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THE INTEGER KNAPSACK COVER POLYHEDRON*

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Abstract. We study the integer knapsack cover polyhedron which is the convex hull of the set of vectors $x \in \mathbb{Z}^n_+$ that satisfy $C^T x \geq b$, with $C \in \mathbb{Z}^n_{++}$ and $b \in \mathbb{Z}_{++}$. We present some general results about the nontrivial facet-defining inequalities. Then we derive specific families of valid inequalities, namely, rounding, residual capacity, and lifted rounding inequalities, and identify cases where they define facets. We also study some known families of valid inequalities called 2-partition inequalities and improve them using sequence-independent lifting.

Key words. integer knapsack cover polyhedron, valid inequalities, facets, sequence-independent lifting

AMS subject classifications. 90C10, 90C57

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1. Introduction.

The purpose of this paper is to study the integer knapsack cover polyhedron. Let $N = \{1, 2, \ldots, n\}$. Item $i \in N$ has capacity c_i. We would like to cover a demand of b using integer amounts of items in N. We assume that b and c_i for $i \in N$ are positive integers.

We are interested in the integer knapsack cover set

$$X = \left\{ x \in \mathbb{Z}^n_+ : \sum_{i \in N} c_i x_i \geq b \right\} \tag{1}$$

and its convex hull $PX = conv(X)$. The constraint $\sum_{i \in N} c_i x_i \geq b$ is called the *cover constraint*.

Set X is a relaxation of the feasible sets of many optimization problems involving demands that may be covered with different types of items. Pochet and Wolsey [15] study a special case to derive valid inequalities for a network design problem. Mazur [11] uses the polyhedral results on PX to generate strong valid inequalities for the multifacility location problem. Yaman [18] uses the same relaxation to strengthen formulations for the heterogeneous vehicle routing problem, which generalizes the well-known capacitated vehicle routing problem by introducing the choice between different vehicle types. Yaman and Sen [19] arrive at the same relaxation in the context of the manufacturer's mixed pallet design problem, where each customer can buy integer numbers of pallets with different configurations to satisfy its demand. Knowledge about polyhedral properties of PX can be used in deriving strong formulations for these problems. For recent work in understanding the structure of simple mixed integer and integer sets, see, e.g., [3, 7, 12, 13, 15].

There has been a lot of work on the polytope of the 0/1 knapsack problem (e.g., [5, 8, 9, 16, 17, 20]). The situation is different for the integer knapsack cover polyhedron. Despite the many application areas where set X may appear as a relaxation, the literature on the polyhedral properties of its convex hull is quite limited.

Pochet and Wolsey [15] study the special case where c_{i+1} is an integer multiple of c_i for all $i = 1, 2, \ldots, n-1$. They derive the partition inequalities and show that

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these inequalities define the convex hull together with the nonnegativity constraints. They derive conditions under which these inequalities are valid in the general case.

Mazur [11] and Mazur and Hall [12] study the general case. They show that $\dim(PX) = n$, $x_i \geq 0$ defines a facet of PX for $i \in N$, and if $\sum_{i\in N} \alpha_i x_i \geq \alpha_0$ is a nontrivial facet-defining inequality of PX, then $\alpha_i > 0$ for all $i \in N$ and $\alpha_0 > 0$. Let $c_1^{'}, \dots, c_m^{'}$ be the distinct c_i values that are less than b. An important result by Mazur [11] is that, if one knows the description of $conv(\{x \in \mathbb{Z}_+^m : \sum_{i=1}^m c_i^{'} x_i \geq b\})$, it is trivial to obtain the description of PX. The inequality $\sum_{i\in N} \alpha_i x_i \geq \alpha_0$ is a nontrivial facet-defining inequality for PX if and only if $\alpha_i = \alpha_j$ for all $i, j \in N$ with $c_i = c_j$, $\alpha_i = \alpha_0$ for all $i \in N$ with $c_i \geq b$, and $\sum_{i=1}^m \alpha_i^{'} x_i \geq \alpha_0$ is a nontrivial facet-defining inequality for $conv(\{x \in \mathbb{Z}_+^m : \sum_{i=1}^m c_i^{'} x_i \geq b\})$, where $\alpha_i^{'} = \alpha_j$ if $c_i^{'} = c_j$ for $i = 1, \cdots, m$ and $j \in N$. So interesting instances satisfy $c_1 < c_2 < \cdots < c_n < b$.

Mazur and Hall [12] also study the integer capacity cover polyhedron defined as the convex hull of the set $\{(y, x) \in \{0,1\}^q \times \mathbb{Z}_+^n : \sum_{i\in N} c_i x_i \geq \sum_{i=1}^q y_i\}$. They use simultaneous lifting to derive facet-defining inequalities for this polyhedron using those of the integer knapsack cover polyhedron. They remark that little is known about the polyhedral properties of the latter polyhedron, and it is difficult to identify its facets.

Atamturk [1] presents a family of facet-defining inequalities and lifting results for the polytope $conv(X \cap \{x \in \mathbb{Z}^n : x \leq u\})$ for $u \in \mathbb{Z}_{++}^n$.

In this paper, we derive several families of valid inequalities and discuss when they define facets of PX. We investigate the domination relations between these families of valid inequalities. Most of our results on facet-defining inequalities are for the special case where $c_1 = 1$.

This work is motivated by the results of Mazur and Hall [12], where valid inequalities for the integer knapsack cover polyhedron are lifted to valid inequalities for a more complicated polyhedron, the integer capacity cover polyhedron. We are also motivated by the positive results in [18, 19], which demonstrate the use of simple valid inequalities based on the integer knapsack cover relaxation in closing the duality gap for complicated mixed integer programming problems studied in these papers.

The paper is organized as follows. In section 2, we give the general properties of nontrivial facet-defining inequalities of PX. In sections 3–6, we introduce four families of valid inequalities, namely, rounding, residual capacity, lifted rounding, and lifted 2-partition inequalities. We compare their relative strengths and give conditions under which they define facets of PX. In section 7, we investigate the use of lifted rounding and lifted 2-partition inequalities in solving the manufacturer's mixed pallet design problem introduced by Yaman and Sen [19]. We conclude in section 8.

2. General results on facet-defining inequalities. In this section, we derive general properties of nontrivial facet-defining inequalities of PX.

In the sequel, we assume that $c_1, \dots, c_n$ and b are positive integers and that they satisfy $c_1 < c_2 < \cdots < c_n < b$ (this assumption is made without loss of generality due to the result of Mazur [11] mentioned above). Let c be the greatest common divisor of c_i's. We replace c_i with $\frac{c_i}{c}$ for each $i \in N$ and b with $\lceil \frac{b}{c} \rceil$. This does not change the set X but strengthens the cover constraint. Let e_i denote the n-dimensional unit vector with 1 at the ith place and 0 elsewhere.

PROPOSITION 1. *Let $\sum_{i\in N} \alpha_i x_i \geq \alpha_0$ be a nontrivial facet-defining inequality for PX. Then*

$$0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \alpha_0 \leq \min_{i\in N} \alpha_i \left\lceil \frac{b}{c_i} \right\rceil.$$

Proof. Suppose that $\sum_{i \in N} \alpha_i x_i \geq \alpha_0$ is a nontrivial facet-defining inequality for PX. The fact that $\alpha_i > 0$ for $i = 0, 1, \ldots, n$ is proved in [11, 12].

Let j and l be such that $j < l$ and $x \in PX$ be such that $\sum_{i \in N} \alpha_i x_i = \alpha_0$, with $x_j \geq 1$. Consider $x' = x - e_j + e_l$. As $c_l > c_j$, $x' \in PX$. Then $\sum_{i \in N} \alpha_i x_i' \geq \alpha_0$, implying that $\alpha_l \geq \alpha_j$. So $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$.

Let $x \in PX$ be such that $\sum_{i \in N} \alpha_i x_i = \alpha_0$, with $x_n \geq 1$. Then $\alpha_n x_n \leq \alpha_0$ and, as $x_n \geq 1$, $\alpha_n \leq \alpha_0$.

For $i \in N$, $x = \lceil \frac{b}{c_i} \rceil e_i$ is in PX, and so $\alpha_i \lceil \frac{b}{c_i} \rceil \geq \alpha_0$. Thus $\alpha_0 \leq \min_{i \in N} \alpha_i \lceil \frac{b}{c_i} \rceil$. ∎

We have a necessary condition for a nontrivial inequality to be facet-defining.

THEOREM 1. *Let $\sum_{i \in N} \alpha_i x_i \geq \alpha_0$ be a nontrivial facet-defining inequality for PX. Let $j \in \arg\max_{i \in N} \frac{c_i}{\alpha_i}$. Then $(\alpha_0 - \alpha_i)\frac{c_j}{\alpha_j} + c_i \geq b$ for all $i \in N \setminus \{j\}$.*

Proof. Assume that there exists $l \in N \setminus \{j\}$ such that $(\alpha_0 - \alpha_l)\frac{c_j}{\alpha_j} + c_l < b$. Let $x \in X$ be such that $\sum_{i \in N} \alpha_i x_i = \alpha_0$. Then $x_j = \frac{\alpha_0 - \sum_{i \in N \setminus \{j\}} \alpha_i x_i}{\alpha_j}$. The left-hand side of the cover constraint evaluated at x is $\sum_{i \in N} c_i x_i = \sum_{i \in N \setminus \{j\}} (c_i - \frac{c_j}{\alpha_j}\alpha_i) x_i + \frac{c_j}{\alpha_j}\alpha_0$. This is less than or equal to $(c_l - \frac{c_j}{\alpha_j}\alpha_l)x_l + \frac{c_j}{\alpha_j}\alpha_0$, since $c_i - \frac{c_j}{\alpha_j}\alpha_i \leq 0$ for all $i \in N \setminus \{j\}$. Now as $(\alpha_0 - \alpha_l)\frac{c_j}{\alpha_j} + c_l < b$ and $c_l - \frac{c_j}{\alpha_j}\alpha_l \leq 0$, whenever $x_l \geq 1$, $(c_l - \frac{c_j}{\alpha_j}\alpha_l)x_l + \frac{c_j}{\alpha_j}\alpha_0 < b$. This proves that, for any $x \in X$ such that $\sum_{i \in N} \alpha_i x_i = \alpha_0$, we have $x_l = 0$. ∎

Next, we give necessary and sufficient conditions for some inequalities to be facet-defining. Later, we use this result to identify specific families of facet-defining inequalities.

THEOREM 2. *Let $\sum_{i \in N} \alpha_i x_i \geq \alpha_0$ be a valid inequality for PX, with $\alpha_i > 0$ and integer for all $i \in N \cup \{0\}$ and $\alpha_1 = 1$. Let j be the largest index, with $\alpha_j = 1$. If $\alpha_i \geq \frac{c_i}{c_j}$ for all $i = j+1, \ldots, n$, then the inequality $\sum_{i \in N} \alpha_i x_i \geq \alpha_0$ is facet-defining for PX if and only if $(\alpha_0 - \alpha_i)c_j + c_i \geq b$ for $i = j+1, \ldots, n$ and $(\alpha_0 - 1)c_j + c_1 \geq b$.*

Proof. If the conditions of the theorem are satisfied, then $\alpha_0 e_j$, $(\alpha_0 - 1)e_j + e_i$ for $i = 1, \ldots, j-1$, and $(\alpha_0 - \alpha_i)e_j + e_i$ for $i = j+1, \ldots, n$ are in PX; they satisfy $\sum_{i \in N} \alpha_i x_i = \alpha_0$ and are affinely independent. This proves that the inequality $\sum_{i \in N} \alpha_i x_i \geq \alpha_0$ is facet-defining for PX.

The necessity of the conditions are implied by Theorem 1. ∎

To conclude this section, we investigate when the cover constraint is facet-defining for PX. If c_j divides b for all $j \in N$, then the nonnegativity constraints and the cover constraint describe the polyhedron PX, i.e., $PX = \{x \in \mathbb{R}^n_+ : \sum_{j \in N} c_j x_j \geq b\}$.

Using Theorem 2, we identify another case where the cover constraint is facet-defining.

COROLLARY 1. *If $c_1 = 1$, then the cover constraint is facet-defining for PX.*

The conclusion of Theorem 1 is trivially satisfied for the cover constraint. But the cover constraint is not necessarily facet-defining for PX. The following simple example proves this statement.

Example 1. Let $X^1 = \{x \in \mathbb{Z}^2_+ : 3x_1 + 4x_2 \geq 14\}$. The polyhedron $conv(X^1) = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 + x_2 \geq 4, 2x_1 + 3x_2 \geq 10\}$.

3. Rounding inequalities.

In this section, we derive a family of valid inequalities, called the *rounding inequalities*, and identify some cases where they are facet-defining for PX.

For $\lambda > 0$, the rounding inequality

$$\sum_{i \in N} \left\lceil \frac{c_i}{\lambda} \right\rceil x_i \geq \left\lceil \frac{b}{\lambda} \right\rceil \tag{2}$$

is a valid inequality for PX. It is obtained using the well-known Chvatal–Gomory procedure (see, e.g., Nemhauser and Wolsey [14]). These inequalities have been used by Yaman [18]. Here we investigate under which conditions these inequalities are facet-defining for PX. The inequality for $\lambda = c_n$ is $\sum_{i\in N} x_i \geq \lceil \frac{b}{c_n} \rceil$. Mazur [11] proves that this inequality is facet-defining for PX if and only if $b \leq \left(\lceil \frac{b}{c_n} \rceil - 1\right)c_n + c_1$. Inequality (2) for any $\lambda > c_n$ is dominated by the corresponding inequality for c_n. So we are interested in $\lambda < c_n$.

The result below is a corollary to Theorem 2.

COROLLARY 2. *Let λ be such that $c_j \leq \lambda < c_{j+1}$ for some $j \in \{1, \dots, n-1\}$. If $\lceil \frac{c_i}{\lambda} \rceil \geq \frac{c_i}{c_j}$ for all $i = j+1, \dots, n$, then inequality* (2) *is facet-defining if and only if $\left(\lceil \frac{b}{\lambda} \rceil - 1\right)c_j + c_1 \geq b$ and $\left(\lceil \frac{b}{\lambda} \rceil - \lceil \frac{c_i}{\lambda} \rceil\right)c_j + c_i \geq b$ for all $i = j+1, \dots, n$.*

Proof. As $\lceil \frac{c_i}{\lambda} \rceil$ for $i \in N$ and $\lceil \frac{b}{\lambda} \rceil$ are positive integers, $\lceil \frac{c_1}{\lambda} \rceil = 1$, j is the largest index with coefficient 1 in inequality (2), and $\lceil \frac{c_i}{\lambda} \rceil \geq \frac{c_i}{c_j}$ for all $i = j+1, \dots, n$, Theorem 2 applies. ∎

We have a necessary condition as a corollary to Theorem 1.

COROLLARY 3. *Let $\lambda > 0$. If there exists $j \in N$ such that c_j is divisible by λ and if inequality* (2) *is facet-defining for PX, then $(\lceil \frac{b}{\lambda} \rceil - \lceil \frac{c_i}{\lambda} \rceil)\lambda + c_i \geq b$ for all $i \in N \setminus \{j\}$.*

Proof. For $i \in N$, $\frac{c_i}{\lceil \frac{c_i}{\lambda} \rceil} \leq \lambda$. So, if $j \in N$ is such that λ divides c_j, $j \in \arg\max_{i\in N} \frac{c_i}{\lceil \frac{c_i}{\lambda} \rceil}$, and we can apply Theorem 1. ∎

We consider the subset of inequalities (2) defined by λ equal to $c_1, \dots, c_n$. In the following corollary, we generalize the result by Mazur [11].

COROLLARY 4. *For $j \in N$, the inequality*

$$\sum_{i\in N} \left\lceil \frac{c_i}{c_j} \right\rceil x_i \geq \left\lceil \frac{b}{c_j} \right\rceil \tag{3}$$

is facet-defining for PX if and only if $\left(\lceil \frac{b}{c_j} \rceil - 1\right)c_j + c_1 \geq b$ and $\left(\lceil \frac{b}{c_j} \rceil - \lceil \frac{c_i}{c_j} \rceil\right)c_j + c_i \geq b$ for all $i = j+1, \dots, n$.

Proof. Take $\lambda = c_j$. As $\lceil \frac{c_i}{c_j} \rceil \geq \frac{c_i}{c_j}$ for all $i = j+1, \dots, n$, we apply Corollary 2 to obtain the result. ∎

Atamturk [1] studies the polytope $conv(X \cap \{x \in \mathbb{Z}^n : x \leq u\})$ for $u \in \mathbb{Z}^n_{++}$ and proves that inequality (3) for $j \in N$ such that $u_j c_j \geq b$ is facet-defining if and only if the conditions of Corollary 4 are satisfied.

We go back to Example 1 and see if rounding inequalities are facet-defining.

Example 2. Consider set X^1 defined in Example 1. The rounding inequality for $\lambda = c_1$ is not facet-defining since $\left(\lceil \frac{14}{3} \rceil - \lceil \frac{4}{3} \rceil\right)3 + 4 = 13 < 14 = b$. The inequality is $x_1 + 2x_2 \geq 5$ and is dominated by $2x_1 + 3x_2 \geq 10$. We can obtain the latter inequality by lifting inequality $x_1 \geq 5$, which is a rounding inequality when $x_2 = 0$ with variable x_2 (see section 5).

The rounding inequality for $\lambda = c_2$ is facet-defining since $\left(\lceil \frac{14}{4} \rceil - 1\right)4 + 3 = 15 \geq 14 = b$. This is the inequality $x_1 + x_2 \geq 4$.

The convex hull of X^1 is described by the nonnegativity constraints, a rounding inequality ($x_1 + x_2 \geq 4$), and a lifted rounding inequality ($2x_1 + 3x_2 \geq 10$).

In the next example, we see two sets that are defined by parameters which differ only in the right-hand side of the cover constraint. The rounding inequalities for $\lambda = c_2, c_3, \dots, c_n$ are facet-defining for the polyhedron when the right-hand side is b, and none are facet-defining when the right-hand side is $b + 1$.

Example 3. Consider the set $X^2 = \{x \in \mathbb{Z}_+^4 : x_1 + 4x_2 + 5x_3 + 6x_4 \geq 61\}$. The convex hull of X^2 is described by the nonnegativity constraints and the following inequalities (these results are obtained using PORTA [6]):

$$x_1 + 4x_2 + 5x_3 + 6x_4 \geq 61, \tag{4}$$

$$x_1 + 2x_2 + 3x_3 + 3x_4 \geq 31, \tag{5}$$

$$x_1 + x_2 + 2x_3 + 2x_4 \geq 16, \tag{6}$$

$$x_1 + x_2 + x_3 + 2x_4 \geq 13, \tag{7}$$

$$x_1 + x_2 + x_3 + x_4 \geq 11. \tag{8}$$

Inequality (4) is the cover constraint. By Corollary 1, as $c_1 = 1$, we know that the cover constraint is facet-defining. Inequalities (6)–(8) are rounding inequalities. It is easy to verify that the conditions of Corollary 4 are satisfied. Note that inequality (5) is the rounding inequality for $\lambda = 2$, and the conditions of Corollary 3 are satisfied.

Now consider the set $X^3 = \{x \in \mathbb{Z}_+^4 : x_1 + 4x_2 + 5x_3 + 6x_4 \geq 62\}$. The following inequalities together with the nonnegativity constraints describe the convex hull of X^3:

$$x_1 + 4x_2 + 5x_3 + 6x_4 \geq 62, \tag{9}$$

$$x_1 + 2x_2 + 3x_3 + 4x_4 \geq 32, \tag{10}$$

$$x_1 + 2x_2 + 2x_3 + 3x_4 \geq 26, \tag{11}$$

$$x_1 + 2x_2 + 2x_3 + 2x_4 \geq 22. \tag{12}$$

The cover constraint (9) is facet-defining, but the rounding inequalities for $\lambda = c_2, c_3, c_4$ do not define facets. Inequality (10) dominates the rounding inequality for $\lambda = c_2$, which is $x_1 + x_2 + 2x_3 + 2x_4 \geq 16$, (11) dominates inequality $x_1 + x_2 + x_3 + 2x_4 \geq 13$, which is the rounding inequality for $\lambda = c_3$, and (12) dominates $x_1 + x_2 + x_3 + x_4 \geq 11$, which is the rounding inequality for $\lambda = c_4$. In the following section, we will identify these inequalities (10)–(12).

4. Residual capacity inequalities. Residual capacity inequalities are introduced by Magnanti, Mirchandani, and Vachani [10] for the single arc design problem. Here we present inequalities that are based on a similar idea.

Assume that the demand b is covered using some item $j \in N$. Then at least $\lceil \frac{b}{c_j} \rceil$ units of item j need to be used. If $\lceil \frac{b}{c_j} \rceil - 1$ units are used to full capacity, then the capacity of the last unit to be used is $r_j = b - (\lceil \frac{b}{c_j} \rceil - 1)c_j$. If only $\lceil \frac{b}{c_j} \rceil - 1$ units of item j are used, then the remaining items should cover a demand equal to r_j. This is expressed in the following valid inequality.

For $j \in N$, define $N_j = \{1, 2, \ldots, j\}$ and $N_j^{'} = \{i \in N_j : c_i \geq r_j\}$. For $N^0 \subset N$ and $N^1 = N \setminus N^0$, let $X_h(N^1) = \{x \in \mathbb{Z}_+^n : \sum_{i \in N} c_i x_i \geq h, x_i = 0 \text{ for all } i \in N^0\}$.

THEOREM 3. *For $j \in N$, the inequality*

$$\sum_{i=1}^{j} \min\{c_i, r_j\} x_i + \sum_{i=j+1}^{n} c_i x_i \geq r_j \left\lceil \frac{b}{c_j} \right\rceil \tag{13}$$

is valid for PX.

Proof. If $\sum_{i \in N_j'} x_i = \lceil \frac{b}{c_j} \rceil$, then the inequality is satisfied. If $\sum_{i \in N_j'} x_i = \lceil \frac{b}{c_j} \rceil - p$ for some $p \geq 1$, then the feasibility of x implies $\sum_{i \in N_j \setminus N_j'} c_i x_i + \sum_{i=j+1}^{n} c_i x_i \geq$

$b-\sum_{i\in N_j'} c_ix_i \geq b-c_j\sum_{i\in N_j'} x_i = r_j+(p-1)c_j$. As $r_j+(p-1)c_j \geq r_jp$, inequality (13) is satisfied. ∎

For $j\in N$, if $r_j=c_j$, then b is divisible by c_j and inequality (13) is the same as the cover constraint.

THEOREM 4. *If $c_1=1$ for $j\in N$, the inequality*

$$\sum_{i=1}^{j} \min\{c_i,r_j\}x_i \geq r_j\left\lceil \frac{b}{c_j}\right\rceil \tag{14}$$

is facet-defining for $conv(X_b(N_j))$.

Proof. Let $F=\{x\in X_b(N_j): \sum_{i=1}^{j}\min\{c_i,r_j\}x_i = r_j\lceil\frac{b}{c_j}\rceil\}$. Assume that all $x\in F$ satisfy $\sum_{i=1}^{j}\alpha_ix_i=\alpha_0$. As $\lceil\frac{b}{c_j}\rceil e_j\in F$, we need $\alpha_0=\lceil\frac{b}{c_j}\rceil\alpha_j$. For $i\in N_j'$, $(\lceil\frac{b}{c_j}\rceil-1)e_j+e_i\in F$, implying that $\alpha_i=\alpha_j$. As $c_1=1$, we have $(\lceil\frac{b}{c_j}\rceil-1)e_j+r_je_1\in F$. So $\alpha_1=\frac{\alpha_j}{r_j}$. Finally, for $i\in N_j\setminus(N_j'\cup\{1\})$, $(\lceil\frac{b}{c_j}\rceil-1)e_j+e_i+(r_j-c_i)e_1\in F$. Hence, $\alpha_i=\frac{\alpha_jc_i}{r_j}$. Then $\sum_{i=1}^{j}\alpha_ix_i=\alpha_0$ is a $\frac{\alpha_j}{r_j}$ multiple of $\sum_{i=1}^{j}\min\{c_i,r_j\}x_i = r_j\lceil\frac{b}{c_j}\rceil$. ∎

For $j\in N$, if $r_j=1$, then inequality (14) is $\sum_{i=1}^{j}x_i\geq\lceil\frac{b}{c_j}\rceil$ and is the same as the rounding inequality for $\lambda=c_j$ for $conv(X_b(N_j))$. By Corollary 4, it is facet-defining since $\left(\lceil\frac{b}{c_j}\rceil-1\right)c_j+c_1=b-r_j+c_1\geq b$.

For $j=n$, $conv(X_b(N_n))=PX$, and the following result can be deduced from Theorem 4.

COROLLARY 5. *If $c_1=1$, inequality* (13) *for $j=n$ is facet-defining for PX.*

Example 4. Consider the set X^3 given in Example 3. For item 2, $r_2=2$ and $\lceil\frac{b}{c_2}\rceil=16$. Inequality (13) for item 2 is $x_1+2x_2+5x_3+6x_4\geq 32$ and is dominated by inequality (10). For item 3, $r_3=2$ and $\lceil\frac{b}{c_3}\rceil=13$. The corresponding inequality (13) is $x_1+2x_2+2x_3+6x_4\geq 26$ and is dominated by inequality (11). For item 4, $r_4=2$ and $\lceil\frac{b}{c_4}\rceil=11$. Inequality (13) is $x_1+2x_2+2x_3+2x_4\geq 22$ and is the same as inequality (12). In the remaining of this section, we will try to identify inequalities (10) and (11).

We can generalize inequality (13) as follows.

THEOREM 5. *For $j\in N$, let $\mu\geq 0$ be such that $\lceil\frac{r_j(r_j+\mu)}{c_j}+\mu\rceil\geq r_j$ and $r_j+\mu\leq c_j$. The inequality*

$$\sum_{i=1}^{j}\min\{c_i,r_j\}x_i+\sum_{i=j+1}^{n}\left\lceil\frac{c_i(r_j+\mu)}{c_j}\right\rceil x_i\geq r_j\left\lceil\frac{b}{c_j}\right\rceil \tag{15}$$

is valid for PX.

Proof. If $\sum_{i\in N_j'}x_i=\lceil\frac{b}{c_j}\rceil$, then the inequality is satisfied. If $\sum_{i\in N_j'}x_i=\lceil\frac{b}{c_j}\rceil-1$, then inequality (15) simplifies to $\sum_{i\in N_j\setminus N_j'}c_ix_i+\sum_{i=j+1}^{n}\lceil\frac{c_i(r_j+\mu)}{c_j}\rceil x_i\geq r_j$. By feasibility, we need to have $\sum_{i\in N_j\setminus N_j'}c_ix_i+\sum_{i=j+1}^{n}c_ix_i\geq r_j$. Using coefficient reduction, we obtain $\sum_{i\in N_j\setminus N_j'}c_ix_i+\sum_{i=j+1}^{n}r_jx_i\geq r_j$. As $\lceil\frac{c_i(r_j+\mu)}{c_j}\rceil\geq r_j$ for all $i=j+1,\ldots,n$, inequality (15) is satisfied.

If $\sum_{i\in N_j'}x_i=\lceil\frac{b}{c_j}\rceil-p$ for some $p\geq 2$, then inequality (15) simplifies to $\sum_{i\in N_j\setminus N_j'}c_ix_i+\sum_{i=j+1}^{n}\lceil\frac{c_i(r_j+\mu)}{c_j}\rceil x_i\geq r_jp$. The feasibility of x implies that $\sum_{i\in N_j\setminus N_j'}c_ix_i+\sum_{i=j+1}^{n}c_ix_i\geq r_j+(p-1)c_j$. We multiply this inequality with $\frac{r_j+\mu}{c_j}$

and obtain $\sum_{i\in N_j\setminus N_j'} c_i\frac{r_j+\mu}{c_j}x_i + \sum_{i=j+1}^{n} c_i\frac{r_j+\mu}{c_j}x_i \geq \frac{r_j(r_j+\mu)}{c_j} + (p-1)(r_j+\mu)$. Now, as $r_j+\mu \leq c_j$ and so $\sum_{i\in N_j\setminus N_j'} c_ix_i + \sum_{i=j+1}^{n}\lceil\frac{c_i(r_j+\mu)}{c_j}\rceil x_i \geq \sum_{i\in N_j\setminus N_j'} c_i\frac{(r_j+\mu)}{c_j}x_i + \sum_{i=j+1}^{n} c_i\frac{(r_j+\mu)}{c_j}x_i$, we have $\sum_{i\in N_j\setminus N_j'} c_ix_i + \sum_{i=j+1}^{n}\lceil\frac{c_i(r_j+\mu)}{c_j}\rceil x_i \geq \frac{r_j(r_j+\mu)}{c_j} + (p-1)(r_j+\mu)$. Since the left-hand side is always an integer, we round up the right-hand side and get $\lceil\frac{r_j(r_j+\mu)}{c_j} + (p-1)\mu\rceil + (p-1)r_j$. As $\lceil\frac{r_j(r_j+\mu)}{c_j}+\mu\rceil \geq r_j$, $\mu \geq 0$, and $p \geq 2$, we obtain $\sum_{i\in N_j\setminus N_j'} c_ix_i + \sum_{i=j+1}^{n}\lceil\frac{c_i(r_j+\mu)}{c_j}\rceil x_i \geq r_jp$. So x satisfies inequality (15). ∎

For $\mu = c_j - r_j$, inequality (15) is the same as inequality (13).

As μ increases, inequality (15) gets weaker. So for given $j \in N$, we are interested in inequality (15) defined by the smallest μ that satisfies the condition $\lceil\frac{r_j(r_j+\mu)}{c_j}+\mu\rceil \geq r_j$. Let $\epsilon > 0$ be very small. We take $\mu_j = \frac{c_j(r_j-1)-r_j^2}{r_j+c_j} + \epsilon$, if $\lceil\frac{r_j^2}{c_j}\rceil < r_j$, and $\mu_j = 0$, otherwise.

Observe that nondominated residual capacity inequalities (15) are defined per item, so there are $O(n)$ of them.

Example 5. Consider again the set X^3 of Example 3. For item 2, $r_2 = 2$. As $\lceil\frac{r_2^2}{c_2}\rceil = 1 < 2 = r_2$, $\mu_2 = \frac{4(2-1)-4}{2+4} + \epsilon = \epsilon$. The corresponding inequality (15) is $x_1 + 2x_2 + 3x_3 + 4x_4 \geq 32$ and is the same as inequality (10). For item 3, $r_3 = 2$. As $\lceil\frac{r_3^2}{c_3}\rceil = 1 < 2 = r_3$, $\mu_3 = \frac{5(2-1)-4}{2+5} + \epsilon = \frac{1}{7} + \epsilon$. The corresponding inequality (15) is $x_1 + 2x_2 + 2x_3 + 3x_4 \geq 26$ and is the same as inequality (11).

If $r_j = 1$, then $\mu_j = 0$ and inequality (15) is the same as the rounding inequality (3) for $\lambda = c_j$.

If $r_j = c_j$, then again $\mu_j = 0$. This time inequality (15) is the same as the cover constraint.

We have a necessary condition for inequality (15) to be facet-defining.

Corollary 6. *For $j \in N$, if inequality* (15) *is facet-defining for PX and $r_j < c_j$, then $c_i + \lceil\frac{b}{c_j}\rceil c_j - \frac{c_j}{r_j}\lceil\frac{c_i(r_j+\mu_j)}{c_j}\rceil \geq b$ for all $i = j+1,\ldots,n$.*

Proof. As $c_i - \frac{c_j}{r_j}\min\{c_i, r_j\} \leq 0$ for all $i = 1,\ldots,j-1$ and $\left(c_i - \frac{c_j}{r_j}\lceil\frac{c_i(r_j+\mu_j)}{c_j}\rceil\right) \leq 0$ for all $i = j+1,\ldots,n$, we apply Theorem 1. So, if inequality (15) is facet-defining for PX, then $\lceil\frac{b}{c_j}\rceil c_j - \min\{c_i, r_j\}\frac{c_j}{r_j} + c_i \geq b$ for $i = 1,\ldots,j-1$ and $\lceil\frac{b}{c_j}\rceil c_j - \frac{c_j}{r_j}\lceil\frac{c_i(r_j+\mu_j)}{c_j}\rceil + c_i \geq b$ for all $i = j+1,\ldots,n$.

For $i \in N_j^{'}$, the condition is $\lceil\frac{b}{c_j}\rceil c_j - c_j + c_i \geq b$. The left-hand side is equal to $\lfloor\frac{b}{c_j}\rfloor c_j + c_i \geq \lfloor\frac{b}{c_j}\rfloor c_j + r_j = b$. For $i \in N_j \setminus N_j^{'}$, the condition is $\lceil\frac{b}{c_j}\rceil c_j - c_i\frac{c_j}{r_j} + c_i \geq b$. The left-hand side is equal to $b - r_j + c_j - c_i\frac{c_j-r_j}{r_j} = b + (c_j - r_j)\frac{(r_j-c_i)}{r_j} \geq b$ since $c_j \geq r_j$ and $r_j \geq c_i$. So the conditions of Theorem 1 are always satisfied for $i \in N_j$. ∎

5. Lifted rounding inequalities. In this section, we derive valid inequalities using lifting. For $N^0 \subset N$ and $N^1 = N\setminus N^0$, let $\sum_{i\in N^1}\alpha_ix_i \geq \alpha_0$ be a valid inequality for $X_b(N^1)$.

Suppose we lift inequality $\sum_{i\in N^1}\alpha_ix_i \geq \alpha_0$, with x_l with $l \in N^0$. The optimal lifting coefficient of x_l is

$$\alpha_l = \max\frac{\alpha_0 - \sum_{i\in N^1}\alpha_ix_i}{x_l}$$
$$\text{s.t. } x_l \geq 1$$
$$x \in X_b(N^1 \cup \{l\}).$$

Consider the case where $\alpha_i = 1$ for all $i \in N^1$, $j = \arg\max_{i\in N^1} c_i$, and $\alpha_0 = \lceil \frac{b}{c_j} \rceil$. For $l \in N^0$, the nonlinear lifting problem simplifies to

$$\alpha_l = \max_{x_l \in \mathbb{Z}_{++}} \frac{\lceil \frac{b}{c_j} \rceil - \lceil \frac{(b-c_l x_l)^+}{c_j} \rceil}{x_l}.$$

Clearly, a maximizing x_l cannot be larger than $\lceil \frac{b}{c_l} \rceil$. Hence, we obtain

$$\alpha_l = \max_{x_l \in \{1,2,\dots,\lceil \frac{b}{c_l} \rceil\}} \frac{\lceil \frac{b}{c_j} \rceil - \lceil \frac{(b-c_l x_l)^+}{c_j} \rceil}{x_l},$$

and we can compute α_l by enumeration.

Example 6. Consider the set X^1 defined in Example 1. Inequality $x_1 \geq 5$ is facet-defining for $conv(X^1 \cap \{x \in \mathbb{Z}^2_+ : x_2 = 0\})$. We lift inequality $x_1 \geq 5$ with variable x_2. The optimal lifting coefficient $\alpha_2 = \max_{x_2 \in \{1,2,3,4\}} \frac{5 - \lceil \frac{(14-4x_2)^+}{3} \rceil}{x_2} = \max\{1, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}\} = \frac{3}{2}$. The corresponding inequality is $2x_1 + 3x_2 \geq 10$ and is facet-defining for $conv(X^1)$.

Computation of the optimal lifting coefficients of variables that are lifted in later in the sequence may become harder. So we are interested in sequence-independent lifting.

Atamturk [4] studies sequence-independent lifting for mixed integer programming. The following can be derived from his results. Consider the lifting function $\Phi(a) = \alpha_0 - \min_{x \in X_{b-a}(N^1)} \sum_{i \in N^1} \alpha_i x_i$. If this function is subadditive, i.e., if $\Phi(a) + \Phi(d) \geq \Phi(a+d)$ for all $a, d \in \mathbb{R}$, then the lifting is sequence-independent. In this case, the inequality $\sum_{i\in N^1} \alpha_i x_i + \sum_{i \in N^0} \Phi(c_i) x_i \geq \alpha_0$ is a valid inequality for PX. In the general case, let Θ be a subadditive function, with $\Theta \geq \Phi$. Then the inequality $\sum_{i\in N^1} \alpha_i x_i + \sum_{i\in N^0} \Theta(c_i) x_i \geq \alpha_0$ is a valid inequality for PX. If the inequality $\sum_{i \in N^1} \alpha_i x_i \geq \alpha_0$ is facet-defining for $conv(X_b(N^1))$ and $\Theta(c_i) = \Phi(c_i)$ for all $i \in N^0$, then inequality $\sum_{i\in N^1} \alpha_i x_i + \sum_{i\in N^0} \Theta(c_i) x_i \geq \alpha_0$ is facet-defining for PX.

THEOREM 6. *Let $N^1 \subset N$ and $\sum_{i\in N^1} \alpha_i x_i \geq \alpha_0$ be a valid inequality for $X_b(N^1)$. If there exists $j \in N^1$ such that $\alpha_i \geq \alpha_j \lceil \frac{c_i}{c_j} \rceil$ for all $i \in N^1 \setminus \{j\}$, then the lifting function is*

$$\Phi(a) = \alpha_0 - \alpha_j \left\lceil \frac{(b-a)^+}{c_j} \right\rceil.$$

Proof. Suppose there exists $j \in N^1$ such that $\alpha_i \geq \alpha_j \lceil \frac{c_i}{c_j} \rceil$ for all $i \in N^1 \setminus \{j\}$. The lifting function is $\Phi(a) = \alpha_0 - \min_{x\in X_{b-a}(N^1)} \sum_{i\in N^1} \alpha_i x_i$. Let x be an optimal solution to the minimization problem. Consider $\overline{x} = x - \sum_{i \in N^1 \setminus \{j\}} x_i e_i + \lceil \frac{\sum_{i\in N^1\setminus\{j\}} c_i x_i}{c_j} \rceil e_j$. Clearly, $\overline{x} \in X_{b-a}(N^1)$. The objective function evaluated at $\overline{x}$ is equal to

$$\begin{aligned}\sum_{i\in N^1} \alpha_i \overline{x}_i &= \sum_{i \in N^1} \alpha_i x_i - \sum_{i\in N^1\setminus\{j\}} \alpha_i x_i + \alpha_j \left\lceil \frac{\sum_{i\in N^1\setminus\{j\}} c_i x_i}{c_j} \right\rceil \\ &\leq \sum_{i \in N^1} \alpha_i x_i - \sum_{i\in N^1\setminus\{j\}} \alpha_i x_i + \alpha_j \sum_{i\in N^1\setminus\{j\}} \left\lceil \frac{c_i}{c_j} \right\rceil x_i.\end{aligned}$$

As $\alpha_i \geq \alpha_j \lceil \frac{c_i}{c_j} \rceil$ for all $i \in N^1 \setminus \{j\}$, $\sum_{i\in N^1} \alpha_i \overline{x}_i \leq \sum_{i \in N^1} \alpha_i x_i$, and so $\overline{x}$ is also optimal. Hence $\lceil \frac{(b-a)^+}{c_j} \rceil e_j$ is also optimal and the optimal value is $\alpha_j \lceil \frac{(b-a)^+}{c_j} \rceil$. $\square$

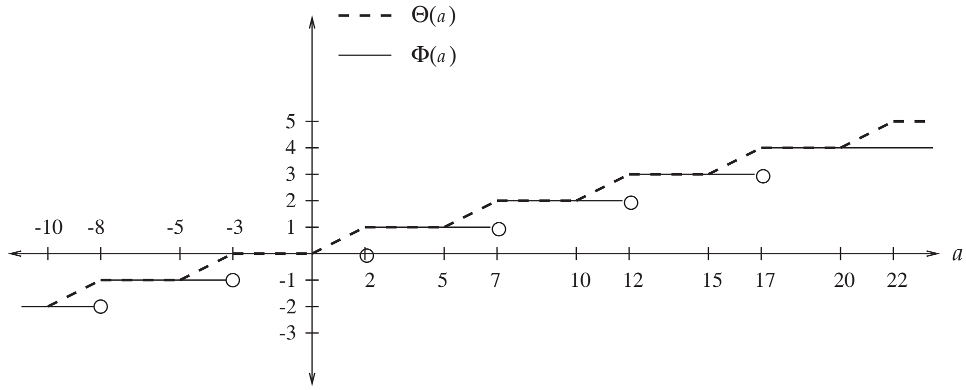

FIG. 1. *Lifting function Φ and subadditive function Θ for $b = 17$ and $c_j = 5$.*

Suppose there exists $j \in N^1$ such that $\alpha_i \geq \alpha_j \lceil \frac{c_i}{c_j} \rceil$ for all $i \in N^1 \setminus \{j\}$, $\alpha_j = 1$, and $\alpha_0 = \lceil \frac{b}{c_j} \rceil$. The lifting function for the inequality $\sum_{i \in N^1} \alpha_i x_i \geq \lceil \frac{b}{c_j} \rceil$ is $\Phi(a) = \lceil \frac{b}{c_j} \rceil - \lceil \frac{(b-a)^+}{c_j} \rceil$. The function Φ is not subadditive. An example where $b = 17$ and $c_j = 5$ is depicted in Figure 1. Here for $a = 11$ and $d = 6$, we have $\lceil \frac{b}{c_j} \rceil - \lceil \frac{b-a}{c_j} \rceil + \lceil \frac{b}{c_j} \rceil - \lceil \frac{b-d}{c_j} \rceil = 4 - 2 + 4 - 3 = 3 < \lceil \frac{b}{c_j} \rceil - \lceil \frac{b-a-d}{c_j} \rceil = 4 - 0 = 4$.

For $j \in N$ and $a \in \mathbb{R}$, define

$$\rho_j(a) = a - \left\lfloor \frac{a}{c_j} \right\rfloor c_j.$$

LEMMA 1. *For $j \in N$, if $\rho_j(b) > 0$, the function $\Theta(a) = \lfloor \frac{a}{c_j} \rfloor + \min\{\frac{\rho_j(a)}{\rho_j(b)}, 1\}$ (see Figure 1) is subadditive.*

Proof. Let $a, d \in \mathbb{R}$. Then $\Theta(a) + \Theta(d) = \lfloor \frac{a}{c_j} \rfloor + \min\{\frac{\rho_j(a)}{\rho_j(b)}, 1\} + \lfloor \frac{d}{c_j} \rfloor + \min\{\frac{\rho_j(d)}{\rho_j(b)}, 1\}$. There are two cases: (i) $\rho_j(a) + \rho_j(d) = \rho_j(a+d)$ and (ii) $\rho_j(a) + \rho_j(d) = \rho_j(a+d) + c_j$. In case (i), since $\rho_j(a) + \rho_j(d) = \rho_j(a+d)$, we have $\lfloor \frac{a}{c_j} \rfloor + \lfloor \frac{d}{c_j} \rfloor = \lfloor \frac{a+d}{c_j} \rfloor$. If $\min\{\frac{\rho_j(a)}{\rho_j(b)}, 1\} = 1$ or $\min\{\frac{\rho_j(d)}{\rho_j(b)}, 1\} = 1$, then $\Theta(a) + \Theta(d) \geq \lfloor \frac{a+d}{c_j} \rfloor + 1 \geq \Theta(a+d)$. Otherwise, $\min\{\frac{\rho_j(a)}{\rho_j(b)}, 1\} = \frac{\rho_j(a)}{\rho_j(b)}$ and $\min\{\frac{\rho_j(a)}{\rho_j(b)}, 1\} = \frac{\rho_j(a)}{\rho_j(b)}$. Then $\Theta(a) + \Theta(d) = \lfloor \frac{a+d}{c_j} \rfloor + \frac{\rho_j(a)}{\rho_j(b)} + \frac{\rho_j(d)}{\rho_j(b)} = \lfloor \frac{a+d}{c_j} \rfloor + \frac{\rho_j(a+d)}{\rho_j(b)} \geq \Theta(a+d)$. In case (ii), as $\rho_j(a) + \rho_j(d) = \rho_j(a+d) + c_j$, $\lfloor \frac{a}{c_j} \rfloor + \lfloor \frac{d}{c_j} \rfloor = \lfloor \frac{a+d}{c_j} \rfloor - 1$. If $\min\{\frac{\rho_j(a)}{\rho_j(b)}, 1\} = 1$ and $\min\{\frac{\rho_j(d)}{\rho_j(b)}, 1\} = 1$, then $\Theta(a) + \Theta(d) = \lfloor \frac{a+d}{c_j} \rfloor + 1 \geq \Theta(a+d)$. If $\min\{\frac{\rho_j(a)}{\rho_j(b)}, 1\} = \frac{\rho_j(a)}{\rho_j(b)}$ and $\min\{\frac{\rho_j(d)}{\rho_j(b)}, 1\} = 1$, then $\Theta(a) + \Theta(d) = \lfloor \frac{a+d}{c_j} \rfloor + \frac{\rho_j(a)}{\rho_j(b)}$. Since $\rho_j(d) \leq c_j$, $\rho_j(a) \geq \rho_j(a+d)$. So $\lfloor \frac{a+d}{c_j} \rfloor + \frac{\rho_j(a)}{\rho_j(b)} \geq \Theta(a+d)$. The case where $\min\{\frac{\rho_j(a)}{\rho_j(b)}, 1\} = 1$ and $\min\{\frac{\rho_j(d)}{\rho_j(b)}, 1\} = \frac{\rho_j(d)}{\rho_j(b)}$ is similar. Finally, if $\min\{\frac{\rho_j(a)}{\rho_j(b)}, 1\} = \frac{\rho_j(a)}{\rho_j(b)}$ and $\min\{\frac{\rho_j(a)}{\rho_j(b)}, 1\} = \frac{\rho_j(a)}{\rho_j(b)}$, $\Theta(a) + \Theta(d) = \lfloor \frac{a+d}{c_j} \rfloor - 1 + \frac{\rho_j(a)}{\rho_j(b)} + \frac{\rho_j(d)}{\rho_j(b)} = \lfloor \frac{a+d}{c_j} \rfloor - 1 + \frac{\rho_j(a+d)}{\rho_j(b)} + \frac{c_j}{\rho_j(b)}$. Since $c_j \geq \rho_j(b)$, $\lfloor \frac{a+d}{c_j} \rfloor - 1 + \frac{\rho_j(a+d)}{\rho_j(b)} + \frac{c_j}{\rho_j(b)} \geq \lfloor \frac{a+d}{c_j} \rfloor + \frac{\rho_j(a+d)}{\rho_j(b)} \geq \Theta(a+d)$. This proves that Θ is subadditive. $\square$

Now we will lift the inequality $\sum_{i \in N^1} \alpha_i x_i \geq \lceil \frac{b}{c_j} \rceil$ using the function Θ.

THEOREM 7. *Let $N^0 \subset N$, $N^1 = N \setminus N^0$, and $\sum_{i \in N^1} \alpha_i x_i \geq \alpha_0$ be a valid inequality for $X_b(N^1)$. If there exists $j \in N^1$ such that $\alpha_j = 1$, $\alpha_i \geq \lceil \frac{c_i}{c_j} \rceil$ for all*

$i \in N^1 \setminus \{j\}$, $\alpha_0 = \lceil \frac{b}{c_j} \rceil$, and $\rho_j(b) > 0$, then the inequality

$$\sum_{i \in N^1} \rho_j(b)\alpha_i x_i + \sum_{i \in N^0} \left(\rho_j(b)\left\lfloor \frac{c_i}{c_j} \right\rfloor + \min\{\rho_j(c_i), \rho_j(b)\}\right)x_i \geq \rho_j(b)\left\lceil \frac{b}{c_j} \right\rceil \tag{16}$$

is a valid inequality for PX.

Proof. The inequality $\sum_{i \in N^1} \alpha_i x_i \geq \lceil \frac{b}{c_j} \rceil$ is valid for $X_b(N_1)$. Consider the subadditive function $\Theta(a) = \lfloor \frac{a}{c_j} \rfloor + \min\{\frac{\rho_j(a)}{\rho_j(b)}, 1\}$ given in Lemma 1. We will show that $\Theta \geq \Phi$. If $a < b$ and $\rho_j(a) < \rho_j(b)$, then $\rho_j(b-a) = \rho_j(b) - \rho_j(a) > 0$. So $\Phi(a) = \lceil \frac{b}{c_j} \rceil - \lceil \frac{b-a}{c_j} \rceil = \frac{b - \rho_j(b) + c_j}{c_j} - \frac{b - a - \rho_j(b) + \rho_j(a) + c_j}{c_j} = \frac{a - \rho_j(a)}{c_j} = \lfloor \frac{a}{c_j} \rfloor \leq \Theta(a)$. If $a < b$ and $\rho_j(a) \geq \rho_j(b)$, then $\Theta(a) = \lceil \frac{a}{c_j} \rceil \geq \lceil \frac{b}{c_j} \rceil - \lceil \frac{b-a}{c_j} \rceil = \Phi(a)$. If $a \geq b$, then $\Phi(a) = \lceil \frac{b}{c_j} \rceil$. If $\lceil \frac{a}{c_j} \rceil = \lceil \frac{b}{c_j} \rceil$, then $\rho_j(a) \geq \rho_j(b)$. So $\Theta(a) = \lceil \frac{a}{c_j} \rceil = \Phi(a)$. If $\lceil \frac{a}{c_j} \rceil \geq \lceil \frac{b}{c_j} \rceil + 1$, then $\Theta(a) \geq \lfloor \frac{a}{c_j} \rfloor \geq \lceil \frac{b}{c_j} \rceil = \Phi(a)$. So the inequality $\sum_{i \in N^1} \alpha_i x_i + \sum_{i \in N^0} \left(\lfloor \frac{c_i}{c_j} \rfloor + \min\{\frac{\rho_j(c_i)}{\rho_j(b)}, 1\}\right)x_i \geq \lceil \frac{b}{c_j} \rceil$ is a valid inequality for PX. Multiplying both sides with $\rho_j(b)$, we obtain inequality (16). ∎

Some of the inequalities (16) are dominated by others. Indeed, as given in the following proposition, the number of nondominated inequalities (16) is polynomial.

PROPOSITION 2. *For $j \in N$ with $\rho_j(b) > 0$, the inequality*

$$\sum_{i=1}^{j} \min\{c_i, \rho_j(b)\}x_i + \sum_{i=j+1}^{n} \left(\rho_j(b)\left\lfloor \frac{c_i}{c_j} \right\rfloor + \min\{\rho_j(c_i), \rho_j(b)\}\right)x_i \geq \rho_j(b)\left\lceil \frac{b}{c_j} \right\rceil \tag{17}$$

is valid and dominates inequality (16) *for $N^0 \subset N$, $N^1 = N \setminus N^0$ such that $j \in N^1$, $\alpha_j = 1$, $\alpha_i \geq \lceil \frac{c_i}{c_j} \rceil$ for all $i \in N^1 \setminus \{j\}$ and $\alpha_0 = \lceil \frac{b}{c_j} \rceil$.*

Proof. Inequality (17) is valid since it is the same as inequality (16) for $N^1 = \{j\}$.

Let $N^0 \subset N$, $N^1 = N \setminus N^0$ such that $j \in N^1$, $\alpha_j = 1$, $\alpha_i \geq \lceil \frac{c_i}{c_j} \rceil$ for all $i \in N^1 \setminus \{j\}$, and $\alpha_0 = \lceil \frac{b}{c_j} \rceil$. For $i \in N^1$, $\rho_j(b)\lfloor \frac{c_i}{c_j} \rfloor + \min\{\rho_j(c_i), \rho_j(b)\} \leq \rho_j(b)\lceil \frac{c_i}{c_j} \rceil \leq \rho_j(b)\alpha_i$. So the coefficient of x_i in (17) is less than or equal to its coefficient in (16). The coefficients of x_i for $i \in N^0$ and the right-hand sides are the same in both inequalities. Hence inequality (17) dominates inequality (16). ∎

We call inequalities (17) *lifted rounding inequalities*. The number of lifted rounding inequalities that are not dominated is $O(n)$.

It is interesting to note that even though inequalities (16) are not, inequalities (17) are special cases of the multifacility cut-set inequalities derived by Atamturk [2] for the single commodity-multifacility network design problem.

For $j \in N$ such that $\rho_j(b) > 0$, consider the inequality $x_j \geq \lceil \frac{b}{c_j} \rceil$, which is facet-defining for $conv(X_b(\{j\}))$. If $c_1 \geq \rho_j(b)$, then, for $i < j$, $c_i \geq \rho_j(b)$. So $\Phi(c_i) = \Theta(c_i) = 1$. For $i > j$, if $\rho_j(c_i) = 0$ or $\rho_j(c_i) \geq \rho_j(b)$, then $\Phi(c_i) = \Theta(c_i) = \lceil \frac{c_i}{c_j} \rceil$. By Theorem 5 in Atamturk [4], the resulting inequality

$$\sum_{i=1}^{j} x_i + \sum_{i=j+1}^{n} \left\lceil \frac{c_i}{c_j} \right\rceil x_i \geq \left\lceil \frac{b}{c_j} \right\rceil \tag{18}$$

is facet-defining for PX. Notice that this is the same inequality as the rounding inequality (2) for $\lambda = c_j$. The condition $c_1 \geq \rho_j(b)$ implies that $\left(\lceil \frac{b}{c_j} \rceil - 1\right)c_j + c_1 \geq b$. For $i < j$, if $\rho_j(c_i) = 0$, then $\left(\lceil \frac{b}{c_j} \rceil - \lceil \frac{c_i}{c_j} \rceil\right)c_j + c_i = \lceil \frac{b}{c_j} \rceil c_j \geq b$. If $\rho_j(c_i) \geq \rho_j(b)$, then $\left(\lceil \frac{b}{c_j} \rceil - \lceil \frac{c_i}{c_j} \rceil\right) c_j + c_i = \left(\frac{b+c_j-\rho_j(b)}{c_j} - \frac{c_i+c_j-\rho_j(c_i)}{c_j}\right) c_j + c_i = b - \rho_j(b) + \rho_j(c_i) \geq b$. As a result, the conditions stated above are the same as the conditions of Corollary 4. However, Corollary 4 is a stronger result, since it states that these conditions are both necessary and sufficient.

Now we compare inequalities (17) and (3). The two following propositions are easy to prove.

PROPOSITION 3. *For $j \in N$ with $\rho_j(b) = 1$, inequalities* (17) *and* (3) *are the same.*

PROPOSITION 4. *For $j \in N$ with $\rho_j(b) \geq 2$, inequality* (17) *dominates inequality* (3).

If, for $j \in N$, $\rho_j(b) > 0$ (or, equivalently, $r_j < c_j$), then $\rho_j(b) = r_j$. So residual capacity inequalities (15) and inequalities (17) look very similar. Coefficients of variables x_i, with $i \in \{1, \ldots, j\}$, are the same in both inequalities. The right-hand sides are also the same. Only coefficients of variables x_i, with $i \in \{j+1, \ldots, n\}$, may be different.

PROPOSITION 5. *For $j \in N$, if $r_j < c_j$ and $\lceil \frac{r_j^2}{c_j} \rceil \geq r_j$, then inequality* (15) *for $\mu = 0$ and inequality* (17) *are the same.*

Proof. If $\lceil \frac{r_j^2}{c_j} \rceil \geq r_j$, then the coefficient of x_i, with $i \in \{j+1, \ldots, n\}$, is $\lceil \frac{c_i r_j}{c_j} \rceil$ in inequality (15) with $\mu = 0$. This is equal to

$$\left\lceil \frac{(\lfloor \frac{c_i}{c_j} \rfloor c_j + \rho_j(c_i)) r_j}{c_j} \right\rceil = \left\lfloor \frac{c_i}{c_j} \right\rfloor r_j + \left\lceil \frac{\rho_j(c_i) r_j}{c_j} \right\rceil.$$

Since $\rho_j(c_i) \leq c_j$ and $r_j \leq c_j$, $\lceil \frac{\rho_j(c_i) r_j}{c_j} \rceil \leq \min\{\rho_j(c_i), r_j\}$. So the coefficient of x_i in (15) is less than or equal to its coefficient in (17).

If $\rho_j(c_i) \geq r_j$, then $\lceil \frac{\rho_j(c_i) r_j}{c_j} \rceil \geq \lceil \frac{r_j^2}{c_j} \rceil \geq r_j$. Now assume that $\rho_j(c_i) < r_j$ and $\lceil \frac{\rho_j(c_i) r_j}{c_j} \rceil < \rho_j(c_i)$. Then $\rho_j(c_i) r_j \leq (\rho_j(c_i) - 1) c_j$. This is equivalent to $c_j \leq (c_j - r_j) \rho_j(c_i)$. Since $\lceil \frac{r_j^2}{c_j} \rceil \geq r_j$, $r_j^2 > (r_j - 1) c_j$. So $c_j > (c_j - r_j) r_j > (c_j - r_j) \rho_j(c_i)$. This contradicts $c_j \leq (c_j - r_j) \rho_j(c_i)$. Hence if $\rho_j(c_i) < r_j$, then $\lceil \frac{\rho_j(c_i) r_j}{c_j} \rceil \geq \rho_j(c_i)$. So the coefficients of variable x_i in inequalities (15) and (17) are the same. ∎

PROPOSITION 6. *For $j \in N$, if $\lceil \frac{r_j^2}{c_j} \rceil < r_j$, then inequality* (17) *dominates inequality* (15) *for $\mu = \mu_j$.*

Proof. If $\lceil \frac{r_j^2}{c_j} \rceil < r_j$, then the coefficient of x_i, with $i > j$, in (15) for $\mu = \mu_j$ is $\lceil \frac{c_i (r_j + \mu_j)}{c_j} \rceil = \lfloor \frac{c_i}{c_j} \rfloor r_j + \lceil \lfloor \frac{c_i}{c_j} \rfloor \mu_j + \frac{\rho_j(c_i)(r_j + \mu_j)}{c_j} \rceil$. If $\rho_j(c_i) \geq r_j$, then $\lceil \lfloor \frac{c_i}{c_j} \rfloor \mu_j + \frac{\rho_j(c_i)(r_j + \mu_j)}{c_j} \rceil \geq \lceil \lfloor \frac{c_i}{c_j} \rfloor \mu_j + \frac{r_j(r_j + \mu_j)}{c_j} \rceil$. Since $c_i \geq c_j$, $\lceil \lfloor \frac{c_i}{c_j} \rfloor \mu_j + \frac{r_j(r_j + \mu_j)}{c_j} \rceil \geq \lceil \mu_j + \frac{r_j(r_j + \mu_j)}{c_j} \rceil \geq r_j$.

Assume that $\rho_j(c_i) < r_j$ and $\lceil \lfloor \frac{c_i}{c_j} \rfloor \mu_j + \frac{\rho_j(c_i)(r_j + \mu_j)}{c_j} \rceil < \rho_j(c_i)$. Then $\rho_j(c_i)(r_j + \mu_j) \leq c_j(\rho_j(c_i) - 1 - \lfloor \frac{c_i}{c_j} \rfloor \mu_j)$ or, equivalently, $c_j \leq \rho_j(c_i)(c_j - r_j) - \mu_j c_i$. Since $\frac{r_j(r_j + \mu_j)}{c_j} + \mu_j > r_j - 1$, we have that $c_j > r_j(c_j - r_j - \mu_j) - \mu_j c_j$, and now, since $r_j > \rho_j(c_i)$, $c_j > \rho_j(c_i)(c_j - r_j - \mu_j) - \mu_j c_j$. Putting together with $c_j \leq \rho_j(c_i)(c_j - r_j) - \mu_j c_i$, we obtain $\rho_j(c_i)(c_j - r_j) - \mu_j c_i > \rho_j(c_i)(c_j - r_j - \mu_j) - \mu_j c_j$. This is equivalent to $\rho_j(c_i) + c_j > c_i$ since $\mu_j > 0$. But this is impossible. So if $\rho_j(c_i) < r_j$, then $\lceil \lfloor \frac{c_i}{c_j} \rfloor \mu_j + \frac{\rho_j(c_i)(r_j + \mu_j)}{c_j} \rceil \geq \rho_j(c_i)$. This proves that the coefficient of x_i in (15) is greater than or equal to its coefficient in (17). ∎

These four propositions show that, for $j \in N$ with $\rho_j(b) > 0$, the lifted rounding inequality (17) dominates the rounding inequality (2) for $\lambda = c_j$ and the residual capacity inequality (15) for $\mu = \mu_j$. For a special case, these inequalities (17) are facet-defining for PX.

THEOREM 8. *For $j \in N$ such that $\rho_j(b) > 0$, if $c_1 = 1$, then inequality* (17) *is facet-defining for PX.*

Proof. Suppose that $\rho_j(b) > 0$ and $c_1 = 1$. Assume that all points in X which satisfy inequality (17) at equality also satisfy $\sum_{i=1}^n \alpha_i x_i = \alpha_0$. The point $\lceil \frac{b}{c_j} \rceil e_j$ is in X and satisfies inequality (17) at equality. So $\alpha_0 = \alpha_j \lceil \frac{b}{c_j} \rceil$.

Notice that, if we remove one item j, the remaining demand to be covered is $\rho_j(b)$. For $i < j$, if $c_i > \rho_j(b)$, then consider the point $e_i + (\lceil \frac{b}{c_j} \rceil - 1)e_j$. It is easy to verify that this point is also in X and that inequality (17) is tight at this point. Then we have $\alpha_i = \alpha_j$.

For $i < j$, if $c_i \leq \rho_j(b)$, then the point $e_i + (\lceil \frac{b}{c_j} \rceil - 1)e_j + (\rho_j(b) - c_i)e_1$ is in X and inequality (17) is tight at this point. So $\alpha_i = \alpha_j - (\rho_j(b) - c_i)\alpha_1$. Since $c_1 = 1 \leq \rho_j(b)$, we obtain $\alpha_1 = \frac{\alpha_j}{\rho_j(b)}$. Then $\alpha_i = c_i \frac{\alpha_j}{\rho_j(b)}$. Hence for $i < j$, $\alpha_i = \min\{c_i, \rho_j(b)\} \frac{\alpha_j}{\rho_j(b)}$.

For $i > j$, if $\rho_j(c_i) = 0$, consider point $e_i + (\lceil \frac{b}{c_j} \rceil - \frac{c_i}{c_j})e_j$. The left-hand side of inequality (17) at this point is equal to $\rho_j(b)\frac{c_i}{c_j} + (\lceil \frac{b}{c_j} \rceil - \frac{c_i}{c_j})\rho_j(b) = \lceil \frac{b}{c_j} \rceil \rho_j(b)$. So inequality (17) is tight. The left-hand side of the cover constraint is equal to $c_i + (\lceil \frac{b}{c_j} \rceil - \frac{c_i}{c_j})c_j = \lceil \frac{b}{c_j} \rceil c_j \geq b$. Thus this point is in X. Then we have $\alpha_i = \alpha_j \frac{c_i}{c_j}$.

Finally, for $i > j$, with $\rho_j(c_i) > 0$, consider $e_i + (\lceil \frac{b}{c_j} \rceil - \lceil \frac{c_i}{c_j} \rceil)e_j + (\rho_j(b) - \rho_j(c_i))^+ e_1$. The left-hand side of inequality (17) evaluated at this point is equal to $\rho_j(b)\lfloor \frac{c_i}{c_j} \rfloor + \min\{\rho_j(c_i), \rho_j(b)\} + (\lceil \frac{b}{c_j} \rceil - \lceil \frac{c_i}{c_j} \rceil)\rho_j(b) + (\rho_j(b) - \rho_j(c_i))^+ = \rho_j(b)\lfloor \frac{c_i}{c_j} \rfloor + \rho_j(b) + (\lceil \frac{b}{c_j} \rceil - \lceil \frac{c_i}{c_j} \rceil)\rho_j(b)$. Since $\rho_j(c_i) > 0$, this is equal to $\rho_j(b) + (\lceil \frac{b}{c_j} \rceil - 1)\rho_j(b) = \lceil \frac{b}{c_j} \rceil \rho_j(b)$, showing that inequality (17) is tight at this point. The left-hand side of the cover constraint is equal to

$$c_i + \left(\left\lceil \frac{b}{c_j} \right\rceil - \left\lceil \frac{c_i}{c_j} \right\rceil \right) c_j + (\rho_j(b) - \rho_j(c_i))^+. \tag{19}$$

If $\rho_j(c_i) > \rho_j(b)$, then (19) is equal to $c_i + (\lceil \frac{b}{c_j} \rceil - \lceil \frac{c_i}{c_j} \rceil)c_j = c_i + (\lceil \frac{b}{c_j} \rceil - \lfloor \frac{c_i}{c_j} \rfloor - 1)c_j = \rho_j(c_i) + (\lceil \frac{b}{c_j} \rceil - 1)c_j > \rho_j(b) + (\lceil \frac{b}{c_j} \rceil - 1)c_j = b$. If $\rho_j(c_i) \leq \rho_j(b)$, then (19) is equal to $c_i + (\lceil \frac{b}{c_j} \rceil - \lceil \frac{c_i}{c_j} \rceil)c_j + \rho_j(b) - \rho_j(c_i) = c_i + (\lfloor \frac{b}{c_j} \rfloor - \lfloor \frac{c_i}{c_j} \rfloor)c_j + \rho_j(b) - \rho_j(c_i) = b$. So this point is in X. This proves that $\alpha_i = \alpha_j \lceil \frac{c_i}{c_j} \rceil - (\rho_j(b) - \rho_j(c_i))^+ \alpha_1 = \alpha_j \lceil \frac{c_i}{c_j} \rceil - (\rho_j(b) - \rho_j(c_i))^+ \frac{\alpha_j}{\rho_j(b)}$. If $\rho_j(b) \leq \rho_j(c_i)$, then $\alpha_i = \alpha_j \lceil \frac{c_i}{c_j} \rceil = \alpha_j(\lfloor \frac{c_i}{c_j} \rfloor + 1)$. If $\rho_j(b) > \rho_j(c_i)$, then $\alpha_i = \alpha_j \lceil \frac{c_i}{c_j} \rceil - (\rho_j(b) - \rho_j(c_i))\frac{\alpha_j}{\rho_j(b)} = \alpha_j(\lfloor \frac{c_i}{c_j} \rfloor + 1) - \alpha_j + \rho_j(c_i)\frac{\alpha_j}{\rho_j(b)} = \alpha_j(\lfloor \frac{c_i}{c_j} \rfloor + \frac{\rho_j(c_i)}{\rho_j(b)})$. So, for $i < j$, $\alpha_i = \alpha_j(\lfloor \frac{c_i}{c_j} \rfloor + \min\{\frac{\rho_j(c_i)}{\rho_j(b)}, 1\}) = \frac{\alpha_j}{\rho_j(b)}(\lfloor \frac{c_i}{c_j} \rfloor \rho_j(b) + \min\{\rho_j(c_i), \rho_j(b)\})$.

Hence $\sum_{i=1}^n \alpha_i x_i = \alpha_0$ has the form

$$\sum_{i=1}^{j-1} \min\{c_i, \rho_j(b)\} \frac{\alpha_j}{\rho_j(b)} x_i + \alpha_j x_j + \sum_{j+1}^{n} \frac{\alpha_j}{\rho_j(b)} \left(\left\lfloor \frac{c_i}{c_j} \right\rfloor \rho_j(b) + \min\{\rho_j(c_i), \rho_j(b)\} \right) x_i = \alpha_j \left\lceil \frac{b}{c_j} \right\rceil.$$

This is $\frac{\alpha_j}{\rho_j(b)}$ times $\sum_{i=1}^{j} \min\{c_i, \rho_j(b)\} x_i + \sum_{i=j+1}^{n} \left(\rho_j(b) \lfloor \frac{c_i}{c_j} \rfloor + \min\{\rho_j(c_i), \rho_j(b)\}\right) x_i = \rho_j(b) \lceil \frac{b}{c_j} \rceil$. ∎

Example 7. Consider the set $X^4 = \{x \in \mathbb{Z}^7_+ : x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 + 7x_7 \geq 38\}$. The convex hull of X^4 is described by the nonnegativity constraints and the following inequalities (obtained using PORTA [6]):

$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 + 7x_7 \geq 38, \tag{20}$$
$$2x_1 + 2x_2 + 4x_3 + 4x_4 + 5x_5 + 6x_6 + 6x_7 \geq 34, \tag{21}$$
$$x_1 + 2x_2 + 3x_3 + 3x_4 + 4x_5 + 5x_6 + 5x_7 \geq 28, \tag{22}$$
$$x_1 + 2x_2 + 2x_3 + 3x_4 + 4x_5 + 4x_6 + 5x_7 \geq 26, \tag{23}$$
$$x_1 + 2x_2 + 3x_3 + 3x_4 + 3x_5 + 4x_6 + 5x_7 \geq 24, \tag{24}$$
$$x_1 + 2x_2 + 2x_3 + 2x_4 + 3x_5 + 4x_6 + 4x_7 \geq 20, \tag{25}$$
$$x_1 + 2x_2 + 3x_3 + 3x_4 + 3x_5 + 3x_6 + 3x_7 \geq 18, \tag{26}$$
$$x_1 + x_2 + 2x_3 + 2x_4 + 2x_5 + 3x_6 + 3x_7 \geq 16, \tag{27}$$
$$x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5 + 2x_6 + 3x_7 \geq 14, \tag{28}$$
$$x_1 + x_2 + 2x_3 + 2x_4 + 2x_5 + 2x_6 + 2x_7 \geq 12. \tag{29}$$

As $c_1 = 1$, the cover constraint (20) is facet-defining for $conv(X^4)$. None of the rounding inequalities for items $\lambda = c_2, \ldots, c_7$ is facet-defining for $conv(X^4)$. For item 2, $\rho_2(38) = 0$. For item 3, $\rho_3(38) = 2$. Inequality (17) for 3, $x_1+2x_2+2x_3+3x_4+4x_5+4x_6+5x_7 \geq 26$, is a valid inequality and is facet-defining since $c_1 = 1$ and $\rho_3(38) > 0$. Indeed, it is the same as inequality (23). For item 4, $\rho_4(38) = 2$. Inequality (17) reads $x_1+2x_2+2x_3+2x_4+3x_5+4x_6+4x_7 \geq 20$ and is a valid inequality. This is the same as inequality (25) and is facet-defining. Note here that $\mu_4 = \epsilon$ and inequality (15) for item 4, $x_1 + 2x_2 + 2x_3 + 3x_4 + 3x_5 + 4x_6 + 4x_7 \geq 20$, is dominated by inequality (25). For item 5, $\rho_5(38) = 3$. Inequality (17), $x_1 + 2x_2 + 3x_3 + 3x_4 + 3x_5 + 4x_6 + 5x_7 \geq 24$, is the same as inequality (24). For item 6, $\rho_6(38) = 2$. The corresponding inequality (17) is $x_1+2x_2+2x_3+2x_4+2x_5+2x_6+3x_7 \geq 14$ and is the same as inequality (28). For item 7, $\rho_7(38) = 3$. The inequality $x_1 + 2x_2 + 3x_3 + 3x_4 + 3x_5 + 3x_6 + 3x_7 \geq 18$ is valid and facet-defining for $conv(X^4)$. This is the same as inequality (26).

6. Lifted 2-partition inequalities. Pochet and Wolsey [15] derive partition inequalities for PX where c_i divides c_{i+1} for all $i = 1, \ldots, n-1$. Then they prove that these inequalities are valid for PX in general under some conditions. Let $(i_1, \ldots, j_1), \ldots, (i_p, \ldots, j_p)$ be a partition of N such that $i_1 = 1$, $j_p = n$, and $i_t = j_{t-1} + 1$ for all $t = 2, \ldots, p$. Let $\beta_p = b$. For $t = p, \ldots, 1$, compute $\kappa_t = \lceil \frac{\beta_t}{c_{i_t}} \rceil$ and $\beta_{t-1} = \beta_t - (\kappa_t - 1)c_{i_t}$. The inequality

$$\sum_{t=1}^{p} \left(\prod_{s=1}^{t-1} \kappa_s \right) \sum_{j=i_t}^{j_t} \min \left\{ \left\lceil \frac{c_j}{c_{i_t}} \right\rceil, \kappa_t \right\} x_j \geq \prod_{s=1}^{p} \kappa_s \tag{30}$$

is called the *partition inequality*. Pochet and Wolsey [15] prove that the partition inequality is valid for PX if $\kappa_{t-1} \leq \lfloor \frac{c_{i_t}}{c_{i_{t-1}}} \rfloor$ for all $t = 2, \ldots, p$. If c_i divides c_{i+1} for all $i = 1, \ldots, n-1$, then the partition inequalities are valid without any condition, and they describe PX together with nonnegativity constraints.

Consider the case where $i_1 = 1$ and $j_1 = n$. Then inequality (30) reduces to the inequality $\sum_{j=1}^{n} \min \{ \lceil \frac{c_j}{c_1} \rceil, \kappa_1 \} x_j \geq \kappa_1$. This is the same as the rounding inequality (2) for $\lambda = c_1$ since $\kappa_1 = \lceil \frac{b}{c_1} \rceil$ and $c_j < b$ for all $j \in N$.

The next special case is when $i_1 = 1$, $j_1 = j - 1$, $i_2 = j$, and $j_2 = n$. Then $\kappa_2 = \lceil \frac{b}{c_j} \rceil$, $\beta_1 = b - (\lceil \frac{b}{c_j} \rceil - 1)c_j$. Notice that $\beta_1 = r_j$. Finally, $\kappa_1 = \lceil \frac{r_j}{c_1} \rceil$. Inequality

(30) becomes

$$\sum_{i=1}^{j-1}\min\left\{\left\lceil\frac{c_i}{c_1}\right\rceil,\left\lceil\frac{r_j}{c_1}\right\rceil\right\}x_i+\left\lceil\frac{r_j}{c_1}\right\rceil\sum_{i=j}^{n}\left\lceil\frac{c_i}{c_j}\right\rceil x_i\geq\left\lceil\frac{r_j}{c_1}\right\rceil\left\lceil\frac{b}{c_j}\right\rceil \tag{31}$$

and is valid if $\lceil\frac{r_j}{c_1}\rceil\leq\lfloor\frac{c_j}{c_1}\rfloor$. We refer to these inequalities as 2-*partition inequalities.*

PROPOSITION 7. *For $j\in N$, if $c_1=1$, inequality* (31) *is dominated by the cover constraint or inequality* (17).

Proof. If $c_1=1$, then the inequality simplifies to

$$\sum_{i=1}^{j}\min\{c_i,r_j\}x_i+r_j\sum_{i=j+1}^{n}\left\lceil\frac{c_i}{c_j}\right\rceil x_i\geq r_j\left\lceil\frac{b}{c_j}\right\rceil \tag{32}$$

and is always valid. If, moreover, $r_j=c_j$, then the inequality becomes $\sum_{i=1}^{j}c_ix_i+\sum_{i=j+1}^{n}c_j\lceil\frac{c_i}{c_j}\rceil x_i\geq b$ and is dominated by the cover constraint. If $r_j<c_j$, then $r_j=\rho_j(b)$ and $\rho_j(b)>0$. For $i>j$, if c_i is divisible by c_j, then $r_j\lceil\frac{c_i}{c_j}\rceil=\rho_j(b)\lfloor\frac{c_i}{c_j}\rfloor+\rho_j(c_i)$ since $\rho_j(c_i)=0$. If c_i is not divisible by c_j, then $r_j\lceil\frac{c_i}{c_j}\rceil=\rho_j(b)\lfloor\frac{c_i}{c_j}\rfloor+\rho_j(b)$. So the coefficient of x_i in (32) is greater than or equal to its coefficient in inequality (17). For $i\leq j$, the variable x_i has the same coefficient in (32) and (17). Also, the right-hand sides of (32) and (17) are the same. Hence if $c_1=1$ and $r_j<c_j$, inequality (17) dominates inequality (32). ∎

If $\lceil\frac{c_i}{c_1}\rceil\geq\lceil\frac{r_j}{c_1}\rceil$ for all $i<j$, then inequality (31) simplifies to $\sum_{i=1}^{j}x_i+\sum_{i=j+1}^{n}\lceil\frac{c_i}{c_j}\rceil x_i\geq\lceil\frac{b}{c_j}\rceil$, which is the rounding inequality (2) for $\lambda=c_j$.

Now we will improve the 2-partition inequalities (31) using lifting. Let $N^0\subset N$, $N^1=N\setminus N^0$, $j_{min}=\arg\min_{i\in N^1}c_i$, and $j\in N^1$, with $j_{min}\neq j$. The 2-partition inequality for the partition $N^-=\{i\in N^1:i<j\}$ and $N^+=\{i\in N^1:i\geq j\}$ is

$$\sum_{i\in N^-}\min\left\{\left\lceil\frac{c_i}{c_{j_{min}}}\right\rceil,\left\lceil\frac{r_j}{c_{j_{min}}}\right\rceil\right\}x_i+\left\lceil\frac{r_j}{c_{j_{min}}}\right\rceil\sum_{i\in N^+}\left\lceil\frac{c_i}{c_j}\right\rceil x_i\geq\left\lceil\frac{r_j}{c_{j_{min}}}\right\rceil\left\lceil\frac{b}{c_j}\right\rceil \tag{33}$$

and is valid when $x_i=0$ for all $i\in N^0$ if $\lceil\frac{r_j}{c_{j_{min}}}\rceil\leq\lfloor\frac{c_j}{c_{j_{min}}}\rfloor$.

The lifting function for inequality (33) is

$$\begin{aligned}\beta(a)=&\left\lceil\frac{r_j}{c_{j_{min}}}\right\rceil\left\lceil\frac{b}{c_j}\right\rceil\\&-\min_{x\in X_{b-a}(N^1)}\left(\sum_{i\in N^-}\min\left\{\left\lceil\frac{c_i}{c_{j_{min}}}\right\rceil,\left\lceil\frac{r_j}{c_{j_{min}}}\right\rceil\right\}x_i+\left\lceil\frac{r_j}{c_{j_{min}}}\right\rceil\sum_{i\in N^+}\left\lceil\frac{c_i}{c_j}\right\rceil x_i\right).\end{aligned}$$

LEMMA 2. *If $r_j\leq c_j-1$ and $\lceil\frac{r_j}{c_{j_{min}}}\rceil\leq\lfloor\frac{c_j}{c_{j_{min}}}\rfloor$, for $a\in\mathbb{R}$,*

$$\beta(a)=\begin{cases}\lceil\frac{r_j}{c_{j_{min}}}\rceil\lceil\frac{a}{c_j}\rceil-\lceil\frac{\rho_j(b-a)}{c_{j_{min}}}\rceil & \text{if } a<b \text{ and } 0<\rho_j(a)<r_j,\\ \lceil\frac{r_j}{c_{j_{min}}}\rceil\lceil\frac{a}{c_j}\rceil & \text{if } a<b \text{ and } \rho_j(a)\geq r_j \text{ or } \rho_j(a)=0,\\ \lceil\frac{r_j}{c_{j_{min}}}\rceil\lceil\frac{b}{c_j}\rceil & \text{if } a\geq b.\end{cases}$$

Proof. For $d\in\mathbb{R}$, let

$$z(d)=\min_{x\in X_d(N^1)}\left(\sum_{i\in N^-}\min\left\{\left\lceil\frac{c_i}{c_{j_{min}}}\right\rceil,\left\lceil\frac{r_j}{c_{j_{min}}}\right\rceil\right\}x_i+\left\lceil\frac{r_j}{c_{j_{min}}}\right\rceil\sum_{i\in N^+}\left\lceil\frac{c_i}{c_j}\right\rceil x_i\right).$$

If $d \leq 0$, then $z(d) = 0$. If $d > 0$, Pochet and Wolsey [15] prove that there exists an optimal solution where $x_i = 0$, for $i \neq j_{min}$ and $i \neq j$, and $x_{j_{min}} \leq \lceil \frac{r_j}{c_{j_{min}}} \rceil - 1$. Consider such optimal solutions. If $d < c_j$, then e_j or $\lceil \frac{d}{c_{j_{min}}} \rceil e_{j_{min}}$ is optimal. Hence $z(d) = \min\{\lceil \frac{r_j}{c_{j_{min}}} \rceil, \lceil \frac{d}{c_{j_{min}}} \rceil\}$. If $d \geq c_j$, then $x_j \geq \lfloor \frac{d}{c_j} \rfloor$ since otherwise $x_{j_{min}} \geq \lceil \frac{c_j}{c_{j_{min}}} \rceil$. So $\lfloor \frac{d}{c_j} \rfloor e_j + \lceil \frac{\rho_j(d)}{c_{j_{min}}} \rceil e_{j_{min}}$ or $\lceil \frac{d}{c_j} \rceil e_j$ is optimal, and $z(d) = \min\{\lceil \frac{r_j}{c_{j_{min}}} \rceil \lfloor \frac{d}{c_j} \rfloor + \lceil \frac{\rho_j(d)}{c_{j_{min}}} \rceil, \lceil \frac{r_j}{c_{j_{min}}} \rceil \lceil \frac{d}{c_j} \rceil\}$. So if $a < b$, then

$$\beta(a) = \left\lceil \frac{r_j}{c_{j_{min}}} \right\rceil \left\lceil \frac{b}{c_j} \right\rceil - \left\lceil \frac{r_j}{c_{j_{min}}} \right\rceil \left\lfloor \frac{b-a}{c_j} \right\rfloor - \min\left\{ \left\lceil \frac{r_j}{c_{j_{min}}} \right\rceil, \left\lceil \frac{\rho_j(b-a)}{c_{j_{min}}} \right\rceil \right\}.$$

Consider $a < b$. If $\rho_j(b-a) = \rho_j(b) - \rho_j(a)$ and $\rho_j(a) > 0$, then

$$\begin{aligned}
\beta(a) &= \left\lceil \frac{r_j}{c_{j_{min}}} \right\rceil \left(\left\lceil \frac{b}{c_j} \right\rceil - \left\lfloor \frac{b-a}{c_j} \right\rfloor \right) - \left\lceil \frac{\rho_j(b-a)}{c_{j_{min}}} \right\rceil \\
&= \left\lceil \frac{r_j}{c_{j_{min}}} \right\rceil \left(\frac{b-\rho_j(b)+c_j}{c_j} - \frac{b-a-\rho_j(b-a)}{c_j} \right) - \left\lceil \frac{\rho_j(b-a)}{c_{j_{min}}} \right\rceil \\
&= \left\lceil \frac{r_j}{c_{j_{min}}} \right\rceil \left(\frac{b-\rho_j(b)+c_j}{c_j} - \frac{b-a-\rho_j(b)+\rho_j(a)}{c_j} \right) - \left\lceil \frac{\rho_j(b-a)}{c_{j_{min}}} \right\rceil \\
&= \left\lceil \frac{r_j}{c_{j_{min}}} \right\rceil \left(\frac{a-\rho_j(a)+c_j}{c_j} \right) - \left\lceil \frac{\rho_j(b-a)}{c_{j_{min}}} \right\rceil \\
&= \left\lceil \frac{r_j}{c_{j_{min}}} \right\rceil \left\lceil \frac{a}{c_j} \right\rceil - \left\lceil \frac{\rho_j(b-a)}{c_{j_{min}}} \right\rceil.
\end{aligned}$$

If $\rho_j(b-a) = \rho_j(b) - \rho_j(a) + c_j$, then

$$\begin{aligned}
\beta(a) &= \left\lceil \frac{r_j}{c_{j_{min}}} \right\rceil \left(\left\lceil \frac{b}{c_j} \right\rceil - \left\lfloor \frac{b-a}{c_j} \right\rfloor \right) - \left\lceil \frac{r_j}{c_{j_{min}}} \right\rceil \\
&= \left\lceil \frac{r_j}{c_{j_{min}}} \right\rceil \left(\frac{b-\rho_j(b)+c_j}{c_j} - \frac{b-a-\rho_j(b-a)}{c_j} - 1 \right) \\
&= \left\lceil \frac{r_j}{c_{j_{min}}} \right\rceil \left(\frac{b-\rho_j(b)+c_j}{c_j} - \frac{b-a-\rho_j(b)+\rho_j(a)-c_j}{c_j} - 1 \right) \\
&= \left\lceil \frac{r_j}{c_{j_{min}}} \right\rceil \frac{a-\rho_j(a)+c_j}{c_j} \\
&= \left\lceil \frac{r_j}{c_{j_{min}}} \right\rceil \left\lceil \frac{a}{c_j} \right\rceil.
\end{aligned}$$

If $\rho_j(a) = 0$, then

$$\begin{aligned}
\beta(a) &= \left\lceil \frac{r_j}{c_{j_{min}}} \right\rceil \left(\left\lceil \frac{b}{c_j} \right\rceil - \left\lfloor \frac{b-a}{c_j} \right\rfloor \right) - \left\lceil \frac{r_j}{c_{j_{min}}} \right\rceil \\
&= \left\lceil \frac{r_j}{c_{j_{min}}} \right\rceil \left(\frac{b-\rho_j(b)+c_j}{c_j} - \frac{b-a-\rho_j(b)}{c_j} - 1 \right) \\
&= \left\lceil \frac{r_j}{c_{j_{min}}} \right\rceil \frac{a}{c_j}. \qquad \square
\end{aligned}$$

Function β is not subadditive in general. Consider $b = 18$, $c_j = 5$, and $c_{j_{min}} = 2$. Let $a = 2.5$ and $b = 5.5$. Then $\beta(2.5) = 1$, $\beta(5.5) = 2$, and $\beta(8) = 4$. So, $\beta(2.5) +$

$\beta(5.5) < \beta(8)$. So, to do lifting, we need a subadditive function which is greater than or equal to β. We first study the case where $c_{j_{min}}$ divides r_j. Notice that, in this case, $\lceil \frac{r_j}{c_{j_{min}}} \rceil \leq \lfloor \frac{c_j}{c_{j_{min}}} \rfloor$ is always satisfied.

THEOREM 9. *Let $N^0 \subset N$, $N^1 = N \setminus N^0$, $j_{min} = \arg\min_{i \in N^1} c_i$, $j \in N^1$, with $j_{min} < j$, $r_j \leq c_j - 1$, and $\rho_{j_{min}}(r_j) = 0$, $N^- = \{i \in N^1 : i < j\}$, and $N^+ = \{i \in N^1 : i \geq j\}$. The inequality*

$$\sum_{i \in N^-} \min\left\{\left\lceil \frac{c_i}{c_{j_{min}}} \right\rceil, \frac{r_j}{c_{j_{min}}}\right\} x_i + \frac{r_j}{c_{j_{min}}} \sum_{i \in N^+} \left\lceil \frac{c_i}{c_j} \right\rceil x_i + \sum_{i \in N^0} \left(\frac{r_j}{c_{j_{min}}} \left\lfloor \frac{c_i}{c_j} \right\rfloor + \min\left\{\frac{\rho_j(c_i)}{c_{j_{min}}}, \frac{r_j}{c_{j_{min}}}\right\}\right) x_i \geq \frac{r_j}{c_{j_{min}}} \left\lceil \frac{b}{c_j} \right\rceil \tag{34}$$

is valid for PX.

Proof. Consider the function $\sigma(a) = \frac{r_j}{c_{j_{min}}} \lfloor \frac{a}{c_j} \rfloor + \min\{\frac{\rho_j(a)}{c_{j_{min}}}, \frac{r_j}{c_{j_{min}}}\}$. Notice that $\sigma(a) = \frac{r_j}{c_{j_{min}}} \Theta(a)$ for all $a \in \mathbb{R}$. Since Θ is subadditive (see Lemma 1) and $\frac{r_j}{c_{j_{min}}} > 0$, σ is subadditive. So, to prove the validity of (34), we need to show that $\sigma(a) \geq \beta(a)$ for all $a \in \mathbb{R}$.

If $a \geq b$ and $\lceil \frac{a}{c_j} \rceil = \lceil \frac{b}{c_j} \rceil$, then $\rho_j(a) \geq \rho_j(b)$. So $\sigma(a) = \frac{r_j}{c_{j_{min}}} \lceil \frac{a}{c_j} \rceil = \beta(a)$. If $a > b$ and $\lceil \frac{a}{c_j} \rceil \geq \lceil \frac{b}{c_j} \rceil + 1$, then $\sigma(a) \geq \frac{r_j}{c_{j_{min}}} \lfloor \frac{a}{c_j} \rfloor \geq \beta(a)$. If $a < b$ and $0 < \rho_j(a) < r_j$, then $\sigma(a) = \frac{r_j}{c_{j_{min}}} \lfloor \frac{a}{c_j} \rfloor + \frac{\rho_j(a)}{c_{j_{min}}}$ and $\beta(a) = \frac{r_j}{c_{j_{min}}} \lceil \frac{a}{c_j} \rceil - \lceil \frac{\rho_j(b-a)}{c_{j_{min}}} \rceil = \frac{r_j}{c_{j_{min}}} \lceil \frac{a}{c_j} \rceil - \frac{r_j}{c_{j_{min}}} - \lceil \frac{-\rho_j(a)}{c_{j_{min}}} \rceil = \frac{r_j}{c_{j_{min}}} \lfloor \frac{a}{c_j} \rfloor + \lfloor \frac{\rho_j(a)}{c_{j_{min}}} \rfloor \leq \sigma(a)$. If $a < b$ and $\rho_j(a) \geq r_j$ or $\rho_j(a) = 0$, then $\sigma(a) = \beta(a)$. Hence $\sigma(a) \geq \beta(a)$ for all $a \in \mathbb{R}$. ∎

These inequalities are not useful as they are dominated by the lifted rounding inequalities.

PROPOSITION 8. *For $j \in N$ with $r_j \leq c_j - 1$, inequality (17) dominates inequality (34) for all choices of $N^0 \subset N$, $N^1 = N \setminus N^0$, with $j \in N^1$, $j_{min} = \arg\min_{i \in N^1} c_i$, $j_{min} \neq j$, and $\rho_{j_{min}}(r_j) = 0$.*

Proof. Let $N^0 \subset N$, $N^1 = N \setminus N^0$, with $j \in N^1$, $j_{min} = \arg\min_{i \in N^1} c_i$, $j_{min} \neq j$, and $\rho_{j_{min}}(r_j) = 0$. If we divide inequality (17) by $c_{j_{min}}$, we obtain

$$\sum_{i=1}^{j} \min\left\{\frac{c_i}{c_{j_{min}}}, \frac{r_j}{c_{j_{min}}}\right\} x_i + \sum_{i=j+1}^{n} \left(\frac{r_j}{c_{j_{min}}} \left\lfloor \frac{c_i}{c_j} \right\rfloor + \min\left\{\frac{\rho_j(c_i)}{c_{j_{min}}}, \frac{r_j}{c_{j_{min}}}\right\}\right) x_i \geq \frac{r_j}{c_{j_{min}}} \left\lceil \frac{b}{c_j} \right\rceil. \tag{35}$$

In inequality (34), variable x_i has the coefficient $\min\{\lceil \frac{c_i}{c_{j_{min}}} \rceil, \frac{r_j}{c_{j_{min}}}\} \geq \min\{\frac{c_i}{c_{j_{min}}}, \frac{r_j}{c_{j_{min}}}\}$ if $i \in N^-$. For $i \in N^+$, the variable x_i has the coefficient $\frac{r_j}{c_{j_{min}}} \lceil \frac{c_i}{c_j} \rceil \geq \frac{r_j}{c_{j_{min}}} \lfloor \frac{c_i}{c_j} \rfloor + \min\{\frac{\rho_j(c_i)}{c_{j_{min}}}, \frac{r_j}{c_{j_{min}}}\}$. The coefficient of x_i for $i \in N^0$ and the right-hand sides are equal in inequalities (17) and (34). ∎

Now we are interested in cases where $c_{j_{min}}$ does not divide r_j.

LEMMA 3. *If $r_j \leq c_j - 1$, $\lceil \frac{r_j}{c_{j_{min}}} \rceil \leq \lfloor \frac{c_j}{c_{j_{min}}} \rfloor$, and $\rho_{j_{min}}(r_j) > 0$, then the function*

$$\gamma(a) = \left\lceil \frac{r_j}{c_{j_{min}}} \right\rceil \left\lfloor \frac{a}{c_j} \right\rfloor + \min\left\{\frac{\rho_j(a)}{\rho_{j_{min}}(r_j)}, \left\lceil \frac{r_j}{c_{j_{min}}} \right\rceil\right\}$$

for $a \in \mathbb{R}$ is subadditive.

Proof. For $a, d \in \mathbb{R}$, if $\rho_j(a) + \rho_j(d) = \rho_j(a + d)$, then $\lfloor \frac{a}{c_j} \rfloor + \lfloor \frac{d}{c_j} \rfloor = \lfloor \frac{a+d}{c_j} \rfloor$. If $\min\{\frac{\rho_j(a)}{\rho_{j_{min}}(r_j)}, \lceil \frac{r_j}{c_{j_{min}}} \rceil\} = \lceil \frac{r_j}{c_{j_{min}}} \rceil$ or $\min\{\frac{\rho_j(d)}{\rho_{j_{min}}(r_j)}, \lceil \frac{r_j}{c_{j_{min}}} \rceil\} = \lceil \frac{r_j}{c_{j_{min}}} \rceil$, then $\gamma(a) + \gamma(d) \geq \lceil \frac{r_j}{c_{j_{min}}} \rceil \lfloor \frac{a+d}{c_j} \rfloor + \lceil \frac{r_j}{c_{j_{min}}} \rceil \geq \gamma(a + d)$. Otherwise, $\gamma(a) + \gamma(d) = \lceil \frac{r_j}{c_{j_{min}}} \rceil \lfloor \frac{a+d}{c_j} \rfloor + \frac{\rho_j(a+d)}{\rho_{j_{min}}(r_j)} \geq \gamma(a + d)$. If $\rho_j(a) + \rho_j(d) = \rho_j(a + d) + c_j$, then $\lfloor \frac{a}{c_j} \rfloor + \lfloor \frac{d}{c_j} \rfloor = \lfloor \frac{a+d}{c_j} \rfloor - 1$. If $\min\{\frac{\rho_j(a)}{\rho_{j_{min}}(r_j)}, \lceil \frac{r_j}{c_{j_{min}}} \rceil\} = \lceil \frac{r_j}{c_{j_{min}}} \rceil$ and $\min\{\frac{\rho_j(d)}{\rho_{j_{min}}(r_j)}, \lceil \frac{r_j}{c_{j_{min}}} \rceil\} = \lceil \frac{r_j}{c_{j_{min}}} \rceil$, then $\gamma(a) + \gamma(d) = \lceil \frac{r_j}{c_{j_{min}}} \rceil \lfloor \frac{a+d}{c_j} \rfloor + \lceil \frac{r_j}{c_{j_{min}}} \rceil \geq \gamma(a + d)$. If $\min\{\frac{\rho_j(a)}{\rho_{j_{min}}(r_j)}, \lceil \frac{r_j}{c_{j_{min}}} \rceil\} = \frac{\rho_j(a)}{\rho_{j_{min}}(r_j)}$ and $\min\{\frac{\rho_j(d)}{\rho_{j_{min}}(r_j)}, \lceil \frac{r_j}{c_{j_{min}}} \rceil\} = \lceil \frac{r_j}{c_{j_{min}}} \rceil$, then $\gamma(a) + \gamma(d) = \lceil \frac{r_j}{c_{j_{min}}} \rceil \lfloor \frac{a+d}{c_j} \rfloor + \frac{\rho_j(a)}{\rho_{j_{min}}(r_j)} \geq \lceil \frac{r_j}{c_{j_{min}}} \rceil \lfloor \frac{a+d}{c_j} \rfloor + \frac{\rho_j(a+d)}{\rho_{j_{min}}(r_j)} \geq \gamma(a + d)$. The case where $\min\{\frac{\rho_j(a)}{\rho_{j_{min}}(r_j)}, \lceil \frac{r_j}{c_{j_{min}}} \rceil\} = \lceil \frac{r_j}{c_{j_{min}}} \rceil$ and $\min\{\frac{\rho_j(d)}{\rho_{j_{min}}(r_j)}, \lceil \frac{r_j}{c_{j_{min}}} \rceil\} = \frac{\rho_j(d)}{\rho_{j_{min}}(r_j)}$ is similar. Finally, if we have $\min\{\frac{\rho_j(a)}{\rho_{j_{min}}(r_j)}, \lceil \frac{r_j}{c_{j_{min}}} \rceil\} = \frac{\rho_j(a)}{\rho_{j_{min}}(r_j)}$ and $\min\{\frac{\rho_j(d)}{\rho_{j_{min}}(r_j)}, \lceil \frac{r_j}{c_{j_{min}}} \rceil\} = \frac{\rho_j(d)}{\rho_{j_{min}}(r_j)}$, then $\gamma(a) + \gamma(d) = \lceil \frac{r_j}{c_{j_{min}}} \rceil (\lfloor \frac{a+d}{c_j} \rfloor - 1) + \frac{\rho_j(a+d)+c_j}{\rho_{j_{min}}(r_j)}$. Since $\lceil \frac{r_j}{c_{j_{min}}} \rceil \leq \lfloor \frac{c_j}{c_{j_{min}}} \rfloor$ and $\frac{c_j}{\rho_{j_{min}}(r_j)} \geq \frac{c_j}{c_{j_{min}}} \geq \lfloor \frac{c_j}{c_{j_{min}}} \rfloor$, $\gamma(a) + \gamma(d) \geq \lceil \frac{r_j}{c_{j_{min}}} \rceil \lfloor \frac{a+d}{c_j} \rfloor + \frac{\rho_j(a+d)}{\rho_{j_{min}}(r_j)} \geq \gamma(a + d)$. So γ is subadditive. ∎

Using function γ, we will lift inequality (33).

THEOREM 10. *Let $N^0 \subset N$, $N^1 = N \setminus N^0$, $j_{min} = \arg\min_{i \in N^1} c_i$, $j \in N^1$, with $j_{min} < j$, $r_j \leq c_j - 1$, $\rho_{j_{min}}(r_j) > 0$, and $\lceil \frac{r_j}{c_{j_{min}}} \rceil \leq \lfloor \frac{c_j}{c_{j_{min}}} \rfloor$, $N^- = \{i \in N^1 : i < j\}$, and $N^+ = \{i \in N^1 : i \geq j\}$. The lifted 2-partition inequality*

$$\sum_{i \in N^-} \min\left\{\left\lceil \frac{c_i}{c_{j_{min}}} \right\rceil, \left\lceil \frac{r_j}{c_{j_{min}}} \right\rceil\right\} x_i + \left\lceil \frac{r_j}{c_{j_{min}}} \right\rceil \sum_{i \in N^+} \left\lceil \frac{c_i}{c_j} \right\rceil x_i$$
$$+ \sum_{i \in N^0} \left(\left\lceil \frac{r_j}{c_{j_{min}}} \right\rceil \left\lfloor \frac{c_i}{c_j} \right\rfloor + \min\left\{ \frac{\rho_j(c_i)}{\rho_{j_{min}}(r_j)}, \left\lceil \frac{r_j}{c_{j_{min}}} \right\rceil \right\} \right) x_i \geq \left\lceil \frac{r_j}{c_{j_{min}}} \right\rceil \left\lceil \frac{b}{c_j} \right\rceil \tag{36}$$

is valid for PX.

Proof. To prove the validity of (36), we need to show that $\gamma(a) \geq \beta(a)$ for all $a \in \mathbb{R}$. For $a < b$, with $0 < \rho_j(a) < r_j$, if $\min\{\frac{\rho_j(a)}{\rho_{j_{min}}(r_j)}, \lceil \frac{r_j}{c_{j_{min}}} \rceil\} = \frac{\rho_j(a)}{\rho_{j_{min}}(r_j)}$, then

$$\begin{aligned}
\gamma(a) - \beta(a) &= \lceil \frac{r_j}{c_{j_{min}}} \rceil \lfloor \frac{a}{c_j} \rfloor + \frac{\rho_j(a)}{\rho_{j_{min}}(r_j)} - \lceil \frac{r_j}{c_{j_{min}}} \rceil \lceil \frac{a}{c_j} \rceil + \lceil \frac{\rho_j(b-a)}{c_{j_{min}}} \rceil \\
&= \frac{\rho_j(a)}{\rho_{j_{min}}(r_j)} - \lceil \frac{r_j}{c_{j_{min}}} \rceil + \lceil \frac{\rho_j(b-a)}{c_{j_{min}}} \rceil \\
&= \frac{\rho_j(a)}{\rho_{j_{min}}(r_j)} - \lceil \frac{r_j}{c_{j_{min}}} \rceil + \lceil \frac{r_j - \rho_j(a)}{c_{j_{min}}} \rceil \\
&= \frac{\rho_j(a)}{\rho_{j_{min}}(r_j)} - \lceil \frac{r_j}{c_{j_{min}}} \rceil \\
&\qquad + \lceil \frac{r_j - \rho_{j_{min}}(r_j) + c_{j_{min}} - \rho_j(a) + \rho_{j_{min}}(r_j) - c_{j_{min}}}{c_{j_{min}}} \rceil \\
&= \frac{\rho_j(a)}{\rho_{j_{min}}(r_j)} + \lceil \frac{-\rho_j(a) + \rho_{j_{min}}(r_j) - c_{j_{min}}}{c_{j_{min}}} \rceil.
\end{aligned}$$

If $\rho_j(a) < \rho_{j_{min}}(r_j)$, then $\lceil \frac{-\rho_j(a)+\rho_{j_{min}}(r_j)-c_{j_{min}}}{c_{j_{min}}} \rceil = 0$ and $\gamma(a) - \beta(a) = \frac{\rho_j(a)}{\rho_{j_{min}}(r_j)} \geq 0$. If $\rho_j(a) \geq \rho_{j_{min}}(r_j)$, then $\gamma(a) - \beta(a) = \frac{\rho_j(a)}{\rho_{j_{min}}(r_j)} - 1 + \lceil \frac{-\rho_j(a)+\rho_{j_{min}}(r_j)}{c_{j_{min}}} \rceil \geq \frac{\rho_j(a)-\rho_{j_{min}}(r_j)}{c_{j_{min}}} - \lfloor \frac{\rho_j(a)-\rho_{j_{min}}(r_j)}{c_{j_{min}}} \rfloor \geq 0$.

If $\min\left\{\frac{\rho_j(a)}{\rho_{j_{min}}(r_j)}, \lceil\frac{r_j}{c_{j_{min}}}\rceil\right\} = \lceil\frac{r_j}{c_{j_{min}}}\rceil$, $\gamma(a) = \lceil\frac{r_j}{c_{j_{min}}}\rceil\lceil\frac{a}{c_j}\rceil \geq \beta(a)$. For $a < b$, with $\rho_j(a) = 0$, $\gamma(a) = \lceil\frac{r_j}{c_{j_{min}}}\rceil\lfloor\frac{a}{c_j}\rfloor = \beta(a)$. For $a < b$, with $\rho_j(a) \geq r_j$,

$$\begin{aligned}\frac{\rho_j(a)}{\rho_{j_{min}}(r_j)} - \left\lceil\frac{r_j}{c_{j_{min}}}\right\rceil &= \frac{\rho_j(a)}{\rho_{j_{min}}(r_j)} - \frac{r_j - \rho_{j_{min}}(r_j) + c_{j_{min}}}{c_{j_{min}}}\\ &= \frac{\rho_j(a)c_{j_{min}} - \rho_{j_{min}}(r_j)(r_j - \rho_{j_{min}}(r_j) + c_{j_{min}})}{\rho_{j_{min}}(r_j)c_{j_{min}}}\\ &\geq \frac{r_jc_{j_{min}} - \rho_{j_{min}}(r_j)(r_j - \rho_{j_{min}}(r_j) + c_{j_{min}})}{\rho_{j_{min}}(r_j)c_{j_{min}}}\\ &= \frac{r_j(c_{j_{min}} - \rho_{j_{min}}(r_j)) - \rho_{j_{min}}(r_j)(-\rho_{j_{min}}(r_j) + c_{j_{min}})}{\rho_{j_{min}}(r_j)c_{j_{min}}}\\ &= \frac{(r_j - \rho_{j_{min}}(r_j))(c_{j_{min}} - \rho_{j_{min}}(r_j))}{\rho_{j_{min}}(r_j)c_{j_{min}}} \geq 0.\end{aligned}$$

So $\gamma(a) = \lceil\frac{r_j}{c_{j_{min}}}\rceil\lceil\frac{a}{c_j}\rceil = \beta(a)$. For $a \geq b$, if $\lceil\frac{a}{c_j}\rceil = \lceil\frac{b}{c_j}\rceil$, then $\rho_j(a) \geq r_j$ and $\gamma(a) = \lceil\frac{r_j}{c_{j_{min}}}\rceil\lceil\frac{a}{c_j}\rceil = \beta(a)$. Otherwise, $\lceil\frac{a}{c_j}\rceil = \lceil\frac{b}{c_j}\rceil + 1$, and so $\gamma(a) \geq \beta(a)$. Hence $\gamma(a) \geq \beta(a)$ for all $a \in \mathbb{R}$. ☐

As in the case of lifted rounding inequalities, the lifted 2-partition inequalities are also dominated by a subset of them which is polynomial in size.

PROPOSITION 9. *Let* $\{j_{min}, j\} \subseteq N$, *with* $j_{min} < j$, $r_j \leq c_j - 1$, $\rho_{j_{min}}(r_j) > 0$, *and* $\lceil\frac{r_j}{c_{j_{min}}}\rceil \leq \lfloor\frac{c_j}{c_{j_{min}}}\rfloor$. *The inequality*

$$\sum_{i=1}^{j_{min}-1} \min\left\{\frac{c_i}{\rho_{j_{min}}(r_j)}, \left\lceil\frac{r_j}{c_{j_{min}}}\right\rceil\right\}x_i + \sum_{i=j_{min}}^{j-1} \min\left\{\left\lceil\frac{c_i}{c_{j_{min}}}\right\rceil, \frac{c_i}{\rho_{j_{min}}(r_j)}, \left\lceil\frac{r_j}{c_{j_{min}}}\right\rceil\right\}x_i$$
$$+ \sum_{i=j}^{n}\left(\left\lceil\frac{r_j}{c_{j_{min}}}\right\rceil\left\lfloor\frac{c_i}{c_j}\right\rfloor + \min\left\{\frac{\rho_j(c_i)}{\rho_{j_{min}}(r_j)}, \left\lceil\frac{r_j}{c_{j_{min}}}\right\rceil\right\}\right)x_i \geq \left\lceil\frac{r_j}{c_{j_{min}}}\right\rceil\left\lceil\frac{b}{c_j}\right\rceil \quad (37)$$

is valid and dominates inequality (36) *for* $N^0 \subset N$, $N^1 = N\setminus N^0$, *with* $\{j_{min}, j\} \subset N^1$ *and* $j_{min} = \arg\min_{i\in N^1} c_i$.

Proof. Let $\{j_{min}, j\} \subseteq N$, with $j_{min} < j$, $r_j \leq c_j - 1$, $\rho_{j_{min}}(r_j) > 0$, and $\lceil\frac{r_j}{c_{j_{min}}}\rceil \leq \lfloor\frac{c_j}{c_{j_{min}}}\rfloor$. Consider $N^- = \{j_{min} \leq i < j : \lceil\frac{c_i}{c_{j_{min}}}\rceil \leq \frac{c_i}{\rho_{j_{min}}(r_j)}\}$, $N^+ = \{j\}$, $N^1 = N^- \cup N^+$, and $N^0 = N \setminus N^1$. For this choice of subsets, inequality (36) is the same as inequality (37).

Let $N^1 \subset N$, with $\{j_{min}, j\} \subset N^1$ and $j_{min} = \arg\min_{i\in N^1} c_i$. In inequality (36), for $i \in N^1$, if $i < j$, then x_i has the coefficient $\min\left\{\lceil\frac{c_i}{c_{j_{min}}}\rceil, \lceil\frac{r_j}{c_{j_{min}}}\rceil\right\}$, and if $i \in N^0$, then it has the coefficient $\min\left\{\frac{c_i}{\rho_{j_{min}}(r_j)}, \lceil\frac{r_j}{c_{j_{min}}}\rceil\right\}$. In both cases, its coefficient in inequality (36) is greater than or equal to its coefficient in inequality (37). If $i > j$ and $i \in N^1$, then the coefficient of x_i in inequality (36) is $\lceil\frac{r_j}{c_{j_{min}}}\rceil\lceil\frac{c_i}{c_j}\rceil$ and is greater than or equal to its coefficient in inequality (37). Other variables have the same coefficients in both inequalities. As the right-hand sides are also the same, we can conclude that inequality (37) dominates inequality (36). ☐

The number of lifted 2-partition inequalities that are not dominated is $O(n^2)$.

7. Preliminary computational results. We mentioned in the introduction that the inequalities presented in this paper could be used to solve some hard mixed integer programming problems such as the *heterogeneous vehicle routing problem* (see

[18]) and the *manufacturer's mixed pallet design problem* (MPD) (see [19]). Some preliminary results with the rounding inequalities and the lifted rounding inequalities are presented in [18] and [19], respectively.

In this section, we investigate the effect of the lifted rounding inequalities and the lifted 2-partition inequalities in solving the MPD instances. The rounding inequalities for $\lambda = c_j$ for some $j \in N$ and the residual capacity inequalities are not included in this study as they are the same as or dominated by the lifted rounding inequalities.

We first give a brief definition of the MPD. For details, we refer the reader to [19]. Let C be the set of customers, N be the set of products, and $T = \{1, 2, \ldots, \tau\}$ be the set of periods. Each customer $k \in C$ has a demand of d_{kit} units for product $i \in N$ in period $t \in T$. Products are of identical dimensions and are sold in pallets. Each pallet has Q_1 rows, and, in each row, there are Q_2 units of a product. A pallet which contains more than one product type is called a mixed pallet. Let P denote the set of potential mixed pallet designs and q_{ij} denote the number of rows of product $i \in N$ in pallet design $j \in P$. The manufacturer also offers full pallets for each product $i \in N$, which consists of Q_1Q_2 units of product i. We denote by h_{kit} and π_{kit} the unit inventory holding cost and the unit backlogging cost, respectively, for product $i \in N$ and customer $k \in C$ at the end of period $t \in T$. No backlogging is permitted at the end of period τ. The problem is to select at most m mixed pallet designs from set P to minimize the sum of customers' inventory holding and backlogging costs in periods $1, 2, \ldots, \tau$.

Let p_j be 1, if mixed pallet design $j \in P$ is offered, and 0, otherwise. Let P_k denote the set of mixed pallets that customer $k \in C$ can buy. Define y_{kjt} to be the number of pallets of type $j \in P_k$ that customer $k \in C$ buys in period $t \in T$ and f_{kit} to be the number of full pallets of product type $i \in N$ that customer $k \in C$ buys in period $t \in T$. In addition, define I_{kit} and B_{kit} to be the amount of product $i \in N$ that remains in inventory and that is backlogged at the end of period $t \in T$ for customer $k \in C$, respectively. Let M be a very large number. The MPD is formulated as follows in [19]:

$$\min \sum_{k\in C}\sum_{i\in N}\sum_{t\in T}(\pi_{kit}B_{kit} + h_{kit}I_{kit}) \tag{38}$$

$$\text{s.t.} \sum_{j\in P} p_j \leq m, \tag{39}$$

$$I_{kit-1} - B_{kit-1} + Q_1Q_2 f_{kit} + \sum_{j\in P_k} Q_2 q_{ij} y_{kjt} = d_{kit} + I_{kit} - B_{kit} \quad \forall k \in C, i \in N, t \in T, \tag{40}$$

$$y_{kjt} \leq Mp_j \quad \forall k \in C, j \in P_k, t \in T, \tag{41}$$

$$I_{ki0} = B_{ki0} = B_{ki\tau} = 0 \quad \forall k \in C, i \in N, \tag{42}$$

$$I_{kit}, B_{kit} \geq 0 \quad \forall k \in C, i \in N, t \in T, \tag{43}$$

$$f_{kit} \geq 0 \text{ and integer} \quad \forall k \in C, i \in N, t \in T, \tag{44}$$

$$y_{kjt} \geq 0 \text{ and integer} \quad \forall k \in C, j \in P_k, t \in T, \tag{45}$$

$$p_j \in \{0, 1\} \quad \forall j \in P. \tag{46}$$

The objective function (38) is the sum of inventory holding and backlogging costs over all periods. At most m mixed pallet designs can be offered due to constraint (39). Constraints (40) are the balance equations. Constraints (41) ensure that customers do

Table 1
Results with and without valid inequalities.

	Model1		Model2				Model3		
Problem	Nodes	CPU	(17)	% gap	Nodes	CPU	(37)	Nodes	CPU
1	1040094	168.38	33	96.57	84039	15.06	4	49348	10.05
2	3158201	662.17	39	97.18	189635	40.64	53	67257	16.24
3	29531186	6774.68	43	97.53	621224	159.62	59	248578	68.50
4	25242255	5800.38	48	95.43	600664	152.93	65	266693	76.80
5	2008508	1535.85	54	97.96	42476	34.12	77	87575	75.31
6	7650540	6310.95	58	98.30	395894	329.14	83	175031	150.45
7	110344292	7751.75	63	96.65	148494	121.37	89	48285	45.20

not buy mixed pallets that are not offered. Constraints (42) are beginning and ending conditions. Constraints (43)–(46) are nonnegativity and integrality constraints.

Yaman and Sen prove that the optimal value of the linear programming relaxation of MPD is zero. As a result it is important to derive strong valid inequalities for this problem to be able to improve the linear programming-based lower bounds.

For $k \in C$ and $i \in N$, let $D_{ki} = \lceil \sum_{t \in T} d_{kit}/Q_2 \rceil$. The inequality

$$\sum_{t \in T} \left(\min\{Q_1, D_{ki}\} f_{kit} + \sum_{j \in P_k} \min\{q_{ij}, D_{ki}\} y_{kjt} \right) \geq D_{ki} \tag{47}$$

is satisfied by all feasible solutions of MPD. Remark that the set of nonnegative integer solutions satisfying inequality (47) is an integer knapsack cover set. Hence we can generate valid lifted rounding and lifted 2-partition inequalities for the MPD based on inequalities (47).

We test the use of these valid inequalities on seven problem instances. We start with two base instances. In the first instance the number of products is two, and in the second instance the number of products is three. In both base instances, the number of periods is three, and the maximum number of mixed pallet designs to be offered is one. Using the first base instance, we generated four problems where the number of customers takes values 4, 5, 6, and 7. Using the second base instance, we generated three problems with 5, 6, and 7 customers.

For each problem instance, we first solve the model without valid inequalities. We call this *Model*1. We report the number of nodes in the branch and bound tree (in column *node*) and the CPU time in seconds (in column *CPU*). Then we form *Model*2 by adding the nondominated lifted rounding inequalities (17) to Model1. For Model2, we report the number of inequalities (17) added (in column (17)), the percentage duality gap (in column *%gap*, where $\%gap = \frac{opt - lp}{opt} * 100$, *opt* is the optimal value, and *lp* is the lower bound obtained from the linear programming relaxation), the number of nodes in the branch and bound tree, and the CPU time in seconds. Finally, we form *Model*3 by adding the nondominated lifted 2-partition inequalities (37) to Model2. We report here the number of inequalities (37) added (in column (37)), the number of nodes in the branch and bound tree, and the CPU time in seconds. The percentage duality gaps remained the same as the ones of Model2 and so are not reported. We solve the models using the mixed integer programming (MIP) solver of CPLEX 8.1 on an AMD Opteron 252 processor (2.6 GHz) with 2 GB of RAM. The results are given in Table 1.

The results show that both families of valid inequalities have been useful in decreasing the number of nodes in the branch and bound tree and the solution times for these instances. The solution time for Model3 is larger than the one of Model2 for instance five, but still it is about twenty times less than the one of Model1. The averages of percentage improvements obtained in the number of nodes and CPU time with the addition of inequalities (17) are 96.29% and 95.85%, respectively. The averages of percentage improvements obtained in the number of nodes and CPU time compared to Model2 with the addition of inequalities (37) are 34.07% and 28.07%, respectively.

8. Conclusion. We studied the polyhedral properties of the convex hull of the integer knapsack cover set which appears as a relaxation of many optimization problems that concern covering a given demand using integer numbers of different types of items. We derived four families of valid inequalities, investigated when they dominate each other, and gave some conditions under which some are facet-defining. We used sequence-independent lifting to derive that last two families of valid inequalities. These inequalities can be used to solve problems such as those investigated in [11, 18, 19].

Except the rounding inequalities for arbitrary λ values, the valid inequalities derived in this paper share some common features. There exists always an item $j \in N$ such that the right-hand side of the inequality is equal to the coefficient of x_j times $\lceil \frac{b}{c_j} \rceil$. We know that this is an upper bound on the value of the right-hand side (see Proposition 1). Clearly, there are facet-defining inequalities which do not follow this rule. For instance, the cover constraint is facet-defining for $conv(\{x \in \mathbb{Z}^3_+ : 3x_1 + 4x_2 + 5x_3 \geq 13\})$.

Again excluding rounding inequalities, another common feature is that the number of inequalities that are nondominated within a family is polynomial even when the family has an exponential number of inequalities. These inequalities can be further lifted or modified to define larger families of valid inequalities for more complicated problems in consideration. For instance, an exponential number of valid inequalities can be derived for the integer capacity cover polyhedron using the inequalities of this paper and the lifting results of Mazur and Hall [12].

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