Quantification of entanglement via uncertainties

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We show that entanglement of pure multiparty states can be quantified by means of quantum uncertainties of certain basic observables through the use of a measure that was initially proposed by Klyachko *et al.* [Appl. Phys. Lett. **88**, 124102 (2006)] for bipartite systems.

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I. INTRODUCTION

Recent success in the realization of quantum key distribution has been achieved through the use of quantum correlations between the parts in two-qubit systems, which are peculiar to entangled states (see [1–3] and references therein). Further development of practical implementations of quantum-information technologies requires sources of robust entangled states and reliable methods of detection of the amount of entanglement carried by those states (e.g., see [4,5] and references therein).

There is no universal measure of entanglement suitable for all systems even in the case of pure states. For example, entanglement of two qubits is measured by means of the *concurrence* [6] for both pure and mixed states. In the case of pure states, the definition of concurrence has been extended to bipartite systems with any dimension of the single-party Hilbert state $d \geq 2$ [7,8]. At the same time, this definition does not work for systems with the number of parts larger than two. In particular, concurrence is incapable of measuring the three-party entanglement in three-qubit systems [9].

In our previous paper [10], we found a representation of concurrence valid for pure states of an arbitrary bipartite system which coincides with the Wootters concurrence [6] for the case of pure two-qubit states. A logical advantage of this representation is that it expresses the amount of entanglement in terms of variances (quantum uncertainties) of certain observables. In a sense, this reflects the physical nature of entanglement as a manifestation of quantum uncertainties at their extreme [11–14] (see also the discussion in Refs. [15–17]).

The main objective of this paper is to prove the validity of the measure of Ref. [10] for pure states in general settings.

The paper is organized as follows. We start by giving a definition of the basic observables specifying a given physical system. We further connect the notion of total variance with the measure of entanglement. Then we discuss application of this measure to pure states of two and three qubits. Further, we briefly consider how this measure works in the case of mixed states. Finally, in Appendix A, we put the proof of the validity of our measure in general settings.

II. QUANTUM DYNAMICAL SYSTEMS

An idealized von Neumann approach to quantum mechanics, based on the assumption that all Hermitian operators represent measurable quantities, was first put into question by Wick, Wightman, and Wigner [18] in 1952. Later, Hermann [19] argued soundly that the basic principles of quantum mechanics require that measurable observables should form a Lie algebra $\mathcal{L}$ of (skew) Hermitian operators acting in the Hilbert space $\mathcal{H}$ of the quantum system in question. We refer to $\mathcal{L}$ as the Lie algebra of observables and to the corresponding Lie group $G=\exp(i\mathcal{L})$ as the dynamical symmetry group of the quantum system.

Restrictions on available observations are of fundamental importance for physics in general, and for quantum information specifically. The latter case usually deals with correlated states of a quantum system with macroscopically separated spatial components, where only local measurements are feasible. For example, the dynamical group of the bipartite system $\mathcal{H}=\mathcal{H}_A \otimes \mathcal{H}_B$ with full access to local degrees of freedom amounts to $\mathrm{SU}(\mathcal{H}_A)\times \mathrm{SU}(\mathcal{H}_B)$. Without such restrictions, the dynamical group $G=\mathrm{SU}(\mathcal{H})$ would act transitively on pure states $\psi \in \mathcal{H}$, which makes them all equivalent. In this case there would be no place for entanglement and other subtle quantum phenomena based on intrinsic differences between quantum states.

III. TOTAL VARIANCE

Recall that the uncertainty of an observable $X \in \mathcal{L}$ in the state $\psi \in \mathcal{H}$ is given by the variance

$$V(X,\psi) = \langle \psi | X^2 | \psi \rangle - \langle \psi | X | \psi \rangle^2. \qquad (1)$$

Let us now choose an orthonormal basis X_α of the algebra of observables $\mathcal{L}$ with respect to its Cartan-Killing form $(X,Y)_K$ [20] and define the total variance by equation

$$\mathbb{V}(\psi) = \sum_\alpha (\langle \psi | X_\alpha^2 | \psi \rangle - \langle \psi | X_\alpha | \psi \rangle^2). \qquad (2)$$

For example, for a two-qubit system $\mathcal{H}_A \otimes \mathcal{H}_B$ one can take the basis of $\mathcal{L}=\mathrm{su}(\mathcal{H}_A)+\mathrm{su}(\mathcal{H}_B)$, consisting of Pauli operators σ_i^A and σ_j^B that act on components A and B, respectively. For a general multipartite system, the sum (2) is extended over orthonormal bases of traceless local operators for all parties of the system.

The total variance (2) can be understood as the trace of the quadratic form

$$Q(X) = \langle \psi | X^2 | \psi \rangle - \langle \psi | X | \psi \rangle^2, \quad X \in \mathcal{L},$$

on the Lie algebra $\mathcal{L}$, and therefore it is independent of the basis X_α. It measures the overall level of quantum fluctuations of the system in state ψ.

The first sum in the total variance (2) contains the Casimir operator $C=\Sigma_\alpha X_\alpha^2$, which acts as a scalar $C_{\mathcal{H}}$ in every irreducible representation $G:\mathcal{H}$. As a result we get

$$\mathbb{V}(\psi) = C_{\mathcal{H}} - \sum_\alpha \langle\psi|X_\alpha|\psi\rangle^2. \quad (3)$$

To clarify the second sum, consider the average of the basic observables X_α in state ψ,

$$X_\psi = \sum_\alpha \langle\psi|X_\alpha|\psi\rangle X_\alpha. \quad (4)$$

It can be understood as the center of quantum fluctuations of the system in state ψ. For example, in a spin system it is given by a suitably scaled spin projection onto the mean spin direction in state ψ. The operator X_ψ is also independent of the basis X_α. This can be seen from the following property:

$$\langle\psi|X|\psi\rangle = (X,X_\psi)_K, \quad \forall\ X \in \mathcal{L}, \quad (5)$$

which holds for basic observables $X=X_\alpha$ by orthogonality $(X_\alpha,X_\beta)_K=\delta_{\alpha\beta}$, and hence by linearity for all $X\in\mathcal{L}$. Since the Killing form is nondegenerate, Eq. (5) uniquely determines X_ψ and provides for it a coordinate-free definition. We show in Appendix A that the operator X_ψ is closely related to orthogonal projection of $\rho=|\psi\rangle\langle\psi|$ into the Lie algebra $\mathcal{L}$. The operator X_ψ allows one to recast the total variance (2) into the form

$$\mathbb{V}(\psi) = C_{\mathcal{H}} - \langle\psi|X_\psi|\psi\rangle. \quad (6)$$

In Appendix A, we explain how the total variance can be calculated and give an explicit formula for the multicomponent system $\mathcal{H}=\otimes_A\mathcal{H}_A$ with full access to local degrees of freedom in terms of reduced states ρ_A

$$\mathbb{V}(\psi) = \sum_A [\dim \mathcal{H}_A - \mathrm{Tr}_{\mathcal{H}_A}(\rho_A^2)]. \quad (7)$$

IV. COMPLETELY ENTANGLED STATES

We can infer from (3) the inequality

$$\mathbb{V}(\psi) \le C_{\mathcal{H}} \quad (8)$$

which turns into an equation if and only if

$$\langle\psi|X|\psi\rangle = 0, \quad \forall\ X \in \mathcal{L}. \quad (9)$$

For multiparty systems $\mathcal{H}=\otimes_A\mathcal{H}_A$, the latter equation means that all one-party reduced states are completely disordered. In other words, there exists some local basis such that the reduced state is given by a diagonal matrix ρ_A, corresponding to a uniform probability distribution (that is, ρ_A are scalar operators). This is a well-known characterization of maximally entangled states. In general we refer to (9) as the entanglement equation and call the corresponding state ψ completely entangled.

The completely entangled states are characterized by maximality of the total variance. Therefore one may be tempted to consider entanglement as a manifestation of quantum fluctuations in a state where they come to their extreme. The entanglement equation (9) just states that, in a completely entangled state ψ, the quantum system is at the center of its quantum fluctuations, that is $X_\psi=0$.

V. MEASURE OF ENTANGLEMENT

States opposite to entangled ones, to wit those with a minimal total level of quantum fluctuations $\mathbb{V}(\psi)$, for a long time were known as coherent states [21] (see also Refs. [11,22]). For multicomponent systems like $\mathcal{H}_A\otimes\mathcal{H}_B$ coherent states are just decomposable or unentangled states $\psi=\psi_A\otimes\psi_B$.

Observe [10] that the square of the concurrence $C(\psi)$ for a two-component system coincides with the total variance $\mathbb{V}(\psi)$ reduced to the interval $[0,1]$

$$C^2(\psi) = \frac{\mathbb{V}(\psi) - \mathbb{V}_{\mathrm{coh}}}{\mathbb{V}_{\mathrm{ent}} - \mathbb{V}_{\mathrm{coh}}}, \quad (10)$$

where $\mathbb{V}_{\mathrm{ent}}$ and $\mathbb{V}_{\mathrm{coh}}$ are the total levels of quantum fluctuations in completely entangled and coherent states, respectively. This clarifies the physical meaning of the concurrence as a measure of overall quantum fluctuations in the system and leads us to the natural measure of entanglement of pure states [10]

$$\mu(\psi) = \sqrt{\frac{\mathbb{V}(\psi) - \mathbb{V}_{\mathrm{coh}}}{\mathbb{V}_{\mathrm{ent}} - \mathbb{V}_{\mathrm{coh}}}} \quad (11)$$

valid for an arbitrary quantum system. It coincides with the concurrence for two-component systems, but we refrain from using this term in general, to avoid confusion with other multicomponent versions of this notion introduced in [23]. We explain how this measure can be calculated in Appendix A. For a multicomponent system $\mathcal{H}=\otimes_A\mathcal{H}_A$, it can be expressed via local data, encoded in reduced states ρ_A,

$$\mu^2(\psi) = \frac{\sum_A (1 - \mathrm{Tr}\,\rho_A^2)}{\sum_A \left(1 - \frac{1}{\dim \mathcal{H}_A}\right)}. \quad (12)$$

For example, in a two-component system $\mathcal{H}=\mathcal{H}_A\otimes\mathcal{H}_B$ the reduced states ρ_A and ρ_B are isospectral. Hence $\mathrm{Tr}\,\rho_A^2=\mathrm{Tr}\,\rho_B^2$ and for a system of square format $d\times d$ we arrive at the familiar formula for concurrence [7]

$$C(\psi) = \sqrt{\frac{d}{d-1}(1 - \mathrm{Tr}\,\rho_A^2)} \quad (13)$$

(in [7] the normalization factor is left adjustable). The isospectrality of single-party reduced states means that entanglement can be measured locally. For example, in the case of a bipartite spin-s system, measurement of only three observables (spin operators for either party) completely specifies the concurrence (see also the discussion in [24]).

An important application for the case of two qubits is provided by the polarization of photon twins (biphotons) that are created by type-II down-conversion [25]. The spin operators S_j can be associated with the Stokes operators

$$S_x \sim (a_H^\dagger a_V + a_V^\dagger a_H)/\sqrt{2},$$

$$S_y \sim i(a_H^\dagger a_V - a_V^\dagger a_H)/\sqrt{2},$$

$$S_z \sim a_H^\dagger a_H - a_V^\dagger a_V, \tag{14}$$

so that the measurement of concurrence (11) assumes measurement of three Stokes operators for either outgoing photon beam. Here a_H (a_V) denotes the photon annihilation operator with horizontal (vertical) polarization. The polarization of photons is known to be measured by means of either a standard six-state or a minimal four-state ellipsometer [26].

Nevertheless, there is a certain problem with simultaneous measurement of polarization for one of two photons created at once and forming an entangled couple. Because of the commutation relation

$$[S_j, S_k] = i\epsilon_{jkm}S_m, \quad j,k,m = x,y,z,$$

the three projections of spin (or three Stokes operators) cannot be measured independently. The minimal uncertainty relation by Schrödinger [27] states

$$V(\psi;S_j)V(\psi;S_k) - [\mathrm{Cov}(S_j,S_k)]^2 \geq \frac{1}{4}|\langle\psi|[S_j,S_k]|\psi\rangle|^2, \tag{15}$$

where $V(\psi;S_j)$ denotes the variance (uncertainty) of observable S_j in the state ψ and the covariance $\mathrm{Cov}(S_j,S_k)$ has the form

$$\mathrm{Cov}(S_j,S_k) = \frac{1}{2}\langle\psi|S_jS_k + S_kS_j|\psi\rangle - \langle\psi|S_j|\psi\rangle\langle\psi|S_k|\psi\rangle.$$

It is a straightforward matter to see that the uncertainty relation is simply reduced to the following one:

$$0 \leq \langle\psi|X_\psi|\psi\rangle \leq 1/4, \tag{16}$$

where X_ψ is defined by Eq. (4). Thus, the uncertainty relation (15) becomes an exact equality when $\psi = \psi_{coh}$ with $\langle\psi|X_\psi|\psi\rangle = 1/4$. In other words, this is an unentangled biphoton state in which each photon has well-defined polarization.

In the case of a completely entangled biphoton state, the quantity $\langle\psi|X_\psi|\psi\rangle$ has zero value [due to the condition (9)]. In this case, the measurement performed on a single photon raises an additional question: how to distinguish between entanglement and classical unpolarized state.

Since Eq. (16) is the only relation, connecting different components of the average spin vector in either party, the local quantity $\langle\psi|X_\psi|\psi\rangle$ cannot be detected by either a single or even two measurements.

VI. MEASURE $\mu(\psi)$ BEYOND TWO-PARTITE STATES

Postponing consideration of the measure $\mu(\psi)$ in general settings to Appendix A, we now note that, in the case of a multipartite system, it gives the total amount of entanglement carried by all types of interparty correlations.

For example, the Greenberger-Horne-Zeilinger (GHZ) state of three qubits

$$|G\rangle = x|000\rangle + \sqrt{1-|x|^2}|111\rangle, \quad |x| \in [0,1], \tag{17}$$

carries only three-party entanglement. This means that any two parties are not entangled. In fact, any reduced two-qubit state, say,

$$\rho_{AB} = \mathrm{Tr}_C|G\rangle\langle G| = |x|^2|00\rangle\langle 00| + (1-|x|^2)|11\rangle\langle 11|,$$

clearly has zero concurrence. The amount of three-part entanglement in (17) is measured by the three-tangle τ [9] or Cayley hyperdeterminant [28] (for the definition of the three-tangle, see Appendix B). It is easily seen that

$$\tau(G) = \mu^2(G) = 4|x|^2(1-|x|^2).$$

Thus, the squared measure (11), calculated for the three-qubit state (17), gives the same result as three-tangle.

Another interesting example is provided by the so-called W state of three qubits,

$$|W\rangle = \frac{1}{\sqrt{3}}(|011\rangle + |010\rangle + |110\rangle). \tag{18}$$

This is a nonseparable state in three-qubit Hilbert space. Nevertheless, it does not manifest three-party entanglement because the corresponding three-tangle $\tau(W)=0$ [28]. At the same time, the measure (11) gives

$$\mu(W) = \frac{2\sqrt{2}}{3} \approx 0.94 \tag{19}$$

because $\mathbb{V}(W) = 8 + 2/3$ and $\mathbb{V}_{\mathrm{coh}} = 6$ in this case. The point is that there is a two-qubit entanglement in the state (18). To justify that the difference $2+2/3$ is caused just by quantum pairwise correlations, let us calculate the total covariance

$$\mathrm{Cov}(W) = \sum_{i=x,y,z}\sum_{J\neq J'}(\langle W|\sigma_i^J\sigma_i^{J'}|W\rangle - \langle W|\sigma_i^J|W\rangle\langle W|\sigma_i^{J'}|W\rangle). \tag{20}$$

Here $J, J' = A, B, C$ label the parties. It is a straightforward matter to see that $\mathbb{V}(W) - \mathbb{V}_{\mathrm{coh}} = \mathrm{Cov}(W)$. Similar results can be obtained for the so-called biseparable states of three qubits

$$(|001\rangle + |010\rangle), \quad (|001\rangle + |100\rangle), \quad (|010\rangle + |100\rangle), \tag{21}$$

which also manifest entanglement of two qubits and no entanglement of all three parts.

Examining entanglement of multiqubit systems in general (the number of parts is greater than two), it is necessary first to determine classes of states with different types of entanglement (including the class of unentangled states). It is assumed that those classes are nonequivalent with respect to stochastic local operations assisted by classical communication (SLOCC) [29]. The point is that entanglement of a given type cannot be created or destroyed under action of SLOCC. In the case of three qubits, such a classification has been considered in Refs. [28,30]. In the case of four qubits, the number of classes is much higher [31]. A useful approach to classification is based on investigation of geometrical invariants for a given system (e.g., see Refs. [11,32]).

For example, the class of four-qubit entangled states can be specified by the generic GHZ-type state

$$x|0000\rangle \pm \sqrt{1-|x|^2}|1111\rangle, \quad |x| \in [0,1], \tag{22}$$

which becomes completely entangled at $|x|=1/\sqrt{2}$. In general, four-qubit completely entangled states can be defined by means of the condition (9) (see Appendix C). For the state (22), the measure (11) gives the amount of entanglement $\mu = \sqrt{1-(2|x|^2-1)^2}$, which becomes complete entanglement at $|x|=1/\sqrt{2}$ as expected.

At the same time, there is another class of pairwise separable four-qubit states

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \tag{23}$$

in which the first two pairs and the last two pairs separately manifest complete two-party entanglement, while there is no four-qubit entanglement [compare with the biseparable states of three qubits (21)]. In this case, the measure (11) again gives the total amount of entanglement carried by the parts of the system.

VII. MIXED ENTANGLEMENT

The measure (11) cannot be directly applied to calculation of entanglement of mixed states because it is incapable of separation of classical and quantum contributions into the total variance (2). Therefore, $\mu(\rho)$ always gives an estimation from above for the entanglement of mixed states. This can be easily checked for some characteristic states like the Werner state [33] and the so-called maximally entangled mixed state of Ref. [34].

As far as we know, nowadays there is no universally recognized protocol for separation of classical and quantum uncertainties in mixed states except for the case of two qubits [6]. A promising approach proposed in Refs. [8,23] consists in the representation of concurrence of a mixed state ρ as $\inf\Sigma_i C(\psi_i)$ of all properly normalized states ψ such that $\rho = \Sigma_i |\psi_i\rangle\langle\psi_i|$.

VIII. SUMMARY

We have shown that the description of entanglement in a given system requires pre-definition of basic observables and that the entanglement of pure states can be adequately quantified in terms of the total variance of all basic observables. Unlike the conventional concurrence and three-tangle, which measure the amount of entanglement of different groups of correlated parties, our measure gives the total amount of multipartite entanglement carried by a given state. Other evident virtues of the measure (11) are its simple physical meaning, its applicability beyond bipartite systems, and its operational character caused by measurement of quantum uncertainties of well-defined physical observables.

At the same time, this measure cannot be directly applied to calculation of entanglement in mixed states. However, it may be used in the way that has been discussed in Refs. [8,23] as follows:

$$\mu(\rho) = \inf \sum_i \mu(\psi_i).$$

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APPENDIX A

Here we calculate the total variance $\mathbb{V}(\psi)$ and the entanglement measure $\mu(\psi)$.

Let $\mathrm{Herm}(\mathcal{H})$ be the space of all Hermitian operators acting in the Hilbert space $\mathcal{H}$ with trace metric $\mathrm{Tr}_{\mathcal{H}}(XY)$. For the simple algebra $\mathcal{L}$, restriction of the trace metric onto $\mathcal{L}$ is proportional to the Cartan-Killing form

$$\mathrm{Tr}_{\mathcal{H}}(XY) = D_{\mathcal{H}}(X,Y)_K, \quad X,Y \in \mathcal{L}$$

with the coefficient $D_{\mathcal{H}}$ known as the Dynkin index [20]. Consider now the orthogonal projection $\rho_{\mathcal{L}}$ of $\rho := |\psi\rangle\langle\psi| \in \mathrm{Herm}(\mathcal{H})$ into the subalgebra $\mathcal{L} \subset \mathrm{Herm}(\mathcal{H})$, so that $\mathrm{Tr}_{\mathcal{H}}(\rho X) = \mathrm{Tr}_{\mathcal{H}}(\rho_{\mathcal{L}} X)$, $\forall\, X \in \mathcal{L}$. The projection $\rho_{\mathcal{L}}$ is closely related to the mean operator (4)

$$\begin{aligned} X_\psi &= \sum_\alpha \mathrm{Tr}_{\mathcal{H}}(\rho X_\alpha) X_\alpha \\ &= \sum_\alpha \mathrm{Tr}_{\mathcal{H}}(\rho_{\mathcal{L}} X_\alpha) X_\alpha \\ &= D_{\mathcal{H}} \sum_\alpha (\rho_{\mathcal{L}}, X_\alpha)_K X_\alpha \\ &= D_{\mathcal{H}} \rho_{\mathcal{L}}. \end{aligned}$$

Therefore

$$\langle\psi|X_\psi|\psi\rangle = \mathrm{Tr}_{\mathcal{H}}(\rho X_\psi) = \mathrm{Tr}_{\mathcal{H}}(\rho_{\mathcal{L}} X_\psi) = D_{\mathcal{H}} \mathrm{Tr}_{\mathcal{H}}(\rho_{\mathcal{L}}^2)$$

and the total variance (2) can be written in the form

$$\mathbb{V}(\psi) = C_{\mathcal{H}} - \langle\psi|X_\psi|\psi\rangle = C_{\mathcal{H}} - D_{\mathcal{H}} \mathrm{Tr}_{\mathcal{H}}(\rho_{\mathcal{L}}^2). \tag{A1}$$

For simple algebra the Casimir $C_{\mathcal{H}}$ and Dynkin index $D_{\mathcal{H}}$ are given by equations

$$C_{\mathcal{H}} = (\lambda, \lambda + 2\delta), \quad D_{\mathcal{H}} = \frac{\dim \mathcal{H}}{\dim \mathcal{L}}(\lambda, \lambda + 2\delta), \tag{A2}$$

where λ denotes the highest weight of the irreducible representation $\mathcal{H}$ and 2δ is the sum of positive roots of $\mathcal{L}$. For example, for full algebra of traceless Hermitian operators $\mathcal{L} = \mathrm{su}(\mathcal{H})$ we have

$$C_{\mathcal{H}} = \dim \mathcal{H} - \frac{1}{\dim \mathcal{H}}, \quad D_{\mathcal{H}} = 1. \tag{A3}$$

In general, the algebra $\mathcal{L}$ splits into simple components $\mathcal{L}=\oplus_A \mathcal{L}_A$ and its irreducible representation $\mathcal{H}$ into tensor product $\mathcal{H}=\otimes_A \mathcal{H}_A$. In this case Eq. (A1) should be modified as follows:

$$\mathbb{V}(\psi) = \sum_A [C_{\mathcal{H}_A} - D_{\mathcal{H}_A} \mathrm{Tr}_{\mathcal{H}_A}(\nu_A^2 \rho_{\mathcal{L}_A}^2)], \tag{A4}$$

where $\nu_A = \dim \mathcal{H}/\dim \mathcal{H}_A$.

In the quantum-information setting $\mathcal{L}_A$ is the full algebra of traceless Hermitian operators $X_A:\mathcal{H}_A \to \mathcal{H}_A$. In this case everything can be done explicitly.

By definition of the reduced states ρ_A we have

$$\mathrm{Tr}_{\mathcal{H}}(\rho X_A) = \mathrm{Tr}_{\mathcal{H}_A}(\rho_A X_A) = \nu_A^{-1} \mathrm{Tr}_{\mathcal{H}}(\rho_A X_A).$$

Comparing this with the equation $\mathrm{Tr}_{\mathcal{H}}(\rho X_A) = \mathrm{Tr}_{\mathcal{H}}(\rho_{\mathcal{L}_A} X_A)$, $\forall\, X_A \in \mathcal{L}_A$, characterizing the projection $\rho_{\mathcal{L}_A} \in \mathcal{L}_A$ we infer

$$\rho_{\mathcal{L}_A} = \nu_A^{-1} \rho_A^0,$$

where $\rho_A^0 = \rho_A - (1/\dim \mathcal{H}_A)\mathbb{I}$ is the traceless part of ρ_A. This allows us to calculate the trace

$$\mathrm{Tr}_{\mathcal{H}_A}(\rho_{\mathcal{L}_A}^2) = \nu^{-2}\left(\mathrm{Tr}_{\mathcal{H}_A}(\rho_A^2) - \frac{1}{\dim \mathcal{H}_A}\right).$$

Plugging this into Eq. (A4) and using (A3) we finally get

$$\mathbb{V}(\psi) = \sum_A [\dim \mathcal{H}_A - \mathrm{Tr}_{\mathcal{H}_A}(\rho_A^2)]. \tag{A5}$$

As an example, consider the completely entangled state ψ for which $\rho_A = (1/\dim \mathcal{H}_A)\mathbb{I}$. This gives the maximum of the total variance,

$$\mathbb{V}_{\max} = \mathbb{V}_{\mathrm{ent}} = \sum_A \left(\dim \mathcal{H}_A - \frac{1}{\dim \mathcal{H}_A}\right).$$

The minimum of the total variance is attained for the coherent (=separable) state ψ, for which reduced states ρ_A are pure. Hence

$$\mathbb{V}_{\min} = \mathbb{V}_{\mathrm{coh}} = \sum_A (\dim \mathcal{H}_A - 1).$$

Combining these equations we can write down our measure of entanglement (11) explicitly for a multicomponent system $\mathcal{H}=\otimes_A \mathcal{H}_A$ of arbitrary format

$$\mu^2(\psi) = \frac{\sum_A [1 - \mathrm{Tr}(\rho_A^2)]}{\sum_A \left(1 - \frac{1}{\dim \mathcal{H}_A}\right)}. \tag{A6}$$

APPENDIX B

For an arbitrary normalized state of three qubits

$$|\psi\rangle = \sum_{\ell,m,n=0}^{1} \psi_{\ell mn} |\ell mn\rangle$$

the three-tangle has the form [9,28]

$$\begin{aligned}\tau(\psi) = 4\big|&\psi_{000}^2\psi_{111}^2 + \psi_{001}^2\psi_{110}^2 + \psi_{010}^2\psi_{101}^2 + \psi_{100}^2\psi_{011}^2 \\ &- 2(\psi_{000}\psi_{001}\psi_{110}\psi_{111} + \psi_{000}\psi_{010}\psi_{101}\psi_{111} \\ &+ \psi_{000}\psi_{100}\psi_{011}\psi_{111} + \psi_{001}\psi_{010}\psi_{101}\psi_{110} \\ &+ \psi_{001}\psi_{100}\psi_{011}\psi_{110} + \psi_{010}\psi_{100}\psi_{011}\psi_{101}) \\ &+ 4(\psi_{000}\psi_{011}\psi_{101}\psi_{110} + \psi_{001}\psi_{010}\psi_{100}\psi_{111})\big|.\end{aligned}$$

APPENDIX C

A general pure state of four qubits can be written in the form

$$|\psi\rangle = \sum_{k,\ell,m,n=0}^{1} \psi_{k\ell mn}|k,\ell,m,n\rangle \tag{C1}$$

with the normalization condition $\Sigma_{k,\ell,m,n=0}^{1}|\psi_{k\ell mn}|^2=1$. Thus, there are 31 real parameters, defining any state. Condition (9) gives 12 equations for the coefficients $\psi_{k\ell mn}$ in (C1):

$$\begin{aligned}\langle\sigma_x^{(A)}\rangle = (&\psi_{0000}^*\psi_{1000} + \psi_{0100}^*\psi_{1100} + \psi_{0010}^*\psi_{1010} + \psi_{0001}^*\psi_{1001} \\ &+ \psi_{0110}^*\psi_{1110} + \psi_{0101}^*\psi_{1101} + \psi_{0011}^*\psi_{1011} + \psi_{0111}^*\psi_{1111}) \\ &+ (\mathrm{c.c.}) = 0,\end{aligned}$$

$$\begin{aligned}\langle\sigma_x^{(B)}\rangle = (&\psi_{0000}^*\psi_{0100} + \psi_{1000}^*\psi_{1100} + \psi_{0010}^*\psi_{0110} + \psi_{0001}^*\psi_{0101} \\ &+ \psi_{1010}^*\psi_{1110} + \psi_{1001}^*\psi_{1101} + \psi_{0011}^*\psi_{0111} + \psi_{1011}^*\psi_{1111}) \\ &+ (\mathrm{c.c.}) = 0,\end{aligned}$$

$$\begin{aligned}\langle\sigma_x^{(C)}\rangle = (&\psi_{0000}^*\psi_{0010} + \psi_{1000}^*\psi_{1010} + \psi_{0100}^*\psi_{0110} + \psi_{0001}^*\psi_{0011} \\ &+ \psi_{1100}^*\psi_{1110} + \psi_{1001}^*\psi_{1011} + \psi_{0101}^*\psi_{0111} + \psi_{1101}^*\psi_{1111}) \\ &+ (\mathrm{c.c.}) = 0,\end{aligned}$$

$$\begin{aligned}\langle\sigma_x^{(D)}\rangle = (&\psi_{0000}^*\psi_{0001} + \psi_{1000}^*\psi_{1001} + \psi_{0100}^*\psi_{0101} + \psi_{0010}^*\psi_{0011} \\ &+ \psi_{1100}^*\psi_{1101} + \psi_{1010}^*\psi_{1011} + \psi_{0110}^*\psi_{0111} + \psi_{1110}^*\psi_{1111}) \\ &+ (\mathrm{c.c.}) = 0,\end{aligned}$$

$$\begin{aligned}\langle\sigma_y^{(A)}\rangle = i(&\psi_{1000}^*\psi_{0000} + \psi_{1100}^*\psi_{0100} + \psi_{1010}^*\psi_{0010} + \psi_{1001}^*\psi_{0001} \\ &+ \psi_{1110}^*\psi_{0110} + \psi_{1101}^*\psi_{0101} + \psi_{1011}^*\psi_{0011} + \psi_{1111}^*\psi_{0111}) \\ &+ (\mathrm{c.c.}) = 0,\end{aligned}$$

$$\begin{aligned}\langle\sigma_y^{(B)}\rangle = i(&\psi_{0100}^*\psi_{0000} + \psi_{1100}^*\psi_{1000} + \psi_{0110}^*\psi_{0010} + \psi_{0101}^*\psi_{0001} \\ &+ \psi_{1110}^*\psi_{1010} + \psi_{1101}^*\psi_{1001} + \psi_{0111}^*\psi_{0011} + \psi_{1111}^*\psi_{1011}) \\ &+ (\mathrm{c.c.}) = 0,\end{aligned}$$

$$\begin{aligned}\langle\sigma_y^{(C)}\rangle = i(&\psi_{0010}^*\psi_{0000} + \psi_{1010}^*\psi_{1000} + \psi_{0110}^*\psi_{0100} + \psi_{0011}^*\psi_{0001} \\ &+ \psi_{1110}^*\psi_{1100} + \psi_{1011}^*\psi_{1001} + \psi_{0111}^*\psi_{0101} + \psi_{1111}^*\psi_{1101}) \\ &+ (\mathrm{c.c.}) = 0,\end{aligned}$$

$$\langle\sigma_y^{(D)}\rangle = i(\psi_{0001}^*\psi_{0000} + \psi_{1001}^*\psi_{1000} + \psi_{0101}^*\psi_{0100} + \psi_{0011}^*\psi_{0010} + \psi_{1101}^*\psi_{1100} + \psi_{1011}^*\psi_{1010} + \psi_{0111}^*\psi_{0110} + \psi_{1111}^*\psi_{1110}) + (\text{c.c.}) = 0,$$

$$\langle\sigma_z^{(A)}\rangle = |\psi_{0000}|^2 - |\psi_{1000}|^2 + |\psi_{0100}|^2 + |\psi_{0010}|^2 + |\psi_{0001}|^2 - |\psi_{1100}|^2 - |\psi_{1010}|^2 - |\psi_{1001}|^2 + |\psi_{0110}|^2 + |\psi_{0101}|^2 + |\psi_{0011}|^2 - |\psi_{1011}|^2 - |\psi_{1101}|^2 - |\psi_{1110}|^2 + |\psi_{0111}|^2 - |\psi_{1111}|^2 = 0,$$

$$\langle\sigma_z^{(B)}\rangle = |\psi_{0000}|^2 + |\psi_{1000}|^2 - |\psi_{0100}|^2 + |\psi_{0010}|^2 + |\psi_{0001}|^2 - |\psi_{1100}|^2 + |\psi_{1010}|^2 + |\psi_{1001}|^2 - |\psi_{0110}|^2 - |\psi_{0101}|^2 + |\psi_{0011}|^2 + |\psi_{1011}|^2 - |\psi_{1101}|^2 - |\psi_{1110}|^2 - |\psi_{0111}|^2 - |\psi_{1111}|^2 = 0,$$

$$\langle\sigma_z^{(C)}\rangle = |\psi_{0000}|^2 + |\psi_{1000}|^2 + |\psi_{0100}|^2 - |\psi_{0010}|^2 + |\psi_{0001}|^2 + |\psi_{1100}|^2 - |\psi_{1010}|^2 + |\psi_{1001}|^2 - |\psi_{0110}|^2 + |\psi_{0101}|^2 - |\psi_{0011}|^2 - |\psi_{1011}|^2 + |\psi_{1101}|^2 - |\psi_{1110}|^2 - |\psi_{0111}|^2 - |\psi_{1111}|^2 = 0,$$

$$\langle\sigma_z^{(D)}\rangle = |\psi_{0000}|^2 + |\psi_{1000}|^2 + |\psi_{0100}|^2 + |\psi_{0010}|^2 - |\psi_{0001}|^2 + |\psi_{1100}|^2 + |\psi_{1010}|^2 - |\psi_{1001}|^2 + |\psi_{0110}|^2 - |\psi_{0101}|^2 - |\psi_{0011}|^2 - |\psi_{1011}|^2 - |\psi_{1101}|^2 + |\psi_{1110}|^2 - |\psi_{0111}|^2 - |\psi_{1111}|^2 = 0,$$

where $\langle\sigma_\alpha^{(i)}\rangle = \langle\psi_{\text{ent}}|\sigma_\alpha^{(i)}|\psi_{\text{ent}}\rangle$ and c.c. denotes the complex conjugate. Thus, there are infinitely many completely entangled states and the state (22) at $|x| = 1/\sqrt{2}$ is among them.

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