[4] A. J. Krener and W. Respondek, "Nonlinear observer with linearizable error dynamics," *SIAM J. Control Optim.*, vol. 23, pp. 197–216, 1985.
[5] F. E. Thau, "Observing the state of nonlinear dynamic systems," *Int. J. Control*, vol. 17, pp. 471–479, 1973.
[6] S. R. Kou, D. L. Elliott, and T. J. Tarn, "Exponential observers for nonlinear dynamic systems," *Inf. Control*, vol. 29, pp. 204–216, 1975.
[7] S. Raghavan, "Observers and compensators for nonlinear systems, with application to flexible joint orbots," Ph.D. dissertation, Univ. of Calif., Berkeley, 1992.
[8] R. Rajamani, "Observers for Lipschitz nonlinear systems," *IEEE Trans. Autom. Control*, vol. 43, pp. 397–401, 1998.
[9] G. Besancon and H. Hammouri, "On uniform observation of nonuniformly observable systems," *Syst. Control Lett.*, vol. 29, pp. 9–19, 1996.
[10] F. Zhu and Z. Han, "A note on observers for Lipschitz nonlinear systems," *IEEE Trans. Autom. Control*, vol. 47, pp. 1751–1754, 2002.
[11] B. L. Walcott and S. H. Zak, "State observation of nonlinear uncertain dynamical systems," *IEEE Trans. Autom. Control*, vol. 32, pp. 166–170, 1987.
[12] S. H. Zak, "On the stabilization and observation of nonlinear/uncertain dynamical systems," *IEEE Trans. Autom. Control*, vol. 35, pp. 604–607, 1990.
[13] D. M. Dawson, Z. Qu, and J. C. Carroll, "On state observation and output feedback problems for nonlinear uncertain dynamic sytems," *Syst. Control Lett.*, vol. 18, pp. 217–222, 1992.
[14] K. K. Busawon and M. Saif, "A state observer for nonlinear systems," *IEEE Trans. Autom. Control*, vol. 44, pp. 2098–2103, 1999.
[15] J. P. Gauthier, H. Hammouri, and S. Othman, "A simple observer for nonlinear systems, applications to bioreactors," *IEEE Trans. Automat. Control*, vol. 37, pp. 875–880, 1992.
[16] J. C. Doyle and G. Stein, "Robustness with observers," *IEEE Trans. Autom. Control*, vol. AC-24, pp. 607–611, 1979.
[17] G. Stein and M. Athans, "The LQG/LTR procedure for multivariable feedback control design," *IEEE Trans. Autom. Control*, vol. 32, pp. 105–113, 1987.
[18] M. Arcak and P. Kokotovic, "Nonlinear observers: A circle criterion design and robustness analysis," *Automatica*, vol. 37, pp. 885–892, 2003.
[19] H. Shim, J. H. Seo, and A. R. Teel, "Nonlinear observer design via passivation of error dynamics," *Automatica*, vol. 39, pp. 1923–1930, 2001.
[20] A. Alessandri, "Observer design for nonlinear systems by using input-to-state stability," in *Proc. IEEE Conf. Decision Control*, 2004, pp. 3892–3897.
[21] A. Saberi, P. Sannuti, and B. M. Chen, *H_2 Optimal Control*. Englewood Cliffs, NJ: Prentice-Hall, 1995, ch. 4, Theorem 4.1.2.
[22] F. L. Lewis, *Applied Optimal Control and Estimation*. Englewood Cliffs, NJ: Prentice-Hall, 1992, ch. 3, Theorem 2.
[23] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*. New York: Wiley, 1972, ch. 3, Theorem 3.14.
[24] H. K. Khalil, *Nonlinear Systems*. Upper Saddle River, NJ: Prentice-Hall, 2000, ch. 14.5.
[25] R. Marino and P. Tomei, *Nonlinear Control Design*. Upper Saddle River, NJ: Prentice-Hall, 1995, p. 27.

Parameterization of Suboptimal Solutions of the Nehari Problem for Infinite-Dimensional Systems

Kenji Kashima, *Member, IEEE*, Yutaka Yamamoto, *Fellow, IEEE*, and Hitay Özbay, *Senior Member, IEEE*

***Abstract*—The Nehari problem plays an important role in H^∞ control theory. It is well known that H^∞ control problem can be reduced to solving this problem. This note gives a parameterization of all suboptimal solutions of the Nehari problem for a class of infinite-dimensional systems. Many earlier solutions of this problem are seen to be special cases of this new parameterization. It is also shown that for finite impulse response systems this parameterization takes a particularly simple form.**

***Index Terms*—Delay systems, H^∞-control, infinite-dimensional systems, Nehari problem.**

I. Introduction

It is well known that many interesting H^∞ control problems can be transformed to the so-called one-block problem; see for example, [1]–[3] and references therein. The one-block problem can be seen as a model matching problem where stable approximation(s), in the sense of L^∞, of a given unstable system is sought. This is precisely the Nehari problem which can be stated as follows: Given $F \in L^\infty$, find all $\phi \in H^\infty$ such that

$$\|F + \phi\|_{L^\infty} < 1. \tag{1}$$

(We have chosen to write $F + \phi$ in place of $F - \phi$, which is more conventional, for the convenience of later developments.) Nehari's theorem states that a solution $\phi \in H^\infty$ satisfying (1) exists if and only if $\|\Gamma_F\| < 1$, where Γ_F is the Hankel operator associated with symbol F; see Section II and [2]. In this note, we assume that $F \in L^\infty$ with $\|\Gamma_F\| < 1$ is given and we derive a parameterization of solutions $\phi \in H^\infty$ of (1). We approach this problem from an operator theoretic viewpoint.

The Nehari problem has been studied in the control community for various classes of F, and many different solution techniques have been developed, depending on the assumptions of F. In the finite-dimensional case where F is rational, the solution can be obtained easily by solving Lyapunov equations derived from a state space realization of F. However, for the infinite-dimensional case where F is irrational, state space approaches require solutions of operator equations (instead of matrix Lyapunov equations), see, e.g., [4].

Manuscript received October 9, 2006; revised February 27, 2007. Recommended by Associate Editor D. Dochain. This work was supported in part by the Ministry of Education, Culture, Sports, Science, and Technology of Japan under a Grant-in-Aid for Young Scientists (B) 18760316, Grant-in-Aid for Exploratory Research 17656138, Grant-in-Aid for Scientific Research (B) 18360203, and by the European Commission under Contract MIRG-CT-2004-006666, and by TÜBİTAK under Grant EEEAG-105E156.

K. Kashima is with the Graduate School of Information Science and Engineering, Tokyo Institute of Technology, Tokyo 152-8552, Japan (e-mail: kashima@mei.titech.ac.jp; kashima@ieee.org).
Y. Yamamoto is with the Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan (e-mail: yy@i.kyoto-u.ac.jp).
H. Özbay is with the Department of Electrical and Electronics Engineering, Bilkent University, Bilkent, Ankara TR-06800, Turkey (e-mail: hitay@bilkent.edu.tr).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.
Digital Object Identifier 10.1109/TAC.2007.910725

One of the most interesting solutions of the Nehari problem is a characterization due to Adamjan, Arov and Krein (abbr. AAK hereafter) [5]. The AAK solution was originally given for H^∞ functions defined on the unit disk [5]. Several parameterization for the continuous-time suboptimal Nehari problem have also been derived; see, e.g. [4] and references therein. Among them, the following two independently derived results played an important role in the frequency-domain approaches for infinite-dimensional H^∞ control theory:

- Toker and Özbay [6] derived a solution by directly converting AAK theory to the continuous-time domain by using a conformal mapping between the unit disk and the right half plane. This result involves a redundant variable, conformal map parameter, which blurs the structure. We will further comment on this issue in Section III.
- Meinsma *et al.* [7] gave a parameterization for the Nehari problem with a continuous-time finite impulse response (FIR) system F (see Section IV) by constructing J-spectral factor via solving 2 matrix Riccati equations. However, its structure and relation with the AAK theory is not very clear.

Our goal in this note is to clarify the relationship between these two results and represent AAK theory in a unified way for infinite-dimensional systems. To this end, we first derive Theorem 1, a direct continuous-time counterpart of AAK theory. While this result is in a similar form to that in [6], no redundant variable is introduced in Theorem 1. We then show that Theorem 1 includes [7] as a special case. This way we establish a clear connection between the chain scattering approach taken by [7] and the AAK theory used in [6].

The remaining parts of this note are organized as follows: The next section summarizes notational conventions and preliminary results. The main contribution, Theorem 1, is given in Section III. In Section IV, we investigate a special case which is crucial for standard H^∞ control problems for a class of infinite-dimensional system including systems with time delays.

II. Preliminary Results

A. Notation and Convention

As usual, H^p and H^p_- denote the Hardy spaces on the open right- and left-half complex planes, respectively. The spaces L^∞ and $L^2(j\mathbb{R})$ denote, respectively, the space of essentially bounded functions and square integrable functions on the imaginary axis. The orthogonal projections from $L^2(j\mathbb{R}) = H^2 \oplus H^2_-$ to $H^2(H^2_-)$ are denoted by $\pi^+[\cdot](\pi^-[\cdot])$. Let $q^\sim(s) := \overline{q(-\bar{s})}^\top$ where $M^\top$ denotes the transpose of a matrix M. For state-space realization of rational transfer matrix we write

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := D + C(sI - A)^{-1}B.$$

The size of matrices is omitted for brevity. For a normed space X, the open unit ball is denoted by BX

$$BX := \{x \in X : \|x\|_X < 1\}.$$

For $F \in L^\infty$, the Hankel operator is defined by

$$\Gamma_F : H^2 \to H^2_- : x \mapsto \pi^-[Fx]$$

and its operator norm $\|\Gamma_F\|$ is called the Hankel norm of F. In what follows, Γ_F is denoted by Γ for simplicity.

B. Chain Scattering Representation

Let G, K be transfer matrices of appropriate dimensions. Then the *chain scattering* of G and K is defined by

$$C_r(G, K) := H_1 H_2^{-1}, \quad \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = G \begin{bmatrix} K \\ I \end{bmatrix}$$

provided that $H_2 \not\equiv 0$; see also [8].

Definition 1: Let Θ be a (2×2)-block matrix with square diagonal blocks. Define

$$J = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array}\right]$$

where J is partitioned accordingly to Θ. Then Θ is said to be J*-unitary* if $\Theta^\sim J \Theta = J$.

The following lemma reduces the Nehari problem to an equivalent problem of finding a J-unitary matrix.

Lemma 1: Given $F \in L^\infty$ such that $\|\Gamma\| < 1$, define

$$G := \left[\begin{array}{c|c} I & F \\ \hline 0 & I \end{array}\right]. \tag{2}$$

Suppose that a J-unitary matrix $\Theta \in L^\infty$ satisfies $G^{-1}\Theta$, $\Theta^{-1}G \in H^\infty$. Then all $\phi \in H^\infty$ satisfying (1) is given by

$$\phi = C_r(G^{-1}\Theta, U) \tag{3}$$

where $U \in BH^\infty$ but otherwise arbitrary.

Proof: For J-unitary matrix Θ in L^∞, its inverse is given by $\Theta^{-1} = J\Theta^\sim J$. It can be easily verified that $\Theta^{-1}G$ is a J-spectral factor for $G^\sim JG$, i.e., bistable matrix $\Theta^{-1}G$ satisfies $G^\sim JG = (\Theta^{-1}G)^\sim J(\Theta^{-1}G)$. Therefore, the same discussion as that in [9, Theorem 3.1] yields the bistability of Θ_{22} and the desired result; see also [10, Appendix], [11]. ■

It should be noted that, in many cases, $\|\Gamma\| < 1$ actually imply the existence of such Θ; see the following sections and [4], [9], and [10].

III. Main Result

In this section, we attempt to construct Θ satisfying the properties required in Lemma 1, i.e., Θ should be J-unitary and $G^{-1}\Theta$ should be bistable. It should be stressed that AAK theory was derived in a similar way and that such a Θ was given in terms of the Hankel operator. While we will not go into further details of AAK theory, it would be informative to consider an equivalent form in the continuous-time domain as follows. Let us partition Θ as

$$\Theta := \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \tag{4}$$

and assume that it satisfies

$$\begin{cases} (I - \Gamma\Gamma^*)\Theta_{11} = I \\ \Gamma^*\Theta_{11} = \Theta_{21} \end{cases}$$

and

$$\begin{cases} (I - \Gamma^*\Gamma)\Theta_{22} = I \\ \Gamma\Theta_{22} = \Theta_{12} \end{cases}$$

where Γ^* denotes the adjoint operator of Γ

$$\Gamma^* : H^2_- \to H^2 : y \mapsto \pi^+[F^\sim y].$$

We may try to construct Θ from the above. However, this function may not satisfy the required properties nor be well-defined either. To see this, let us focus our attention only on

$$(I - \Gamma^*\Gamma)\Theta_{22} = I. \tag{5}$$

Recall that $\Gamma^*\Gamma$ is a bounded operator in H^2. Obviously, the left-hand side belongs to H^2 whenever so does Θ_{22}. Therefore, this equation is meaningless because the constant I never belongs to H^2. This shows a clear contrast with the discrete-time domain case where any constant function is square integrable on the unit circle.

In [6], an extra variable was introduced to fill the gap of measures between the imaginary axis and the unit circle; that is, the right-hand side of (5) was simply replaced by $(1/(s+\alpha))\cdot I$ with $\alpha > 0$. While this modification succeeded in deriving a solution, the additional variable α seems redundant but not easily removable. To circumvent this problem, let us assume that F is square integrable on the imaginary axis, i.e., $F \in L^2(j\mathbb{R})$. Under this assumption, $\pi^-[F]$ is well-defined. Moreover, if $\|\Gamma\| < 1$ then both $I - \Gamma^*\Gamma$ and $I - \Gamma\Gamma^*$ are invertible in H^2 and H^2_-, respectively, and consequently we can define

$$\eta := (I - \Gamma\Gamma^*)^{-1}\left(\pi^-[F]\right).$$

In other words, there exist unique $\xi \in H^2$ and $\eta \in H^2_-$ such that

$$\begin{cases} \Gamma\xi + \pi^-[F] = \eta \\ \Gamma^*\eta = \xi. \end{cases} \tag{6}$$

The point here is that defining $\Theta_{22} := I + \xi$ yields

$$\Theta_{22} - \pi^+\left[F^\sim \pi^-[F\Theta_{22}]\right] = I.$$

It seems natural that we use this equality instead of (5). Dually, there exist unique $\acute{\xi} \in H^2$ and $\acute{\eta} \in H^2_-$ such that

$$\begin{cases} \Gamma^*\acute{\eta} + \pi^+[F^\sim] = \acute{\xi} \\ \Gamma\acute{\xi} = \acute{\eta}. \end{cases} \tag{7}$$

Under these definitions, the desired J-unitary matrix Θ can be given as follows.

Assumption 1: Let $F \in L^\infty \cap L^2(j\mathbb{R})$ such that $\|\Gamma\| < 1$. Suppose that the unique solutions to operator equations (6) and (7), $\xi, \acute{\xi} \in H^2$ and $\eta, \acute{\eta} \in H^2_-$, belong to L^∞.

Theorem 1: Let Assumption 1 hold. Then all $\phi \in H^\infty$ such that $F + \phi \in BL^\infty$ is given by

$$\left\{\phi = C_r(G^{-1}\Theta, U) = C_r(\Theta, U) - F : U \in BH^\infty\right\}$$

where $\Theta \in L^\infty$ is defined by

$$\Theta := \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} := \begin{bmatrix} \acute{\eta} + I & \eta \\ \acute{\xi} & \xi + I \end{bmatrix}. \tag{8}$$

The proof of this result will be given below. First, note that from Lemma 1, it is sufficient to show that Θ in (8) is J-unitary and that $G^{-1}\Theta, \Theta^{-1}G$ belong to H^∞. We will also need the following lemma on shift operators.

Lemma 2: For $h \geq 0$, let $\sigma_h[\cdot]$ be the left-shift operator on H^2, i.e.

$$\sigma_h : x(s) \mapsto \pi^+\left[e^{hs}x(s)\right].$$

Then for arbitrary $F \in L^\infty, h \geq 0, y \in H^2_-$, we have

$$\sigma_h[\Gamma^* y] = \Gamma^*(e^{hs}y).$$

Proof: This can be shown by direct calculation. ■

Proof of Theorem 1: First we show that Θ in (8) is J-unitary, or equivalently

a) $\Theta_{11}^\sim\Theta_{11} - \Theta_{21}^\sim\Theta_{21} = I$;
b) $\Theta_{22}^\sim\Theta_{22} - \Theta_{12}^\sim\Theta_{12} = I$;
c) $\Theta_{11}^\sim\Theta_{12} - \Theta_{21}^\sim\Theta_{22} = 0$.

We prove b) only, because a) and c) can be shown similarly. By the definition of Θ, b) is equivalent to

$$\xi^\sim\xi - \eta^\sim\eta + \xi + \xi^\sim = 0.$$

By the assumption $\Theta \in L^\infty$, the left hand side belongs to $L^2(j\mathbb{R})$. Moreover, because of its symmetry, it suffices to show that

$$\pi^+[\xi^\sim\xi - \eta^\sim\eta + \xi + \xi^\sim] = \pi^+[\xi^\sim\xi - \eta^\sim\eta + F^\sim\eta] = 0.$$

This is equivalent to saying that, in the time domain

$$\int_{-\infty}^{\infty} (\xi^\sim\xi - \eta^\sim\eta + F^\sim\eta)(j\omega)e^{j\omega h}d\omega = 0 \tag{9}$$

holds for every $h \geq 0$. Under the following definition

$$(x, y) := \int_{-\infty}^{\infty} (y^\sim x)(j\omega)d\omega$$

for $x, y \in L^2(j\mathbb{R})$, (9) can be rewritten as

$$(e^{hs}\xi, \xi) - (e^{hs}\eta, \eta) + (e^{hs}\eta, F) = 0.$$

Recall that $(y, x) = 0$ and $(y, \Gamma x) = (\Gamma^* y, x)$ for any $x \in H^2$ and $y \in H^2_-$. From Lemma 2 and (6), for any $h \geq 0$

$$\begin{aligned} (e^{hs}\eta, \eta) - (e^{hs}\eta, F) &= (e^{hs}\eta, \eta) - \left(e^{hs}\eta, \pi^-[F]\right) \\ &= (e^{hs}\eta, \Gamma\xi) = \left(\Gamma^*(e^{hs}\eta), \xi\right) \\ &= (\sigma_h[\Gamma^*\eta], \xi) \\ &= (\sigma_h\xi, \xi) = (e^{hs}\xi, \xi) - \left(\pi^-[e^{hs}\xi], \xi\right) \\ &= (e^{hs}\xi, \xi). \end{aligned}$$

Hence, b) follows.

By substituting $\Theta^{-1} = J\Theta^\sim J$ to $\Theta^{-1}G$, we have

$$\begin{aligned} G^{-1}\Theta &= \begin{bmatrix} I + \pi^+[F\acute{\xi}] & -\pi^+[F\xi] - \pi^+[F] \\ \acute{\xi} & \xi + I \end{bmatrix} \\ \Theta^{-1}G &= \begin{bmatrix} I + \acute{\eta}^\sim & \left(\pi^-[F^\sim\acute{\eta}] + \pi^-[F^\sim]\right)^\sim \\ -\eta^\sim & I - \left(\pi^-[F^\sim\eta]\right)^\sim \end{bmatrix}. \end{aligned}$$

Both of these are analytic on the open right half plane. Thus $G^{-1}\Theta$, $\Theta^{-1}G$ belong to H^∞. ■

Theorem 1 can be viewed as a direct continuous-time counterpart of AAK theory. Notice that no additional variable is introduced at the cost of Assumption 1.

IV. Continuous-Time FIR Case

For an arbitrary irrational F it is not easy to obtain suboptimal ϕ from Theorem 1. When F has a certain special structure, our result can be more effectively utilized. In this section, we illustrate this fact for so-called FIR systems. These type of problems are crucial in deriving an explicit realization of suboptimal H^∞ controllers for a class of standard H^∞ control problems for time delay systems; see [3], [7], [10], [12]–[15] and references therein for earlier work on FIR systems and H^∞-control.

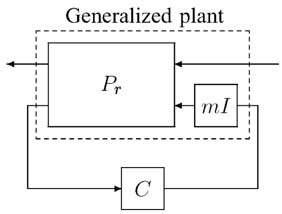

Fig. 1. H^∞ control problem for systems with finitely many unstable modes.

A. Extension of FIR Systems

For a scalar complex function $f(s)$, the set of square matrices M such that $f^{\sim}(s)$ is analytic in a neighborhood of every eigenvalue of M is denoted by $\mathcal{M}_f$. For $M \in \mathcal{M}_f$, the matrix function $f^{\sim}(M)$ is well-defined [16, Section 11.1.1]. Hereafter we confine ourselves to the following class of infinite-dimensional systems including FIR systems:

Lemma 3: Let $m(s)$ be a scalar inner function, and $(A, B, C, 0)$ with $A \in \mathcal{M}_m$ be a realization of a rational matrix W. Then $\pi^m[W]$ defined by

$$\pi^m[W] := C(sI - A)^{-1}\left(I - m(s)m^{\sim}(A)\right)B \tag{10}$$

is in both H^2 and H^∞. Moreover, $m^{\sim}\pi^m[W]$ belongs to H^2_-.

Proof: See [3]. ∎

When $m(s) = e^{-hs}$, $\pi^m[W]$ is a continuous-time FIR system, i.e., the Laplace transform of

$$\begin{cases} Ce^{At}B, & t \in [0, h] \\ 0, & t \notin [0, h] \end{cases}$$

with compact support $[0, h]$. For general inner functions $m(s)$, $\pi^m[W]$ does not necessarily have finite impulse response. However, since all poles of $\pi^m[W]$ are shown to be those of $m(s)$, $\pi^m[W]$ is always stable.

In [3], the standard H^∞ control problem for systems in Fig. 1 was studied. Here the generalized plant is given as the series connection of a rational transfer matrix P_r and a scalar inner function m. This problem covers a wide class of practical control problems for infinite-dimensional plants with finitely many unstable modes, and was shown to be reducible to the Nehari problem in (1), via solving a couple of matrix Riccati equations. Moreover, the symbol associated with the resulting Nehari problem is always given in the form of

$$F := m^{\sim}\pi^m[W] \tag{11}$$

where W is an appropriately defined strictly proper rational transfer matrix. For such an F, its Hankel norm can be computed by analyzing singularity of a matrix of finite size as seen in Theorem 2 below. If the norm conditions are satisfied, all suboptimal solutions can be given by Theorem 1 and Fig. 1.

Theorem 2: Let $m(s)$ be an inner function, $(A, B, C, 0)$ with $A \in \mathcal{M}_m$ a minimal realization of a rational matrix W, and $F := m^{\sim}\pi^m[W]$. Suppose that the essential norm of Γ is less than 1 and that for any $\rho \geq 1$

$$H_\rho := \begin{bmatrix} A & \rho^{-1}BB^\top \\ -\rho^{-1}C^\top C & -A^\top \end{bmatrix} \in \mathcal{M}_m$$

and the (2,2)-block of $m^{\sim}(H_\rho)$ is of full-rank. Then, Θ in (8) belongs to L^∞ and is given by

$$\Theta = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} m^{\sim}I & 0 \\ 0 & I \end{bmatrix}\pi^m[\Theta_r]$$

$$\Theta_r = \left[\begin{array}{cc|cc} A & BB^\top & 0 & B \\ -C^\top C & -A^\top & \Sigma_{22}^{-1}C^\top & -\Sigma_{22}^{-1}\Sigma_{21}B \\ \hline C & 0 & 0 & 0 \\ 0 & B^\top & 0 & 0 \end{array}\right]$$

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} := m^{\sim}\left(\begin{bmatrix} A & BB^\top \\ -C^\top C & -A^\top \end{bmatrix}\right).$$

Proof: Since this theorem can be shown by the same computations as that in [3], we only give the outline. First, the assumptions in this theorem is necessary and sufficient conditions for $\|\Gamma\| < 1$, [3]. In what follows, we derive state space forms of ξ and η in (6) explicitly. It can be verified that solutions ξ, η to operator equations (6) can be represented by two finite-dimensional vectors:

$$\begin{bmatrix} m\eta \\ \xi \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & B^\top \end{bmatrix}(sI - H)^{-1}\left(\begin{bmatrix} B \\ -\psi_2 \end{bmatrix} - m(s)\begin{bmatrix} \psi_1 \\ 0 \end{bmatrix}\right)$$

where

$$H := \begin{bmatrix} A & BB^\top \\ -C^\top C & -A^\top \end{bmatrix}$$

and ψ_1, ψ_2 satisfies

$$\begin{bmatrix} \psi_1 \\ 0 \end{bmatrix} = \Sigma\begin{bmatrix} B \\ -\psi_2 \end{bmatrix}.$$

Since Σ_{22} is nonsingular by the assumption, ψ_1, ψ_2 are uniquely determined and $\psi_2 = \Sigma_{22}^{-1}\Sigma_{21}B$. Therefore, by (10), we have

$$\begin{bmatrix} m\eta \\ \xi \end{bmatrix} = \pi^m\left[\left[\begin{array}{c|c} H & \begin{matrix} B \\ -\Sigma_{22}^{-1}\Sigma_{21}B \end{matrix} \\ \hline \begin{matrix} C & 0 \\ 0 & B^\top \end{matrix} & 0 \end{array}\right]\right].$$

Similarly

$$\begin{bmatrix} m\acute{\eta} \\ \acute{\xi} \end{bmatrix} = \pi^m\left[\left[\begin{array}{c|c} H & \begin{matrix} 0 \\ \Sigma_{22}^{-1}C^\top \end{matrix} \\ \hline \begin{matrix} C & 0 \\ 0 & B^\top \end{matrix} & 0 \end{array}\right]\right]$$

follows from (7). Finally, by (8), we obtain

$$\begin{bmatrix} mI & 0 \\ 0 & I \end{bmatrix}\Theta = \begin{bmatrix} mI & 0 \\ 0 & I \end{bmatrix} + \pi^m[\Theta_r].$$

Trivially, Θ is in L^∞ since so are $m^{\sim}$ and $\pi^m[\Theta_r]$. ∎

The realization of Θ in Theorem 2 is exactly the same as that in [7] when we take $m(s) = e^{-hs}$. In this sense, Theorem 1 includes the existing result as a special case and provides us with an AAK theoretic interpretation of the chain scattering based results.

Remark 1: Let us go back to the standard H^∞ control problem in Fig. 1. By substituting the parameterization in Theorem 2 to suboptimal H^∞ controller obtained in [3], we can show that all suboptimal H^∞ controllers are given in the form of *modified Smith predictor*. This fact can be proven by simply replacing e^{-hs} by a general inner function $m(s)$ in [12]; see also [17].

We close this section with a remark on the assumption $F \in L^2(j\mathbb{R})$. A possible relaxation of Assumption 1 follows.

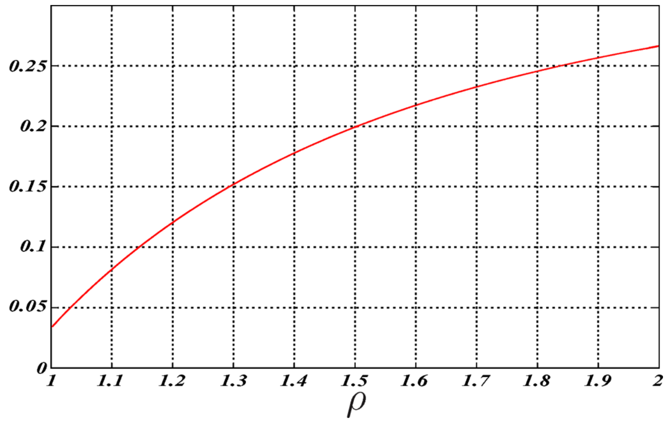

Fig. 2. Minimal singular values of $m\tilde{}(H_\rho)|_{22}$.

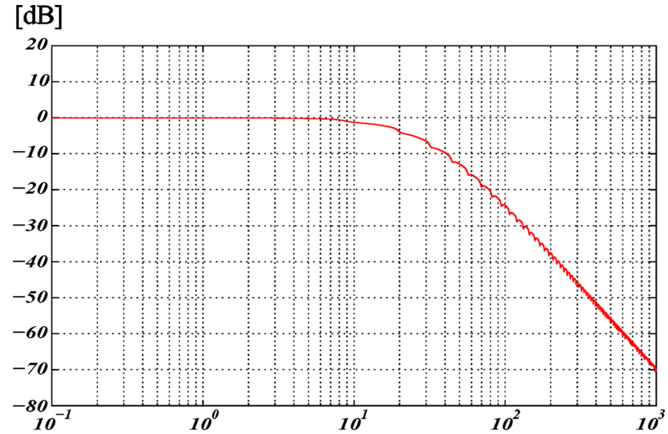

Fig. 3. Bode gain plot of $m\tilde{}\pi^m[W] + \phi_0$.

Assumption 2: $F \in L^\infty$ can be given by

$$F = K + m\tilde{}D$$

where $K \in L^2(j\mathbb{R})$, $m(s)$ is an scalar inner function and D is a constant matrix such that $mD^\top K$ and $mKD^\top$ are analytic on the open right half plane.

The present authors have derived a solution for H^∞ control problem for systems with infinitely many unstable modes (and finite-dimensional inner part) [18]. The Nehari problem to which this standard problem reduces satisfies Assumption 2 only. We can derive similar results to Theorem 1 and 2 under this relaxed assumption. Details are omitted since it can be proven straightforwardly.

B. Example

We demonstrate the above result numerically on problem data derived originally from a weighted mixed sensitivity optimization for a delayed feedback system. Let us consider the Nehari problem with $F = m\tilde{}\pi^m[W]$, where W is the unstable rational function

$$W = \frac{1}{s-1} = \left[\begin{array}{c|c} 1 & 1 \\ \hline 1 & 0 \end{array}\right]$$

and m is the inner function with infinitely many unstable zeros

$$m = \frac{2(s-8)e^{-0.5s} + (s+1)}{2(s+8) + (s-1)e^{-0.5s}}.$$

It should be noted that the original mixed sensitivity optimization is a so-called *two*-block problem for an infinite-dimensional system. However, it can be described by the system in Fig. 1, and consequently can be reduced to a *one*-block problem that is equivalent to the Nehari problem of the above form via solving a couple of Riccati equations; see [3] and [18] for details of the reduction procedure.

Fig. 2 shows minimal singular values of $m\tilde{}(H_\rho)|_{22}$ for $\rho \geq 1$. Since this matrix is nonsingular for any $\rho \geq 1$ and consequently $\|\Gamma_{m\tilde{}\pi^m[W]}\| < 1$, we can apply Theorem 2 to obtain Θ in Theorem 1. Let ϕ_0 be the central solution ($U = 0$), i.e., $\phi_0 := \Theta_{21}\Theta_{22}^{-1} - F$. We can show the stability of ϕ_0 by using Nyquist plot. Furthermore, Fig. 3 shows the Bode gain plot of

$$F + \phi_0 = \frac{m\tilde{}(s)(s+30) + 30(s+1)}{(s-1)(s+30) - 30m(s)}$$

which is less than 1 over all frequencies. Therefore, a solution ϕ_0 to the Nehari problem is obtained without solving any operator equations. Of course, all solutions are also given by exhausting $U \in BH^\infty$.

V. Conclusion

In this note, we derived a parameterization of all suboptimal solutions of the Nehari problem for a class of infinite dimensional systems. The key additional assumption was that F is square integrable on the imaginary axis. This can be viewed as a continuous-time counterpart of the AAK theory, and enables us to interpret in a unified way some existing results that use chain scattering approach.

References

[1] B. A. Francis, *A Course in H_∞ Control Theory*. London, U.K.: Springer-Verlag, 1987, vol. 88, Lecture Notes in Control and Information Science.
[2] C. Foias, H. Özbay, and A. Tannenbaum, *Robust Control of Infinite Dimensional Systems: Frequency Domain Methods*. London, U.K.: Springer-Verlag, 1996, vol. 209, Lecture Notes in Control and Inf. Sci..
[3] K. Kashima and Y. Yamamoto, "General solution to standard H^∞ control problems for infinite-dimensional systems," in *Proc. Joint 44th IEEE Conf. Decision Control and Europ. Control Conf. 2005*, 2005, pp. 2457–2462.
[4] R. F. Curtain and H. J. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*. New York: Springer-Verlag, 1995.
[5] V. M. Adamjan, D. Z. Arov, and M. G. Krein, "Analytic properties of Schmidt pairs for a Hankel operator and the generalized Shur-Takagi problem," *Math. USSR Sbornik*, vol. 15, pp. 31–73, 1971.
[6] O. Toker and H. Özbay, "H^∞ optimal and suboptimal controllers for infinite dimensional SISO plants," *IEEE Trans. Autom. Control*, vol. 40, pp. 751–755, Apr. 1995.
[7] G. Meinsma, L. Mirkin, and Q. C. Zhong, "Control of systems with I/O delay via reduction to a one-block problem," *IEEE Trans. Autom. Control*, vol. 47, pp. 1890–1895, Nov. 2002.
[8] H. Kimura, *Chain-Scattering Approach to H_∞-Control*. Boston, MA: Birkhäuser, 1996.
[9] O. Iftime and H. Zwart, "Nehari problems and equalizing vectors for infinite-dimensional systems," *Syst. Contr. Lett.*, vol. 45, pp. 217–225, Mar. 2002.
[10] G. Meinsma and H. Zwart, "On H_∞ control of dead-time systems," *IEEE Trans. Autom. Control*, vol. 45, pp. 272–285, Feb. 2000.
[11] K. Clancey and I. Gohberg, *Factorization of Matrix Functions and Singular Integral Oprators*. Boston, MA: Birkäuser, 1981.
[12] L. Mirkin, "On the extraction of dead-time controllers and estimators from delay-free parameterizations," *IEEE Trans. Autom. Control*, vol. 48, pp. 543–553, Apr. 2003.
[13] G. Meinsma and L. Mirkin, "H^∞ control of systems with multiple I/O delays via decomposition to adobe problems," *IEEE Trans. Autom. Control*, vol. 50, pp. 199–211, Feb. 2005.

[14] S. Gümüşsoy and H. Özbay, "Remarks on H^∞ controller design for SISO plants with time delays," in *Preprints of the 5th IFAC Symp. Robust Control Design*, 2006, CD-ROM.
[15] H. Fujioka, "Quadratic performance analysis for finite-horizon systems," in *Proc. 16th. IFAC World Congress*, Prague, Czech Repbulic, Jul. 2005, pp. 991–996.
[16] G. H. Golub and C. F. van Loan*, Matrix Computations*. Baltimore, MD: The Johns Hopkins Univ. Press, 1989.
[17] K. Kashima, T. Yamamoto, and Y. Yamamoto, "A Smith-type predictor for non-minimum phase infinite-dimensinal systems and its dual structure," in *Proc. 45th IEEE Conf. Decision and Control*, 2006, pp. 4706–4711.
[18] K. Kashima and Y. Yamamoto, "On standard H^∞ control problems for systems with infinitely many unstable poles," *Syst. Control Lett.*, 2007, to be published.

Attitude Tracking With Adaptive Rejection of Rate Gyro Disturbances

Pierluigi Pisu and Andrea Serrani

***Abstract*—The classical attitude control problem for a rigid body is revisited under the assumption that measurements of the angular rates obtained by means of rate gyros are corrupted by harmonic disturbances, a setup of importance in several aerospace applications. This note extends previous methods developed to compensate bias in angular rate measurements by accounting for a more general class of disturbances, and by allowing uncertainty in the inertial parameters. By resorting to adaptive observers designed on the basis of the internal model principle, it is shown how converging estimates of the angular velocity can be used effectively in a passivity-based controller yielding global convergence within the chosen parametrization of the group of rotations. Since a persistence of excitation condition is not required for the convergence of the state estimates, only an upper bound on the number of distinct harmonic components of the disturbance is needed for the applicability of the method.**

***Index Terms*—Adaptive observers, aerospace control, nonlinear systems.**

I. Problem Definition

Consider the rotational dynamics of a rigid body

$$\begin{aligned}\dot{R} &= RS(\omega)\\ J(\mu)\dot{\omega} &= S\left(J(\mu)\omega\right)\omega + u\end{aligned} \tag{1}$$

with state $(R,\omega) \in SO(3) \times \mathbb{R}^3$, representing the orientation and angular velocity of a body-fixed frame with respect to an inertial frame, and control input $u \in \mathbb{R}^3$. The matrix $S(\cdot)$ denotes the skew-symmetric operator $S(v)w := v \times w$, where $v,\ w \in \mathbb{R}^3$. The inertia matrix $J(\mu) = J^T(\mu) > 0$ is assumed to depend continuously on a vector of unknown parameters μ ranging over a given compact set $\mathcal{K}_\mu \subset \mathbb{R}^p$. The desired reference trajectory $(R_d, \omega_d) \in SO(3) \times \mathbb{R}^3$ for the body-fixed frame of (1) is provided by a smooth autonomous system of the form

$$\begin{aligned}\dot{\varpi}_d &= s(\varpi_d)\\ \dot{R}_d &= R_d S(\omega_d)\\ \omega_d &= r(\varpi_d)\end{aligned} \tag{2}$$

with state (ϖ_d, R_d) evolving on a compact invariant subset $\mathcal{K}_\varpi \times SO(3)$ of $\mathbb{R}^{n_d} \times SO(3)$. This setup, while obviously not the most general, encompasses many configurations of importance in aerospace applications [1]. The attitude error $R_e := R_d^T R \in SO(3)$ satisfies the kinematic equation $\dot{R}_e = R_e S(\omega_e)$, where $\omega_e := \omega - R_e^T \omega_d$ denotes the angular velocity error resolved in the body frame.

The classic attitude control problem [2] is loosely defined as that of finding a feedback control law such that all trajectories of the closed-loop system are bounded, and the tracking error satisfies $(R_e(t), \omega_e(t)) \to (I_3, 0)$ as $t \to \infty$, for any given reference trajectory in the considered family of solutions of (2), and for all $\mu \in \mathcal{K}_\mu$. In this note, the problem in question is revisited under the assumption that measurements of the rotation matrix $R(t)$ are available, while measurements of $\omega(t)$ obtained by means of rate gyros are corrupted by additive harmonic noise. The considered setup arises frequently in the control of aerospace vehicles with significant aeroelastic effects [3], [4], where structural vibrations are transmitted to the rate gyros through the coupling with the airframe, or in the attitude control of rigid of flexible satellites, where harmonic disturbance in the angular velocity measurements are produced by imbalance or mechanical defects in gyroscopes [5]–[7]. Dealing with uncertainties on the natural frequencies is a fundamental issue in applications to control of hypersonic vehicles, where the vibrational modes change in response to mass variation and heating effects [8].

Building upon the results of [9], in this study the disturbance is modeled as an exogenous signal containing a finite number of harmonics of unknown amplitude, phase and frequency. While the formulation of the problem falls in principle within the scope of output regulation theory (see [1] and [10] for related applications), the occurrence of the disturbance at the sensor input poses unique challenges, as the error to be regulated is not directly available to the controller [11]. For the problem at issue, it will be shown first that a converging estimate of the angular velocity can be obtained using an observer endowed with a nonlinear adaptive internal model of the exogenous disturbance. The design of the adaptive observer extends (nontrivially) the approach proposed in [7] to the more general situation discussed here. A remarkable feature of our approach is that only an upper bound on the number of distinct harmonics of the disturbance is required for the implementation of the adaptive observer, since persistence of excitation of the regressor is not needed for the convergence of the state estimates. Then, it will be shown that the availability of converging estimates of the angular velocity suffices to obtain global tracking (with respect to the chosen parametrization of the attitude error in $SO(3)$) by means of a certainty-equivalence robust redesign of the adaptive attitude regulator of Egeland and Godhavn [12]. Since the design of the regulator is independent from that of the observer, the result yields a form of separation principle for attitude regulation that may be applicable to more general situations.

The note is organized as follows. The disturbance model is briefly described in Section II, whereas the design of the adaptive observer and the certainty-equivalence controller are presented in Section III and Section IV, respectively. Simulation results are illustrated in Section V.

Manuscript received December 5, 2006; revised March 9, 2007. Recommended by Associate Editor D. Dochain. This work was supported in part by the NSF under Grant 0220180, by AFOSR, and by AFRL/VA through the Collaborative Center of Control Science (Contract F33615-01-2-3154).

P. Pisu is with the Department of Mechanical Engineering, Clemson University, Clemson, SC 29634 USA.

A. Serrani is with the Department of Electrical and Computer Engineering, The Ohio State University, Columbus, OH 43210 USA (e-mail: serrani@ece.osu.edu).

Color versions of one or more of the figures in this note are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2007.910726