Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 250 (2007) 483–520
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Carleson measures for Besov spaces on the ball with applications ☆

H. Turgay Kaptanoğlu

Department of Mathematics, Bilkent University, Ankara 06800, Turkey

Received 20 November 2006; accepted 28 December 2006

Available online 27 February 2007

Communicated by Paul Malliavin

Abstract

Carleson and vanishing Carleson measures for Besov spaces on the unit ball of $\mathbb{C}^N$ are characterized in terms of Berezin transforms and Bergman-metric balls. The measures are defined via natural imbeddings of Besov spaces into Lebesgue classes by certain combinations of radial derivatives. Membership in Schatten classes of the imbeddings is considered too. Some Carleson measures are not finite, but the results extend and provide new insight to those known for weighted Bergman spaces. Special cases pertain to Arveson and Dirichlet spaces, and a unified view with the usual Hardy-space Carleson measures is presented by letting the order of the radial derivatives tend to 0. Weak convergence in Besov spaces is also characterized, and weakly 0-convergent families are exhibited. Applications are given to separated sequences, operators of Forelli–Rudin type, gap series, characterizations of weighted Bloch, Lipschitz, and growth spaces, inequalities of Fejér–Riesz and Hardy–Littlewood type, and integration operators of Cesàro type.

Keywords: Carleson measure; Berezin transform; Bergman metric; Bergman projection; Weak, ultraweak convergence; Schatten–von Neumann ideal; Besov, Bergman, Dirichlet, Hardy, Arveson, Bloch, Lipschitz, growth space; Separated sequence; Forelli–Rudin-type operator; Lacunary series; Fejér–Riesz, Hardy–Littlewood inequality; Cesàro-type operator

☆ The research of the author is partially supported by a Fulbright grant.
E-mail address: kaptan@fen.bilkent.edu.tr.
URL: http://www.fen.bilkent.edu.tr/~kaptan/.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2006.12.016

1. Introduction

We let $\mathbb{B}$ be the unit ball of $\mathbb{C}^N$ and $H(\mathbb{B})$ the space of holomorphic functions on $\mathbb{B}$. When $N=1$, we have the unit disc $\mathbb{D}$. Unless otherwise specified, our main parameters and their range of values are

$$q \in \mathbb{R}, \qquad 0 < p < \infty, \qquad s \in \mathbb{R}, \qquad t \in \mathbb{R}, \qquad 0 < r < \infty;$$

and given q and p, we often choose t to satisfy

$$q + pt > -1. \tag{1}$$

Let ν be the volume measure on $\mathbb{B}$ normalized with $\nu(\mathbb{B}) = 1$. We define on $\mathbb{B}$ also the measures

$$d\nu_q(z) = \left(1 - |z|^2\right)^q \, d\nu(z), \tag{2}$$

which are finite only for $q > -1$, where $|z|^2 = \langle z, z\rangle$ and $\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_N\overline{w}_N$. The corresponding Lebesgue classes are L^p_q. We also let $d\mu_q(z) = (1 - |z|^2)^q \, d\mu(z)$ for a general measure μ on $\mathbb{B}$.

Consider the linear transformation I^t_s defined for $f \in H(\mathbb{B})$ by

$$I^t_s f(z) = \left(1 - |z|^2\right)^t D^t_s f(z),$$

where D^t_s is a bijective radial differential operator on $H(\mathbb{B})$ of order t for any s, and every I^0_s is the identity I. The following definition is known to be independent of s, t, where the term norm is used even when $0 < p < 1$; see [23, Theorem 4.1] or [11, Theorem 5.12(i)], for example.

Definition 1.1. The *Besov space* B^p_q consists of all $f \in H(\mathbb{B})$ for which the function $I^t_s f$ belongs to L^p_q for some s, t satisfying (1). The L^p_q norms of $I^t_s f$ are all equivalent. We call any one of them the B^p_q norm of f and denote it by $\|f\|_{B^p_q}$.

So I^t_s is an imbedding of B^p_q into L^p_q. The necessary background for B^p_q spaces is given in Section 3. They are all complete, Banach spaces for $p \geqslant 1$, and Hilbert spaces for $p = 2$. They include many known spaces as special cases.

Definition 1.2. We call a positive Borel measure μ on $\mathbb{B}$ a *Carleson measure for* B^p_q provided some I^t_s maps B^p_q into $L^p(\mu)$ continuously.

Now we are ready to state our main result. Here, the $b(w, r)$ is the ball in the Bergman metric with center $w \in \mathbb{B}$ and radius r, and an r-lattice is defined by Lemma 2.5. As is commonly used, C is a finite positive constant whose value might be different at each occurrence. The context makes it clear what each C depends on, but C never depends on the functions in the formula in which it appears.

Theorem 1.3. *Let q be fixed but unrestricted. Let p and r, and also s be given. The following conditions are equivalent for a positive Borel measure μ on $\mathbb{B}$.*

(i) *There is a C such that*

$$\sup_{w\in\mathbb{B}} \frac{\mu(b(w,r))}{\nu_q(b(w,r))} \leqslant C.$$

(ii) *There is a C such that if $\{a_n\}$ is an r-lattice in $\mathbb{B}$, then*

$$\sup_{n\in\mathbb{N}} \frac{\mu(b(a_n,r))}{\nu_q(b(a_n,r))} \leqslant C.$$

(iii) *There is a C such that if t satisfies* (1), *then*

$$\int_{\mathbb{B}} \left|I_s^t f\right|^p \, d\mu \leqslant C\|f\|_{B_q^p}^p \quad \left(f \in B_q^p\right).$$

(iv) *There is a C such that if t satisfies* (1), *then*

$$\sup_{w\in\mathbb{B}} \left(1-|w|^2\right)^{N+1+q+pt} \int_{\mathbb{B}} \frac{(1-|z|^2)^{pt}}{|1-\langle z,w\rangle|^{(N+1+q+pt)2}} \, d\mu(z) \leqslant C.$$

Condition (iii) is the statement that μ is a Carleson measure for B_q^p.

As is common with Carleson-measure theorems, the property of being a Carleson measure is independent of p or r, and now also of s, t as long as (1) holds, because (i) is true for any p, s, t and (iii) is true for any r. However, all conditions depend on q. So for a fixed q, a Carleson measure for one B_q^p with one suitable s, t is a Carleson measure for all B_q^p with the same q with any other such s, t. And we conveniently call such a μ also a *q-Carleson measure*. So setting

$${}_q\hat{\mu}_r(w) = \frac{\mu(b(w,r))}{\nu_q(b(w,r))} \quad (w \in \mathbb{B}),$$

a q-Carleson measure is a positive Borel measure on $\mathbb{B}$ for which the *averaging function* ${}_q\hat{\mu}_r$ is bounded on $\mathbb{B}$ for some r. Thus Theorem 1.3 gives a full characterization of q-Carleson measures for *all* real q.

We can draw some immediate conclusions from Theorem 1.3. Clearly the model q-Carleson measure is ν_q. So Carleson measures need not be finite for $q \leqslant -1$. By Lemma 2.2, $\nu_q(b(w,r))$ is of order $(1-|w|^2)^{N+1+q}$. Thus by (i), any ν_{q_1} with $q_1 > q$ is also a q-Carleson measure while no ν_{q_2} with $q_2 < q$ is. Further, by (i) again, any finite Borel measure is a q-Carleson measure for $q \leqslant -(N+1)$. And for $q = -(N+1)$, q-Carleson measures are precisely those Borel measures that are finite on Bergman balls of a fixed radius. On the question of finiteness, with $w = 0$ and $b = pt$, (iv) immediately implies the following.

Corollary 1.4. *If μ is a q-Carleson measure, then the measure μ_β is finite for any β with $\beta + q > -1$.*

Theorem 1.3 is better appreciated when we restrict to $q > -1$. Then $t = 0$ satisfies (1) for any p, and by Definition 1.1, the space B_q^p coincides with the weighted Bergman space A_q^p. In this case Theorem 1.3 becomes a well-known result, and Corollary 1.4 implies that a Carleson measure must then be finite; see [14, Theorem 2.36] for $N = 1$. But it is possible to take $t \neq 0$ also with $q > -1$ as long as t satisfies (1); then Theorem 1.3 extends known results for weighted Bergman spaces by giving equivalences also with I_s^t in place of the inclusion map.

Moreover, the space B_{-1}^2 is the Hardy space H^2. Now (1) requires a $t > 0$, no matter how small. It follows that Definition 1.2 and Theorem 1.3 are about Carleson measures different from the usual Carleson measures on H^2. However, as $t \to 0^+$, we show that we indeed obtain the usual Carleson measures on H^2, and hence on H^p. Therefore we unify the theory of Carleson measures on weighted Bergman, Besov, and Hardy spaces simultaneously.

Theorem 1.3 depends on an imbedding of B_q^p into a Lebesgue class via I_s^t which involve certain combinations of radial derivatives of functions in B_q^p. Using derivatives to imbed holomorphic function spaces into Lebesgue classes is not uncommon; see [4, Theorem 13], [27] and its references, and [13]. On the other hand, descriptions of Carleson measures defined using the inclusion map on Besov spaces are limited to certain values of q and p and to $N = 1$. For example, $q = -(N + 1) = -2$ in [5] although their Besov spaces are defined with a more general weight than $1 - |z|^2$. In other places, the equivalent conditions are not uniform over the values of q, p considered; for example, see [36] for $q + p > -1$ with $N = 1$.

It is still possible to strengthen the characterization of q-Carleson measures by relaxing their dependence on Besov spaces and weakening the condition in Theorem 1.3(iv) after a relabeling of the parameters. The following result seems to be new in its generality also for Bergman-space Carleson measures and even in the most classical case $q = 0$. Recalling that ν_q is the model q-Carleson measure and in view of [32, Proposition 1.4.10], its conditions are as natural as can be hoped for.

Theorem 1.5. *Let μ be a positive Borel measure on $\mathbb{B}$. If*

$$U_{\alpha,\beta,q}\mu(w) = \left(1 - |w|^2\right)^{\alpha} \int_{\mathbb{B}} \frac{(1 - |z|^2)^{\beta}}{|1 - \langle z, w\rangle|^{N+1+\alpha+\beta+q}} \, d\mu(z) \quad (w \in \mathbb{B})$$

is bounded for some real α, β, and q, then μ is a q-Carleson measure. If μ is a q-Carleson measure, $\alpha > 0$, and $\beta + q > -1$, then $U_{\alpha,\beta,q}\mu$ is bounded on $\mathbb{B}$.

The idea of this theorem leads to a characterization of Hardy-space Carleson measures which also seems new in its generality.

Theorem 1.6. *Let μ be a positive Borel measure on $\mathbb{B}$. If $U_{\alpha,0,-1}\mu(w)$ is bounded on $\mathbb{B}$ for some real α, then μ is a Hardy-space Carleson measure. If μ is a Hardy-space Carleson measure and $\alpha > 0$, then $U_{\alpha,0,-1}\mu(w)$ is bounded on $\mathbb{B}$.*

Some of the results in this paper have been announced in [24].

All our results are valid when s and t are complex numbers too; we just need to replace them with their real parts in inequalities as done in [17,23].

The proof of Theorem 1.3 is in Section 5 along with a discussion of related Berezin transforms. The little oh version of this theorem that connects the compactness of I_s^t to vanishing Carleson measures is Theorem 5.3. This section contains also the proof of Theorem 1.5 and its little oh version. An immediate application is given to separated sequences in $\mathbb{B}$. We further give equivalent conditions for the imbedding $I_s^t : B_q^2 \to L^2(\mu)$ to belong to the Schatten ideal S^c with $c \geqslant 2$ in Theorem 5.12. Compact operators require a characterization of ultraweak convergence in Besov spaces which is in Section 4. We next give examples of ultraweakly convergent families in Besov spaces in Example 4.7 that are instrumental in the proof of the implication (iii) $\Rightarrow$ (iv) of Theorem 1.3. We gather some basic facts about Bergman geometry in Section 2, and review Besov spaces in Section 3. Later in Section 6, we show how the Hardy-space Carleson measures come into the picture as the order t of the derivative D_s^t tends to 0 when $q = -1$. The proof of Theorem 1.6 is also here.

The remaining sections are for applications. In Section 7, we apply Theorem 1.5 to an analysis of integral operators on L^∞ inspired by Forelli–Rudin estimates. In Section 8, we characterize functions in weighted Bloch and little Bloch spaces $\mathcal{B}^\alpha$ and $\mathcal{B}_0^\alpha$ for all $\alpha \in \mathbb{R}$, which include the Lipschitz classes and the growth spaces. In Section 9, we develop a finiteness criterion for positive Borel measures imbedding Bloch spaces into Lebesgue classes using I_s^t, and we construct Carleson measures from functions in Besov spaces, using gap series for both. In Section 10, we generalize to Besov spaces two classical inequalities of Fejér–Riesz and Hardy–Littlewood for Hardy spaces, which are reobtained in a limiting case. In Section 11, we investigate integration operators companion to a Cesàro-type operator.

As for notation, if X is a set, then $\overline{X}$ denotes its closure and ∂X its boundary. The surface measure on $\partial\mathbb{B}$ is denoted σ and normalized with $\sigma(\partial\mathbb{B}) = 1$. Bounded measurable and bounded holomorphic functions on $\mathbb{B}$ are denoted by L^∞ and H^∞, and $\|f\|_{H^\infty} = \sup_{\partial\mathbb{B}} |f|$. Note that $L_q^\infty = L^\infty$ for any q. We let $\mathcal{C}$ be the space of continuous functions on $\overline{\mathbb{B}}$ and $\mathcal{C}_0$ its subspace whose members vanish on $\partial\mathbb{B}$.

We use the convenient *Pochhammer symbol* defined by

$$(x)_y = \frac{\Gamma(x+y)}{\Gamma(x)}$$

when x and $x+y$ are off the pole set $-\mathbb{N}$ of the gamma function Γ. For fixed x, y, Stirling formula gives

$$\frac{\Gamma(c+x)}{\Gamma(c+y)} \sim c^{x-y} \quad \text{and} \quad \frac{(x)_c}{(y)_c} \sim c^{x-y} \quad (c \to \infty), \tag{3}$$

where $x \sim y$ means that $|x/y|$ is bounded above and below by two positive constants that are independent of any parameter present (c here).

We use multi-index notation in which $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{N}^N$ is an N-tuple of nonnegative integers, $|\lambda| = \lambda_1 + \cdots + \lambda_N$, $\lambda! = \lambda_1! \cdots \lambda_N!$, $z^\lambda = z_1^{\lambda_1} \cdots z_N^{\lambda_N}$, and $0^0 = 1$.

2. Bergman geometry

We collect here some standard facts on balls in the Bergman metric, and prove some subharmonicity results with respect to these balls.

The biholomorphic automorphism group $\mathrm{Aut}(\mathbb{B})$ of the ball is generated by unitary mappings of $\mathbb{C}^n$ and the involutive *Möbius transformations* φ_a that exchange 0 and $a \in \mathbb{B}$. A most useful property of φ_a is

$$1 - \langle \varphi_a(z), \varphi_a(w) \rangle = \frac{(1-|a|^2)(1-\langle z, w\rangle)}{(1-\langle z, a\rangle)(1-\langle a, w\rangle)} \quad (a, z, w \in \mathbb{B}); \tag{4}$$

the real Jacobian of the transformation $w = \varphi_a(z)$ is

$$J_{\mathbb{R}}\varphi_a(z) = \left(\frac{1 - |a|^2}{|1 - \langle z, a\rangle|^2}\right)^{N+1}; \tag{5}$$

see [32, Section 2.2]. The *Bergman metric* on $\mathbb{B}$ is

$$d(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|} = \tanh^{-1}\bigl|\varphi_z(w)\bigr|,$$

where $|\varphi_z(w)| = d_\psi(z, w)$ is the *pseudohyperbolic metric* on $\mathbb{B}$. These metrics are invariant under the automorphisms of $\mathbb{B}$; that is, $d(\psi(z), \psi(w)) = d(z, w)$ and $d_\psi(\psi(z), \psi(w)) = d_\psi(z, w)$ for $\psi \in \operatorname{Aut}(\mathbb{B})$.

The balls centered at w of radius r in the Bergman (hyperbolic), pseudohyperbolic, and Euclidean metrics are denoted by $b(w, r)$, $b_\psi(w, r)$, and $b_e(w, r)$, respectively. A pseudohyperbolic ball is a Bergman ball rescaled by the hyperbolic tangent, and a Euclidean ball is a pseudohyperbolic ball translated by an automorphism of $\mathbb{B}$, as explicitly displayed by the relations

$$b(w, r) = b_\psi(w, \tanh r) = \varphi_w\bigl(b_e(0, \tanh r)\bigr), \tag{6}$$

where $0 < \tanh r < 1$. The automorphism invariance of the two metrics d and d_ψ shows that $\varphi_a(b(w, r)) = b(\varphi_a(w), r)$ and $\varphi_a(b_\psi(w, r)) = b_\psi(\varphi_a(w), r)$.

Lemma 2.1. *Given $r_1 > 0$ and $w \in \mathbb{B}$, we have*

$$1 - \langle z_1, z_2\rangle \sim 1 - |w|^2$$

for all $z_1, z_2 \in b(w, r)$ and $r \leqslant r_1$. Hence

$$1 - |z|^2 \sim 1 - |w|^2 \quad \textit{and} \quad 1 - \langle z, w\rangle \sim 1 - |w|^2$$

for all $z \in b(w, r)$ and $r \leqslant r_1$.

Proof. If $z_j \in b(w, r)$, then $z_j = \varphi_w(v_j)$ for some v_j with $|v_j| < \tanh r$ for $j = 1, 2$ by (6). Then

$$1 - \langle z_1, z_2\rangle = 1 - \bigl\langle \varphi_w(v_1), \varphi_w(v_2)\bigr\rangle = \frac{(1 - |w|^2)(1 - \langle v_1, v_2\rangle)}{(1 - \langle v_1, w\rangle)(1 - \langle w, v_2\rangle)} \sim 1 - |w|^2,$$

because the other factors are ~ 1 since $|v_j| < \tanh r \leqslant \tanh r_1$, $j = 1, 2$. $\square$

Lemma 2.2. *Given q and $r_1 > 0$, we have*

$$\nu_q\bigl(b(w, r)\bigr) \sim \bigl(1 - |w|^2\bigr)^{N+1+q} \quad (w \in \mathbb{B},\ r \leqslant r_1).$$

Proof. Computing using (6), (4), and (5),

$$\begin{aligned}\nu_q\big(b(w,r)\big) &= \int_{b(w,r)} \big(1-|z|^2\big)^q \, d\nu(z) = \int_{\varphi_w(b_e(0,\tanh r))} \big(1-|z|^2\big)^q \, d\nu(z)\\ &= \int_{b_e(0,\tanh r)} \big(1-\big|\varphi_w(z)\big|^2\big)^q J_{\mathbb{R}}\varphi_w(z)\, d\nu(z)\\ &= \big(1-|w|^2\big)^{N+1+q} \int_{|z|<\tanh r} \frac{(1-|z|^2)^q}{|1-\langle z,w\rangle|^{2(N+1+q)}}\, d\nu(z).\end{aligned}$$

The last integral is equivalent to 1 since $\tanh r \leqslant \tanh r_1$. $\square$

Corollary 2.3. *Given q and $r_1, r_2, r_3, r_4, r_5 > 0$, we have*

$$\frac{\nu_q(b(z,r))}{\nu_q(b(w,\rho))} \sim 1$$

for all $r \leqslant r_1$, $\rho \leqslant r_2$, $r_3 \leqslant r/\rho \leqslant r_4$ and $z, w \in \mathbb{B}$ with $d(z,w) \leqslant r_5$.

Definition 2.4. A sequence $\{a_n\}$ in $\mathbb{B}$ is called *separated* (or *uniformly discrete*) if there is a constant $\tau > 0$, called the *separation constant*, such that $d(a_n, a_m) \geqslant \tau$ for all $n \neq m$.

The disc version of the following covering lemma is in [9, Lemma 3.5]. A sequence $\{a_n\}$ satisfying its conditions is called an *r-lattice* in $\mathbb{B}$ in the literature.

Lemma 2.5. *There is a positive integer M such that for any given r, there exists a sequence $\{a_n\}$ in $\mathbb{B}$ with $|a_n| \to 1$ satisfying the following conditions*:

(i) $\mathbb{B} = \bigcup_{n=1}^{\infty} b(a_n, r)$;
(ii) *$\{a_n\}$ is separated with separation constant $r/2$*;
(iii) *any point in $\mathbb{B}$ belongs to at most M of the balls $b(a_n, 2r)$*.

It is common to use Carleson windows in theorems and proofs on Carleson measures. These windows are extensions to $\mathbb{D}$ of arcs on $\partial\mathbb{D}$, and their higher-dimensional generalizations. The arcs are the balls of the natural metric on $\partial\mathbb{D}$, which is the natural domain for the Hardy spaces. However, when considering Bergman or Besov spaces on $\mathbb{D}$ and especially on $\mathbb{B}$, it is much more natural to use balls of the relevant metric, which is the Bergman metric. Certain details of proofs using Carleson windows involve a decomposition of $\mathbb{D}$ into windows that get smaller as they get closer to $\partial\mathbb{D}$. As a matter of fact, they do so in such a way that their size in the Bergman metric remain roughly fixed. In Lemma 2.5 instead, we use a decomposition of $\mathbb{B}$ into balls of a fixed radius that does the same job in a much less complicated manner.

Lastly we obtain two generalized subharmonicity properties with respect to each of the measures ν_Q on Bergman balls. The proofs given in [42] for $Q = 0$ work equally well for other Q too. A final use of Jensen inequality in the second extends the result to $p \neq 1$.

Lemma 2.6. *Given $Q \in \mathbb{R}$ and $r_1 > 0$, there is a constant C such that for all p, $g \in H(\mathbb{B})$, $w \in \mathbb{B}$, and $r \leqslant r_1$, we have*

$$\left|g(w)\right|^p \leqslant \frac{C}{\nu_Q(b(w,r))} \int\limits_{b(w,r)} |g|^p \, d\nu_Q.$$

Lemma 2.7. *Given q and $r_1 > 0$, there is a constant C such that for all p, positive Borel measure μ on $\mathbb{B}$, $w \in \mathbb{B}$, and $r \leqslant r_1$, we have*

$$\mu\big(b(w,r)\big)^p \leqslant \frac{C}{\nu_q(b(w,r))} \int\limits_{b(w,r)} \mu\big(b(z,r)\big)^p \, d\nu_q(z).$$

3. Besov spaces

There are several different ways to define Besov spaces on $\mathbb{B}$. All require one kind of derivative or another, but the easiest one to use is the radial derivative. The particular description started in [22] and continued in [23] suits best our interests. We review their relevant points here. Another major source of information is [11]. For comparison, our B_q^p space is their $A^p_{1+q+pt,t}$ space.

Let $f \in H(\mathbb{B})$ be given by its *homogeneous expansion* $f = \sum_{k=0}^{\infty} F_k$, where F_k is a homogeneous polynomial of degree k. Then its *radial derivative* at z is $\mathcal{R}f(z) = \sum_{k=1}^{\infty} kF_k(z)$. In [23, Definition 3.1], for any s, t, the radial differential operator D_s^t is defined on $H(\mathbb{B})$ by $D_s^t f = \sum_{k=0}^{\infty} {}_s^t d_k F_k$, where

$${}_s^t d_k = \begin{cases} \dfrac{(N+1+s+t)_k}{(N+1+s)_k}, & \text{if } s > -(N+1),\ s+t > -(N+1); \\ \dfrac{(N+1+s+t)_k(-(N+s))_{k+1}}{(k!)^2}, & \text{if } s \leqslant -(N+1),\ s+t > -(N+1); \\ \dfrac{(k!)^2}{(N+1+s)_k(-(N+s+t))_{k+1}}, & \text{if } s > -(N+1),\ s+t \leqslant -(N+1); \\ \dfrac{(-(N+s))_{k+1}}{(-(N+s+t))_{k+1}}, & \text{if } s \leqslant -(N+1),\ s+t \leqslant -(N+1). \end{cases}$$

What is important is that

$${}_s^t d_k \neq 0 \quad (k = 0, 1, 2, \ldots) \quad \text{and} \quad {}_s^t d_k \sim k^t \quad (k \to \infty) \tag{7}$$

for any s, t. It turns out that each D_s^t is a continuous invertible operator of order t on $H(\mathbb{B})$ with two-sided inverse

$$\big(D_s^t\big)^{-1} = D_{s+t}^{-t}. \tag{8}$$

Other useful properties are that D_s^0 is the identity for any s, $D_{-N}^1 = I + \mathcal{R}$, $D_{s+t}^u D_s^t = D_s^{u+t}$, $D_s^t(1) = {}_s^t d_0 > 0$, and $D_s^t(z^\lambda) = {}_s^t d_{|\lambda|} z^\lambda$.

The next result, reproduced from [23, Theorem 4.1], justifies Definition 1.1. It is equivalent to [11, Theorem 5.12(i)].

Proposition 3.1. *The space B^p_q is independent of the particular choice of s, t as long as* (1) *holds. The L^p_q norms of $I^{t_1}_{s_1} f$ and $I^{t_2}_{s_2} f$ are equivalent as long as* (1) *is satisfied by t_1 and t_2.*

So the norm in Definition 1.1 represents a whole family of equivalent norms. The same is true in B^2_q for the inner product

$$[f,g]_q = \int_{\mathbb{B}} I^t_s f \, \overline{I^t_s g} \, d\nu_q \tag{9}$$

with s, t satisfying (1) with $p = 2$.

Each B^2_q space is a reproducing kernel Hilbert space with reproducing kernel

$$K_q(z,w) = \begin{cases} \dfrac{1}{(1-\langle z,w\rangle)^{N+1+q}} = \displaystyle\sum_{k=0}^{\infty} \frac{(N+1+q)_k}{k!} \langle z,w\rangle^k, & \text{if } q > -(N+1); \\ \dfrac{{}_2F_1(1,1;1-N-q;\langle z,w\rangle)}{-N-q} = \displaystyle\sum_{k=0}^{\infty} \frac{k!\langle z,w\rangle^k}{(-N-q)_{k+1}}, & \text{if } q \leqslant -(N+1), \end{cases}$$

where ${}_2F_1$ is the hypergeometric function; see [11, p. 13]. Thus B^2_q spaces are nothing but Dirichlet spaces, B^2_{-1} the Hardy space H^2, B^2_{-N} the Arveson space $\mathcal{A}$ (see [2,6]), and $B^2_{-(N+1)}$ the classical Dirichlet space $\mathcal{D}$, the last due to the fact that $K_{-(N+1)}(z,w) = -\langle z,w\rangle^{-1} \log(1-\langle z,w\rangle)$.

Monomials $\{z^\lambda\}$ form a dense orthogonal set in B^2_q. Moreover, by (3),

$$K_q(z,w) \sim \sum_{k=0}^{\infty} k^{N+q} \langle z,w\rangle^k = \sum_{\lambda} \frac{|\lambda|^{N+q}|\lambda|!}{\lambda!} z^\lambda \overline{w}^\lambda \tag{10}$$

for any q, because

$$\langle z,w\rangle^k = \sum_{|\lambda|=k} \frac{k!}{\lambda!} z^\lambda \overline{w}^\lambda. \tag{11}$$

This shows that K_q is bounded for $q < -(N+1)$, and that for all q,

$$\|z^\lambda\|^2_{B^2_q} \sim \frac{\lambda!}{|\lambda|^{N+q}|\lambda|!} \quad (\lambda \in \mathbb{N}^N).$$

The reproducing property of K_q is that $[f, K_q(\cdot,w)]_q = Cf(w)$ with any s, t satisfying (1). Since $K_q(\cdot,w) \in B^2_q$ for any $w \in \mathbb{B}$, we also have

$$\|K_q(\cdot,w)\|^2_{B^2_q} = \big[K_q(\cdot,w), K_q(\cdot,w)\big]_q = CK_q(w,w) \tag{12}$$

with any s, t satisfying (1). Although the results on reproducing property follow directly from considerations in reproducing kernel Hilbert spaces, they can be checked as well by using the integral forms of the inner product and the norm and [32, Proposition 1.4.10]. Differentiation in the holomorphic variable z and the series expansion of K_q show that always

$$D_q^t K_q(z,w) = K_{q+t}(z,w).$$

Almost as easily, we have the following for the spaces; but a proof can be found in [25, Proposition 3.1].

Proposition 3.2. *For any q, p, s, t, $D_s^t(B_q^p) = B_{q+pt}^p$ is an isometric isomorphism under the equivalence of norms.*

Lemma 3.3. *Given q, p, s, t, there is a constant C such that if $f \in B_q^p$, then for $z \in \mathbb{B}$,*

$$\left|D_s^t f(z)\right| \leqslant C\|f\|_{B_q^p} \begin{cases} (1-|z|^2)^{-(N+1+q+pt)/p}, & \text{if } q > -(N+1+pt); \\ \log(1-|z|^2)^{-1}, & \text{if } q = -(N+1+pt); \\ 1, & \text{if } q < -(N+1+pt). \end{cases}$$

Proof. See [11, Lemma 5.6]. □

Definition 3.4. Extended *Bergman projections* are the linear transformations

$$P_s f(z) = \int_{\mathbb{B}} K_s(z,w) f(w)\, d\nu_s(w) \quad (z \in \mathbb{B})$$

defined for suitable f and all s.

The following result is contained in [23, Theorem 1.2].

Theorem 3.5. *For $1 \leqslant p < \infty$, P_s is a bounded operator from L_q^p onto B_q^p if and only if*

$$q + 1 < p(s+1). \tag{13}$$

Given an s satisfying (13)*, if t satisfies* (1)*, then*

$$\left(P_s \circ I_s^t\right) f = \frac{N!}{(1+s+t)_N} f \quad \left(f \in B_q^p\right).$$

Together (13) and (1) imply $s + t > -1$ so that $1 + s + t$ does not hit a pole of Γ. If $q > -1$, we can take $t = 0$, and Theorem 3.5 reduces to the classical result on Bergman spaces. When $p = \infty$ for fixed q, the inequalities (1) and (13) turn into

$$t > 0 \quad \text{and} \quad s > -1. \tag{14}$$

Then the spaces B_q^∞ are all the same and called the *Bloch space* $\mathcal{B}$, which is the space of all $f \in H(\mathbb{B})$ for which some $I_s^t f$ with $t > 0$ is bounded on $\mathbb{B}$. Its subspace the *little Bloch space*

$\mathcal{B}_0$ consists of those $f \in \mathcal{B}$ for which some $I_s^t f$ with $t > 0$ vanishes on $\partial\mathbb{B}$. The norm on these spaces is the *Bloch norm*

$$\|f\|_{\mathcal{B}} = \sup_{\mathbb{B}} \left| I_s^t f \right|$$

valid for any $t > 0$.

Theorem 3.6. *The operator P_s maps L^∞ boundedly onto $\mathcal{B}$ if and only if $s > -1$; and it maps either of $\mathcal{C}$ or $\mathcal{C}_0$ boundedly onto $\mathcal{B}_0$ if and only if $s > -1$. Given such an s, if also $t > 0$, then $(P_s \circ I_s^t) f = Cf$ for $f \in \mathcal{B}$, and hence for $f \in \mathcal{B}_0$.*

Proof. See [25, Theorem 5.3]. □

A consequence of Bergman projections is that for $1 \leqslant p < \infty$, the dual of B_q^p can be identified with $B_q^{p'}$, where $p' = p/(p-1)$ is the exponent conjugate to p, under each of the pairings

$$_q[f, g]_{s,q+t}^{t,-q+s} = \int_{\mathbb{B}} I_s^t f \, \overline{I_{q+t}^{-q+s} g} \, d\nu_q, \tag{15}$$

where s, t satisfy (13) and (1), or (14), and $f \in B_q^p$, $g \in B_q^{p'}$. Similarly, the dual of $\mathcal{B}_0$ can be identified with any B_q^1 under each of the same pairings with $f \in \mathcal{B}_0$, $g \in B_q^1$. The details can be found in [23, Section 7].

4. Compact operators and ultraweak convergence

This section has dual purpose. First we give a characterization of compact I_s^t acting on Besov spaces that leads to a little oh version of Theorem 1.3 for all p. Then we construct (ultra)weakly convergent families in Besov spaces that makes the proof of Theorem 1.3 possible. These are still normalized reproducing kernels, but the kernel and the normalization are of different spaces.

Definition 4.1. Let X and Y be F-spaces, that is, topological vector spaces whose topologies are induced by complete translation-invariant metrics. A linear operator $T : X \to Y$ is called *compact* if the images of balls of X under T have compact closures in Y.

Compactness of T is equivalent to that the image under T of a bounded sequence in X has a subsequence convergent in Y. We also know that if X and Y be Banach spaces and X is reflexive, a linear operator $T : X \to Y$ is compact if and only if $f_k \to 0$ weakly in X implies $\|T f_k\|_Y \to 0$.

The only F-spaces we consider that are not Banach spaces are $L^p(\mu)$ and B_q^p for $0 < p < 1$. For the latter, we have a family of equivalent invariant metrics $\|f - g\|_{B_q^p}^p$ for each s, t satisfying (1).

Extending a concept defined in [45, p. 61], we make the following definition.

Definition 4.2. Let s, t satisfy (1). A sequence $\{f_k\}$ converges (s,t)-*ultraweakly* to 0 in B_q^p if $\{\|f_k\|_{B_q^p}\}$ is bounded and $\{I_s^t f_k\}$ converges to 0 uniformly on compact subsets of $\mathbb{B}$.

The next result is essential for the proof of Theorem 5.3. A similar result holds for composition operators on similar spaces too; see [14, Proposition 3.11] and [35, Lemmas 3.7 and 3.8].

Theorem 4.3. *Let μ be a positive Borel measure on $\mathbb{B}$, and let s, t satisfy* (1). *The operator $I_s^t : B_q^p \to L^p(\mu)$ is compact if and only if for any sequence $\{f_k\}$ in B_q^p converging (s,t)-ultraweakly to* 0, *we have $\|I_s^t f_k\|_{L^p(\mu)} \to 0$.*

Proof. Suppose I_s^t is compact, and let $\{f_k\}$ converge (s,t)-ultraweakly to 0 in B_q^p. Assume that there is an $\varepsilon > 0$ and a subsequence $\{f_{k_j}\}$ such that $\|I_s^t f_{k_j}\|_{L^p(\mu)}^p \geqslant \varepsilon$ for all j. By the compactness of I_s^t, there is another subsequence $\{f_{k_{j_m}}\}$ such that $I_s^t f_{k_{j_m}} \to h$ in $L^p(\mu)$. And there is a further subsequence $\{f_{k_{j_{m_l}}}\}$ such that $I_s^t f_{k_{j_{m_l}}}(z) \to h(z)$ a.e. in $\mathbb{B}$. But $I_s^t f_k(z) \to 0$ for all $z \in \mathbb{B}$ by uniform convergence on compact subsets. Thus $h(z) = 0$ a.e. in $\mathbb{B}$ and $I_s^t f_{k_{j_m}} \to 0$ in $L^p(\mu)$. This contradicts the assumption.

Conversely, suppose $\{\|f_k\|_{B_q^p}^p\}$ is bounded. By Lemma 3.3, for all k and R with $0 < R < 1$, we have $\sup\{|D_s^t f_k(z)|^p \colon |z| \leqslant R\} \leqslant C\|f_k\|_{B_q^p}^p \leqslant C$. Hence $\{D_s^t f_k\}$ is a normal family and has a subsequence $\{f_{k_j}\}$ such that $D_s^t f_{k_j}$ converges uniformly on compact subsets of $\mathbb{B}$ to a function in $H(\mathbb{B})$ which we can take as $D_s^t f$ for some $f \in H(\mathbb{B})$. Then also $I_s^t f_{k_j} \to I_s^t f$ uniformly on compact subsets of $\mathbb{B}$. Then by Fatou lemma,

$$\int_{\mathbb{B}} |I_s^t f|^p \, d\nu_q = \int_{\mathbb{B}} \lim_{j\to\infty} |I_s^t f_{k_j}|^p \, d\nu_q \leqslant \liminf_{j\to\infty} \int_{\mathbb{B}} |I_s^t f_{k_j}|^p \, d\nu_q = \liminf_{j\to\infty} \|f_{k_j}\|_{B_q^p}^p \leqslant C,$$

which implies $f \in B_q^p$. Thus $\{f_{k_j} - f\}$ is a sequence converging (s,t)-ultraweakly to 0 in B_q^p. It follows that $\|I_s^t(f_{k_j} - f)\|_{L^p(\mu)}^p \to 0$ and $\{I_s^t f_{k_j}\}$ converges in $L^p(\mu)$. □

Considering the characterization of compactness on reflexive spaces, the following result is no surprise. It applies to weighted Bergman spaces by taking $q > -1$ and $t = 0$. But we can take other s, t as long as they satisfy (13) and (1) also with $q > -1$. Thus we obtain some new conditions for weak convergence on weighted Bergman spaces equivalent to the known ones.

Theorem 4.4. *For $1 < p < \infty$, a sequence $\{f_k\}$ converges to* 0 *weakly in B_q^p if and only if it converges (s,t)-ultraweakly to* 0 *in B_q^p for some s, t satisfying* (13) *and* (1).

Proof. Suppose $\{f_k\}$ converges (s,t)-ultraweakly to 0 with s, t of the form given. Then $D_s^t f_k \to 0$ uniformly on compact subsets of $\mathbb{B}$. Since functions with compact support are dense in L_q^p, it suffices to consider the following. Let $0 < R < 1$, χ be the characteristic function of the Euclidean ball $b_{\mathrm{e}}(0,R)$, and $g = P_{q+t}\chi$. Now (13) is satisfied with $q+t$ and p' replacing s and p because of (1); hence $g \in B_q^{p'}$ by Theorem 3.5. Then by (15), differentiation under the integral, and Fubini theorem, we obtain

$$\begin{aligned}
{}_q[f_k, g]_{s,q+t}^{t,-q+s} &= \int_{\mathbb{B}} I_s^t f_k \overline{I_{q+t}^{-q+s} g} \, d\nu_q \\
&= \int_{\mathbb{B}} I_s^t f_k(z) \big(1 - |z|^2\big)^s \int_{\mathbb{B}} \frac{(1-|w|^2)^{q+t} \chi(w)}{(1 - \langle w, z\rangle)^{N+1+s+t}} \, d\nu(w) \, d\nu(z)
\end{aligned}$$

$$= \int_{\mathbb{B}} \left(1-|w|^2\right)^{q+t} \chi(w) \int_{\mathbb{B}} \frac{(1-|z|^2)^{s+t}}{(1-\langle w,z\rangle)^{N+1+s+t}} D_s^t f_k(z)\, d\nu(z)\, d\nu(w)$$

$$= \int_{|w|<R} \left(1-|w|^2\right)^{q+t} P_{s+t} D_s^t f_k(w)\, d\nu(w).$$

Now by Proposition 3.2, $D_s^t f_k \in B^p_{q+pt} = A^p_{q+pt}$, and hence $P_{s+t}(D_s^t f_k) = C D_s^t f_k$ by Theorem 3.5. Then

$${}_q[f_k, g]^{t,-q+s}_{s,q+t} = C \int_{|w|<R} \left(1-|w|^2\right)^{q+t} D_s^t f_k(w)\, d\nu(w) = C \int_{|w|<R} I_s^t f_k(w)\, d\nu_q(w).$$

Thus $|{}_q[f_k, g]^{t,-q+s}_{s,q+t}| \leqslant C \sup\{|I_s^t f_k(w)|\colon |w| \leqslant R\}$, and $f_k \to 0$ weakly.

Suppose $f_k \to 0$ weakly in B^p_q, and s, t satisfy (13) and (1). Then $\{\|f_k\|_{B^p_q}\}$ is bounded. Lemma 3.3 yields that $\sup\{|D_s^t f_k(z)|\colon |z| \leqslant R\} \leqslant C\|f_k\|_{B^p_q} \leqslant C$ for all k and R with $0 < R < 1$. Then $\{D_s^t f_k\}$ is a normal family and has a subsequence $\{D_s^t f_{k_j}\}$ that converges uniformly on compact subsets. Putting $h_k = I_s^t f_k$, this forces $\{h_{k_j}\}$ also to converge uniformly on compact subsets, say, to h. But f_{k_j} then converges weakly to $f \equiv 0$ and $h = I_s^t f$. Hence $h \equiv 0$. If $\{h_k\}$ had another subsequence $\{h_{k_l}\}$ that stayed bounded away from 0, then since $f_{k_l} \to 0$ weakly, this subsequence would in turn yield a subsubsequence $\{h_{k_{l_m}}\}$ as above that would converge uniformly on compact subsets to 0, contradicting the defining property of $\{h_{k_l}\}$. Therefore the full sequence $h_k \to 0$ uniformly on compact subsets of $\mathbb{B}$. □

Theorem 4.5. *A sequence $\{g_k\}$ converges to 0 weak-* in $B^1_q = (\mathcal{B}_0)^*$ if and only if it converges (s,t)-ultraweakly to 0 in B^1_q for some s, t satisfying* (13) *and* (1) *with $p = 1$. A sequence $\{g_k\}$ converges to 0 weak-* in $\mathcal{B} = (B^1_q)^*$ if and only if it converges (s,t)-ultraweakly to 0 in $\mathcal{B}$ for some s, t satisfying* (14).

Proof. The only differences from the proof of Theorem 4.4 are that we use a continuous χ for the first statement and Theorem 3.6 for the second statement. □

Example 4.6. It is well known [43, Section 6.1] that if $q > -1$, then the normalized reproducing kernels $g_w(z) = K_q(z,w)/\|K_q(\cdot,w)\|_{A^2_q}$ converge to 0 weakly as $|w| \to 1$ in the Bergman spaces A^2_q. More generally, $g_w^{2/p}$ converges to 0 as $|w| \to 1$ weakly in A^p_q for $p > 1$ and weak-* in A^1_q.

In Besov spaces B^p_q with $-(N+1) < q \leqslant -1$ when the associated reproducing kernel is binomial, the same idea gives ultraweakly 0-convergent families. We show the details, because derivatives have to be taken care of in the computations of norms. By (12) we have

$$g_w(z) \sim \left(\frac{1-|w|^2}{(1-\langle z,w\rangle)^2}\right)^{(N+1+q)/p}$$

$$\sim \left(1-|w|^2\right)^{(N+1+q)/p} \sum_{k=0}^{\infty} k^{(N+1+q)2/p-1} \langle z,w\rangle^k \quad \left(|w| \to 1\right).$$

If s, t satisfy (1), then by (7) and (10),

$$I_s^t g_w(z) \sim \left(1-|w|^2\right)^{(N+1+q)/p}\left(1-|z|^2\right)^t \sum_{k=0}^{\infty} k^{(N+1+q)2/p-1+t}\langle z, w\rangle^k$$
$$\sim \frac{(1-|w|^2)^{(N+1+q)/p}(1-|z|^2)^t}{(1-\langle z, w\rangle)^{(N+1+q)2/p+t}} \quad \left(|w| \to 1\right).$$

So if $|z| \leqslant R < 1$, then $|I_s^t g_w(z)| \leqslant C(1-|w|^2)^{(N+1+q)/p} \to 0$ as $|w| \to 1$. Further

$$\|g_w\|_{B_q^p}^p = \int_{\mathbb{B}} \left|I_s^t g_w\right|^p d\nu_q \sim \left(1-|w|^2\right)^{N+1+q} \int_{\mathbb{B}} \frac{(1-|z|^2)^{q+pt}}{|1-\langle z, w\rangle|^{(N+1+q)2+pt}}\, d\nu(z)$$
$$\sim 1 \quad \left(|w| \to 1\right)$$

by [32, Proposition 1.4.10]. Thus $g_w \to 0$ as $|w| \to 1$ (s,t)-ultraweakly.

In particular, the normalized reproducing kernels of the Hardy space $H^2 = B_{-1}^2$ and the Arveson space $\mathcal{A} = B_{-N}^2$ are weakly 0-convergent families in their own spaces.

Even when $q = -(N+1)$, when the associated reproducing kernel is logarithmic and hence unbounded, the same procedure gives weakly 0-convergent families in $B_{-(N+1)}^p$, but seems unlikely to work for $q < -(N+1)$ when the reproducing kernels are bounded. We need a modification.

Example 4.7. We now explicitly construct ultraweakly 0-convergent families in all Besov spaces B_q^p. Our construction works in Bergman spaces too and gives us such families that are not necessarily normalized reproducing kernels.

Fix q. Let t satisfy (1); then also $N+1+q+pt > 0$. Pick complex numbers c_k such that $c_k \sim k^{(N+1+q+pt)2/p-1-t}$ as $k \to \infty$, and put

$$f_w(z) = \sum_{k=0}^{\infty} c_k \langle z, w\rangle^k.$$

Similar to Example 4.6,

$$I_s^t f_w(z) \sim \frac{(1-|z|^2)^t}{(1-\langle z, w\rangle)^{(N+1+q+pt)2/p}} \quad \left(|w| \to 1\right). \tag{16}$$

If $|z| \leqslant R < 1$, then $|I_s^t f_w(z)| \leqslant C$ for any $w \in \mathbb{B}$. Again similar to Example 4.6,

$$\|f_w\|_{B_q^p}^p \sim \frac{1}{(1-|w|^2)^{N+1+q+pt}}. \tag{17}$$

Set $g_w(z) = f_w(z)/\|f_w\|_{B_q^p}$ so that each $\|g_w\|_{B_q^p} = 1$. Moreover, if $|z| \leqslant R$, then we have $|I_s^t g_w(z)| \leqslant C(1-|w|^2)^{(N+1+q+pt)/p} \to 0$ as $|w| \to 1$. The (s,t)-ultraweak convergence follows.

Remark 4.8. Consider the case of a Hilbert space, $p = 2$, in Example 4.7. Let s satisfy (13) in which case $t = -q + s$ satisfies (1) since $-q + 2s > -1$. Then $c_k \sim k^{N+s}$, and by (10) we can take $f_w(z) = K_s(z, w) = D_q^{-q+s} K_q(z, w)$ in B_q^2. Thus

$$g_w(z) = \frac{K_s(z, w)}{\|K_s(\cdot, w)\|_{B_q^2}} = \sqrt{\frac{(1-q+2s)_{N+1}}{N!}} \left(1 - |w|^2\right)^{(N+1-q+2s)/2} K_s(z, w)$$
$$\sim \frac{K_s(z, w)}{\sqrt{K_{-q+2s}(w, w)}} \in B_q^2$$

is a normalized reproducing kernel indeed, but it is the kernel of B_s^2 normalized so that its B_q^2 norm is 1, and is considered an element of B_q^2. The second equality above follows from the proof of [32, Proposition 1.4.10] using $t = -q + s$. It is interesting that

$$g_w(z) = D_{-q+2s}^{q-s} \frac{K_{-q+2s}(z, w)}{\|K_{-q+2s}(\cdot, w)\|_{B_{-q+2s}^2}}.$$

It is possible to take $s = q$ if and only if $q > -1$, the Bergman-space case. For $q \leqslant -1$, (13) requires $s > q$. For such q, $s = -q$ works for $p \geqslant 1$ and $s = 0$ works for all p.

Specifically, the Bergman kernel $K_0(\cdot, w)$ is a weakly 0-convergent family in H^2 or $\mathcal{A}$ as $|w| \to 1$ when it is normalized by dividing it by its norm in H^2 or $\mathcal{A}$.

It is easy to see that f_w is the kernel in B_q^2 for the evaluation of the derivative $D_s^{-q+s} f$ of $f \in B_q^2$ at $w \in \mathbb{B}$ in the sense that $[f, f_w]_q = C D_s^{-q+s} f(w)$, where $C = N!/(1-q+2s)_N$ when $[\cdot,\cdot]_q = {}_q[\cdot,\cdot]_{s,s}^{-q+s,-q+s}$, and g_w is this kernel normalized in B_q^2. A similar weak-convergence result can be found in [14, Proposition 7.13].

Example 4.9. We lastly obtain weak-$*$ 0-convergent families in the Bloch space $\mathcal{B}$. Let s and t satisfy (14), pick $c_k \sim k^{t-1}$ as $k \to \infty$, and define f_w as in Example 4.7. Then we have $\|f_w\|_{\mathcal{B}} \leqslant C(1 - |w|^2)^{-2t}$. Setting $g_w(z) = f_w(z)/\|f_w\|_{\mathcal{B}}$, we obtain that $g_w \to 0$ weak-$*$ in $\mathcal{B}$ as $|w| \to 1$ by Theorem 4.5 as in Example 4.7. By taking t close to 0, we find families $\{g_w\}$ in $\mathcal{B}$ that converge weak-$*$ to 0 arbitrarily slowly.

5. Carleson measures and separated sequences

In this section we prove Theorems 1.3 and 1.5 and their little oh counterparts on vanishing Carleson measures, and relate their conditions to Berezin transforms and averaging functions. We also prove an associated result on Schatten ideal criteria for I_s^t. Our results naturally extend well-known results on q-Carleson measures for $q > -1$ on weighted Bergman spaces in two directions; $q \leqslant -1$, and for $q > -1$, imbeddings that are not inclusion. They are readily applied to separated sequences.

Lemma 5.1. *Let q, r, also $\alpha, \beta \in \mathbb{R}$, and a positive Borel measure μ on $\mathbb{B}$ be given. Then*

$${}_q\hat{\mu}_r(w) \leqslant U_{\alpha,\beta,q}\mu(w) \quad (w \in \mathbb{B}),$$

where $U_{\alpha,\beta,q}\mu$ is defined in Theorem 1.5.

Proof. By Lemmas 2.2 and 2.1,

$$
\begin{aligned}
{}_q\hat{\mu}_r(w) &\sim \frac{1}{(1-|w|^2)^{N+1+q}} \int\limits_{b(w,r)} d\mu \\
&\sim \left(1-|w|^2\right)^{\alpha} \int\limits_{b(w,r)} \frac{(1-|z|^2)^{\beta}}{|1-\langle z,w\rangle|^{N+1+\alpha+\beta+q}}\, d\mu(z) \\
&\leqslant \left(1-|w|^2\right)^{\alpha} \int\limits_{\mathbb{B}} \frac{(1-|z|^2)^{\beta}}{|1-\langle z,w\rangle|^{N+1+\alpha+\beta+q}}\, d\mu(z) = U_{\alpha,\beta,q}\mu(w)
\end{aligned}
$$

for every $w \in \mathbb{B}$. $\square$

After all the preparation, the proof of our main theorem goes very smoothly.

Proof of Theorem 1.3. (i) $\Rightarrow$ (ii). There is nothing to prove.

(ii) $\Rightarrow$ (iii). Suppose (ii) holds for some r. We start by applying Lemma 2.5(i) to an integral of $I_s^t f$.

$$
\begin{aligned}
\int\limits_{\mathbb{B}} \left|I_s^t f\right|^p d\mu &\leqslant \sum_{n=1}^{\infty} \int\limits_{b(a_n,r)} \left|I_s^t f\right|^p d\mu \\
&\leqslant \sum_{n=1}^{\infty} \mu\big(b(a_n,r)\big) \sup\big\{\left|I_s^t f(w)\right|^p\colon\ w \in b(a_n,r)\big\}.
\end{aligned}
$$

If $w \in b(a_n,r)$, we apply Lemma 2.6 with $g = D_s^t f$ and $Q = q + pt$ bearing in mind Lemma 2.2 to obtain

$$
\left(1-|w|^2\right)^{pt} \left|D_s^t f(w)\right|^p \leqslant \frac{C}{\nu_q(b(w,r))} \int\limits_{b(w,r)} \left(1-|z|^2\right)^{pt} \left|D_s^t f(z)\right|^p d\nu_q(z).
$$

Then by Corollary 2.3,

$$
\begin{aligned}
\left|I_s^t f(w)\right|^p &\leqslant \frac{C}{\nu_q(b(a_n,r))} \frac{\nu_q(b(a_n,r))}{\nu_q(b(w,r))} \int\limits_{b(w,r)} \left|I_s^t f\right|^p d\nu_q \\
&\leqslant \frac{C}{\nu_q(b(a_n,r))} \int\limits_{b(a_n,2r)} \left|I_s^t f\right|^p d\nu_q,
\end{aligned}
$$

because $b(w,r) \subset b(a_n,2r)$. The right-hand side is now independent of $w \in b(a_n,r)$, and we can take the supremum of the left-hand side on all such w. By assumption (ii) and Lemma 2.5(iii), it follows that

$$\int_{\mathbb{B}} \left|I_s^t f\right|^p d\mu \leqslant C \sum_{n=1}^{\infty} \frac{\mu(b(a_n, r))}{\nu_q(b(a_n, r))} \int_{b(a_n,2r)} \left|I_s^t f\right|^p d\nu_q$$

$$\leqslant C \sum_{n=1}^{\infty} \int_{b(a_n,2r)} \left|I_s^t f\right|^p d\nu_q \leqslant CM \int_{\mathbb{B}} \left|I_s^t f\right|^p d\nu_q,$$

and we obtain (iii).

(iii) $\Rightarrow$ (iv). Suppose (iii) holds for some p. We obtain this implication by picking a special f, one for each $w \in \mathbb{B}$, namely, $f = g_w$ of Example 4.7. Equations (16) and (17), and assumption (iii) imply that

$$\left(1 - |w|^2\right)^{N+1+q+pt} \int_{\mathbb{B}} \frac{(1 - |z|^2)^{pt}}{|1 - \langle z, w\rangle|^{(N+1+q+pt)2}} \, d\mu(z)$$

$$\leqslant C \int_{\mathbb{B}} \left|I_s^t g_w(z)\right|^p d\mu(z) \leqslant C \|g_w\|^p_{B^p_q} \leqslant C,$$

which is (iv).

(iv) $\Rightarrow$ (i). This is covered by Lemma 5.1 by picking $\alpha = N + 1 + q + pt$ and $\beta = pt$. □

Definition 5.2. We call a Carleson measure for B^p_q a *vanishing Carleson measure for* B^p_q whenever some I_s^t mapping B^p_q into $L^p(\mu)$ is further compact.

Theorem 5.3. *Let q be fixed. Let p and r, and also s be given. The following are equivalent for a positive Borel measure μ on $\mathbb{B}$.*

(i) *It holds that*

$$\lim_{|w|\to 1} \frac{\mu(b(w, r))}{\nu_q(b(w, r))} = 0.$$

(ii) *If $\{a_n\}$ is an r-lattice in $\mathbb{B}$, then*

$$\lim_{n\to\infty} \frac{\mu(b(a_n, r))}{\nu_q(b(a_n, r))} = 0.$$

(iii) *The measure μ is a vanishing Carleson measure for B^p_q with respect to I_s^t, where t satisfies* (1).

(iv) *If t satisfies* (1), *then*

$$\lim_{|w|\to 1} \left(1 - |w|^2\right)^{N+1+q+pt} \int_{\mathbb{B}} \frac{(1 - |z|^2)^{pt}}{|1 - \langle z, w\rangle|^{(N+1+q+pt)2}} \, d\mu(z) = 0.$$

Proof. (ii) $\Rightarrow$ (iii). Suppose (ii) holds for some r. Let $\{f_k\}$ be a sequence in B_q^p converging ultraweakly to 0. Then $\{\|f_k\|_{B_q^p}\}$ is bounded, and $I_s^t f_k \to 0$ uniformly on compact subsets of $\mathbb{B}$. Let $\varepsilon > 0$. By assumption (ii), there is an n_0 such that

$$\frac{\mu(b(a_n, r))}{\nu_q(b(a_n, r))} < \varepsilon \quad (n > n_0).$$

Then as in the proof of the corresponding implication of Theorem 1.3, for all k,

$$\begin{aligned}\sum_{n=n_0+1}^{\infty} \int_{b(a_n,2r)} \left|I_s^t f_k\right|^p d\mu &\leqslant C\varepsilon \sum_{n=n_0+1}^{\infty} \int_{b(a_n,2r)} \left|I_s^t f_k\right|^p d\nu_q \\ &\leqslant CM\varepsilon \int_{\mathbb{B}} \left|I_s^t f_k\right|^p d\nu_q \leqslant CM\varepsilon \|f_k\|_{B_q^p}^p \leqslant CM\varepsilon.\end{aligned}$$

On the other hand,

$$\lim_{k\to\infty} \sum_{n=1}^{n_0} \int_{b(a_n,2r)} \left|I_s^t f_k\right|^p d\mu = 0$$

by uniform convergence on compact subsets. Then

$$\lim_{k\to\infty} \int_{\mathbb{B}} \left|I_s^t f_k\right|^p d\mu = \lim_{k\to\infty} \left\|I_s^t f_k\right\|_{L^p(\mu)}^p \leqslant CM\varepsilon$$

by Lemma 2.5(i). Since $\varepsilon > 0$ is arbitrary, this is (iii) by Theorem 4.3.

The proofs of the implications (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) are entirely similar to the proofs of the corresponding implications of Theorem 1.3 and are omitted. $\square$

So for fixed q, a vanishing Carleson measure is independent of p, r, the r-lattice, and s, t satisfying (1). And we call such a μ also a *vanishing q-Carleson measure*. So a vanishing q-Carleson measure is a positive Borel measure on $\mathbb{B}$ for which the averaging function ${}_q\mu_r(w)$ has limit 0 as $|w| \to 1$ for some r. No ν_q or ν_{q_2} with $q_2 < q$ is a vanishing q-Carleson measure, but any ν_{q_1} with $q_1 > q$ is by Lemma 2.2.

Remark 5.4. The discussion after Theorem 1.3 yields that $I_s^t : B_q^p \to L_{q_1}^p$ with s, t satisfying (1) is bounded if and only if $q_1 \geqslant q$. The previous discussion now yields that $I_s^t : B_q^p \to L_{q_1}^p$ with s, t satisfying (1) is compact if and only if $q_1 > q$. That $I_s^t : B_q^p \to L_q^p$ cannot be compact is also clear from the fact that it is an isometry and L_q^p is not finite-dimensional.

Corollary 5.5. *A positive Borel measure μ on $\mathbb{B}$ is a q-Carleson* (*respectively, vanishing q-Carleson*) *measure if and only if μ_Q is a $(q+Q)$-Carleson* (*respectively, vanishing $(q+Q)$-Carleson*) *measure.*

Corollary 5.6. *If μ is a q-Carleson (respectively, vanishing q-Carleson) measure, then μ is a Q-Carleson (respectively, vanishing Q-Carleson) measure too for any $Q \leqslant q$. Equivalently, if μ is a q-Carleson (respectively, vanishing q-Carleson) measure and $Q \geqslant 0$, then μ_Q is also a q-Carleson (respectively, vanishing q-Carleson) measure.*

Proof. Both corollaries follow from Theorems 1.3(i) and 5.3(i) and Lemma 2.2. □

Definition 5.7. We call an operator $\mathbf{B}_q^u$ that takes a function f on $\mathbb{B}$ to

$$\mathbf{B}_q^u f(w) = \left(1-|w|^2\right)^{N+1+u} \int\limits_{\mathbb{B}} \frac{(1-|z|^2)^{-q+u}}{|1-\langle z,w\rangle|^{(N+1+u)2}} f(z)\,d\nu_q(z) \quad (w \in \mathbb{B})$$

or a measure μ on $\mathbb{B}$ to

$$\mathbf{B}_q^u \mu(w) = \left(1-|w|^2\right)^{N+1+u} \int\limits_{\mathbb{B}} \frac{(1-|z|^2)^{-q+u}}{|1-\langle z,w\rangle|^{(N+1+u)2}}\,d\mu(z) \quad (w \in \mathbb{B})$$

a *Berezin transform.*

The Berezin transform $\mathbf{B}_q^u f$ makes sense for $f \in L_u^1$, for example. The $\mathbf{B}_\alpha$ defined on functions for $\alpha > -1$ in [19, Section 2.1] is the $\mathbf{B}_\alpha^\alpha$ defined here. We use two parameters in order to accommodate measures and values of $q \leqslant -1$.

Now Theorems 1.3 and 5.3 on Carleson measures can be reformulated in terms of Berezin transforms and averaging functions.

Theorem 5.8. *Fix q. Let r and an r-lattice $\{a_n\}$ in $\mathbb{B}$, p and s, t satisfying* (1), *and also $u > -1$ be given. The following conditions are equivalent for a positive Borel measure μ on $\mathbb{B}$.*

(i) *The measure μ is a q-Carleson (vanishing q-Carleson, respectively) measure; that is, the averaging function ${}_q\hat{\mu}_r$ is bounded on $\mathbb{B}$ (in $\mathcal{C}_0$, respectively).*
(ii) *The sequence $\{{}_q\hat{\mu}_r(a_n)\}$ is bounded (has limit 0, respectively).*
(iii) *The operator $I_s^t : B_q^p \to L^p(\mu)$ is bounded (compact, respectively).*
(iv) *The Berezin transform $\mathbf{B}_q^u\mu$ is bounded on $\mathbb{B}$ (in $\mathcal{C}_0$, respectively).*

Theorem 1.5 at first sight does not seem to offer anything new other than rewriting the equivalence of (iv) and (i) of Theorem 1.3 using the new parameters $\alpha = N+1+q+pt$ and $\beta = pt$. However, Theorem 1.3(iv) has the restriction $\alpha > N$, and lowering this to $\alpha > 0$ needs some work.

Proof of Theorem 1.5. One direction is again covered by Lemma 5.1.

Conversely, suppose μ is a q-Carleson measure, that is, ${}_q\hat{\mu}_r$ is bounded for some r. We follow the proof of (ii) ⇒ (iii) of Theorem 1.3; only now it is simpler. We take $p = N+1+\alpha+\beta+q$, $Q = \beta + q$, and recall that Lemma 2.6 applies to $g_w(z) = (1-\langle z,w\rangle)^{-1} \in H(\mathbb{B})$ too. Note that $p > 0$. By Corollary 5.5, μ_β is a $(\beta + q)$-Carleson measure. After doing the usual trick of replacing this measure by $\nu_{\beta+q}$ with the help of the open cover in Lemma 2.5, we obtain

$$\int_{\mathbb{B}} |g_w|^p \, d\mu_\beta \leqslant CM \int_{\mathbb{B}} \frac{(1 - |\zeta|^2)^{\beta+q}}{|1 - \langle \zeta, w\rangle|^{N+1+\alpha+\beta+q}} \, d\nu(\zeta) \sim \frac{1}{(1 - |w|^2)^\alpha}$$

since $\alpha > 0$, where we have used [32, Proposition 1.4.10], which requires $\beta + q > -1$. This is equivalent to the boundedness of $U_{\alpha,\beta,q}\mu$. $\square$

Corollary 5.9. *Let μ be a positive Borel measure on $\mathbb{B}$. If $U_{\alpha,\beta,q}\mu(w)$ has limit 0 as $|w| \to 1$ for some real α, β, and q, then μ is a vanishing q-Carleson measure. If μ is a vanishing q-Carleson measure, $\alpha > 0$, and $\beta + q > -1$, then $U_{\alpha,\beta,q}\mu(w)$ has limit 0 as $|w| \to 1$.*

Theorem 1.5 not only provides a general description of q-Carleson measures, but also generalizes the case $c > 0$ of [32, Proposition 1.4.10] from the Lebesgue measure to arbitrary positive Borel measures on $\mathbb{B}$. See [33] for some other generalizations. Versions of Theorem 1.5 and Corollary 5.9 using Carleson windows when $N = 1$ are in [31, Proposition 2.1] with additional restrictions such as $\beta > -1$ and $q > 0$.

We give an early application to separated sequences. Here the *counting function* $n_r^Z(w)$ counts the number of points of a sequence $Z = \{z_n\}$ that happens to fall in the ball $b(w, r)$, and δ_a denotes the unit point mass at a.

Theorem 5.10. *Let q, r, $\alpha > 0$, β with $\beta + q > -1$ be given. The following are equivalent for a sequence $Z = \{z_n\}$ of distinct points in $\mathbb{B}$.*

(i) *The sequence Z is a disjoint union of finitely many separated sequences.*
(ii) *The counting function $n_r^Z(w)$ is bounded in $\mathbb{B}$.*
(iii) *The measure $\mu = \sum_{n=1}^\infty (1 - |z_n|^2)^{N+1+q} \delta_{z_n}$ is a q-Carleson measure.*
(iv) *There is a constant C such that*

$$\sup_{w \in \mathbb{B}} \bigl(1 - |w|^2\bigr)^\alpha \sum_{n=1}^\infty \frac{(1 - |z_n|^2)^\beta}{|1 - \langle z_n, w\rangle|^{\alpha+\beta}} \leqslant C.$$

(v) *There is a constant C such that*

$$\sup_m \bigl(1 - |z_m|^2\bigr)^\alpha \sum_{m \neq n=1}^\infty \frac{(1 - |z_n|^2)^\beta}{|1 - \langle z_n, z_m\rangle|^{\alpha+\beta}} \leqslant C.$$

This theorem is contained in [21, Lemma 4.1] to a large extent. What is new here is that we remove restrictions such as $q > -1$ and weaken others such as $\beta > N$ present in this reference. We also aim to show that the proof requires almost no extra effort once we have strong results for q-Carleson measures for all q. As usual, the equivalences are independent of the values of q, r, α, β under the conditions in the statement of the theorem, and also of the values of parameters like p, s, t under (1) that would come from the equivalences in Theorem 1.3.

Proof. (i) $\Rightarrow$ (iii). Let first Z be one separated sequence with separation constant τ. Let $f \in B_q^p$. As in the proof of (ii) $\Rightarrow$ (iii) of Theorem 1.3, applying Lemma 2.6 with the choice $g = D_s^t f$ and $Q = q + pt > -1$ on the disjoint balls $b(z_n, \tau/2)$ gives

$$\left(1 - |z_n|^2\right)^{N+1+q+pt} \left|D_s^t f(z_n)\right|^p \leqslant C \int\limits_{b(z_n,\tau/2)} \left(1 - |z|^2\right)^{pt} \left|D_s^t f(z)\right|^p \, d\nu_q(z).$$

Summing on n yields

$$\sum_{n=1}^{\infty} \left|I_s^t f(z_n)\right|^p \mu(z_n) \leqslant C \|f\|_{B_q^p}^p.$$

In general, if Z is a union of finitely many separated sequences, we add them on the left and reach the same conclusion.

(iii) $\Rightarrow$ (iv). This is one direction of Theorem 1.5 applied to our measure μ.

(iv) $\Rightarrow$ (v). This is obvious.

(v) $\Rightarrow$ (ii). Take an arbitrary $b(w, r)$. By assumption and Lemma 2.1, we have

$$C \geqslant \sup_{z_m \in b(w,r)} \left(1 - |z_m|^2\right)^{\alpha} \sum_{z_n \in b(w,r)} \frac{(1 - |z_n|^2)^{\beta}}{|1 - \langle z_n, z_m \rangle|^{\alpha+\beta}} \sim \sum_{z_n \in b(w,r)} 1 = n_r^Z(w).$$

(i) $\Leftrightarrow$ (ii). This is in [16, Section 2.11]. $\square$

Separated sequences can be viewed as one way of constructing q-Carleson measures for any q. Here is another construction.

Example 5.11. Suppose μ is a positive Borel measure on $\mathbb{B}$ and μ_β is finite for some real β. Assume that the monomials z^{λ_1} and z^{λ_2} are orthogonal in the space $L^2(\mu_\beta)$ for $\lambda_1 \neq \lambda_2$. Rotation invariance of μ would imply this for example. Put $\kappa = (N + 1 + \alpha + \beta + q)/2$ and assume also $\kappa > 0$ for some real α and q. (The case $\kappa \leqslant 0$ can likewise be investigated.) Then using an expansion like the one in (10), the orthogonality assumption, and (11), we have

$$\begin{aligned} \frac{U_{\alpha,\beta,q}\mu(w)}{(1 - |w|^2)^{\alpha}} &= \int\limits_{\mathbb{B}} \frac{d\mu_\beta(z)}{|(1 - \langle z, w \rangle)^{\kappa}|^2} = \sum_{\lambda} \frac{((\kappa)_{|\lambda|})^2}{(\lambda!)^2} \left|w^\lambda\right|^2 \int\limits_{\mathbb{B}} \left|z^\lambda\right|^2 d\mu_\beta(z) \\ &= \sum_{k=0}^{\infty} \frac{((\kappa)_k)^2}{(k!)^2} \sum_{|\lambda|=k} \frac{(k!)^2}{(\lambda!)^2} \left|w^\lambda\right|^2 \int\limits_{\mathbb{B}} \left|z^\lambda\right|^2 d\mu_\beta(z) \\ &\leqslant \sum_{k=0}^{\infty} \frac{((\kappa)_k)^2}{(k!)^2} |w|^{2k} \sum_{|\lambda|=k} \frac{k!}{\lambda!} \int\limits_{\mathbb{B}} \left|z^\lambda\right|^2 d\mu_\beta(z) = \sum_{k=0}^{\infty} \frac{((\kappa)_k)^2}{(k!)^2} m_k |w|^{2k}, \end{aligned} \tag{18}$$

where $m_k = \int_{\mathbb{B}} |z|^{2k}\, d\mu_\beta(z)$ is the (k,k) "moment" of μ_β. Assume further, for some real η, that $m_k \leqslant C\, k^{-(N+\beta+q+\eta)}$. Then by (3), (18) is

$$\sim \sum_{k=1}^{\infty} k^{2\kappa-2} k^{-(N+\beta+q+\eta)} |w|^{2k} \sim \sum_{k=1}^{\infty} k^{\alpha-\eta-1} |w|^{2k}. \tag{19}$$

If one of the three pairs of inequalities $0 \leqslant \eta < \alpha$, $0 < \eta = \alpha$, $0 \leqslant \eta > \alpha$ is satisfied, then $U_{\alpha,\beta,q}\,\mu$ is bounded by (10). Note that (19) is binomial in the first case, logarithmic in the second, and bounded in the third. Therefore μ is a q-Carleson measure by Theorem 1.5.

For the model q-Carleson measure ν_q, we have $\eta = 0$ by [23, Proposition 2.1], and we could choose $\alpha = \beta + q = -1/2$ to get $\kappa > 0$. So this example is instructive in showing that the sufficiency part of Theorem 1.5 can hold as stated without the conditions on α or β required by the proof of the necessity part.

Our final purpose in this section is to investigate the conditions under which the Hilbert space operator $I_s^t : B_q^2 \to L^2(\mu)$ belongs to the *Schatten–von Neumann ideal* $\mathcal{S}^c$. We refer to [29, Chapter 16] for definitions and basic properties of singular numbers and Schatten ideals of operators from one Hilbert space into another. Recall that $\nu_{-(N+1)}$ is the Möbius-invariant measure on $\mathbb{B}$; see [32, Section 2.2].

Theorem 5.12. *Fix q. Let r and an r-lattice $\{a_n\}$ in $\mathbb{B}$, t satisfying* (1) *with $p = 2$, $u > -1$, and also $1 \leqslant c < \infty$ be given. The following conditions are equivalent for a q-Carleson measure μ on $\mathbb{B}$.*

(i) *The averaging function ${}_q\hat{\mu}_r$ belongs to $L^c_{-(N+1)}$.*
(ii) *The sequence $\{{}_q\hat{\mu}_r(a_n)\}$ belongs to ℓ^c.*
(iii) *The operator $I_{q+t}^t : B_q^2 \to L^2(\mu)$ belongs to $\mathcal{S}^{2c}$.*
(iv) *The Berezin transform $\mathbf{B}_q^u\mu$ belongs to $L^c_{-(N+1)}$.*

Proof. Let the singular number sequence of I_{q+t}^t be $\{s_n\}$. By [29, Proposition 16.3], this is equivalent to saying that $\{s_n^2\}$ is the singular number sequence of the operator $(I_{q+t}^t)^* I_{q+t}^t$ on B_q^2. But this composite operator is the generalized Toeplitz operator ${}_{q+t}T_\mu$ on B_q^2 as shown in [3, Theorem 4.6]. Thus I_{q+t}^t belongs to $\mathcal{S}^{2c}$ if and only if ${}_{q+t}T_\mu$ belongs to $\mathcal{S}^c$. Then the equivalence of (i), (iii), and (iv) follows immediately from the corresponding equivalence involving positive Toeplitz operators proved independently in [3, Theorem 6.13]. It is now clear why we need I_{q+t}^t in $\mathcal{S}^{2c}$ rather than in $\mathcal{S}^c$.

(i) $\Rightarrow$ (ii). Lemma 2.7 with $w = a_n$, $p = c$, and $q = -(N+1)$, and once again Lemma 2.1 yield

$$\big({}_q\hat{\mu}_r(a_n)\big)^c \leqslant C \int_{b(a_n,r)} ({}_q\hat{\mu}_r)^c \, d\nu_{-(N+1)}.$$

Then

$$\sum_{n=1}^{\infty}\big({}_q\hat{\mu}_r(a_n)\big)^c \leqslant C\sum_{n=1}^{\infty}\int\limits_{b(a_n,r)}({}_q\hat{\mu}_r)^c\,d\nu_{-(N+1)} \leqslant CM\int\limits_{\mathbb{B}}({}_q\hat{\mu}_r)^c\,d\nu_{-(N+1)}$$

by Lemma 2.5(iii).

(ii) $\Rightarrow$ (i). Repeated use of Lemmas 2.1, 2.5, and that $b(w,r)\subset b(a_n,2r)$ for $w\in b(a_n,r)$ yield

$$\begin{aligned}\int\limits_{\mathbb{B}}({}_q\hat{\mu}_r)^c\,d\nu_{-(N+1)} &\leqslant C\sum_{n=1}^{\infty}\int\limits_{b(a_n,r)}\frac{\mu(b(w,r))^c}{(1-|w|^2)^{N+1+(N+1+q)c}}\,d\nu(w)\\ &\leqslant C\sum_{n=1}^{\infty}\frac{1}{(1-|a_n|^2)^{N+1+(N+1+q)c}}\int\limits_{b(a_n,r)}\mu\big(b(a_n,2r)\big)^c\,d\nu\\ &\sim\sum_{n=1}^{\infty}\frac{(1-|a_n|^2)^{N+1}}{(1-|a_n|^2)^{N+1+(N+1+q)c}}\mu\big(b(a_n,2r)\big)^c\\ &\sim\sum_{n=1}^{\infty}\frac{\mu(b(a_n,2r))^c}{\nu_q(b(a_n,2r))^c}\sim\sum_{n=1}^{\infty}\big({}_q\hat{\mu}_r(a_n)\big)^c,\end{aligned}$$

because ${}_q\hat{\mu}_r(a_n)$ and ${}_q\hat{\mu}_{2r}(a_n)$ are equivalent by the proof of Theorem 1.3. $\square$

It is clear that the equivalences in Theorems 5.8 and 5.12 are independent of p, u, s, t, r, and $\{a_n\}$ under the stated conditions. These theorems for $q=0$ and $t=0$ are in [19, Theorems 2.15 and 2.16] and partly in [43, Exercise 6.7].

Remark 5.13. By Lemma 5.1 once again, if $U_{\alpha,\beta,q}\mu$ lies in $L^c_{-(N+1)}$ for some d with $0<d<\infty$, then so does ${}_q\hat{\mu}_r$. It is also easy to show the reverse implication for $d=1$, but we do not have it for $d>1$.

6. Hardy-space limit

Our aim is to show that (-1)-Carleson measures reduce to Hardy-space Carleson measures in an appropriate limit. But let us first highlight the important case of $(-N)$-Carleson measures covered by Theorem 1.3.

Corollary 6.1. *Let $q=-N$ and $p=2$, whence $B^2_{-N}=\mathcal{A}$, the Arveson space. The following conditions are equivalent for a positive Borel measure μ on $\mathbb{B}$, called a Carleson measure for $\mathcal{A}$.*

(i) *Given $r>0$, there is a C such that*

$$\mu\big(b(w,r)\big)\leqslant C\big(1-|w|^2\big)\quad(w\in\mathbb{B}).$$

(ii) *Given $r > 0$, there is a C such that if $\{a_n\}$ is an r-lattice in $\mathbb{B}$, then*

$$\mu\big(b(a_n, r)\big) \leqslant C\big(1 - |a_n|^2\big) \quad (n \in \mathbb{N}).$$

(iii) *There is a C such that for all s and t with $-N + 2t > -1$, we have*

$$\int_{\mathbb{B}} \big(1 - |z|^2\big)^{2t} \big|D_s^t f(z)\big|^2 \, d\mu(z) \leqslant C \int_{\mathbb{B}} \big(1 - |z|^2\big)^{-N+2t} \big|D_s^t f(z)\big|^2 \, d\nu(z) \quad (f \in \mathcal{A}).$$

(iv) *There is a C such that for all t with $-N + 2t > -1$, we have*

$$\int_{\mathbb{B}} \left(\frac{(1 - |z|^2)^t}{|1 - \langle z, w\rangle|^{1+2t}}\right)^2 d\mu(z) \leqslant C\big(1 - |w|^2\big)^{-(1+2t)} \quad (w \in \mathbb{B}).$$

Analogous statements can be obtained for the Dirichlet space $\mathcal{D} = B^2_{-(N+1)}$ or for vanishing Carleson measures.

Now let $N = 1$, when the Arveson space is the Hardy space H^2 on $\mathbb{D}$. Then Corollary 6.1 seems contrary to what is known for usual Carleson measures for Hardy spaces, because powers in (iv) do not seem right, and we have a characterization of Carleson measures on a Hardy space using Bergman discs in (i); cf. [43, Section 8.2.1]. But (iii) depends on an imbedding of H^2 in L^2_{-1} by way of I_s^t as described in Definition 1.1 in contrast to the usual imbedding of H^2 in $L^2(\partial\mathbb{D})$ by way of inclusion. The imbedding I_s^t and the equivalent norm

$$\|f\|^2_{B^2_{-1}} = \int_{\mathbb{D}} \big(1 - |z|^2\big)^{-1+2t} \big|D_s^t f(z)\big|^2 \, d\nu(z) \quad (t > 0)$$

for H^2 in (iii) require a positive-order radial derivative, where ν now is the area measure. Thus the Carleson measures defined here are different from the usual Carleson measures for Hardy spaces.

Remark 6.2. However, we indeed obtain the usual Hardy-space Carleson measures by taking limits as $t \to 0^+$ in the case $q = -1$ for all N and p in Theorem 1.3(iii) and (iv). The limit of the norm on the right-hand side of (iii) does not exist even for polynomials. But let us replace the right-hand side of (iii) by the equivalent quantity

$$\frac{(pt)_N}{N!} \|\cdot\|^p_{B^p_{-1}}, \tag{20}$$

where the role of the coefficient is to normalize the measure ν_{-1+pt} in $\|\cdot\|^p_{B^p_{-1}}$ with weight 1. Using weak-* convergence of measures, it is noted in [11, Section 0.3] and a detailed proof is given in [25, Section 3] that

$$\lim_{t\to 0^+} \frac{(pt)_N}{N!} \|f\|^p_{B^p_{-1}} = \|f\|^p_{H^p} = \int_{\partial\mathbb{B}} |f|^p \, d\sigma \quad \big(f \in B^p_{-1}\big).$$

Hence the limit of (iii) is the definition of a usual Carleson measure on H^p. More importantly, by Fatou lemma, (iv) becomes equivalent to being a usual Carleson measure on H^p as $t \to 0^+$; see [43, Corollary 8.2.3] for $N = 1$. Although $B^p_{-1} \neq H^p$ for $p \neq 2$ (see [10, p. 840]), the above holds also for $f \in H^p$, because $B^2_{-1} = H^2$, and Carleson measures of either type do not depend on p.

Analogously, the limiting case of parts (iii) and (iv) of Theorem 5.3 as $t \to 0^+$ is [43, Theorem 8.2.5].

We finish this section by a proof of Theorem 1.6.

Proof of Theorem 1.6. The basic ideas are in the standard proofs of theorems on Hardy-space Carleson measures. We outline the few differences for $N \geqslant 1$ and more general α.

If μ is a Carleson measure, then the inclusion map imbeds H^2 into $L^2(\mu)$. As in the proof of the implication (iii) $\Rightarrow$ (iv) of Theorem 1.3, we use the function

$$g_w(z) = \frac{(1 - |w|^2)^{\alpha/2}}{(1 - \langle z, w \rangle)^{(N+\alpha)/2}},$$

which lies in H^2 with norm ~ 1 by [32, Proposition 1.4.10] for $\alpha > 0$.

For the converse, we recall that the Carleson windows in $\mathbb{B}$ are the nonisotropic balls $W(\zeta, \rho) = \{z \in \overline{\mathbb{B}}\colon\ |1 - \langle z, \zeta \rangle| < \rho\}$ for $|\zeta| = 1$ and $0 < \rho < 1$ whose intersections with $\partial\mathbb{B}$ have surface measure $\sim \rho^N$; see [14, pp. 42–43]. It is easy to see that if $w_0 \in W(\zeta, \rho)$, then $1 - |w_0|^2 \sim \rho$ and $1 - \langle z, w_0 \rangle \sim \rho$ for $z \in W(\zeta, \rho)$, which gives rise to a result much like Lemma 5.1. □

Corollary 6.3. *Let μ be a positive Borel measure on $\mathbb{B}$. If $U_{\alpha,0,-1}\mu(w)$ has limit 0 as $|w| \to 1$ for some real α, then μ is a Hardy-space vanishing Carleson measure. If μ is a Hardy-space vanishing Carleson measure and $\alpha > 0$, then $U_{\alpha,0,-1}\mu(w)$ has limit 0 as $|w| \to 1$.*

7. Forelli–Rudin operators

Theorem 1.5 and Corollary 5.9 suggest a consideration of the operators

$$V^\mu_{\alpha,\beta,\gamma} f(w) = \left(1 - |w|^2\right)^\alpha \int\limits_{\mathbb{B}} \frac{(1 - |z|^2)^\beta}{(1 - \langle w, z \rangle)^{N+1+\alpha+\beta+\gamma}} f(z)\, d\mu(z) \quad (w \in \mathbb{B}).$$

Although $U_{\alpha,\beta,q}$ and $V^\mu_{\alpha,\beta,\gamma}$ for $\gamma = q$ are almost the same operators, the emphasis on each is different. The former, $U_{\alpha,\beta,q}$, is more a transform of the variable measure μ, while the latter, $V^\mu_{\alpha,\beta,\gamma}$, for fixed μ is more an operator that acts on a suitable variable function f.

Theorem 7.1. *Let $\alpha > 0$ and $\beta + q > -1$. The operator $V^\mu_{\alpha,\beta,q} : L^\infty(\mu) \to L^\infty$ is bounded if and only if μ is a q-Carleson measure. Further, if μ is a vanishing q-Carleson measure, then $V^\mu_{\alpha,\beta,q} : L^\infty(\mu) \to L^\infty$ is compact.*

Proof. The type of kernels studied in [11, Section 4] shows that replacing the integrand of $V^\mu_{\alpha,\beta,q}$ by its modulus has no effect on our results. Then the claim on boundedness is an immediate consequence of Theorem 1.5.

Next suppose μ is a vanishing q-Carleson measure. Clearly $V^{\mu}_{\alpha,\beta,q}$ is bounded by the first part. Take a bounded sequence $\{f_k\}$ in $L^{\infty}(\mu)$. By Corollary 5.9,

$$\left|V^{\mu}_{\alpha,\beta,q} f_k(w)\right| \leqslant C\left(1-|w|^2\right)^{\alpha} \int\limits_{\mathbb{B}} \frac{(1-|z|^2)^{\beta}}{|1-\langle z,w\rangle|^{N+1+\alpha+\beta+q}}\, d\mu(z),$$

and the right-hand side tends to 0 as $|w| \to 0$ uniformly in k; that is, given $\varepsilon > 0$, there is an $R < 1$ such that for $|w| > R$ and all k, we have $|V^{\mu}_{\alpha,\beta,q} f_k(w)| < \varepsilon$. In particular, every $f_k \in \mathcal{C}_0$. Define $Yf(w) = (1-|w|^2)^{-\alpha} V^{\mu}_{\alpha,\beta,q} f(w)$. Then $Yf_k \in H(\mathbb{B})$ and $Yf_k(w) = o((1-|w|^2)^{-\alpha})$ as $|w| \to 1$ uniformly in k by the above discussion. So $\{Yf_k\}$ is uniformly bounded on each compact subset of $\mathbb{B}$ and a normal family. Thus there exists a subsequence $\{Yf_{k_j}\}$ converging uniformly on compact subsets of $\mathbb{B}$ to $g \in H(\mathbb{B})$. Since $\alpha > 0$, also $V^{\mu}_{\alpha,\beta,q} f_{k_j}(w) \to h(w) = (1-|w|^2)^{\alpha} g(w)$ uniformly on compact subsets of $\mathbb{B}$, and $h \in \mathcal{C}$. So given a compact $E \subset \mathbb{B}$ and $\varepsilon > 0$, there is a j_0 such that for $j > j_0$ and all $w \in E$, we have $|V^{\mu}_{\alpha,\beta,q} f_{k_j}(w) - h(w)| < \varepsilon$. If it were the case that $h(w) \neq o(1)$ as $|w| \to 1$, there would be points $\{w_l\}$ in $\mathbb{B}$ with $|w_l| \to 1$ and an $\eta > 0$ such that $|h(w_l)| \geqslant \eta$ for all l. Taking $\varepsilon = \eta/2$, $|w_{l_0}| > R$, $E = \{w_{l_0}\}$, and $j > j_0$, we would get

$$\left|h(w_{l_0})\right| \leqslant \left|h(w_{l_0}) - V^{\mu}_{\alpha,\beta,q} f_{k_j}(w_{l_0})\right| + \left|V^{\mu}_{\alpha,\beta,q} f_{k_j}(w_{l_0})\right| < \varepsilon + \varepsilon = \eta,$$

contradicting what was just assumed on the order of growth of g. Then $h \in \mathcal{C}_0$, and $|h(w)| < \varepsilon$ for $|w| > R$, picking a larger R than the one above if necessary. Now let $E = \{w\colon |w| \leqslant R\}$ and $j > j_0$. Therefore

$$\begin{aligned}\sup_{w\in\mathbb{B}}\left|V^{\mu}_{\alpha,\beta,q} f_{k_j}(w) - h(w)\right| &\leqslant \sup_{w\in\mathbb{E}}\left|V^{\mu}_{\alpha,\beta,q} f_{k_j}(w) - h(w)\right| \\ &\quad + \sup_{|w|>R}\left|V^{\mu}_{\alpha,\beta,q} f_{k_j}(w)\right| + \sup_{|w|>R}\left|h(w)\right| < 3\varepsilon,\end{aligned}$$

meaning that the subsequence $\{V^{\mu}_{\alpha,\beta,q} f_{k_j}\}$ converges to h in L^{∞}. Thus $V^{\mu}_{\alpha,\beta,q}$ is compact. □

Special cases of $V^{\mu}_{\alpha,\beta,\gamma}$ include the Berezin transforms (take $\alpha = N+1+u$, $\beta = -q+u$, $\gamma = q$, $\mu = \nu_q$, and compare to Definition 5.7), certain cases of Bergman projections (take $\alpha = 0$, $\beta = 0$, $\gamma = s$, $\mu = \nu_s$ with $s > -(N+1)$, and compare to Definition 3.4), and their commonly used simpler version $V^{\nu}_{\alpha,\beta,0}$ with $\gamma = 0$ and $\mu = \nu$.

The boundedness of the operators $V^{\nu}_{\alpha,\beta,\gamma}$ with $\mu = \nu$ on L^p_q for $1 \leqslant p < \infty$ are characterized in [26], but the case $p = \infty$ is missing. We fill in this gap now.

Theorem 7.2. *The operator $V^{\nu}_{\alpha,\beta,\gamma} : L^{\infty} \to L^{\infty}$ is bounded if and only if either $\alpha > 0$, $\beta > -1$, $\gamma \leqslant 0$, or, $\alpha = 0$, $\beta > -1$, $\gamma < 0$. Further, if $\alpha > 0$, $\beta > -1$, and $-(\beta+1) < \gamma < 0$, then $V^{\nu}_{\alpha,\beta,\gamma} : L^{\infty} \to L^{\infty}$ is compact.*

Proof. As noted in the proof of Theorem 7.1, replacing the integrand by its modulus makes no difference.

The measure $\mu = \nu$ is a 0-Carleson measure. If $\alpha > 0$, $\beta > -1$, and $\gamma \leqslant 0$, then $|1 - \langle z, w\rangle|^{N+1+\alpha+\beta+\gamma} \geqslant |1 - \langle z, w\rangle|^{N+1+\alpha+\beta}$. This reduces the problem to the case $q = 0$ of Theorem 7.1, and $V^{\nu}_{\alpha,\beta,\gamma}$ is bounded. If $\alpha = 0$, $\beta > -1$, and $\gamma < 0$, then

$$\left|V^{\nu}_{\alpha,\beta,\gamma} f(w)\right| \leqslant \|f\|_{L^\infty} \int\limits_{\mathbb{B}} \frac{(1-|z|^2)^{\beta}}{|1-\langle z, w\rangle|^{N+1+\beta+\gamma}}\, d\nu(z),$$

which is bounded by [32, Proposition 1.4.10].

Conversely suppose $V^{\nu}_{\alpha,\beta,\gamma}$ is bounded. First take $f \equiv 1$. For the integral in $V^{\nu}_{\alpha,\beta,\gamma}$ to converge, $\beta > -1$ is necessary. With the same f, using the same machinery as in the proof of the if part of Theorem 1.5, we have

$$\left\|V^{\nu}_{\alpha,\beta,\gamma}\right\| \geqslant \left(1-|w|^2\right)^{\alpha} \int\limits_{b(w,r)} \frac{(1-|z|^2)^{\beta}}{|1-\langle z, w\rangle|^{N+1+\alpha+\beta+\gamma}}\, d\nu(z) \sim \frac{1}{(1-|w|^2)^{\gamma}}$$

for any $w \in \mathbb{B}$ and r by Lemmas 2.1 and 2.2. This forces $\gamma \leqslant 0$. Next take $f(z) = (1-|z|^2)^{\eta}$ with $\eta > 0$ so large that $\alpha + \gamma - \eta < 0$. Then $\|f\|_{L^\infty} = 1$, and

$$\left\|V^{\nu}_{\alpha,\beta,\gamma}\right\| \geqslant \left(1-|w|^2\right)^{\alpha} \int\limits_{\mathbb{B}} \frac{(1-|z|^2)^{\beta+\eta}}{|1-\langle z, w\rangle|^{N+1+\alpha+\beta+\gamma}}\, d\nu(z) \sim \left(1-|w|^2\right)^{\alpha}$$

by [32, Proposition 1.4.10]. We must have $\alpha \geqslant 0$. Finally, using $f \equiv 1$ once again in the case $\alpha = 0$, we are led to conclude that $\gamma < 0$ by [32, Proposition 1.4.10].

Under the conditions stated in the last claim, $-\gamma > 0$ so that $\mu = \nu_{-\gamma}$ is a vanishing 0-Carleson measure, and $\beta + \gamma > -1$. Then the operator $V^{\mu}_{\alpha,\beta+\gamma,0}$ is compact on L^∞ by the case $q = 0$ of Theorem 7.1 since now $L^\infty(\mu) = L^\infty$. This operator is just $V^{\nu}_{\alpha,\beta,\gamma}$. □

Corollary 7.3. *The operator $V^{\nu}_{\alpha,\beta,0} : L^\infty \to L^\infty$ is bounded if and only if $\alpha > 0$ and $\beta > -1$.*

This corollary also provides the converse to the if part that has already been shown in [44, Theorem 9]. For $0 < p < 1$, there is a partial result in [23, Theorem 2.4(b)]. We do not know of any earlier results in the literature on the compactness of the operators $V^{\mu}_{\alpha,\beta,\gamma}$ on Lebesgue classes.

8. Weighted Bloch, Lipschitz, and growth spaces

This section is for descriptions of weighted Bloch, Lipschitz, and growth spaces in terms of Carleson measures. As corollaries, we obtain that these spaces can be described using any radial derivative whose order is sufficiently high, and how differentiation transforms one of these spaces to another. We start with the usual Bloch space.

Theorem 8.1. *A function $h \in H(\mathbb{B})$ lies in the Bloch space $\mathcal{B}$ if and only if $d\mu = |I^u_s h|^p\, d\nu_q$ is a q-Carleson measure for some q, p, s, and $u > 0$. Such an h lies in the little Bloch space $\mathcal{B}_0$ if and only if μ is a vanishing q-Carleson measure.*

Proof. First suppose $h \in \mathcal{B}$. Then for any s and $u > 0$, $|I_s^u h|^p$ is bounded, say, by C by (14). Choose $\alpha > 0$ and β with $\beta + q > -1$. Then

$$U_{\alpha,\beta,q}\mu(w) \leqslant C\big(1-|w|^2\big)^{\alpha} \int_{\mathbb{B}} \frac{(1-|z|^2)^{\beta+q}}{|1-\langle z,w\rangle|^{N+1+\alpha+\beta+q}}\, d\nu(z) \quad (w \in \mathbb{B}).$$

Then $U_{\alpha,\beta,q}\mu$ is bounded by [32, Proposition 1.4.10], and consequently μ is a q-Carleson measure by Theorem 1.5.

Conversely, suppose μ is a q-Carleson measure for some $u > 0$. Then $U_{\alpha,\beta,q}\mu$ is bounded on $\mathbb{B}$ by Theorem 1.5. Taking $w = 0$ gives

$$\int_{\mathbb{B}} \big(1-|z|^2\big)^{\beta} \big|I_s^u h(z)\big|^p \, d\nu_q(z) < \infty$$

for some β with $\beta + q > -1$, that is, $(D_s^u h)^p$ belongs to A_Q^1 since $Q = q + pu + \beta > -1$. The reproducing property of the Bergman kernel K_Q on A_Q^1 yields

$$\big(D_s^u h\big)^p(w) = C \int_{\mathbb{B}} \frac{(1-|z|^2)^{q+pu+\beta}}{(1-\langle w,z\rangle)^{N+1+q+pu+\beta}} \big(D_s^u h\big)^p(z)\, d\nu(z) \quad (w \in \mathbb{B}).$$

Thus

$$\big|I_s^u h(w)\big|^p \leqslant C\big(1-|w|^2\big)^{pu} \int_{\mathbb{B}} \frac{(1-|z|^2)^{\beta}}{|1-\langle z,w\rangle|^{N+1+pu+\beta+q}}\, d\mu(z) \quad (w \in \mathbb{B}).$$

The right-hand side is bounded by assumption. This shows $h \in \mathcal{B}$.

The second statement is proved similarly using Corollary 5.9 instead. □

The special case $q > 0$, $p = 2$, $N = 1$ of Theorem 8.1 is contained in [7, Theorem 2.2] with $u = 1$ and in [8, Theorem 5] with u a positive integer. The special case $q > -1$, $p = 2$, $u = 1$ of Theorem 8.1 with any N is essentially in [38,39]. The p of all four sources corresponds to our $(N+1+q)/N$, and the Bloch space is obtained when it exceeds 1. See also [12] for a similar result.

Theorem 8.1 can be generalized to cover functions in weighted Bloch spaces with parameter α. These spaces of holomorphic functions f are usually defined on $\mathbb{D}$ for $\alpha > 0$ by requiring that $(1-|z|^2)^{\alpha}|f'(z)|$ is bounded on $\mathbb{D}$. It is also known that $(1-|z|^2)f'(z)$ can be replaced by $(1-|z|^2)^k f^{(k)}(z)$. In the ball, we replace f' by its radial derivative, better yet, by $D_s^1 f$. Then it is easy to see that the weighted Bloch space is the limiting case as $p \to \infty$ of the space $B^p_{p(\alpha-1)}$. Since the lower index of a Besov space can be any real number, now it is clear that the positivity restriction on α is superfluous. For uniformity of notation with Besov spaces, we change the parameter α to $\alpha + 1$ since $D_s^u f(z)$ goes hand in hand with $(1-|z|^2)^u$.

Definition 8.2. For any $\alpha \in \mathbb{R}$, we define the *weighted Bloch space* $\mathcal{B}^{\alpha}$ to consist of all $f \in H(\mathbb{B})$ for which

$$\big(1-|z|^2\big)^{\alpha}\, I_s^u f(z) \tag{21}$$

is bounded on $\mathbb{B}$ for some s, u satisfying

$$\alpha + u > 0. \tag{22}$$

The *weighted little Bloch space* $\mathcal{B}_0^\alpha$ is the subspace of $\mathcal{B}^\alpha$ consisting of those f for which the quantity in (21) vanishes on $\partial\mathbb{B}$ for some s, u satisfying (22).

With this labeling of parameters, the usual Bloch and little Bloch spaces are $\mathcal{B}^0$ and $\mathcal{B}_0^0$. We show below the independence of Definition 8.2 of s, u satisfying (22). Note the similarity of (22) to the first inequality in (14). By [45, Theorems 7.17 and 7.18], the spaces $\mathcal{B}^\alpha$ and $\mathcal{B}_0^\alpha$ for $\alpha < 0$ are the holomorphic *Lipschitz spaces* $\Lambda_{-\alpha}$ and $\Lambda_{-\alpha,0}$ of the ball. By (22), when $\alpha > 0$, no derivative is required to define these spaces. Thus the spaces $\mathcal{B}^\alpha$ for $\alpha > 0$ are the so-called *growth spaces* $\mathcal{A}^{-\alpha}$; see [19, Definition 4.13]. However, for the usual Bloch space and all Lipschitz spaces, as $|\alpha|$ increases, increasingly higher order derivatives are required.

The following characterization of these spaces extends Theorem 8.1 and is proved along the same lines. The α here must not be confused with the α of Theorem 1.5.

Theorem 8.3. *A function $h \in H(\mathbb{B})$ lies in $\mathcal{B}^\alpha$ if and only if $d\mu = |I_s^u h|^p \, d\nu_{\alpha+q}$ is a q-Carleson measure for some q, p, s, and u satisfying* (22). *Such an h lies in $\mathcal{B}_0^\alpha$ if and only if μ is a vanishing q-Carleson measure.*

Corollary 8.4. *Definition* 8.2 *is independent of s, u satisfying* (22).

Proof. Theorem 8.3 does not depend on s or u as long as (22) is fulfilled. □

Thus the L^∞ norms of the quantities $(1-|z|^2)^\alpha I_s^u f(z)$ for different values of s, u satisfying (22) are equivalent, and we put

$$\|f\|_{\mathcal{B}^\alpha} = \|f\|_{\mathcal{B}_0^\alpha} = \sup_{z\in\mathbb{B}} \bigl(1-|z|^2\bigr)^\alpha \bigl|I_s^u f(z)\bigr|.$$

This fact allows us to pass between different weighted Bloch spaces by taking derivatives or integrals as in Proposition 3.2.

Corollary 8.5. *For any α, s, t, $D_s^t(\mathcal{B}^\alpha) = \mathcal{B}^{\alpha+t}$ and $D_s^t(\mathcal{B}_0^\alpha) = \mathcal{B}_0^{\alpha+t}$ are isometric isomorphisms under the equivalence of norms.*

Corollary 8.6. *Suppose α, q, p are related by $q - \alpha p > -1$. Then $\mathcal{B}^\alpha \subset B_q^p$, and the inclusion is continuous.*

Proof. We recall that the Bloch space $\mathcal{B}$ lies in every Bergman space A_Q^p for which $Q > -1$, take derivatives of order α, and use Corollary 8.5 and Proposition 3.2. □

Corollary 8.7. *Suppose α, q, p are related by $q - \alpha p = -(N+1)$. Then $B_q^p \subset \mathcal{B}^\alpha$, and the inclusion is continuous.*

Proof. This time, we recall that each Besov space $B^p_{-(N+1)}$ lies in the Bloch space $\mathcal{B}$ and proceed as in the previous corollary. $\square$

In particular, when $\alpha > 0$, the last two corollaries give inclusion relations between Besov spaces and growth spaces $\mathcal{A}^{-\alpha}$, and between Besov spaces and Lipschitz spaces $\Lambda_{-\alpha}$ when $\alpha < 0$. In fact, precisely the case $\alpha < 0$ of Corollary 8.7 is proved in [11, Theorem 5.14(i)] by other means.

9. Connections with gap series

In this section, gap series are the common thread. We first decide which Borel measures on the ball are finite in terms of the same imbedding we have used in characterizing Carleson measures. We go ahead with constructing Carleson measures from holomorphic functions under some conditions easily entailed by gap series in Besov spaces. We then see that Carleson measures made with holomorphic functions give rise to Besov functions.

Definition 9.1. Let $f = \sum_k F_k \in H(\mathbb{B})$, where F_k is a homogeneous polynomial of degree n_k. The series f is said to have *Hadamard gaps* and we write $f \in \mathrm{HG}$ if there is an $\omega > 1$ such that $n_{k+1}/n_k \geqslant \omega$ for all k.

When $N = 1$, of course $F_k(z) = c_k z^{n_k}$ and $\|F_k\|_{H^\infty} = \|F_k\|_{H^p} = c_k$.

Lemma 9.2. *If $f = \sum_k F_k \in \mathrm{HG}$ and $\|F_k\|_{H^\infty} \leqslant C n_k^\alpha$, where n_k is the degree of F_k, then $f \in \mathcal{B}^\alpha$.*

Proof. This is really a combination of the techniques of [37, Theorem 1] and [34, Proposition 4.16]. Although the former reference considers only $\alpha > -1$, its proof is valid word by word for the case $\alpha \leqslant -1$ too. $\square$

Our first theorem helps identify those Carleson measures that are finite.

Theorem 9.3. *A positive Borel measure μ on $\mathbb{B}$ is finite if and only if some I_s^u with* (22) *satisfied maps the weighted Bloch space $\mathcal{B}^\alpha$ into $L^p(\mu_{\alpha p})$ for some p continuously.*

Proof. The idea of the proof is in [4, Theorem 16]. If μ is a finite measure, then trivially $\int_{\mathbb{B}} |I_s^u f|^p \, d\mu_{\alpha p} \leqslant \|f\|^p_{\mathcal{B}^\alpha} \int_{\mathbb{B}} d\mu < \infty$. Conversely, assume

$$\int_{\mathbb{B}} \left|I_s^u f\right|^p d\mu_{\alpha p} \leqslant C \, \|f\|^p_{\mathcal{B}^\alpha} \quad \left(f \in \mathcal{B}^\alpha\right) \tag{23}$$

for some p, s, u satisfying (22). Pick multi-indices λ_k with $|\lambda_k| = n_k$, where n_k is as in Definition 9.1, such that the components λ_{k_j}, $j = 1, \dots, N$, are all nonzero, and let $F_k(z) = n_k^\alpha z^{\lambda_k}$. Then $f = \sum_k F_k \in \mathrm{HG} \cap \mathcal{B}^\alpha$ by Lemma 9.2. The nonzero condition assures that we have $|F_k(z)| \geqslant C n_k^\alpha |z|^{n_k}$ for some C and all $z \in \mathbb{B}$. By homogeneity and (7),

$$D_s^u f\left(e^{i\theta} z\right) = \sum_k {}_s^u d_{n_k} F_k\left(e^{i\theta} z\right) \sim \sum_k n_k^u e^{i n_k \theta} F_k(z).$$

If we replace $f(z)$ by $f(e^{i\theta}z)$ on the left-hand side of (23), integrate with respect to θ, then apply Fubini theorem and [46, Theorem V.8.20], we obtain

$$\begin{aligned}
\|f\|^p_{\mathcal{B}^\alpha} &\geqslant C\int_{\mathbb{B}}\left(1-|z|^2\right)^{\alpha p+pu}\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_k n_k^u F_k(z)e^{in_k\theta}\right|^p d\theta\,d\mu(z)\\
&\sim\int_{\mathbb{B}}\left(1-|z|^2\right)^{\alpha p+pu}\left(\sum_k n_k^{2u}\left|F_k(z)\right|^2\right)^{p/2}d\mu(z)\\
&\geqslant C\int_{\mathbb{B}}\left(1-|z|^2\right)^{\alpha p+pu}\left(\sum_k n_k^{2\alpha+2u}|z|^{2n_k}\right)^{p/2}d\mu(z)\\
&\geqslant C\int_{\mathbb{B}}\left(1-|z|^2\right)^{\alpha p+pu}\left(\sum_k k^{2\alpha+2u-1}|z|^{2k}\right)^{p/2}d\mu(z)\\
&\sim\int_{\mathbb{B}}\left(1-|z|^2\right)^{\alpha p+pu}\frac{1}{(1-|z|^2)^{\alpha p+pu}}\,d\mu(z)=\mu(\mathbb{B})
\end{aligned}$$

after some obvious inequalities and a binomial expansion. □

If $\alpha=0$ and $q>-1$, then $\mathcal{B}\subset B^p_q=A^p_q$ and $\|f\|_{A^p_q}\leqslant C\|f\|_{\mathcal{B}}$. Then (23) is satisfied automatically for any q-Carleson measure μ and any $f\in\mathcal{B}$ by Theorem 1.3(iii). Thus we recapture the fact that Bergman-space Carleson measures are finite.

An equivalent statement to Theorem 9.3 is this. Given a positive Borel measure μ on $\mathbb{B}$, the measure $\mu_{-\alpha p}$ is finite if and only if some I^u_s with (22) satisfied maps $\mathcal{B}^\alpha$ into $L^p(\mu)$ continuously. In this form, the particular case $N=1$, $\alpha>-1$, and positive integer t of Theorem 9.3 is in [31, Theorem 3.2].

We next give some conditions on functions in Besov spaces that generate q-Carleson measures. Let $J(k)=\{j\in\mathbb{N}\colon\ 2^k\leqslant j<2^{k+1}\}=\{j\in\mathbb{N}\colon\ j/2<2^k\leqslant j\}$ for $k=0,1,2,\ldots.$ The following result is in [25, Theorem 4.2].

Lemma 9.4. *A power series $f(z)=\sum_k F_k\in\mathrm{HG}$ belongs to B^p_q if and only if $\sum_k n_k^{-(1+q)}\|F_k\|^p_{H^p}<\infty$, where n_k is the degree of F_k.*

Lemma 9.5. *Let q, p, t be related by (1). Let $f(z)=\sum_k F_k\in H(\mathbb{B})$ be given by its homogeneous expansion. Suppose that*

$$\sum_{k=0}^{\infty}2^{-(1+q)k}\left(\sum_{j\in J(k)}\|F_j\|_{H^\infty}\right)^p<\infty.$$

Then $d\mu=|I^t_s f|^p\,d\nu_{1+2q}$ is a q-Carleson measure.

Proof. Let $z=R\zeta\in\mathbb{B}$, where $\zeta\in\partial\mathbb{B}$, and put $M_k=\|F_k\|_{H^\infty}$ for simplicity. Then we have $|F_k(R\zeta)|=R^k|F_k(\zeta)|\leqslant R^kM_k$, and $|D^t_s f(z)|\leqslant\sum_k {}_s^t d_k M_k R^k\leqslant C\sum_k k^t M_k R^k$ by (7). Take a u satisfying (1) when substituted for t. Then the left-hand side of Theorem 1.3(iv) is

$$= \sup_{w\in\mathbb{B}} \bigl(1-|w|^2\bigr)^{N+1+q+pu} \int_{\mathbb{B}} \frac{(1-R^2)^{1+2q+pt+pu}}{|1-\langle R\zeta, w\rangle|^{(N+1+q+pu)2}} \bigl|D_s^t f(R\zeta)\bigr|^p \, d\nu(z)$$

$$\leqslant C \sup_{w\in\mathbb{B}} \bigl(1-|w|^2\bigr)^{N+1+q+pu} \int_0^1 \bigl(1-R^2\bigr)^{1+2q+pt+pu} R^{2N-1} \biggl(\sum_k k^t M_k R^k\biggr)^p$$

$$\times \int_{\partial\mathbb{B}} \frac{1}{|1-\langle \zeta, Rw\rangle|^{(N+1+q+pu)2}} \, d\sigma(\zeta)\, dR$$

$$\leqslant C \sup_{w\in\mathbb{B}} \bigl(1-|w|^2\bigr)^{N+1+q+pu} \int_0^1 \frac{(1-R^2)^{1+2q+pt+pu}}{(1-R^2|w|^2)^{N+(1+q+pu)2}} \biggl(\sum_k k^t M_k R^k\biggr)^p dR$$

$$\leqslant C \int_0^1 \bigl(1-R^2\bigr)^{q+pt} \biggl(\sum_k k^t M_k R^k\biggr)^p dR \leqslant C \sum_{k=0}^{\infty} 2^{-(1+q+pt)k} \biggl(\sum_{j\in J(k)} j^t M_j\biggr)^p$$

$$\leqslant C \sum_{k=0}^{\infty} 2^{-(1+q+pt)k} 2^{ptk} \biggl(\sum_{j\in J(k)} M_j\biggr)^p < \infty,$$

where we have used [32, Proposition 1.4.10], [28, Theorem 1], and the definition of $J(k)$. □

Corollary 9.6. *If $f = \sum_k F_k \in B_q^p \cap \mathrm{HG}$ and the sequence $\{\|F_k\|_{H^\infty}/\|F_k\|_{H^p}\}$ is bounded, then $d\mu = |I_s^t f|^p \, d\nu_{1+2q}$ is a q-Carleson measure.*

Note that the boundedness condition is automatic for $N = 1$ since then the sequence constantly is 1. It is also satisfied if each F_k is a function of a single variable that may vary with k.

Proof. Let f have parameter ω of Definition 9.1. It is well known that for this kind of f the number of n_k in $J(k)$ is at most $1 + \lfloor \log_\omega 2 \rfloor$ independently of k. Coupled with Lemma 9.4 and the way $J(k)$ is defined, this yields

$$\infty > \sum_{k=0}^{\infty} n_k^{-(1+q)} \|F_k\|_{H^p}^p \sim \sum_{k=0}^{\infty} 2^{-(1+q)k} \biggl(\sum_{j\in J(k)} \|F_j\|_{H^p}\biggr)^p$$

$$> C \sum_{k=0}^{\infty} 2^{-(1+q)k} \biggl(\sum_{j\in J(k)} \|F_j\|_{H^\infty}\biggr)^p.$$

We are done by Lemma 9.5. □

We have a partial converse.

Proposition 9.7. *Let $f \in H(\mathbb{B})$, q, p, t be related by (1), and $d\mu = |I_s^t f|^p \, d\nu_{1+2q}$. If μ is a q-Carleson measure, then $f \in B_Q^p$ for any $Q > q$.*

Proof. Pick u such that $Q = 1 + 2q + pu$. Then (1) is satisfied with u in place of t. By Corollary 1.4, μ_{pu} is a finite measure. This just means $f \in B^p_Q$. $\square$

There are similar results in [18, Section 3] using Hardy-space Carleson measures and functions in B^p_{-1} when $N = 1$.

10. Fejér–Riesz and Hardy–Littlewood inequalities

We obtain two kinds of inequalities on the behavior of the derivatives of functions in Besov spaces on lower-dimensional subspaces and spheres. They are interesting and new in their own right, even for Bergman spaces. They also reduce to two classical inequalities of Hardy spaces in the limit of a special case, thereby providing simple and direct proofs of them without resorting to interpolation.

First some notation. Write $\mathbb{C}^N = \mathbb{C}_1 \times \cdots \times \mathbb{C}_N$ and $\mathbb{C}_m = \mathbb{R}_m \times i\mathbb{R}_m$. Put

$$L_{j,k} = \mathbb{R}_1 \times \cdots \times \mathbb{R}_j \times \mathbb{C}_{j+1} \times \cdots \times \mathbb{C}_{j+k} \times \{0\} \times \cdots \times \{0\}$$

for $j = 1, \dots, N$ and $k = 0, \dots, N - j$. Put also $L_{0,k} = \mathbb{C}_1 \times \cdots \times \mathbb{C}_k \times \{0\} \times \cdots \times \{0\}$ for $k = 1, \dots, N$. Denote the normalized Lebesgue measure on $L_{j,k}$ by $\nu^{j,k}$, and the one on $L_{0,k}$ by $\nu^{0,k}$. This notation should not be confused with our basic notation ν_q of (2).

Theorem 10.1. *Given q and j, k as above, there is a constant C such that for all p, s and t satisfying* (1)*, we have*

$$\int_{L_{j,k}\cap\mathbb{B}} \left(1 - |z|^2\right)^{N+1+q-(1+j+2k)/2} \left|I_s^t f(z)\right|^p \, d\nu^{j,k}(z) \leqslant C \|f\|^p_{B^p_q} \quad \left(f \in B^p_q\right)$$

and

$$\int_{L_{0,k}\cap\mathbb{B}} \left(1 - |z|^2\right)^{N+1+q-(1+k)} \left|I_s^t f(z)\right|^p \, d\nu^{0,k}(z) \leqslant C \|f\|^p_{B^p_q} \quad \left(f \in B^p_q\right).$$

Proof. First let $d\mu(z) = (1 - |z|^2)^{N+1+q-(1+j+2k)/2} \, d\nu^{j,k}(z)$. By Lemma 2.1 and (6), we have

$$\begin{aligned}\mu\big(b(w,r)\big) &= \int_{b(w,r)} \left(1 - |z|^2\right)^{N+1+q-(1+j+2k)/2} d\nu^{j,k}(z) \\ &\sim \left(1 - |w|^2\right)^{N+1+q-(1+j+2k)/2} \nu^{j,k}\big(b_\psi(w, \tanh r)\big)\end{aligned}$$

for fixed r. It is explained in [32, Section 2.2.7] that $b_\psi(w, \tanh r)$ is an ellipsoid whose intersection with the complex line $[w]$ is a disc of radius $\sim (1 - |w|^2)$ and whose intersection with the real $(2N - 2)$-dimensional space perpendicular to $[w]$ is a ball of radius $\sim \sqrt{1 - |w|^2}$. Then $\nu^{j,k}(b_\psi(w, \tanh r)) \leqslant C(1 - |w|^2)(\sqrt{1 - |w|^2})^{j-1+2k}$ and $\mu(b(w,r)) \leqslant C(1 - |w|^2)^{N+1+q}$. Therefore μ is a q-Carleson measure and the first inequality follows by the equivalence of Theorem 1.3(i) and (iii).

Analogously, we obtain $\nu^{0,k}(b_\psi(w,\tanh r)) \leqslant C(1-|w|^2)^2(\sqrt{1-|w|^2})^{2k-2}$. Then letting $d\mu(z) = (1-|z|^2)^{N+1+q-(1+k)}\, d\nu^{0,k}(z)$ and arguing as above yield the second inequality. □

For $k = 1, \dots, N$, let $\mathbb{B}_k$ denote the unit ball of $\mathbb{C}^k$ imbedded in $\mathbb{C}^N$ in the same way as $L_{0,k}$ is imbedded in $\mathbb{C}^N$. Clearly $L_{0,k} \cap \mathbb{B} = \mathbb{B}_k$. Let also σ^k denote the normalized Lebesgue measure on $\partial\mathbb{B}_k$.

Corollary 10.2. *The restriction to $\mathbb{B}_k$ imbeds B^p_q continuously into $B^p_{N-k+q}(\mathbb{B}_k)$.*

Before stating the next result, for $k = 1, \dots, N$, let us define the generalized integral *means* of an $f \in H(\mathbb{B})$ by

$$\mathcal{M}_p(f,k,R) = \left(\int_{\partial\mathbb{B}_k} \left|f(R\zeta, 0')\right|^p d\sigma^k(\zeta) \right)^{1/p} \quad (0 < R < 1,\ 0 < p < \infty),$$

where $0'$ is an $(N-k)$-tuple of zeros.

Theorem 10.3. *Let q, $0 < p \leqslant p_1 < \infty$, and $k = 1, \dots, N$ be given. Put*

$$q_1 = p_1\left(\frac{N+1+q}{p} - \frac{k+1}{p_1}\right).$$

Then there is a constant C such that for all s, t with $q + pt > -1$, we have

$$\int_0^1 \left(1-R^2\right)^{q_1+p_1 t} \mathcal{M}^{p_1}_{p_1}\left(D^t_s f, k, R\right) dR \leqslant C\|f\|^p_{B^p_q} \quad \left(f \in B^p_q\right). \tag{24}$$

Proof. We have $q_1 \geqslant q$ and $q_1 + p_1 t > -1$ too. Clearly $\nu^{0,k}_{q_1}$ is a q_1-Carleson measure on $\mathbb{B}_k$, and Theorem 1.3(iii) implies (24) with $C\|f\|^{p_1}_{B^{p_1}_{q_1}(\mathbb{B}_k)}$ instead on the right-hand side. By Corollary 10.2, this quantity is $\leqslant C\|f\|^{p_1}_{B^{p_1}_{-N+k+q_1}}$.

But $(N+1+q)/p = (N+1+(-N+k+q_1))/p_1$, which is precisely the condition to be satisfied for [11, Theorem 5.13] translated to our notation by the correspondence $A^p_{1+q+pt,t} = B^p_q$ between their labeling of spaces and ours. Hence B^p_q is continuously included in $B^{p_1}_{-N+k+q_1}$, and this completes the proof. □

Remark 10.4. Now we take $q = -1$, replace the norm $\|\cdot\|_{B^2_{-1}}$ by (20), and then let $t \to 0^+$ in Theorem 10.1 as we did in Remark 6.2. This results in the case $c = 1$ of the two inequalities for Hardy spaces of the ball in [30, Theorem 1]. Further letting $N = 1$ in the resulting first inequality, which forces $j = 1$ and $k = 0$, we arrive at the classical Fejér–Riesz inequality for Hardy spaces. When $k = N$ in the second inequality, we cannot let $t \to 0^+$ without an extra requirement such as $c > 1$ in [30, Theorem 1(2)].

Doing the same norm replacement and taking the same limit for $q = -1$ in Theorem 10.3 give [30, Theorem 4] in the case $\lambda = q$, which is our p_1. Further letting $N = 1$, which forces $k = 1$, we find a classical inequality of Hardy and Littlewood for Hardy spaces; see [15, p. 146].

Although we cannot obtain sharp constants in the classical inequalities, it is worth noting that our proofs follow very easily from Theorem 1.3.

Let $Q = q + pt > -1$ and $g = D_s^t f$ in the last two theorems; then $g \in A_Q^p$, a Bergman space, and $\|f\|_{B_q^p} = \|g\|_{A_Q^p}$. From this point of view, Theorems 10.1 and 10.3 are results for Bergman spaces. In this form, the case $k = N$ of Theorem 10.3 can be found in [11, Theorem 3.2(iii)] more generally with one extra free parameter, yet less generally only for $1 < p < \infty$. All other cases of these two theorems, including the cases without derivatives, seem to be new for Bergman spaces too.

11. Cesàro operators

In [1], the authors consider two operators defined through differentiation, multiplication, and then integration. One of them is a generalization of the Cesàro operator, and they analyze it on weighted Bergman spaces on $\mathbb{D}$. This operator is later generalized to mixed norm spaces on $\mathbb{B}$ using first-order radial derivatives; see [20]. The other (companion) operator is investigated on weighted Bloch and weighted BMOA spaces in [40,41].

Here we generalize this companion operator to arbitrary-order radial differentiation and integration, and analyze it on Besov spaces, hence also on weighted Bergman spaces. We also prove that the exponential of a holomorphic function lies in Bergman spaces if the function gives rise to a Carleson measure.

Let s, t be arbitrary, $g \in H(\mathbb{B})$, and define

$$T_g f = D_{s+t}^{-t} M_g D_s^t f \quad \big(f \in H(\mathbb{B})\big),$$

where M_g represents the operator of multiplication by g. The two radial differential operators are inverses of each other by (8).

This operator, for certain values of s, t, shows up as a generalized Toeplitz operator with a holomorphic symbol on Arveson and general Dirichlet spaces in [3, Section 4]. Ordinarily, a Toeplitz operator with a holomorphic symbol is just a multiplication operator, but the generalization to all Dirichlet spaces requires the introduction of derivatives, and when g is holomorphic, the new Toeplitz operators reduce to T_g. This connection has already been predicted in [1, p. 338], where they call their S_g a "distant cousin of Toeplitz operators," which is essentially our T_g with $t = 1$, and not distant at all.

More interesting is the fact that T_g is the link between the weakly convergent families of functions in B_q^2 spaces in Remark 4.8 and the Berezin transforms of functions in B_q^2. So given q and $p = 2$, let s satisfy (13), set $t = -q + s$, and put

$$H_q(z, w) = \frac{K_s(z, w)}{\sqrt{K_{-q+2s}(w, w)}} \in B_q^2.$$

Recall that H_q is some kind of a normalized kernel. Using (9) and (8) we have

$$\begin{aligned}&\big[T_g H_q(\cdot, w), H_q(\cdot, w)\big]_q \\ &\quad = \big(1 - |w|^2\big)^{N+1-q+2s}\big[I_s^{-q+s} D_{-q+2s}^{q-s} M_g D_s^{-q+s} K_s(\cdot, w), I_s^{-q+s} K_s(\cdot, w)\big]_{L_q^2}\end{aligned}$$

$$= \big(1-|w|^2\big)^{N+1-q+2s}\big[\big(1-|z|^2\big)^{(-q+s)2} g(z) K_{-q+2s}(z,w), K_{-q+2s}(z,w)\big]_{L^2_q}$$
$$= \mathbf{B}_q^{-q+2s} g(w) \quad \big(g \in H(\mathbb{B})\big)$$

since $-q+2s > -1$ and $t = -q+s$. This identity is in accord with the classical definition of the Berezin symbol of an operator. For further applications of this idea, see [3, Section 5].

Theorem 11.1. *Given q, p, pick s, t satisfying* (1), *and let $d\mu = |g|^p \, d\nu_q$, where $g \in H(\mathbb{B})$. The operator $T_g : B^p_q \to B^p_q$ defined using the selected s, t is bounded* (*respectively, compact*) *if and only if μ is a q-Carleson* (*respectively, vanishing q-Carleson*) *measure. For $1 \leqslant c < \infty$, the operator $T_g : B^2_q \to B^2_q$ belongs to the Schatten ideal $\mathcal{S}^{2c}$ if and only if the Berezin transform $\mathbf{B}^u_q \mu$ belongs to $L^c_{-(N+1)}$ for $u > -1$.*

Proof. Consider $I^t_s : B^p_q \to L^p(\mu)$, which is the main operator in Section 5. Let $f \in B^p_q$. All three claims follow at once from the equality $\|I^t_s f\|_{L^p(\mu)} = \|T_g f\|_{B^p_q}$ and Theorems 5.8 and 5.12. It also follows that $\|T_g\| \leqslant \|g\|_{H^\infty}$. □

It is important that the proof would not go through if we did not have our characterization of Carleson measures using I^t_s as an imbedding.

Corollary 11.2. *Given q and $g \in H(\mathbb{B})$, suppose that $|g|^p \, d\nu_q$ is a q-Carleson measure. Then $e^g \in \bigcap_{Q>-1} A^2_Q$.*

Proof. We follow the proof of [1, Corollary 4] and show little detail. By assumption, T_g with $q+2t > 0$ is bounded on B^2_q by Theorem 11.1. It is easily computed that $T_g(1) = {}^t_s d_0 D^{-t}_{s+t}(g)$ and $T^k_g(1) = {}^t_s d_0 D^{-t}_{s+t}(g^k)$ for $k = 1, 2, \ldots$, where 1 is the constant function. Put also $T^0_g = I$. Then the series

$$\frac{1}{{}^t_s d_0} \sum_{k=0}^{\infty} \frac{T^k_g(1)}{k!} = D^{-t}_{s+t} \sum_{k=0}^{\infty} \frac{g^k}{k!}$$

converges in the operator norm topology and has sum $D^{-t}_{s+t}(e^g) \in B^2_q$ by Proposition 3.2. Therefore $e^g \in D^t_s(B^2_q) = A^2_{q+2t}$. But t can be chosen so that $Q = q+2t > -1$ is arbitrary. □

One way that the measure μ in Theorems 8.3 and 11.1 is the same is by having $g = D^u_s h$ and $\alpha + u = 0$. But the second equality contradicts (22). In view of Corollary 8.5, we conclude that the g in Corollary 11.2 need not belong to any of the weighted Bloch spaces.

Acknowledgments

Parts of this work were done while the author's host institution was Middle East Technical University and during his sabbatical visit to the University of Virginia. He thanks Middle East Technical University for granting the sabbatical leave. He also thanks the Department of Mathematics of the University of Virginia, the Operator Theory Group there, and especially Thomas Kriete and Barbara MacCluer, for their hospitality.

References

[1] A. Aleman, A.G. Siskakis, Integration operators on Bergman spaces, Indiana Univ. Math. J. 46 (1997) 337–356.
[2] D. Alpay, H.T. Kaptanoğlu, Some finite-dimensional backward-shift-invariant subspaces in the ball and a related interpolation problem, Integral Equations Operator Theory 42 (2002) 1–21.
[3] D. Alpay, H.T. Kaptanoğlu, Toeplitz operators on Arveson and Dirichlet spaces, Integral Equations Operator Theory (2007), in press.
[4] J. Arazy, S.D. Fisher, J. Peetre, Möbius invariant function spaces, J. Reine Angew. Math. 363 (1985) 110–145.
[5] N. Arcozzi, R. Rochberg, E. Sawyer, Carleson measures for analytic Besov spaces, Rev. Mat. Iberoamericana 43 (1991) 443–510.
[6] W. Arveson, Subalgebras of C^*-algebras III: Multivariable operator theory, Acta Math. 181 (1998) 119–228.
[7] R. Aulaskari, D.A. Stegenga, J. Xiao, Some subclasses of BMOA and their characterization in terms of Carleson measures, Rocky Mountain J. Math. 26 (1996) 485–506.
[8] R. Aulaskari, M. Nowak, R. Zhao, The nth derivative characterisation of Möbius-invariant Dirichlet space, Bull. Austral. Math. Soc. 58 (1998) 43–56.
[9] S. Axler, Bergman spaces and their operators, in: Surveys of Some Recent Results in Operator Theory I, in: Pitman Res. Notes Math. Ser., vol. 271, Longman, Harlow, 1988, pp. 1–50.
[10] A. Baernstein II, D. Girela, J.Á. Peláez, Univalent functions, Hardy spaces and spaces of Dirichlet type, Illinois J. Math. 48 (2004) 837–859.
[11] F. Beatrous, J. Burbea, Holomorphic Sobolev spaces on the ball, Diss. Math. 276 (1986), 57 pp.
[12] J.S. Choa, H.O. Kim, Y.Y. Park, A Bergman–Carleson measure characterization of Bloch functions in the unit ball of $\mathbb{C}^n$, Bull. Korean Math. Soc. 29 (1992) 285–293.
[13] B.R. Choe, H. Koo, H. Yi, Carleson type conditions and weighted inequalities for harmonic functions, Osaka J. Math. 39 (2002) 945–962.
[14] C. Cowen, B.D. MacCluer, Composition Operators on Spaces of Analytic Functions, Stud. Adv. Math., CRC, Boca Raton, FL, 1995.
[15] P.L. Duren, Extension of a theorem of Carleson, Bull. Amer. Math. Soc. 75 (1969) 143–146.
[16] P. Duren, A. Schuster, Bergman Spaces, Math. Surveys Monogr., vol. 100, Amer. Math. Soc., Providence, RI, 2004.
[17] F. Forelli, W. Rudin, Projections on spaces of holomorphic functions in balls, Indiana Univ. Math. J. 24 (1974) 593–602.
[18] D. Girela, J.Á. Peláez, Carleson measures for spaces of Dirichlet type, Integral Equations Operator Theory 55 (2006) 415–427.
[19] H. Hedenmalm, B. Korenblum, K. Zhu, Theory of Bergman Spaces, Grad. Texts in Math., vol. 199, Springer, New York, 2000.
[20] Z. Hu, Extended Cesàro operators on mixed norm spaces, Proc. Amer. Math. Soc. 131 (2003) 2171–2179.
[21] M. Jevtić, X. Massaneda, P.J. Thomas, Interpolating sequences for weighted Bergman spaces of the ball, Michigan Math. J. 43 (1996) 495–517.
[22] H.T. Kaptanoğlu, Besov spaces and Bergman projections on the ball, C. R. Math. Acad. Sci. Paris 335 (2002) 729–732.
[23] H.T. Kaptanoğlu, Bergman projections on Besov spaces on balls, Illinois J. Math. 49 (2005) 385–403.
[24] H.T. Kaptanoğlu, Besov spaces and Carleson measures on the ball, C. R. Math. Acad. Sci. Paris 343 (2006) 453–456.
[25] H.T. Kaptanoğlu, A.E. Üreyen, Analytic properties of Besov spaces via Bergman projections, in: Contemp. Math. Amer. Math. Soc., 2007, in press.
[26] O. Kures, K. Zhu, A class of integral operators on the unit ball of $\mathbb{C}^n$, preprint, 2006.
[27] D.H. Luecking, Embedding theorems for spaces of analytic functions via Khinchine's inequality, Michigan Math. J. 40 (1993) 333–358.
[28] M. Mateljević, M. Pavlović, L^p behavior of power series with positive coefficients and Hardy spaces, Proc. Amer. Math. Soc. 87 (1983) 309–316.
[29] R. Meise, D. Vogt, Introduction to Functional Analysis, Oxf. Grad. Texts Math., vol. 2, Oxford Univ. Press, New York, 1997.
[30] N. Mochizuki, Inequalities of Fejér–Riesz and Hardy–Littlewood, Tohoku Math. J. 40 (1988) 77–86.
[31] F. Pérez-González, J. Rättyä, Forelli–Rudin estimates, Carleson measures and $F(p,q,s)$-functions, J. Math. Anal. Appl. 315 (2006) 394–414.
[32] W. Rudin, Function Theory in the Unit Ball of $\mathbb{C}^n$, Grundlehren Math. Wiss., vol. 241, Springer, New York, 1980.
[33] K. Stroethoff, Generalizations of the Forelli–Rudin estimates, J. Math. Anal. Appl. 252 (2000) 936–950.
[34] R.M. Timoney, Bloch functions in several complex variables, I, Bull. London Math. Soc. 12 (1980) 241–267.

[35] M. Tjani, Compact composition operators on Besov spaces, Trans. Amer. Math. Soc. 355 (2003) 4683–4698.
[36] Z. Wu, Carleson measures and multipliers for Dirichlet spaces, J. Funct. Anal. 169 (1999) 148–163.
[37] S. Yamashita, Gap series and α-Bloch functions, Yokohama Math. J. 28 (1980) 31–36.
[38] W. Yang, Carleson type measure characterization of Q_p spaces, Analysis 18 (1998) 345–349.
[39] W. Yang, Vanishing Carleson type measure characterization of $Q_{p,0}$, C. R. Math. Acad. Sci. Soc. R. Can. 21 (1999) 1–5.
[40] R. Yoneda, Pointwise multipliers from BMOA^{α} to BMOA^{β}, Complex Var. Theory Appl. 49 (2004) 1045–1061.
[41] R. Yoneda, Multiplication operators, integration operators, and companion operators on weighted Bloch space, Hokkaido Math. J. 34 (2005) 135–147.
[42] K. Zhu, Positive Toeplitz operators on weighted Bergman spaces of symmetric domains, J. Operator Theory 20 (1988) 329–357.
[43] K. Zhu, Operator Theory in Function Spaces, Monogr. Textbooks Pure Appl. Math., vol. 139, Dekker, New York, 1990.
[44] K. Zhu, Holomorphic Besov spaces on bounded symmetric domains, Q. J. Math. 46 (1995) 239–256.
[45] K. Zhu, Spaces of Holomorphic Functions in the Unit Ball, Grad. Texts in Math., vol. 226, Springer, New York, 2005.
[46] A. Zygmund, Trigonometric Series I and II, Cambridge Math. Lib., Cambridge Univ. Press, Cambridge, 1988.