} \tag{28}$$

and apply the operator $\mathcal{T}$ to this symmetry we obtain a second symmetry of the hierarchy (8)

$$\begin{pmatrix} p_2 \\ q_2 \\ r_2 \end{pmatrix} = \begin{pmatrix} \varepsilon u_x v^2 - 2\varepsilon u v v_x - 2\varepsilon u^2 w_x - \varepsilon u^2 v_{xx} \\ 2u u_x w + 2u v v_x + 2u^2 w_x + u u_x v_x + u^2 v_{xx} + \varepsilon v^2 v_x + 2\varepsilon u v_x w + \varepsilon u v_x^2 \\ 2u_x v w + 2u v_x w + 2u v w_x - u_x^2 w - 3u u_x w_x - u u_{xx} w - u^2 w_{xx} \\ \quad + 2\varepsilon u_x w^2 + 2\varepsilon v v_x w + \varepsilon u_x v_x w + \varepsilon v^2 w_x + 4\varepsilon u w w_x + \varepsilon u v_x w_x + \varepsilon u v_{xx} w \end{pmatrix}. \qquad \square \tag{29}$$

Remark. In the example above we have constructed the transformation $\mathcal{T}$ for hierarchies with Lax operator of order one. In the same way we can construct the transformation between hierarchies with Lax operator of any order N. The operator $\mathcal{T}$ is not a recursion operator. It maps the symmetries of one system of evolution equations to symmetries of another system of evolution equations.

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