

IRREDUCIBLE HEEGNER DIVISORS IN THE PERIOD SPACE OF ENRIQUES SURFACES

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Reflection walls of certain primitive vectors in the anti-invariant sublattice of the K3 lattice define Heegner divisors in the period space of Enriques surfaces. We show that depending on the norm of these primitive vectors, these Heegner divisors are either irreducible or have two irreducible components. The two components are obtained as the walls orthogonal to primitive vectors of the same norm but of different type as ordinary or characteristic.

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1. Introduction

The deck transformation of the universal covering of an Enriques surface induces an involution on the second integral cohomology of the cover, which in turn, under a marking, defines an involution τ on the K3 lattice Λ . If Λ^- denotes the eigenspace of τ corresponding to the eigenvalue -1 , we define the period domain of Enriques surfaces as $\mathcal{D} = \{[v] \in \mathbb{P}(\Lambda^- \otimes \mathbb{C}) \mid \langle v, v \rangle = 0, \langle v, \bar{v} \rangle > 0\}$, where $\langle \cdot, \cdot \rangle$ denotes the bilinear form of Λ . If $O(\Lambda^-)$ denotes the orthogonal group of Λ^- , then $\mathcal{D}/O(\Lambda^-)$ is the period space of Enriques surfaces.

We define $H_\ell = \{[v] \in \mathbb{P}(\Lambda^- \otimes \mathbb{C}) \mid \langle v, \ell \rangle = 0\}$, for all primitive $\ell \in \Lambda^-$. For any integer n , let \mathcal{D}_n denote $\bigcup H_\ell$ where the union is taken over all primitive $\ell \in \Lambda^-$ with $\langle \ell, \ell \rangle = 2n$. The Heegner divisor \mathcal{H}_n is now defined as $\mathcal{D}_n/O(\Lambda^-)$.

We show that \mathcal{H}_n is irreducible for odd n , and has two disjoint irreducible components for even n . Moreover we have a full description of these components

when n is even. For this recall that an element $\ell \in \Lambda^-$ is called characteristic if $\langle \ell, \nu \rangle \equiv \langle \nu, \nu \rangle \pmod 2$ for all $\nu \in \Lambda^-$, and is called ordinary otherwise. Let \mathcal{D}_n^c (respectively, \mathcal{D}_n^o) denote $\bigcup H_\ell$ where the union is taken over all primitive characteristic (respectively, ordinary) $\ell \in \Lambda^-$ with $\langle \ell, \ell \rangle = 2n$. Define the two Heegner divisors \mathcal{H}_n^c and \mathcal{H}_n^o as $\mathcal{D}_n^c/O(\Lambda^-)$ and $\mathcal{D}_n^o/O(\Lambda^-)$ respectively. It follows that $\mathcal{H}_n = \mathcal{H}_n^c \sqcup \mathcal{H}_n^o$ for even n .

For this we first count the number of orbits in Λ^- under the action of its isometries, see Theorem 2.1, and use this to describe Heegner divisors in Sec. 6. As a byproduct of our proof, we obtain a simple description of even type vectors of Λ^- in Sec. 5.

2. Definitions and the Main Theorem

We consider an Enriques surface S whose K3 cover we denote by \tilde{S} . There is a fixed point free involution $\iota : \tilde{S} \rightarrow \tilde{S}$ which extends to an involution on cohomology $\tau : H^2(\tilde{S}, \mathbb{Z}) \rightarrow H^2(\tilde{S}, \mathbb{Z})$. Combined with a marking $\phi : H^2(\tilde{S}, \mathbb{Z}) \rightarrow \Lambda$, we can consider τ as acting on Λ , where Λ is the K3 lattice $E_8^2 \oplus U^3$. Here E_8 is the unimodular, negative definite even lattice of rank 8, and U is the hyperbolic plane. We denote the (-1) -eigenspace of τ in Λ by Λ^- . It is known that $\Lambda^- \cong E_8(2) \oplus U(2) \oplus U$. We will work with a fixed marking and take $\Lambda^- = E_8(2) \oplus U(2) \oplus U$. We denote the orthogonal group of Λ^- , the group of its self isometries, by $O(\Lambda^-)$.

The lattice product of two elements ω_1, ω_2 in any lattice L will be denoted by $\langle \omega_1, \omega_2 \rangle_L$ where we can drop the subscript L if it is clear from the context. We say that ω is an m -vector, or an element of norm m in L if $\langle \omega, \omega \rangle_L = m$. A nonzero element ω in a lattice L is called primitive if $\frac{1}{n}\omega$ is not an element in L for any nonzero integer $n \neq \pm 1$.

For any integer n , we use (n) to denote the lattice $\mathbb{Z} \cdot e$ where $\langle e, e \rangle = n$. For any lattice L and any positive integer n , L^n denotes the direct sum of n copies of L . For any nonzero integer n , $L(n)$ denotes the same \mathbb{Z} -module L except that the inner product is modified by n as $\langle \omega_1, \omega_2 \rangle_{L(n)} = n \langle \omega_1, \omega_2 \rangle_L$ for any $\omega_1, \omega_2 \in L$. We denote by $I_{s,t}$ the odd unimodular lattice $(1)^s \oplus (-1)^t$, where s and t are nonnegative integers.

We define the type of an element $\omega \in L$, for any lattice L , to be characteristic if $\langle \omega, \eta \rangle \equiv \langle \eta, \eta \rangle \pmod 2$ for all $\eta \in L$, and ordinary otherwise. Furthermore, when L is unimodular and $\omega \in L$ is characteristic, then $\langle \omega, \omega \rangle \equiv n - m \pmod 8$, where (n, m) is the signature of L , see [11, 7].

Theorem 2.1. *If B is an even, unimodular, indefinite lattice, then the number of distinct orbits of primitive $(2n)$ -vectors in $B(2) \oplus U$ under the action of $O(B(2) \oplus U)$ is 1 when n is odd, and 2 when n is even.*

Note that since $\Lambda^- = (E_8 \oplus U)(2) \oplus U$, this theorem applies to Λ^- . The proof of the theorem will be given in Sec. 4, and its implementation to Heegner divisors in the period space of Enriques surfaces will be given in Sec. 6.

We remind that some special cases of the theorem for Λ^- were proven by Namikawa (for $n = -1, -2$, see [8, Theorems 2.13, 2.15]), by Sterk (for $n = 0, -2$, see [10]), and by Allcock (for $n = 0, -1, -2$, see [1]).

3. A Crucial Dilatation

In order to count the orbits of $O(\Lambda^-)$ in Λ^- , we will use a dilatation which can be deduced from the following setup. If $A = B(2) \oplus U$, then $((2^{-1/2})A)^*$ is isomorphic to $B \oplus U(2)$ and this lies isometrically in $B \oplus I_{1,1}$. If the signature of B is $(s-1, t-1)$, then $B \oplus I_{1,1} \cong I_{s,t}$. It can be observed that $O(A) \cong O(I_{s,t})$, see [1, 5].

We will construct a dilatation of $B(2) \oplus U$ into $(B \oplus I_{1,1}) \otimes \mathbb{Q}$ through which we recover the isomorphism of the above orthogonal groups. This has the added advantage of giving explicit description of the components of the Heegner divisors under consideration. We start with a lemma.

Lemma 3.1. *An element $\omega = (a_1, \dots, a_r, m, n) \in B \oplus I_{1,1}$ is characteristic if and only if m and n are odd and all a_i 's are even.*

Proof. One way is straightforward. So take ω to be characteristic. Take $\eta = (0, \dots, 0, 1, 0)$. Since $\langle \omega, \eta \rangle = m$ and $\langle \eta, \eta \rangle = 1$, it follows that m is odd. Similarly we can show n to be odd also. Now take $\eta = (c_1^i, \dots, c_r^i, 0, 0)$ where \vec{c}^i is the i th row of B^{-1} , $i = 1, \dots, r$. If \vec{a} denotes the column vector with a_i in the i th place, $i = 1, \dots, r$, then we have

$$\langle \eta, \omega \rangle = (c_1^i, \dots, c_r^i, 0, 0) \begin{pmatrix} B & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_r \\ m \\ n \end{pmatrix} = \vec{c}^i B \vec{a} = a_i, \quad i = 1, \dots, r.$$

On the other hand, we have $\langle \eta, \eta \rangle \equiv 0 \pmod{2}$ since B is even. This forces all a_i 's to be even since ω is assumed to be characteristic. □

For any lattice L of rank n , fix a basis and define the following \mathbb{Z} -module:

$$\left(\frac{1}{2} \right) L = \left\{ \frac{1}{2} \omega \in \mathbb{R}^n \mid \omega \in L \text{ and coordinates of } \omega \text{ are either all odd or all even} \right\}.$$

Extend the inner product of L to this \mathbb{Z} -module linearly to give it a lattice structure. This description is dependent on the chosen basis but it is a particular intermediate construction we use below to obtain coordinate free isomorphisms.

Lemma 3.2. $O(B \oplus (\frac{1}{2})I_{1,1}) \cong O(B \oplus I_{1,1})$.

Proof. $B \oplus I_{1,1}$ is a submodule of $B \oplus (\frac{1}{2})I_{1,1}$, where both are considered as submodules of \mathbb{Z}^{r+2} . It can be shown that with respect to some basis of $B \oplus (\frac{1}{2})I_{1,1}$ which uses the standard basis in $I_{1,1}$, any isometry of $B \oplus (\frac{1}{2})I_{1,1}$ is represented by an integral matrix which then defines an isometry on $B \oplus I_{1,1}$. This defines an injective morphism from $O(B \oplus (\frac{1}{2})I_{1,1})$ to $O(B \oplus I_{1,1})$. Now take any isometry g in $O(B \oplus I_{1,1})$ and $\omega = (a_1, \dots, a_r, m + \frac{1}{2}, n + \frac{1}{2}) \in B \oplus (\frac{1}{2})I_{1,1}$. It follows that 2ω is a characteristic element in $B \oplus I_{1,1}$ by Lemma 3.1, and $g(2\omega)$ is characteristic, see [11]. Finally by the same lemma, $\frac{1}{2}g(2\omega)$ is an element of $B \oplus (\frac{1}{2})I_{1,1}$ and thus g defines an element in $O(B \oplus (\frac{1}{2})I_{1,1})$. \square

In the following lemma, we construct the crucial dilatation ϕ .

Lemma 3.3. *If B is an even, unimodular lattice of signature $(s - 1, t - 1)$, then $O(B(2) \oplus U) \cong O(B \oplus I_{1,1}) \cong O(I_{s,t})$.*

Proof. Let e_1, \dots, e_r be a basis of $B(2)$, where $r = s + t - 2$. Let u, v be a basis of U such that $\langle u, u \rangle = \langle v, v \rangle = 0$ and $\langle u, v \rangle = 1$. Any element of $B(2) \oplus U$ is of the form $a_1e_1 + \dots + a_re_r + b_1u + b_2v$ for some integers a_i and b_i . We denote this element, as above, with the vector $(a_1, \dots, a_r, b_1, b_2)$. Similarly for the lattice $B \oplus (\frac{1}{2})I_{1,1}$ where now x, y are basis for $I_{1,1}$ with $\langle x, x \rangle = -\langle y, y \rangle = 1$ and $\langle x, y \rangle = 0$. We define a map

$$\begin{aligned} \phi : B(2) \oplus U &\rightarrow B \oplus \left(\frac{1}{2}\right) I_{1,1} \\ (a_1, \dots, a_r, b_1, b_2) &\mapsto \left(a_1, \dots, a_r, \frac{b_1 + b_2}{2}, \frac{b_1 - b_2}{2}\right). \end{aligned}$$

This is a \mathbb{Z} -module isomorphism with

$$\langle \omega_1, \omega_2 \rangle = 2\langle \phi(\omega_1), \phi(\omega_2) \rangle, \quad \text{for } \omega_i \in B(2) \oplus U.$$

At this point it is clear that $O(B(2) \oplus U) \cong O(B \oplus (\frac{1}{2})I_{1,1})$. The previous lemma gives the isomorphism to $O(B \oplus I_{1,1})$, which is in turn isomorphic to $O(I_{s,t})$ via the classification theory of unimodular odd lattices. \square

4. Counting the Orbits

In this section, we give a constructive proof of Theorem 2.1. In what follows, B is an even, unimodular, indefinite matrix of signature $(s - 1, t - 1)$.

Lemma 3.3 reduces the problem of counting orbits in $B(2) \oplus U$ to the same problem in $I_{s,t}$. The number of orbits in $I_{s,t}$ is known [11, Theorem 4].

Theorem 4.1 (C.T.C. Wall). *If s, t are each at least 2, then $O(I_{s,t})$ acts transitively on primitive vectors of given norm and type (i.e. characteristic or ordinary). Moreover, if a vector is characteristic then its norm is congruent to $s - t \pmod 8$.*

Since B is even, indefinite and unimodular, we necessarily have $s - t \equiv 0 \pmod 8$, see [6].

Proof of Theorem 2.1.

Case 1. n is an odd integer.

Let $\omega = (a_1, \dots, a_r, b_1, b_2)$ be a primitive vector in $B(2) \oplus U$. We have $\langle \omega, \omega \rangle = 4k + 2b_1b_2 = 2n$ with n odd, so b_1 and b_2 are odd. Then $\phi(\omega) = (a_1, \dots, a_r, \frac{b_1+b_2}{2}, \frac{b_1-b_2}{2})$ is integral, primitive and since $n \not\equiv 0 \pmod{8}$, it is ordinary. Since $O(I_{s,t}) \cong O(B \oplus I_{1,1})$, all such elements are in the same orbit by Wall's theorem above. And since $O(B(2) \oplus U) \cong O(B \oplus I_{1,1})$, all primitive elements of norm $2n$ lie in the same orbit in $B(2) \oplus U$. This orbit is not empty since $\omega = (0, \dots, 0, 1, n)$ lies in it.

Case 2. n is an even integer.

$\langle \omega, \omega \rangle = 4k + 2b_1b_2 = 2n \equiv 0 \pmod{4}$. In this case, b_1 and b_2 cannot both be odd.

Case 2.1. Only one of b_1 and b_2 is even.

In this case, $\phi(\omega) = (a_1, \dots, a_r, \frac{b_1+b_2}{2}, \frac{b_1-b_2}{2})$ and is fractional. But $2\phi(\omega)$ is integral, primitive and by Lemma 3.1, it is characteristic. To show that this orbit is nonempty, take $\omega = (0, \dots, 0, 1, n)$ in $B(2) \oplus U$. It is primitive, of norm $2n$ with only one of b_1 and b_2 even. So this case contributes one orbit.

Case 2.2. b_1 and b_2 are both even.

In this case, $\phi(\omega)$ is integral, primitive and ordinary. Hence this case contributes one orbit if we can show the existence of a primitive vector $\omega \in B(2) \oplus U$ of norm $2n$, with both b_1 and b_2 even.

Since B is even, unimodular and indefinite, $B \cong U^i \oplus E_8(\pm 1)^j$ where $i > 0$. For any integer k , $(1, k) \in U(2)$ is a primitive element of norm $4k$, so in particular $B(2)$ contains a primitive element ω' of norm $4k$. Let $n = 2k$. Then $\omega = \omega' + (0, 0) \in B(2) \oplus U$ is primitive of norm $2n$ with b_1 and b_2 even. So this orbit is not empty either.

Both Cases 2.1 and 2.2 contribute one orbit each, so when n is even there are two orbits. □

We remark in passing that Sterk uses the isotropic vectors $e, e', e' + f' + \omega, e' + 2f' + \alpha$ and $2e + 2f + \alpha$, for notation see [10, Propositions 4.2.3 and 4.5]. He shows that e belongs to an orbit disjoint from the orbit to which all the others belong. In our point of view, $\phi(e)$ is fractional with $2\phi(e)$ characteristic whereas the ϕ image of the others are integral and ordinary, which explains the existence of two distinct orbits.

5. Even Type Vectors

Regarding Λ^+, Λ^- as sublattices of Λ , a primitive $(2n)$ -vector $\omega \in \Lambda^-$ is defined to be *even* if there is a primitive $(2n)$ -vector $\omega' \in \Lambda^+$ such that $\omega + \omega' \in 2\Lambda$ (cf. [8, Proposition 2.16]). We give an explicit description of even type primitive vectors in terms of the dilatation ϕ .

Theorem 5.1. *Let n be an even integer. A primitive $(2n)$ -vector $\omega \in \Lambda^-$ is of even type if and only if its image $\phi(\omega)$ has integral coordinates.*

Proof. Since any self-isometry of Λ^\pm extends to a self-isometry of Λ (see [8, Theorem 1.4]) without loss of generality we can fix a primitive embedding of $E_8(2) \oplus U(2) \oplus U$ into $E_8^2 \oplus U^3$ and prove the statement for the image of this embedding. Thus, we fix the following embedding

$$E_8(2) \oplus U(2) \oplus U \hookrightarrow E_8^2 \oplus U^3$$

$$(e, u, v) \mapsto (e, -e, u, -u, v)$$

and identify the domain with its image. The orthogonal complement of the image is precisely the image of the primitive embedding

$$E_8(2) \oplus U(2) \hookrightarrow E_8^2 \oplus U^3$$

$$(e, u) \mapsto (e, e, u, u, 0).$$

Moreover, the primitive $(2n)$ -vectors $\omega \in \Lambda^-$ with integral images are transitive by Case 2.2. Therefore, it suffices to prove the statement for particular vectors.

Let $\omega = (0, \dots, 0, k, 1, 0, 0) \in E_8(2) \oplus U(2) \oplus U$ be a primitive vector with $\omega^2 = 2n = 4k$, and identify ω with its image in $E_8^2 \oplus U^3$ with coordinates $(0, \dots, 0, k, 1, -k, -1, 0, 0)$, by the above embedding. Notice that $\phi(\omega)$ has integral coordinates. Now choose $\omega' = (0, \dots, 0, k, 1) \in E_8(2) \oplus U(2)$ which corresponds similarly to $(0, \dots, 0, k, 1, k, 1, 0, 0) \in E_8^2 \oplus U^3$. Now it is clear that $\omega + \omega' \in 2(E_8^2 \oplus U^3)$.

On the other hand, the ϕ image of the vector $\omega = (0, \dots, 0, 2k, 1) \in E_8(2) \oplus U(2) \oplus U$ has fractional coordinates, and for no vector ω' in $E_8(2) \oplus U(2)$ can we have $\omega + \omega' \in 2(E_8^2 \oplus U^3)$, because the last coordinate of $\omega + \omega'$ is always 1. \square

6. Heegner Divisors

Here we discuss the irreducibility of certain Heegner divisors in the period space of Enriques surfaces. The period domain is defined as

$$\mathcal{D} = \{[v] \in \mathbb{P}(\Lambda^- \otimes \mathbb{C}) \mid \langle v, v \rangle = 0, \langle v, \bar{v} \rangle > 0\}$$

and the period space is then $\mathcal{D}/O(\Lambda^-)$, see [3]. For any primitive $\ell \in \Lambda^-$, define

$$H_\ell = \{[v] \in \mathbb{P}(\Lambda^- \otimes \mathbb{C}) \mid \langle v, \ell \rangle = 0\},$$

which is the hyperplane orthogonal to ℓ . For any integer n , define

$$\mathcal{D}_n = \bigcup H_\ell,$$

where the union is taken over all primitive $\ell \in \Lambda^-$ with $\langle \ell, \ell \rangle = 2n$. The Heegner divisor \mathcal{H}_n in the period space is now defined as

$$\mathcal{H}_n = \mathcal{D}_n/O(\Lambda^-).$$

It follows from Theorem 2.1 that \mathcal{H}_n is irreducible for odd n , and has exactly two disjoint irreducible components for even n . Moreover the proof of Theorem 2.1

gives a full description of the components of \mathcal{H}_n for even n . For this define the two Heegner divisors

$$\mathcal{H}_n^c \text{ (respectively } \mathcal{H}_n^o) = \left(\bigcup H_\ell \right) / O(\Lambda^-),$$

where the union is taken over all primitive characteristic (respectively, ordinary) $\ell \in \Lambda^-$ with $\langle \ell, \ell \rangle = 2n$. Then we have

Theorem 6.1.

$$\mathcal{H}_n = \mathcal{H}_n^c \bigsqcup \mathcal{H}_n^o, \text{ for even } n.$$

It is known that the period of an Enriques surface Y lies in H_{-2}^o if and only if Y contains a nonsingular rational curve, see [8, Theorem 6.4]. It will be interesting to find similar geometric descriptions for the other Heegner divisors. Further characterizations of nodal Enriques surfaces were also obtained by Cossec [4]. For a recent investigation of this problem, we refer to [2].

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