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Alcahestic subalgebras of the alchemic algebra and a correspondence of simple modules

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Abstract

The unified treatment of the five module-theoretic notions, transfer, inflation, transport of structure by an isomorphism, deflation and restriction, is given by the theory of biset functors, introduced by Bouc. In this paper, we construct the algebra realizing biset functors as representations. The algebra has a presentation similar to the well-known Mackey algebra. We adopt some natural constructions from the theory of Mackey functors and give two new constructions of simple biset functors. We also obtain a criterion for semisimplicity in terms of the biset functor version of the mark homomorphism. The criterion has an elementary generalization to arbitrary finite-dimensional algebras over a field.
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1. Introduction

In [3] and [4], Bouc introduced two different notions of functors. The first notion, now known as a **biset functor**, is an R-linear functor from a category whose objects are finite groups to the category of modules over the commutative ring R with unity. In this case, the morphisms between finite groups are given by finite bisets and the composition product of bisets is given by the usual

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amalgamated product of bisets, called the Mackey product in [3]. In his paper, Bouc proved that any biset is a composition of five types of bisets, namely composition of a transfer biset, an inflation biset, an isomorphism biset, a deflation biset and a restriction biset. The names of these special bisets comes from their actions on the representation ring functor and the Burnside ring functor where the actions coincide with the transfer map, the inflation map, the transport by an isomorphism, the deflation map and the restriction map respectively. Hence, this approach gives a unified treatment of these five module-theoretic notions which extend the operations available with Mackey functors. An application of the biset functors is to the Dade group of a p-group (see [9,10]).

In [4], Bouc used the category of finite groups with morphisms given by the bisets to classify the functors between the categories of finite G-sets. He proved that any such functor preserving disjoint unions and cartesian products corresponds to a biset, and conversely, any biset induces a functor with these properties. However, the usual amalgamated product of bisets does not correspond to the composition of these functors. Therefore, he obtains a restricted product of bisets and a category of functors defined as above with different composition of morphisms.

Both of these categories of functors are abelian and the corresponding simple functors are classified by Bouc in [3] and [4]. The description of simple objects in both of the categories are given by the same construction but the Mackey product of the first category makes the construction complicated while the restricted product of the second category makes the description explicit.

In [16], Webb considered two other kinds of functors by allowing only certain types of bisets as morphisms. For the first one, he allows only the bisets that are free on both sides. These functors are well known as global Mackey functors. The other case is where the right-free bisets are allowed. He called this kind inflation functors. The main example of the inflation functors is the functor of group cohomology. He described simple functors for both of these cases. As we shall see below, his construction is a special case of one of our main results. A general framework for these functors were also introduced by Bouc in [3]. Given two classes $\mathcal{X}$ and $\mathcal{Y}$ of finite groups having certain properties, Bouc considered R-linear functors $\mathcal{C}_R^{\mathcal{X},\mathcal{Y}} \to R\text{-mod}$, called **globally-defined Mackey functor** in [17]. Here the category $\mathcal{C}_R^{\mathcal{X},\mathcal{Y}}$ consists of all finite groups with morphisms given by all finite bisets with left-point stabilizers in $\mathcal{X}$ and right-point stabilizers in $\mathcal{Y}$. The product is still the amalgamated product of bisets. Now global Mackey functors are obtained by letting both $\mathcal{X}$ and $\mathcal{Y}$ consist of the trivial group and the biset functors are obtained by letting them contain all finite groups, and inflation functors are obtained by letting $\mathcal{X}$ consist of all finite groups and $\mathcal{Y}$ consist of the trivial group.

In this paper, we only consider the biset functors defined over the category restricted to a fixed finite group. That is, we consider the category $\mathcal{C}_G$ of subquotients of a fixed finite group G with morphisms given by the Grothendieck group of bisets between the subquotients. The composition is still given by the Mackey product. Here by a subquotient, we mean a pair (H^*, H_*) of subgroups of G such that $H_* \trianglelefteq H^*$. We always write H for the pair (H^*, H_*). Now a biset functor for G (over a commutative ring R with unity) is an R-linear functor $\mathcal{C}_G \to R\text{-mod}$. Note that although we consider a special case of Bouc's biset functors, it is easy to generalize our results from this finite category to the infinite case since the Grothendieck group of bisets for any two finite groups is independent of their inclusions to any larger group as subquotients.

It follows from a well-known result that the category of biset functors for G over R is equivalent to the module category of the algebra generated by finite bisets appearing as morphisms in $\mathcal{C}_G$. Following Barker [1], we call this algebra the **alchemic algebra**.[2] Description of the alchemic algebra using Bouc's decomposition theorem for bisets produces five types of generators and so many relations that cannot be specified in a tractable way. Our approach to the alchemic algebra is to amalgamate some of the variables in a way that the relations become tractable. In order to do this, we introduce two new variables, namely **tinflation** which is the composition of transfer and inflation, and **destriction**, composition of deflation and restriction. The fifth element remains the same but we call it **isogation**. In this way, the relations become not only tractable but also similar to the relations between the generators of the Mackey algebra.

Note that in [17], Webb specified a list of relations that works for globally-defined Mackey functors for any choice of the classes $\mathcal{X}$ and $\mathcal{Y}$. Our list of relations, although obtained using the Mackey product, can also be obtained from these relations with some straightforward calculations. Also in [1], Barker constructed the alchemic algebra, without specifying the relations explicitly, for a family of finite groups closed under taking subquotients and isomorphisms. This covers our case when the family is the family of subquotients of G.

Having the above description of the alchemic algebra, we are able to adapt some natural constructions from the context of the Mackey functors to the context of biset functors. Our first main result in this direction concerns the construction of simple biset functors using the techniques in [12]. In fact our result extends to classification and description of simple modules of certain unital subalgebras. The study of subalgebras of the alchemic algebra is crucial since some natural constructions, such as group homology, group cohomology and modular representation rings, are only modules over some subalgebras of the alchemic algebra. For instance, Webb's inflation functors are representations of such a subalgebra of the alchemic algebra. This subalgebra realizes the modular character ring as well as the group cohomology. Group homology is a module of the opposite algebra of this subalgebra. In Section 4, we shall prove the following classification theorem which extends both Bouc's and Webb's classification of simple functors.

Theorem 1.1. *Let Π be an alcahestic subalgebra of the alchemic algebra. There is a bijective correspondence between*

(i) *the isomorphism classes S^{Π} of simple Π-modules.*
(ii) *The isomorphism classes S of simple Ω_{Π}-modules.*

Here by an alcahestic subalgebra, we mean a subalgebra of the alchemic algebra containing a certain set of orthogonal idempotents summing up to the identity, see Section 4 for a precise definition. The algebra Ω_{Π} is the subalgebra of Π generated by isogations in Π. Our theorem generalizes Bouc's and Webb's classification theorems because in both cases Ω_{Π} is Morita equivalent to the algebra $\prod_H R\mathrm{Out}(H)$ where the product is over subquotients of G up to isomorphism. Note also that globally-defined Mackey functors are always modules of a suitably chosen alcahestic subalgebra.

This classification theorem follows easily from the correspondence theorem below. Note further that a version of the following theorem, Corollary 4.7, gives the bijection of Theorem 1.1, explicitly.

[2] The name refers to the five elements of nature in alchemy; air, fire, water, earth and the fifth element known as *quintessence* or *aether*. Two elements, transfer and inflation, go upwards as the two elements fire and air. Similarly deflation and restriction go downwards as water and earth. See [1].

Theorem 1.2. *Let $\Pi \subset \Theta$ be alcahestic subalgebras of the alchemic algebra such that all destrictions in Θ are contained in Π. Let S be a simple Π-module. Then the induced Θ-module $\mathrm{ind}_{\Pi}^{\Theta} S := \Theta \otimes_{\Pi} S$ has a unique maximal submodule.*

In Section 5, we apply this result to give two descriptions of the simple biset functors. These descriptions are both alternatives to Bouc's construction. Moreover our construction has an advantage that the induced module in Theorem 1.2 is smaller than a similar one used by Bouc in the sense that it is a quotient of the latter. Also one of our descriptions explicitly gives a formula for the action of the tinflation on the simple module. The action is similar to the action of transfer on a simple Mackey functor, which is given by the relative trace map.

Another natural construction that we adapt from the Mackey functors is the **mark morphism**. Originally the mark morphism is a morphism from the Burnside ring to its ghost ring. Boltje [2] generalized this concept to a morphism connecting his plus constructions. It is shown in [12] that the plus constructions are usual induction and coinduction functors and the mark morphism can be constructed by applying a series of adjunctions. In Section 6, we further generalize the mark morphism to biset functors in a similar way. In particular, we have a mark morphism β^S associated to any simple biset functor S. Here we see another advantage of our alternative descriptions that this mark morphism can only be constructed between the induced and coinduced modules described in Section 5.

Our final main result is a characterization of semisimplicity of the alchemic algebra in terms of the mark morphism. Note that the equivalence of the first two statements below is proved by Barker [1] and Bouc, independently. Our proof is less technical than the previous two proofs since Barker and Bouc compared the dimension of the alchemic algebra with the dimensions of simple functors. Instead we have the following result.

Theorem 1.3. *Let G be a finite group and R be a field of characteristic zero. The followings are equivalent.*

(i) *The alchemic algebra for G over R is semisimple.*
(ii) *The group G is cyclic.*
(iii) *The mark morphism β^S is an isomorphism for any simple biset functor S.*

We end the paper by a generalization of this criterion to an arbitrary finite-dimensional algebra over a field. This is an elementary result and it completes a well-known result on semisimplicity of such algebras.

Theorem 1.4. *Let A be a finite-dimensional algebra over a field and e be an idempotent of A. Let $f = 1 - e$. Then the following are equivalent.*

1. *The algebra A is semisimple.*
2. (a) *The algebras eAe and fAf are semisimple.*
   (b) *For any simple gAg-module V, for $g \in \{e, f\}$, there is an isomorphism of A-modules $Ag \otimes_{gAg} V \cong \mathrm{Hom}_{gAg}(gA, V)$.*

Finally note that our results still hold if we change the composition product of the category $\mathcal{C}_G$ with the restricted product in [4] and also for globally-defined Mackey functors, with appropriate changes.

2. Biset functors, an overview

In this section, we summarize some basic definitions and constructions concerning biset functors. First, we introduce the category of bisets. This is the standard theory of bisets and can be found in [3]. Then we review Bouc's definition of biset functors and Bouc's construction of simple functors [3].

2.1. Bisets

Let H and K be two finite groups. An (H,K)-**biset** is a set with a left H-action and a right K-action such that

$$h(xk) = (hx)k$$

for all elements $h \in H$ and $k \in K$.

An (H,K)-biset X is called **transitive** if for any elements $x, y \in X$ there exists an element $h \in H$ and an element $k \in K$ such that hxk is equal to y.

We can regard any (H,K)-biset as a right $H \times K$-set with the action given by

$$x.(h,k) = h^{-1}xk$$

for all $h \in H$ and $k \in K$. Clearly, X is a transitive (H,K)-biset if and only if X is a transitive $H \times K$-set. Hence there is a bijective correspondence between

(i) isomorphism classes $[X]$ of transitive (H,K)-bisets,
(ii) conjugacy classes $[L]$ of subgroups of $H \times K$

where the correspondence is given by $[X] \leftrightarrow [L]$ if and only if the stabilizer of some point $x \in X$ is in $[L]$.

Hence we can denote a transitive biset by $(H \times K)/L$. Given finite groups H, K, M and transitive bisets $(H \times K)/L$ and $(K \times M)/N$, we define the product of these bisets by the **Mackey product** [3], given by

$$(H \times K)/L \times_K (K \times M)/N = \sum_{x \in p_2(L)\backslash K/p_1(N)} (H \times M)/L *^{(x,1)} N$$

where the subgroup $L * N$ of $H \times M$ is defined by

$$L * N = \{(h,m) \in H \times M\colon\ (h,k) \in L \text{ and } (k,m) \in N \text{ for some } k \in K\}$$

and the subgroups $p_1(N)$ and $p_2(L)$ of K are projections of N and L to K, respectively, that is,

$$p_1(L) = \{l \in H\colon\ (l,k) \in L \text{ for some } k \in K\} \quad \text{and}$$
$$p_2(L) = \{k \in K\colon\ (l,k) \in L \text{ for some } l \in L\}.$$

In [3], Bouc proved that any transitive biset is a Mackey product of the following five types of bisets: Let H be a finite group and $N \trianglelefteq J$ be subgroups of H and let L, M be two isomorphic finite groups with a fixed isomorphism $\phi : L \to M$, then the five bisets are given as follows.

1. **Induction biset:** $\mathrm{ind}_J^H := (H \times J)/T$ where $T = \{(j, j)\colon\ j \in J\}$.
2. **Inflation biset:** $\mathrm{inf}_{J/N}^J := (J \times J/N)/I$ where $I = \{(j, jN)\colon\ j \in J\}$.
3. **Isomorphism biset:** $\mathrm{c}_{M,L}^{\phi} = (M \times L)/C^{\phi}$ where $C^{\phi} = \{(\phi(l), l)\colon\ l \in L\}$.
4. **Deflation biset:** $\mathrm{def}_{J/N}^J = (J/N \times J)/D$ where $D = \{(jN, j)\colon\ j \in J\}$.
5. **Restriction biset:** $\mathrm{res}_J^H = (J \times H)/R$ where $R = \{(j, j)\colon\ j \in J\}$.

The following theorem explicitly shows the decomposition of any transitive biset in terms of these special bisets.

Theorem 2.1. *(See [3].) Let L be any subgroup of $H \times K$. Then*

$$(H \times K)/L = \mathrm{ind}_{p_1(L)}^H \times \mathrm{inf}_{p_1(L)/k_1(L)}^{p_1(L)} \times \mathrm{c}_{p_1(L)/k_1(L), p_2(L)/k_2(L)}^{\phi} \times \mathrm{def}_{p_2(L)/k_2(L)}^{p_2(L)} \times \mathrm{res}_{p_2(L)}^K$$

where the subgroup $k_1(L)$ of H and the subgroup $k_2(L)$ of K are given by

$$k_1(L) = \big\{h \in H\colon\ (h, 1) \in L\big\} \quad \text{and} \quad k_2(L) = \big\{x \in K\colon\ (1, x) \in L\big\}.$$

The isomorphism

$$\phi : p_2(L)/k_2(L) \to p_1(L)/k_1(L)$$

is the one given by associating $lk_2(L)$ to $mk_1(L)$ where for a given element $l \in p_2(L)$ we let m be the unique element in $p_1(L)$ be such that $(m, l) \in L$.

2.2. Biset functors

Let $\mathcal{C}$ be the category whose objects are finite groups and let the set of morphisms between two finite groups H and K be given by

$$\mathrm{Hom}_{\mathcal{C}}(H, K) = R \otimes_{\mathbb{Z}} \Gamma(K, H) =: R\Gamma(K, H)$$

where $\Gamma(K, H)$ denotes the Grothendieck group of the isomorphism classes of finite (K, H)-bisets with addition as the disjoint union and where R is a field. The composition of the morphisms in $\mathcal{C}$ is given by the Mackey product of bisets.

Now a **biset functor** F over R is an R-linear functor $\mathcal{C} \to R$-mod. Defining a morphism of biset functors as a natural transformation of functors, we obtain the category Biset_R of biset functors. Since the category R-mod is abelian, the category Biset_R is also abelian. Simple objects of this category are described by Bouc [3]. We shall review his construction.

Let H be a finite group. We denote by E_H the endomorphism algebra $\mathrm{End}_{\mathcal{C}}(H)$ of H in the category $\mathcal{C}$. It is easy to show that E_H decomposes as an R-module as

$$E_H = I_H \oplus R\mathrm{Out}(H)$$

where $\mathrm{Out}(H) = \mathrm{Aut}(H)/\mathrm{Inn}(H)$ is the group of outer automorphisms of H and $R\mathrm{Out}(H)$ is the group algebra of $\mathrm{Out}(H)$ and I_H is a two-sided ideal of E_H (see [3] for explicit description of I_H). Therefore, we obtain an epimorphism

$$E_H \twoheadrightarrow R\mathrm{Out}(H)$$

of algebras. In particular, we can lift any simple $R\mathrm{Out}(H)$-module V to a simple E_H-module, still denoted by V.

Denote by e_H the evaluation at H functor, that is, let $e_H : \mathrm{Biset}_R \to E_H\text{-mod}$ be the functor sending a biset functor to its value at the group H. Now let $L_{H,V}$ denote the left adjoint of the functor e_H. Explicitly, for a finite group K, we get

$$L_{H,V}(K) = \mathrm{Hom}_{\mathcal{C}}(H, K) \otimes_{E_H} V.$$

The action of a biset on $L_{H,V}$ is given by composition of morphisms. The functor $L_{H,V}$ has a unique maximal subfunctor

$$J_{H,V}(K) = \left\{ \sum_i \phi_i \otimes v_i \;\middle|\; \forall \psi \in \mathrm{Hom}_{\mathcal{C}_G}(K, H),\ \sum_i (\psi\phi_i)v_i = 0 \right\}.$$

Hence taking the quotient of $L_{H,V}$ with this maximal ideal, we obtain a simple biset functor

$$S_{H,V} := L_{H,V}/J_{H,V}.$$

Moreover, we have

Theorem 2.2. *(See Bouc [3].) Any simple biset functor is of the form $S_{H,V}$ for some finite group H and a simple $R\mathrm{Out}(H)$-module V.*

As mentioned in the introduction, the main examples of the biset functors are the functor of the Burnside ring and the functor of the representation ring. Further, over a field of characteristic zero, the rational representation ring is an example of a simple biset functor. More precisely, Bouc proved in [3] that the biset functor of rational representation ring $\mathbb{Q}\mathcal{R}_{\mathbb{Q}}$ over $\mathbb{Q}$ is isomorphic to the simple biset functor $S_{1,\mathbb{Q}}$. Another interesting example of a simple biset functor is the functor $S_{C_p\times C_p,\mathbb{Q}}$ defined only over p-groups where p is a prime number. In [10], it is shown that this functor is isomorphic to the functor of the Dade group $\mathbb{Q}D$ with coefficients extended to the rational numbers $\mathbb{Q}$ and there is an exact sequence of biset functors

$$0 \to \mathbb{Q}D \to \mathbb{Q}B \to \mathbb{Q}\mathcal{R}_{\mathbb{Q}} \to 0.$$

Here the map $\mathbb{Q}B \to \mathbb{Q}\mathcal{R}_{\mathbb{Q}}$ can be chosen as the natural map sending a P-set X to the permutation module $\mathbb{Q}X$. For an improvement of this result to $\mathbb{Z}$, see [7] and for more exact sequences relating these functors, see [11]. Some other well-known examples of biset functors are the functor of units of the Burnside ring [5] and the functor of the group of relative syzygies [6]. For further details also see [1,3,8–10].

For the rest of the paper, we concentrate on the biset functors defined only for subquotients of a fixed finite group G. We introduce some notations that will be used throughout the paper. Recall that a **subquotient** of G is a pair (H^*, H_*) where $H_* \trianglelefteq H^* \leqslant G$. We write the pair (H^*, H_*) as H and denote the subquotient relation by $H \preccurlyeq G$. Here, and afterwards, we regard any group L as the subquotient $(L, 1)$. We write $H \preccurlyeq_G G$ to mean that H is taken up to G-conjugacy and write $H \preccurlyeq_* G$ to mean that H is taken up to isomorphism. Note that we always consider H as the quotient group H^*/H_*. Clearly the relation $\preccurlyeq$ extends to a relation on the set of subquotients of G in the following way. Let J and H be two subquotients of G. Then we write $J \preccurlyeq H$ if and only if $H_* \leqslant J_*$ and $H^* \geqslant J^*$. In this case the pair $(J^*/H_*, J_*/H_*)$ is a subquotient of H. Finally we say that two subquotients H and K of G are isomorphic if and only if they are isomorphic as groups, that is, if $H^*/H_* \cong K^*/K_*$.

In this case, we have the category $\mathcal{C}_G$ of subquotients of G with objects the groups H as H runs over the set of subquotients of G and with the same morphisms and the same composition product as $\mathcal{C}$. Note that since the set $\mathrm{Hom}_{\mathcal{C}}(H, K)$ of (H, K)-bisets depends only on the subquotients of the groups H and K, it is easy to generalize the results from the finite category $\mathcal{C}_G$ to the infinite case.

Now we define a **biset functor** for G over R as an R-linear functor $\mathcal{C}_G \to R$-mod. We also denote by $\mathrm{Biset}_R(G)$ the category of biset functors for G over R.

Finally let us introduce a notation that we will use throughout the paper. Let $H, K \preccurlyeq G$. We define the **intersection** $H \sqcap K$ of the subquotients of H and K as

$$H \sqcap K = \frac{(H^* \cap K^*)H_*}{(H^* \cap K_*)H_*}.$$

Note that, in general, this intersection is neither commutative nor associative. But, there is an isomorphism of groups

$$\lambda : H \sqcap K \to K \sqcap H$$

which we call the **canonical isomorphism** between the groups $H \sqcap K$ and $K \sqcap H$. The isomorphism is the one that comes from the Zassenhaus–Butterfly Lemma.

3. Alchemic algebra

It is clear that the category of $\mathrm{Biset}_R(G)$ of biset functors for G over R is equivalent to the category of modules of the algebra $\Gamma_R(G)$ defined by

$$\Gamma_R(G) = \bigoplus_{H,K \preccurlyeq G} \mathrm{Hom}_{\mathcal{C}}(H, K).$$

It is evident that this algebra has a basis consisting of the isomorphism classes of transitive bisets. Hence by Theorem 2.1, it is generated by the five special types of bisets, namely by transfer, inflation, isomorphism, deflation and restriction bisets. Following [1], we call this algebra the **alchemic algebra** for G over R, written shortly as Γ when G and R are understood.

It is possible to *define* the alchemic algebra by forgetting the bisets altogether. In order to do this, we can consider the algebra generated freely over R by the five types of variables corresponding to the five special types of bisets. Then the alchemic algebra is the quotient of this algebra by the ideal generated by relations between the variables induced by the Mackey product of bisets. But the relations obtained in this way are not tractable. To get the relations in a tractable way, we introduce two new amalgamated variables. Instead of the five variables, we consider the composition of transfer and inflation as the first variable, which we call **tinflation**, and the composition of deflation and restriction as the second one, called **destriction**. The third and final variable is the transport of structure by an isomorphism, which we call **isogation**. In this way, we obtain a set of relations that is very similar to the defining relations of the well-known Mackey algebra. Explicitly, consider the algebra freely generated over R by the following three types of variables.

V1. tin_J^H for each $J \preccurlyeq H \preccurlyeq G$,
V2. des_J^H for each $J \preccurlyeq H \preccurlyeq G$,
V3. $\mathrm{c}_{M,L}^{\phi}$ for each $M, L \preccurlyeq G$ such that $M \cong L$ and for each isomorphism $\phi : M \to L$.

Then we let $\tilde{\Gamma}_R(G)$, written $\tilde{\Gamma}$, be the quotient of this algebra by the ideal generated by the following relations.

R1. Let $h : H \to H$ denotes the inner automorphism of H induced by conjugation by $h \in H$. Then $\mathrm{c}_{H,H}^h = \mathrm{tin}_H^H = \mathrm{des}_H^H$.
R2. Let $L \preccurlyeq J$ and $\psi : M \to S$ be an isomorphism. Then
 (i) $\mathrm{c}_{S,M}^{\psi}\, \mathrm{c}_{M,L}^{\phi} = c_{S,L}^{\psi\circ\phi}$,
 (ii) $\mathrm{tin}_J^H\, \mathrm{tin}_L^J = \mathrm{tin}_L^H$,
 (iii) $\mathrm{des}_L^J\, \mathrm{des}_J^H = \mathrm{des}_L^H$.
R3. Let $K \preccurlyeq G$ and let $\alpha : H \to K$ be an isomorphism and let ${}^{\alpha}J$ denote $\alpha(J^*)/\alpha(J_*)$, then
 (i) $\mathrm{c}_{K,H}^{\alpha}\, \mathrm{tin}_J^H = \mathrm{tin}_{{}^{\alpha}J}^K\, \mathrm{c}_{{}^{\alpha}J,J}^{\alpha}$,
 (ii) $\mathrm{des}_I^K\, \mathrm{c}_{K,H}^{\alpha} = \mathrm{c}_{I,{}^{\alpha^{-1}}I}^{\alpha}\, \mathrm{des}_{{}^{\alpha^{-1}}I}^H$.
R4. (Mackey relation.) Let $I \preccurlyeq H$. Then

$$\mathrm{des}_I^H\, \mathrm{tin}_J^H = \sum_{x \in I^* \backslash H / J^*} \mathrm{tin}_{I \sqcap {}^x J}^I\, \mathrm{c}^{x\circ\lambda}\, \mathrm{des}_{J \sqcap I^x}^J .$$

 Here $\mathrm{c}^{x\circ\lambda} := \mathrm{c}_{I\sqcap {}^x J, J\sqcap I^x}^{x\circ\lambda}$ and λ is the canonical isomorphism introduced in the previous section.
R5. $1 = \sum_{H \preccurlyeq G} \mathrm{c}_H$ where $\mathrm{c}_H := \mathrm{c}_{H,H}^1$.
R6. All other products of the generators are zero.

Remark 3.1. In [9], the amalgamated variables tinflation and destriction are abbreviated as indinf and defres, respectively.

Even it is clear from the construction, the following theorem formally shows that the algebra $\tilde{\Gamma}$ is isomorphic to the alchemic algebra Γ.

Theorem 3.2. *The algebras Γ and $\tilde{\Gamma}$ are isomorphic.*

Proof. The correspondence

$$\mathrm{tin}_J^H\, \mathrm{c}_{J,I}^{\phi}\, \mathrm{des}_I^K \to (H \times K)/A$$

where $A = \{(h,k) \in J^* \times I^*\colon\ hJ_* = \phi(kI_*)\}$ extends linearly to a map $\alpha : \tilde{\Gamma} \to \Gamma$. We must show that α is an algebra isomorphism. Indeed, α is an isomorphism of R-modules by Theorem 2.1. Thus, it suffices to check that it respects the multiplication. We shall only check the Mackey relation. The others can be checked similarly. First note that the images of tinflation and destriction are

$$\alpha\big(\mathrm{tin}_J^H\big) = \mathrm{ind}_{J^*/H_*}^H\, \mathrm{inf}_{(J^*/H_*)/(J_*/H_*)}^{J^*/H_*}\, \mathrm{c}_{(J^*/H_*)/(J_*/H_*),J}^{\lambda} =: (H \times J)/T'$$

and

$$\alpha\big(\mathrm{des}_J^H\big) = \mathrm{c}_{J,(J/H_*)/(J_*/H_*)}^{\lambda^{-1}} \mathrm{def}_{(J/H_*)/(J_*/H_*)}^{J^*/H_*} \mathrm{res}_{J^*/H_*}^H =: (J \times H)/R'$$

where λ is the canonical map $J \to (J^*/H_*)/(J_*/H_*)$ and

$$T' = \big\{\big(jH_*, j'J_*\big) \in J^*/H_* \times J\colon\ (jH_*)J_*/H_* = \lambda\big(j'J_*\big)\big\}$$

and

$$R' = \big\{\big(jJ_*, j'H_*\big) \in J \times J^*/H_*\colon\ \big(j'H_*\big)J_*/H_* = \lambda(jJ_*)\big\}.$$

Hence, we must show that

$$\alpha\big(\mathrm{des}_I^H \,\mathrm{tin}_J^H\big) = (I \times H)/R' \times_H (H \times J)/T'.$$

By the Mackey product formula, we have

$$(I \times H)/R' \times_H (H \times J)/T' = \sum_{x \in p_2(R')\backslash H/p_1(T')} (I \times J)/R' * {}^{(x,1)}T'$$

where

$$R * T = \big\{(iI_*, jJ_*) \in I \times J\colon\ (iI_*, hH_*) \in R \text{ and } (hH_*, jJ_*) \in T \text{ for some } hH_* \in H\big\}.$$

Straightforward calculations show that $p_1(R*T)/k_1(R*T) = I \sqcap J$ and $p_2(R*T)/k_2(R*T) = J \sqcap I$, and hence the Mackey relation. $\square$

Hereafter, we shall identify Γ and $\tilde{\Gamma}$ via the above isomorphism α. Now let us describe the free basis of the alchemic algebra consisting of the isomorphism classes of transitive bisets in terms of the new variables. Clearly any transitive biset corresponds to a product of tinflation, isogation and destriction, in this order. We are aiming to find an equivalence relation on the set $\mathcal{B} = \{\mathrm{tin}_J^H \, \mathrm{c}_{J,I}^{\phi} \, \mathrm{des}_I^K\colon\ J \preccurlyeq H,\ I \preccurlyeq K,\ \phi : I \to J\}$ such that under α, the equivalence classes of the relation correspond to the isomorphism classes of transitive bisets.

Given two subquotients H and K of G. Also given subquotients J, A of H and subquotients I, C of K such that there are isomorphisms $\phi : I \to J$ and $\psi : C \to A$. We say that the triples (J, I, ϕ) and (A, C, ψ) are (H, K)-**conjugate** if there exist $k \in K$ and $h \in H$ such that

1. the equalities ${}^hJ = A$ and ${}^kI = C$ hold and
2. (compatibility of ϕ and ψ) the following diagram commutes.

$$\begin{array}{ccc} I & \xrightarrow{\ \phi\ } & J \\ \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[\ \psi\ ]{} & A \end{array}$$

We denote by $[J, I, \phi]$ the (H, K)-conjugacy class of (J, I, ϕ). Then we obtain

Theorem 3.3. *Letting H and K run over the subquotients of G and $[J, I, \phi]$ run over the (H, K)-conjugacy classes of triples ($J \preccurlyeq H, I \preccurlyeq K, \phi : I \to J$), the elements $\mathrm{tin}_J^H \, \mathrm{c}_{J,I}^{\phi} \, \mathrm{des}_I^K$ run, without repetitions, over a free R-basis of the alchemic algebra Γ.*

Proof. We are to show that (H, K)-conjugacy classes of the triples (J, I, ϕ) are in one-to-one correspondence with the isomorphism classes of transitive (H, K)-bisets. This follows from the following lemma. □

Lemma 3.4. *Let $H, K \preccurlyeq G$. Then there is a one-to-one correspondence between*

(i) *the (H, K)-conjugacy classes $[J, I, \phi]$ of triples (J, I, ϕ),*
(ii) *the isomorphism classes $[X]$ of transitive (H, K)-bisets*

where the correspondence is given by associating $[J, I, \phi]$ to the isomorphism class of the biset $\alpha(\mathrm{tin}_J^H \, \mathrm{c}_{J,I}^{\phi} \, \mathrm{des}_I^K)$.

Proof. Let (J, I, ϕ) and (A, C, ψ) be two (H, K)-conjugate triples. Then we have to show that the transitive bisets $\alpha(\mathrm{tin}_J^H \, \mathrm{c}_{J,I}^{\phi} \, \mathrm{des}_I^K)$ and $\alpha(\mathrm{tin}_A^H \, \mathrm{c}_{A,C}^{\psi} \, \mathrm{des}_C^K)$ are isomorphic. Let us write

$$\alpha\big(\mathrm{tin}_J^H \, \mathrm{c}_{J,I}^{\phi} \, \mathrm{des}_I^K\big) = (H \times K)/\mathfrak{a}$$

and

$$\alpha\big(\mathrm{tin}_A^H \, \mathrm{c}_{A,C}^{\psi} \, \mathrm{des}_C^K\big) = (H \times K)/\mathfrak{b}$$

for some subgroups $\mathfrak{a}, \mathfrak{b} \in H \times K$ given explicitly in the proof of Theorem 3.2. Let $h \in H$ and $k \in K$ such that

$${}^{h}J = A \quad \text{and} \quad {}^{k}I = C.$$

We shall show that ${}^{(h,k)}\mathfrak{a} = \mathfrak{b}$. Let $(j, i) \in \mathfrak{a}$. Then by the definition of α, we have $jJ_* = \phi(iI_*)$. Clearly, $({}^{h}j, {}^{k}i) \in A \times C$. So it suffices to show ${}^{h}jA_* = \psi({}^{k}iC_*)$. But,

$${}^{h}(jJ_*) = hjJ_*h^{-1} = hjh^{-1}hJ_*h^{-1} = {}^{h}jA_*,$$

and

$${}^{h}\phi(iI_*) = h\phi\big(k^{-1}\big(kik^{-1}\big)\big(kI_*k^{-1}\big)k\big)h^{-1} = \psi\big({}^{k}iC_*\big)$$

by the compatibility of ϕ and ψ. Hence ${}^{h}jA_* = \psi({}^{k}iC_*)$, as required.

Conversely, let $\mathfrak{a}, \mathfrak{b} \in H \times K$ be two conjugate subgroups of $H \times K$. Then we are to show that the triples $(p_1(\mathfrak{a})/k_1(\mathfrak{a}), p_2(\mathfrak{a})/k_2(\mathfrak{a}), \phi)$ and $(p_1(\mathfrak{b})/k_1(\mathfrak{b}), p_2(\mathfrak{b})/k_2(\mathfrak{b}), \psi)$ are (H, K)-conjugate. Here ϕ and ψ are the canonical isomorphisms introduced in Theorem 2.1.

Now let ${}^{(h,k)}\mathfrak{a} = \mathfrak{b}$ for some $h \in H$ and $k \in K$. Then, clearly,

$${}^{h}\big(p_1(\mathfrak{a}), k_1(\mathfrak{a})\big) = \big(p_1(\mathfrak{b}), k_1(\mathfrak{b})\big)$$

and

$${}^{k}\big(p_2(\mathfrak{a}), k_2(\mathfrak{a})\big) = \big(p_2(\mathfrak{b}), k_2(\mathfrak{b})\big).$$

So, it suffices to show that the diagram

$$\begin{array}{ccc}
p_2(\mathfrak{a})/k_2(\mathfrak{a}) & \xrightarrow{\ \phi\ } & p_1(\mathfrak{a})/k_1(\mathfrak{a}) \\
\Big\downarrow{\scriptstyle k} & & \Big\downarrow{\scriptstyle h} \\
p_2(\mathfrak{b})/k_2(\mathfrak{b}) & \xrightarrow{\ \psi\ } & p_1(\mathfrak{b})/k_1(\mathfrak{b})
\end{array}$$

commutes.

Let $(a, c) \in \mathfrak{b}$. Then by the definition of ψ, we have

$$\psi\big(ak_2(\mathfrak{b})\big) = ck_1(\mathfrak{b}).$$

But writing $i = a^k$ and j for the unique element $j = c^k$, the left-hand side becomes

$$\psi\big(ak_2(\mathfrak{b})\big) = \psi\big(kik^{-1}kk_2(\mathfrak{a})k^{-1}\big) = \psi\big(k\big(ik_2(\mathfrak{a})\big)k^{-1}\big)$$

and the right-hand side becomes

$$ck_1(\mathfrak{b}) = h\big(jk_1(\mathfrak{a})\big)h^{-1} = h\big(\phi\big(jk_2(\mathfrak{a})\big)\big)h^{-1}.$$

Thus combining these two equality we get $\psi(k(ik_2(\mathfrak{a}))k^{-1}) = h\phi(jk_2(\mathfrak{a}))h^{-1}$, as required. □

Note that the unit

$$1_\Gamma = \sum_{H \preccurlyeq G} \mathrm{c}_H$$

of the alchemic algebra Γ induces a decomposition

$$F = \bigoplus_{H \preccurlyeq G} \mathrm{c}_H F$$

of any biset functor F into R-submodules. We call $F(H) := \mathrm{c}_H F$ the **coordinate module** of F at H.

Clearly the coordinate module $F(H)$ at the subquotient $H \preccurlyeq G$ is a module for the truncated subalgebra $\mathrm{c}_H\, \Gamma\, \mathrm{c}_H$ of the alchemic algebra Γ. Now the actions of the generators can be seen as maps between the coordinate modules. Explicitly, given $J \preccurlyeq H \preccurlyeq G$ and $M, L \preccurlyeq G$ such that there is an isomorphism $\phi : M \to L$, we have the following maps.

M1. Tinflation map $\mathrm{tin}_J^H : F(J) \to F(H)$.
M2. Destriction map $\mathrm{des}_J^H : F(H) \to F(J)$.
M3. Isogation map $\mathrm{c}_{M,L}^{\phi} : F(L) \to F(M)$.

In this case, the maps are subject to the relations (R2)–(R6) of the alchemic algebra together with the following relation.

R1. The maps $\mathrm{c}_{H,H}^h$, tin_H^H and des_H^H where $h : H \to H$ is conjugation by $h \in H$ are all equal to the identity map for all subquotients $H \preccurlyeq G$.

Hence we have defined a biset functor F as a quadruple $(F, \mathrm{tin}, \mathrm{des}, \mathrm{c})$ where F is a family consisting of R-modules $F(H)$ for each $H \preccurlyeq G$ and there are three families of maps between these modules given as above. It is straightforward to prove that this definition is equivalent to any of the other two.

4. Alcahestic subalgebras and simple modules

In this section, we explore the simple modules of the alchemic algebra together with its certain unital subalgebras, each called an *alcahestic subalgebra*. In particular, we describe the simple modules of these subalgebras in terms of the head (or the socle) of induced (or coinduced) simple modules and using some particular choice of these subalgebras, we prove a classification theorem for the simple modules. Note that Bouc's classification of simple biset functors (Theorem 2.2) follows from our classification theorem. Further, each of these subalgebras has two special types of alcahestic subalgebras. These special subalgebras allow us to introduce a triangle having similar properties as the Mackey triangle introduced in [12]. We shall not introduce this triangle-structure in this paper. But all of the results in [12, Section 3] hold in this case with some modifications. Furthermore it is easy to describe the coordinate modules of the functors induced (or coinduced) from these subalgebras. We shall describe these functors for the alchemic algebra in the next section.

To begin with, let Π be a subalgebra of the alchemic algebra Γ. We call Π an **alcahestic subalgebra**[3] of Γ if it contains c_H for each subquotient H of G. Clearly the alchemic algebra is an alcahestic subalgebra. Actually the subalgebras of immediate interest, for example the subalgebras realizing cohomology functors or homology functors or Brauer character ring functor, are all alcahestic. However there is a more basic example of such algebras, defined below, which allows us to parameterize the simple modules.

Let Π be an alcahestic subalgebra of Γ. We denote by Ω_Π the subalgebra of Π generated by all isogations in Π. We call Ω_Π the **isogation algebra associated to** Π. We write Ω for the isogation algebra associated to the alchemic algebra Γ. Clearly, Ω_Π is alcahestic since Π is. The structure of the isogation algebra Ω_Π is easy to describe. We examine the structure since it is crucial in proving our main results.

It is evident that the isogation algebra Ω_Π associated to Π has the following decomposition

$$\Omega_\Pi = \bigoplus_{I,J \preccurlyeq G} \mathrm{c}_I \, \Omega_\Pi \, \mathrm{c}_J.$$

[3] In alchemy, alcahest is the universal solvent. The decomposition of unity into a sum of the elements c_H for $H \preccurlyeq G$ allows us to decompose any module into coordinate modules as in Section 3.

Now for a fixed subquotient $H \preccurlyeq G$, the following isomorphism holds.

$$\bigoplus_{I,J\cong H} \mathrm{c}_I\, \Omega_\Pi\, \mathrm{c}_J \cong \mathrm{Mat}_n(\mathrm{c}_H\, \Omega_\Pi\, \mathrm{c}_H)$$

where n is the number of subquotients of G isomorphic to H. In particular, we see that the isogation algebra Ω_Π is Morita equivalent to the algebra $\bigoplus_{H\preccurlyeq_\star G} \mathrm{c}_H \Omega_\Pi \mathrm{c}_H$. Here the sum is over the representatives of the isomorphism classes of subquotients of G. Note that if Π is the alchemic algebra, there is an isomorphism $\mathrm{c}_H \Omega \mathrm{c}_H \cong R\mathrm{Out}(H)$ of algebras. Recall that $\mathrm{Out}(H)$ is the group of outer automorphisms of H.

Now simple modules of the algebra $\mathrm{Mat}_n(\mathrm{c}_H\, \Omega_\Pi\, \mathrm{c}_H)$ correspond to the simple $\mathrm{c}_H\, \Omega_\Pi\, \mathrm{c}_H$-modules. Hence the simple modules of the isogation algebra Ω_Π are parameterized by the pairs (H, V) where H is a subquotient of G and V is a simple $\mathrm{c}_H\, \Omega_\Pi\, \mathrm{c}_H$-module. We call (H, V) a **simple pair** for Ω_Π and denote by $S^\Omega_{H,V}$ the corresponding simple Ω_Π-module. It is clear that $\mathcal{S}^\Omega_{H,V}$ is the Ω_Π-module defined for any subquotient K of G by

$$\mathcal{S}^\Omega_{H,V}(K) = {}^\phi V \quad \text{if there exists an isomorphism} \quad \phi : K \to H$$

and zero otherwise. Note that the definition does not depend on the choice of the isomorphism $\phi : K \to H$ since any two such isomorphisms differ by an inner automorphism of H and the group $\mathrm{Inn}(H)$ acts trivially on V.

More generally, for any alcahestic subalgebra Π, the following theorem holds.

Theorem 4.1. *Let Π be an alcahestic subalgebra of the alchemic algebra Γ. There is a bijective correspondence between*

(i) *the isomorphism classes of simple Π-modules S^Π,*
(ii) *the isomorphism classes of simple Ω_Π-modules $S^\Omega_{H,V}$*

given by $S^\Pi \leftrightarrow S^\Omega_{H,V}$ if and only if H is minimal such that $S(H) \neq 0$ and $S(H) = V$.

In other words, the above theorem asserts that given an alcahestic subalgebra Ω' of the isogation algebra Ω and given any subalgebra Π of the alchemic algebra such that $\Omega_\Pi \cong \Omega'$, the simple modules of Π are parameterized by simple Ω'-modules.

We prove this theorem in several steps. The first step is to characterize the simple modules in terms of images of tinflation maps and kernels of destriction maps. This characterization is an adaptation of a similar result of Thévenaz and Webb [14] for Mackey functors. In particular, this characterization implies that any simple Π-module has a unique minimal subquotient, up to isomorphism, and the minimal coordinate module is simple. In order to do this, we introduce two submodules of a Π-module F, as follows (cf. [14]). Let H be a minimal subquotient for F, that is, H is a subquotient of G minimal subject to the condition that $F(H) \neq 0$. Define two R-submodules of F by

$$\mathcal{I}_{F,H}(J) = \sum_{I\preccurlyeq J,\, I\cong H} \mathrm{Im}\big(\mathrm{tin}^J_I : F(I) \to F(J)\big)$$

and

$$\mathcal{K}_{F,H}(J) = \bigcap_{I \preccurlyeq J,\, I \cong H} \operatorname{Ker}\bigl(\mathrm{des}_I^J : F(J) \to F(I)\bigr).$$

Proposition 4.2. *The R-modules $\mathcal{I}_{F,H}$ and $\mathcal{K}_{F,H}$ are Π-submodules of F, via the induced actions.*

Proof. Let us prove that $\mathcal{I}_{F,H}$ is a submodule of F. The other claim can be proved similarly. Clearly, $\mathcal{I}_{F,H}$ is closed under isogation. It is also clear that $\mathcal{I}_{F,H}$ is closed under tinflation, because of the transitivity of tinflation. So it suffices to show that $\mathcal{I}_{F,H}$ is closed under destriction, which is basically an application of the Mackey relation. Let $A \preccurlyeq K$ be subquotients of G and let f be an element of $\mathcal{I}_{F,H}(K)$. We are to show that $\mathrm{des}_A^K f$ is an element of $\mathcal{I}_{F,H}(A)$. Write

$$f = \sum_I \mathrm{tin}_I^K f_I$$

for some $f_I \in F(I)$. Here the sum is over all subquotients of K isomorphic to H. Applying the Mackey relation, we get

$$\begin{aligned}\mathrm{des}_A^K f &= \sum_I \mathrm{des}_A^K \mathrm{tin}_I^K f_I \\ &= \sum_I \sum_{y \in A^* \backslash K / I^*} \mathrm{tin}_{A \sqcap {}^y I}^A \mathrm{c}^{y\lambda} \mathrm{des}_{I \sqcap A^y}^I f_I.\end{aligned}$$

Since H is minimal for F, the last sum contains only the terms $\mathrm{tin}_{A \sqcap {}^y I}^A \mathrm{c}^{y\lambda}$ where $A \sqcap {}^y I$ is isomorphic to H. Therefore $\mathrm{des}_A^K f \in \mathcal{I}_{F,H}(A)$, as required. □

The characterization of simple Π-modules via these subfunctors is as follows (cf. [14] and [16]).

Proposition 4.3. *Let Π be an alcahestic subalgebra of the alchemic algebra Γ. Let S be a Π-module. Let H be a minimal subquotient for S and let V denotes the coordinate module of S at H. Then S is simple if and only if*

(i) $\mathcal{I}_{S,H} = S$,
(ii) $\mathcal{K}_{S,H} = 0$,
(iii) *V is a simple $\mathrm{c}_H\, \Omega_\Pi\, \mathrm{c}_H$-module.*

Proof. It is clear that if S is simple then the conditions (i) and (ii) hold. Also since H is minimal such that $S(H) \neq 0$, any map that decomposes through a smaller subquotient is a zero map. Thus $S(H)$ is a module of the algebra $\mathrm{c}_H\, \Omega_\Pi\, \mathrm{c}_H$. But it has to be simple since any decomposition of the minimal coordinate module gives a decomposition of S. Now it remains to show the reverse implication. Suppose the conditions hold. Let T be a subfunctor of S. Since $S(H) = V$ is simple, $T(H)$ is either 0 or V. If $T(H) = V$ then by condition (i), it is equal to S. So, let $T(H) = 0$. Then for any $K \preccurlyeq G$, the module $T(K)$ is a submodule of $\mathcal{K}_{S,H}$, because for any $x \in T(K)$

and $H \cong L \preccurlyeq K$, we have $\mathrm{des}_L^K x \in T(L) = 0$. Thus by condition (ii), $T(K) = 0$, that is, $T = 0$. Thus, any subfunctor of S is either zero or S itself. In other words, S is simple. □

Now it is clear that a simple Π-module S has a unique, up to isomorphism, minimal subquotient, say H. Moreover the coordinate module at H is a simple $\mathrm{c}_H\, \Omega_\Pi\, \mathrm{c}_H$-module. That is to saying that there is a map from the set of isomorphism classes of simple Π-modules to the set of isomorphism classes of the simple pairs (H, V) for Ω_Π, justifying the existence of the correspondence of Theorem 4.1.

The second step in proving Theorem 4.1 is to describe the behavior of simple modules under induction and coinduction to certain alcahestic subalgebras. We need two more definitions.

Let Π still denote an alcahestic subalgebra. Define the **destriction algebra** ∇_Π associated to Π as the subalgebra of Π generated by all destriction maps and isogation maps in the algebra Π. Similarly define Δ_Π, the **tinflation algebra**[4] associated to Π. We write ∇ and Δ for the destriction algebra and the tinflation algebra associated to the alchemic algebra Γ, respectively. Clearly, both ∇_Π and Δ_Π are alcahestic subalgebras since Π is.

Now we are ready to state our main theorem. This theorem is a precise statement of Theorem 1.2.

Theorem 4.4 *(Correspondence Theorem). Let $\Pi \subset \Theta$ be two alcahestic subalgebras of the alchemic algebra and S^Π be a simple Π-module with minimal subquotient H and denote by V the coordinate module of S at H.*

(i) *The Θ-module $\mathrm{ind}_\Pi^\Theta S^\Pi$ has a unique maximal submodule provided that $\nabla_\Pi = \nabla_\Theta$. Moreover the minimal subquotient for the simple quotient is H and the coordinate module of the simple quotient at H is isomorphic to V.*
(ii) *The Θ-module $\mathrm{coind}_\Pi^\Theta S^\Pi$ has a unique minimal submodule provided that $\Delta_\Pi = \Delta_\Theta$. Moreover the minimal subquotient for the minimal submodule is H and the coordinate module of the simple submodule at H is isomorphic to V.*

Proof. We only prove part (i). The second part follows from a dual argument. First, observe that the subquotient H is minimal for the induced module $F := \mathrm{ind}_\Pi^\Theta S_{H,V}^\Pi$ since $\nabla_\Pi = \nabla_\Theta$. Moreover observe that there is an isomorphism $F(H) \cong V$. Therefore the submodule $\mathcal{K}_{F,H}$ is defined. We claim that $\mathcal{K}_{F,H}$ is the unique maximal submodule of F.

To prove this, let T be a proper submodule of F. We are to show that $T \leqslant \mathcal{K}_{F,H}$. Since S is generated by its coordinate module at H, the Θ-module F is generated by its coordinate module at H which is simple. So $T(H)$ must be the zero module. Now let $K \preccurlyeq G$ be such that $T(K) \neq 0$. Then clearly, $T(K) \leqslant \mathcal{K}_{F,H}(K)$ as $\mathrm{des}_L^K f \in T(L) = 0$ for any $f \in T(K)$ and $L \cong H$. Thus $T \leqslant \mathcal{K}_{F,H}$, as required. □

To prove Theorem 4.1, we examine a special case of the Correspondence Theorem. This special case also initiates the process of describing simple Θ-module via induction or coinduction using Theorem 4.4. First we describe simple destriction and tinflation modules. For completeness, we include the description of simple Ω_Π-modules.

[4] Our notation is consistent with that of ancient alchemists. In alchemy, the symbols of fire and water are Δ and ∇, respectively. The symbols of air and earth are the same as the symbols of fire and water, respectively, with an extra horizontal line dividing the symbol into two. Moreover quintessence is also known as spirit which has the symbol Ω.

Proposition 4.5. *Let Π be an alcahestic subalgebra and (H, V) be a simple pair for the isogation algebra Ω_Π associated to Π. Then*

(i) *the Ω_Π-module $\mathcal{S}^{\Omega}_{H,V}$ is simple. Moreover any simple Ω_Π-module is of this form for some simple pair (H, V).*
(ii) *The ∇_Π-module $\mathcal{S}^{\nabla}_{H,V} := \mathrm{inf}^{\nabla_\Pi}_{\Omega_\Pi}\mathcal{S}^{\Omega}_{H,V}$ is simple. Moreover any simple ∇_Π-module is of this form for some simple pair (H, V) for Ω_Π.*
(iii) *The Δ_Π-module $\mathcal{S}^{\Delta}_{H,V} := \mathrm{inf}^{\Delta_\Pi}_{\Omega_\Pi}\mathcal{S}^{\Omega}_{H,V}$ is simple. Moreover any simple Δ_Π-module is of this form for some simple pair (H, V) for Ω_Π.*

Here the inflation functor $\mathrm{inf}^{\nabla_\Pi}_{\Omega_\Pi}$ is the inflation induced by the quotient map $\Pi \to \Pi/\mathcal{J}(\nabla_\Pi)$ where $\mathcal{J}(\nabla_\Pi)$ is the ideal generated by proper destriction maps, that is,

$$\mathcal{J}(\nabla_\Pi) = \left\{\mathrm{c}^{\phi}_{H,K}\,\mathrm{des}^{K}_{I} \in \nabla_\Pi\colon\ I \neq K \text{ or } M \neq Y\right\}.$$

We identify the quotient with the isogation algebra Ω_Π in the obvious way.

Proof. The first part follows from the above discussion of simple isogation modules. Moreover it is clear that the module $\mathcal{S}^{\nabla}_{H,V}$ is simple. To see that any simple is of this form, notice that if a ∇-module D has non-zero coordinates at two non-isomorphic subquotient, then D has a non-zero submodule generated by the coordinate module at the subquotient of minimal order. So any simple ∇_Π-module has a unique, up to isomorphism, non-zero coordinate module. Clearly, this coordinate module should be simple. The same argument applies to the second part. □

Remark 4.6. Alternatively, one can apply the correspondence theorem to obtain simple ∇_Π-modules and simple Δ_Π-modules. In the first case, to identify the submodule $\mathcal{K}$, one should identify des^{J}_{I} with $\mathrm{c}_I\ \mathrm{des}^{J}_{I}\ \mathrm{c}_J$. Similar modification is also needed to identify the submodule $\mathcal{I}$.

Evidently, for any alcahestic subalgebra Π, the destriction algebra ∇_Π and the tinflation algebra Δ_Π associated to Π are the minimal examples of the subalgebras satisfying the conditions of the first and the second part of the Correspondence Theorem, respectively. The following corollary restates the Correspondence Theorem for these special cases. We shall refer to this corollary in proving Theorem 4.1 and also in describing the simple biset functors in the next section.

Corollary 4.7. *Let Π be an alcahestic subalgebra of the alchemic algebra such that the destriction algebra ∇_Π and the tinflation algebra Δ_Π are proper subalgebras. Let (H, V) be a simple pair for the isogation algebra Ω_Π associated to Π.*

(i) *The Π-module $\mathrm{ind}^{\Pi}_{\nabla}\,\mathcal{S}^{\nabla}_{H,V}$ has a unique maximal subfunctor. Moreover H is a minimal subquotient for the simple quotient and the coordinate module of S at H is isomorphic to V.*
(ii) *The Π-module $\mathrm{coind}^{\Pi}_{\Delta}\,\mathcal{S}^{\Delta}_{H,V}$ has a unique minimal subfunctor. Moreover H is a minimal subquotient for the simple subfunctor and the coordinate module of S at H is isomorphic to V.*

Now, we are ready to prove Theorem 4.1. By Corollary 4.7, we associated a simple module $S^{\Pi}_{H,V}$ to each simple pair (H, V) for the isogation algebra Ω_Π. Clearly this is an inverse to the correspondence described above. So it suffices to show that the correspondence is injective. This is equivalent to show that any simple Π-module with minimal subquotient H and $S(H) = V$ is isomorphic to $S^{\Pi}_{H,V}$. So let S be a simple Π-module with this property. Then we are to exhibit a non-zero morphism $S^{\Pi}_{H,V} \to S$. By our construction of $S^{\Pi}_{H,V}$, it suffices to exhibit a morphism $\phi : \mathrm{ind}^{\Gamma}_{\nabla} \mathcal{S}^{\nabla}_{H,V} \to S$ such that ϕ_H is non-zero. The morphism exists since

$$\mathrm{Hom}_{\Pi}\big(\mathrm{ind}^{\Pi}_{\nabla} \mathcal{S}^{\nabla}_{H,V}, S\big) \cong \mathrm{Hom}_{\nabla}\big(\mathcal{S}^{\nabla}_{H,V}, \mathrm{res}^{\Pi}_{\nabla}\ S\big) \cong \mathrm{Hom}_{R\mathrm{Out}(H)}(V, V) \neq 0.$$

Here the first isomorphism holds because induction is left adjoint of restriction. On the other hand, the second isomorphism holds since $\mathcal{S}^{\nabla}_{H,V}$ is non-zero only on the isomorphism class of H. Now the identity morphism $V \to V$ induces a morphism $\phi : \mathrm{ind}^{\Pi}_{\nabla} \mathcal{S}^{\nabla}_{H,V} \to S$. Clearly, ϕ_H is non-zero, as required. Therefore we have established the injectivity, as required.

Having proved the classification theorem, we can restate the Correspondence Theorem more precisely. Let $\Pi \subset \Theta$ be two alcahestic subalgebras of the alchemic algebra Γ and (H, V) be a simple pair for the isogation algebra Ω_{Π}.

(i) The Θ-module $\mathrm{ind}^{\Theta}_{\Pi}\ S^{\Pi}_{H,V}$ has a unique maximal submodule provided that $\nabla_{\Pi} = \nabla_{\Theta}$. Moreover the simple quotient is isomorphic to $S^{\Theta}_{H,V}$.
(ii) The Θ-module $\mathrm{coind}^{\Theta}_{\Pi}\ S^{\Pi}$ has a unique minimal submodule provided that $\Delta_{\Pi} = \Delta_{\Theta}$. Moreover the simple submodule is isomorphic to $S^{\Theta}_{H,V}$.

Finally the next result shows that in both cases of the Correspondence Theorem, the inverse correspondence is given by restriction.

Theorem 4.8. *Let $\Pi \subset \Theta$ be alcahestic subalgebras of the alchemic algebra. Then*

(i) *the Π-module* $\mathrm{res}^{\Theta}_{\Pi}\ S^{\Theta}_{H,V}$ *has a unique maximal submodule, provided that $\Delta_{\Pi} = \Delta_{\Theta}$. Moreover the simple quotient is isomorphic to $S^{\Pi}_{H,V}$.*
(ii) *The Π-module* $\mathrm{res}^{\Theta}_{\Pi}\ S^{\Theta}_{H,V}$ *has a unique minimal submodule, provided that $\nabla_{\Pi} = \nabla_{\Theta}$. Moreover this submodule is isomorphic to $S^{\Pi}_{H,V}$.*

Proof of the first part of (i) is similar to the proof of Theorem 4.4. Indeed the maximal subfunctor of the restricted module $\mathrm{res}^{\Theta}_{\Pi}\ S^{\Theta}_{H,V}$ is generated by intersection of kernels of the destriction maps having range isomorphic to H. Note that this subfunctor is non-zero, in general. The second part of (i) follows from Theorem 4.1. Part (ii) can be proved by a dual argument.

In particular, this theorem shows that restriction of a simple Θ-module to a subalgebra Π is indecomposable provided that Π contains either all destriction maps or tinflation maps in Θ. On the other hand, if none of these conditions holds then the restricted module can be zero, semisimple or indecomposable.

5. An application: simple biset functors

As an application of Theorem 4.7, we shall present two descriptions of the simple biset functors by describing induction (and coinduction) from destriction (and tinflation) algebra. Similar descriptions can be made for other alcahestic subalgebras. In [16], Webb constructed simple functors for the alcahestic subalgebra generated by all tinflation and restriction maps. He called the modules of this algebra inflation functors. His construction is equivalent to the construction by coinduction from the tinflation algebra associated to this subalgebra. In his paper, Webb also constructed simple global Mackey functors. In our terminology, global Mackey functors are modules of the alcahestic subalgebra generated by all transfer, restriction and isogation maps and his construction is, again, equivalent to the construction by coinduction from transfer algebra. Note that another description, from restriction algebra, is also possible.

Turning to the construction of simple biset functors, we see that in general, our construction is more efficient than Bouc's construction in the sense that the biset functor in Theorem 4.7 is smaller than the one in Section 2.2. It is also advantageous to have explicit descriptions of the coordinate modules of the induced (or coinduced) modules.

In the following theorem, we characterize the coordinate modules of the induced module $\mathrm{ind}_{\nabla}^{\Gamma} D$ where D is a ∇-module. The proof is similar to the proof of Theorem 5.1 in [12] but we include the proof to introduce our notation.

Theorem 5.1. *Let D be a ∇-module and H be a subquotient of G. Then there is an isomorphism of R-modules*

$$\big(\mathrm{ind}_{\nabla}^{\Gamma} D\big)(H) \cong \Big(\bigoplus_{J \preccurlyeq H} D(J)\Big)_H$$

where the right-hand side is the maximal H-fixed quotient of the direct sum.

Proof. Let

$$D_+(H) := \Big(\bigoplus_{J \preccurlyeq H} D(J)\Big)_H.$$

We write $[J,a]_H$ for the image of $a \in D(J)$ in $D_+(H)$. Since H acts trivially, it is clear that $[J,a]_H = {}^h[J,a]_H$ for all $h \in H$. Moreover, D_+H is generated as an R-module by $[J,a]_H$ for $J \preccurlyeq_H H$ and $a \in D(J)$. Here $\preccurlyeq_H$ means that we take J up to H-conjugacy. In other words,

$$D_+(H) = \bigoplus_{J \preccurlyeq_H H} \big\{[J,a]_H \colon a \in D(J)\big\}.$$

On the other hand,

$$\big(\mathrm{ind}_{\nabla}^{\Gamma} D\big)(H) = \bigoplus_{J \preccurlyeq_H H} \big\{\mathrm{tin}_J^H \otimes a \colon a \in D(J)\big\}.$$

Now $\mathrm{tin}_J^H \otimes a = 0$ if and only if $a \in I(\mathrm{Out}(H))D(J)$ where $I(\mathrm{Out}(H))$ is the augmentation ideal of $R\mathrm{Out}(H)$. Therefore, the correspondence $\mathrm{tin}_J^H \otimes a \leftrightarrow [J,a]_H$ extends linearly to an isomorphism of R-modules $(\mathrm{ind}_{\nabla}^{\Gamma} D)(H) \cong D_+(H)$. Evidently, this is an $R\mathrm{Out}(H)$-modules isomorphism. $\square$

Let us describe the action of tinflation, destriction and isogation on the generating elements $\mathrm{tin}_J^H \otimes a$ of the biset functor $\mathrm{ind}_{\nabla}^{\Gamma} D$. Note that we obtain these formulae by multiplying from the left with the corresponding generator and use the defining relations of the alchemic algebra. Let $J \preccurlyeq H \preccurlyeq G$ and $T \preccurlyeq H \preccurlyeq K$ and $a \in D(J)$. Finally let $A \preccurlyeq G$ such that $\phi : H \cong A$. Then

Tinflation $$\mathrm{tin}_H^K\big(\mathrm{tin}_J^H \otimes a\big) = \mathrm{tin}_J^K \otimes a.$$

Destriction $$\mathrm{des}_T^H\big(\mathrm{tin}_J^H \otimes a\big) = \sum_{x \in T^* \backslash H / J^*} \mathrm{tin}_{T \sqcap {}^x J}^T \otimes \mathrm{c}^{x\lambda}\, \mathrm{des}_{J \sqcap T^x}^J\, a.$$

Isogation $$\mathrm{c}_{A,H}^{\phi}\big(\mathrm{tin}_J^H \otimes a\big) = \mathrm{tin}_{\phi(J)}^A \otimes \mathrm{c}_{\phi(J),J}^{\phi} a.$$

The other functor, that we will make use of, is coinduction from the tinflation algebra Δ to the alchemic algebra Γ. We can describe the coordinate modules in terms of fixed-points as follows.

Theorem 5.2. *Let E be a Δ-module and $H \preccurlyeq G$. Then*

$$\big(\mathrm{coind}_\Delta^\Gamma E\big)(H) \cong \bigg(\prod_{J \preccurlyeq H} E(J)\bigg)^H$$

where the right-hand side is the H-fixed points of the direct product.

The proof of this theorem is similar to the proof of the above theorem. We shall only describe the actions of tinflation, destriction and isogation on the tuples $(x_J)_{J \preccurlyeq H}$. Let $J \preccurlyeq H \preccurlyeq G$ and $T \preccurlyeq H \preccurlyeq K$ and $A \preccurlyeq G$ such that $\phi : H \cong A$. Then

Tinflation $$\big(\mathrm{tin}_H^K\big((x_J)_{J \preccurlyeq H}\big)\big)_I = \sum_{y \in I^* \backslash K / H^*} \mathrm{tin}_{I \sqcap {}^y H}^I\, \mathrm{c}^{y\lambda}\, x_{H \sqcap I^y}.$$

Here, we write x_L for the Lth coordinate of an element $x \in \mathrm{coind}_\Delta^\Gamma E$.

Destriction $$\mathrm{des}_T^H\big((x_J)_{J \preccurlyeq H}\big) = (x_J)_{J \preccurlyeq T}.$$

Isogation $$\mathrm{c}_{A,H}^{\phi}\big((x_J)_{J \preccurlyeq H}\big) = \big({}^{\phi}x_{\phi^{-1}(J)}\big)_{J \preccurlyeq A}$$

where ${}^{\phi}x = \mathrm{c}_{J,\phi^{-1}(J)}^{\phi} x$.

By these theorems together with Theorem 4.7, we obtain the following two descriptions of the coordinate modules of the simple biset functors. For the first description, recall that $\mathcal{S}_{H,V}^{\nabla}$ denotes the simple ∇-module corresponding to the simple pair (H, V) where $H \preccurlyeq G$ and V is a simple $R\mathrm{Out}(H)$-module. (Note that $R\mathrm{Out}(H) \cong \mathrm{c}_H\, \Omega\, \mathrm{c}_H$.) By Theorem 5.1, the coordinate module at $K \preccurlyeq G$ of the induced module $\mathrm{ind}_\nabla^\Gamma \mathcal{S}_{H,V}^{\nabla}$ is given by

$$\mathrm{ind}_\nabla^\Gamma \mathcal{S}_{H,V}^{\nabla}(K) = \bigg(\bigoplus_{L \preccurlyeq K,\, \phi: L \cong H} {}^{\phi}V\bigg)_K.$$

To obtain the coordinate module at K of the simple biset functor $S_{H,V}$, we take the quotient of the above module by the submodule $\mathcal{K}(K)$, defined in the proof of Theorem 4.4. Explicitly, $\mathcal{K}(K)$ is generated by the elements $x \in \mathrm{ind}_\nabla^\Gamma \mathcal{S}_{H,V}^{\nabla}(K)$ such that $\mathrm{des}_L^K\, x = 0$ for any $L \preccurlyeq K$ with $L \cong H$. More explicit conditions on x can be obtained using the generating set $\{\mathrm{tin}_L^K \otimes v\colon L \preccurlyeq K,\ L \cong H,\ v \in S_{H,V}^{\nabla}(L)\}$.

The second description provides the description of the coordinate modules of $S_{H,V}$ in terms of the images of tinflation maps.

Corollary 5.3. *Let H and K be subquotients of G and suppose that K contains H. Also let V be a simple $R\mathrm{Out}(H)$-module. Then the Kth coordinate of $S^{\Gamma}_{H,V}$ is*

$$S^{\Gamma}_{H,V}(K) = \sum_{I \preccurlyeq K,\, \phi: H \cong I} \mathrm{Im}\big(\mathrm{tin}^K_I\big)$$

where for $v \in S^{\Gamma}_{H,V}(I) \cong V$, the Jth coordinate of $\mathrm{tin}^H_I\, v$ is given by

$$\big(\mathrm{tin}^K_I\, v\big)_J = \sum \mathrm{c}^{y\lambda}_{J,I}\, v$$

where the sum is over the representatives of the double cosets $J^*\backslash K/I^*$ such that $I = I \sqcap J^y$ and $J = J \sqcap {}^yI$.

6. The mark morphism and semisimplicity

In this section, we introduce the mark morphism for biset functors. We also show that it connects the two constructions of simple biset functors in an exact sequence. Furthermore we prove a characterization of semisimplicity of the alchemic algebra in terms of the mark morphism. As a corollary to this theorem, we give an alternative proof of the semisimplicity theorem for the alchemic algebra proved independently by Barker [1] and Bouc. Note further that it is straightforward to generalize this construction and the semisimplicity criterion to the alcahestic subalgebras. The characterization is also valid in more general cases where a mark morphism exists, for example for Mackey functors. Finally we shall show that our criterion has an adaptation to any finite-dimensional algebra over a field.

To introduce the mark morphism, let D be a ∇-module. We denote by π^D the morphism

$$\pi^D : \mathrm{res}^{\Gamma}_{\Delta}\, \mathrm{ind}^{\Gamma}_{\nabla}\, D \to \mathrm{inf}^{\Delta}_{\Omega}\, \mathrm{res}^{\nabla}_{\Omega}\, D$$

that forgets the tensor product. That is, given subquotients $H \preccurlyeq K$ of G and an element a of $D(H)$, then $\pi^D_K(\mathrm{tin}^K_H \otimes a) = \mathrm{tin}^K_H\, a$. Notice that since non-trivial tinflation maps of the Δ-module $\mathrm{inf}^{\Delta}_{\Omega}\, \mathrm{res}^{\nabla}_{\Omega}\, D$ are the zero maps, the morphism π^D_K is actually the projection $\mathrm{ind}^{\Gamma}_{\nabla}\, D(K) \to D(K)$. Now via the following adjunction

$$\mathrm{Hom}_{\Gamma}\big(\mathrm{ind}^{\Gamma}_{\nabla}\, D, \mathrm{coind}^{\Gamma}_{\Delta}\, \mathrm{inf}^{\Delta}_{\Omega}\, \mathrm{res}^{\nabla}_{\Omega}\, D\big) \cong \mathrm{Hom}_{\Delta}\big(\mathrm{res}^{\Gamma}_{\Delta}\, \mathrm{ind}^{\Gamma}_{\nabla}\, D, \mathrm{inf}^{\Delta}_{\Omega}\, \mathrm{res}^{\nabla}_{\Omega}\, D\big)$$

we obtain a map

$$\beta^D : \mathrm{ind}^{\Gamma}_{\nabla}\, D \to \mathrm{coind}^{\Gamma}_{\Delta}\, \mathrm{inf}^{\Delta}_{\Omega}\, \mathrm{res}^{\nabla}_{\Omega}\, D.$$

Explicitly, given a subquotient H of K and an element $\mathrm{tin}^K_H \otimes a$ in $\mathrm{ind}^{\Gamma}_{\nabla}\, D(K)$, the morphism β^D is given by

$$\beta^D_K\big(\mathrm{tin}^K_H \otimes a\big) = \big(\pi_I\big(\mathrm{des}^K_I\, \mathrm{tin}^K_H \otimes a\big)\big)_{I \preccurlyeq K}.$$

Following Boltje [2], we call β^D the **mark morphism** for D. Using the Mackey relation, we can calculate β_K^D more explicitly, as follows (cf. [2]):

$$\beta_K^D\big(\mathrm{tin}_H^K \otimes a\big) = \Bigg(\pi_I\Bigg(\sum_{x\in I^*\backslash K/H^*} \mathrm{tin}^I_{I\sqcap {}^x H}\, \mathrm{c}^{x\circ\lambda}\, \mathrm{des}^H_{H\sqcap I^x \otimes a}\Bigg)\Bigg)_{I\preccurlyeq K}$$
$$= \Bigg(\sum_{x\in I^*\backslash K/H^*,\, I=I\sqcap {}^x H} \mathrm{c}^{x\circ\lambda}\, \mathrm{des}^H_{H\sqcap I^x}\, a\Bigg)_{I\preccurlyeq K}.$$

Remark 6.1. The mark morphism defined as above is a generalization of the well-known mark homomorphism

$$\beta_G : B(G) \to \mathcal{B}(G)$$

where $B(G)$ is the Burnside ring of G and $\mathcal{B}(G)$ is the ghost ring of the Burnside ring. The ghost ring $\mathcal{B}(G)$ is defined as the dual of the Burnside ring and is isomorphic to the space of $\mathbb{Z}$-valued functions constant on conjugacy classes of subgroups of G, i.e. $\mathcal{B}(G) = (\prod_{H\leqslant G}\mathbb{Z})^G$. The mark morphism is now given by associating a finite G-set X to the function $(|X^H|)_{H\leqslant G}$. For appropriate choices, this morphism becomes a special case of the morphism between Boltje's plus constructions. For a detailed explanation, see [2].

Now if we put $D = \mathcal{S}^\nabla_{H,V}$, the simple ∇-module associated to the simple pair (H,V), then clearly $\inf^\Delta_\Omega \mathrm{res}^\nabla_\Omega \mathcal{S}^\nabla_{H,V} = \mathcal{S}^\Delta_{H,V}$. Hence the mark morphism is a morphism between the two dual constructions

$$\beta^{H,V} : \mathrm{ind}^\Gamma_\nabla \mathcal{S}^\nabla_{H,V} \to \mathrm{coind}^\Gamma_\Delta \mathcal{S}^\Delta_{H,V}.$$

Moreover, we have the following exact sequence.

Proposition 6.2. *The following spliced sequence is exact.*

$$0 \longrightarrow \mathcal{K}_{\mathrm{ind}^\Gamma_\nabla \mathcal{S}^\nabla_{H,V},H} \longrightarrow \mathrm{ind}^\Gamma_\nabla \mathcal{S}^\nabla_{H,V} \xrightarrow{\ \beta^{H,V}\ } \mathrm{coind}^\Gamma_\Delta \mathcal{S}^\Delta_{H,V} \longrightarrow \mathcal{C}_{H,V} \longrightarrow 0$$
$$\mathrm{ind}^\Gamma_\nabla \mathcal{S}^\nabla_{H,V} \searrow S^\Pi_{H,V} \nearrow \mathrm{coind}^\Gamma_\Delta \mathcal{S}^\Delta_{H,V}, \qquad 0 \nearrow S^\Pi_{H,V} \searrow 0$$

where $\mathcal{C}_{H,V} = \mathrm{coind}^\Gamma_\Delta \mathcal{S}^\Delta_{H,V} / \mathcal{I}_{\mathrm{coind}^\Gamma_\Delta \mathcal{S}^\Delta_{H,V},H}$.

Proof. It suffices to show that the kernel of the mark morphism is the unique maximal subfunctor of $\mathrm{ind}^\Gamma_\nabla \mathcal{S}^\nabla_{H,V}$. The inclusion $\ker \beta^{H,V} \subset \mathcal{K}_{\mathrm{ind}^\Gamma_\nabla \mathcal{S}^\nabla_{H,V},H}$ is trivial, as the right-hand side is maximal and $\beta^{H,V}$ is non-zero. To show the reverse inclusion, let $K \preccurlyeq G$ and $a \in \mathcal{K}_{\mathrm{ind}^\Gamma_\nabla \mathcal{S}^\nabla_{H,V},H}(K)$. Then

$$\beta_K^{H,V}(a) = \big(\pi_I\big(\mathrm{des}^K_I\, a\big)\big)_{I\preccurlyeq K,\, I\cong H} = 0$$

as by definition of $\mathcal{K}_{\mathrm{ind}^\Gamma_\nabla \mathcal{S}^\nabla_{H,V},H}(K)$, we have $\mathrm{des}^K_I\, a = 0$ for any $I \cong H$. $\square$

The following is immediate from the exactness of the above sequence.

Corollary 6.3. *Let (H, V) be a simple pair for Ω over a field R and $K \preccurlyeq G$. Then*

$$\dim_R S^{\Gamma}_{H,V}(K) = \operatorname{rank} \beta^{H,V}_K.$$

Note that when G is a p-group for some prime p and $V = R$ is the trivial $R\mathrm{Out}(H)$-module, the matrix for the mark morphism is the same as the matrix for the bilinear form introduced by Bouc in [3].

A special case of the mark morphism is the well-known natural morphism, called the **linearization morphism**

$$\mathbb{Q}\mathrm{lin}_G : \mathbb{Q}B(G) \to QR_{\mathbb{Q}}(G)$$

where $B(G)$ is the Burnside group, as above, and $\mathcal{R}_{\mathbb{Q}}(G)$ is the Grothendieck group of rational representation of G. The morphism is given by associating a G-set X to the permutation module $\mathbb{Q}X$. To see that this morphism is a mark morphism, we prove the following identification of the Burnside biset functor B^G.

Proposition 6.4. *There is an isomorphism of biset functors $B^G \cong \mathrm{ind}^{\Gamma}_{\nabla} \mathcal{S}^{\nabla}_{1,1}$ where $\mathcal{S}^{\nabla}_{1,1}$ is the simple ∇-module having one copy of the trivial module at each trivial coordinate.*

Proof. We need to specify an isomorphism $\Phi : B^G \to \mathrm{ind}^{\Gamma}_{\nabla} S^{\nabla}_{1,1}$ of biset functors. To do this, we specify an isomorphism $\Phi_H : B^G(H) \to \mathrm{ind}^{\Gamma}_{\nabla} S^{\nabla}_{1,1}(H)$ of R-modules for each $H \preccurlyeq G$ which is compatible with the actions of tinflation, destriction and isogation. By Theorem 5.1, $\mathrm{ind}^{\Gamma}_{\nabla} S^{G}_{1,1}(H)$ is generated by $\{\mathrm{tin}^H_{L/L} \otimes 1\colon L \leqslant H\}$. Now we define Φ_H by associating $[H/L]$ to $\mathrm{tin}^H_{L/L} \otimes 1$. Straightforward calculations show that Φ is an isomorphism of biset functors. □

Now the image of the mark morphism $\beta^{1,1}$ is the simple biset functor $S_{1,1}$ by Proposition 6.2. It is shown by Bouc [3] that over a field of characteristic zero, the functor of rational representations is simple and isomorphic to $S_{1,1}$. Hence, over a field of characteristic zero, the linearization morphism lin_G and the mark morphism $\beta^{1,1}_G$ coincide since they coincide on the trivial subgroup. Note that the mark morphism is given explicitly by associating $\mathrm{tin}^H_{L/L} \otimes 1$ to the function $(|K\backslash H/L|)_{K/K \preccurlyeq H}$ where $|K\backslash H/L|$ is the number of double coset representatives of K and L in H. Notice that the mark morphism for the Burnside biset functor is different than the usual mark morphism, which indeed is not a morphism of biset functors. Finally, in [9], Bouc describe the biset functor structure of the kernel of the linearization morphism for p-groups where p is a prime number. Over a field of characteristic zero and for a p-group P, the kernel is known to be isomorphic to the rational Dade group $\mathbb{Q} \otimes D(P)$ as mentioned in Section 2.

Another corollary of Proposition 6.4 is the following well-known result, see [3] and [1].

Corollary 6.5. *Let R be a field. Then the Burnside biset functor B^G is the projective cover of the simple biset functor $S_{1,1}$.*

Indeed it is clear that the simple destriction functor $\mathcal{S}^{\nabla}_{1,1}$ is projective, which implies that B^G is projective. Also it is indecomposable since the adjunction isomorphism

$$\mathrm{Hom}_{\Gamma}\big(\mathrm{ind}^{\Gamma}_{\nabla}\,\mathcal{S}^{\nabla}_{1,1}, \mathrm{ind}^{\Gamma}_{\nabla}\,\mathcal{S}^{\nabla}_{1,1}\big) \cong \mathrm{Hom}_{\nabla}\big(\mathcal{S}^{\nabla}_{1,1}, \mathrm{res}^{\Gamma}_{\nabla}\,\mathrm{ind}^{\Gamma}_{\nabla}\,\mathcal{S}^{\nabla}_{1,1}\big)$$

is actually an isomorphism of rings. Further to this, the following dual statement also holds.

Proposition 6.6. *There are isomorphisms of biset functors $\mathcal{B}^G \cong \mathrm{coind}^{\Gamma}_{\Delta}\,\mathcal{S}^{\Delta}_{1,1} \cong I_{1,1}$ where $\mathcal{B}^G$ denotes the functor of the ghost ring of Burnside ring and $I_{1,1}$ denotes the injective hull of the simple biset functor $S_{1,1}$.*

In particular, $S_{1,1}$ is both projective and injective if and only if the mark morphism $\beta^{1,1} : B^G \to \mathcal{B}^G$ is an isomorphism. This observation inspires the following criterion of semisimplicity, which is a precise statement of Theorem 1.3. Note that the equivalence of the first two statements is proved independently by Barker [1] and Bouc.

Theorem 6.7. *Let G be a finite group and R be a field of characteristic zero. The following statements are equivalent.*

(i) *The alchemic algebra $\Gamma_R(G)$ for G over R is semisimple.*
(ii) *The group G is cyclic.*
(iii) *The mark morphism $\beta^{H,V} : \mathrm{ind}^{\Gamma}_{\nabla}\,\mathcal{S}^{\nabla}_{H,V} \to \mathrm{coind}^{\Gamma}_{\Delta}\,\mathcal{S}^{\Delta}_{H,V}$ is an isomorphism for any simple pair (H, V) for Ω.*

We prove the theorem in two steps. The first step is to prove the equivalence of (i) and (iii). This follows from the following criterion of projectiveness and injectivity of the simple biset functors.

Theorem 6.8. *Let (H, V) be a simple pair for Ω such that V is projective. Then the following statements are equivalent.*

(i) *The simple biset functor $S^{\Gamma}_{H,V}$ is both projective and injective.*
(ii) *The mark morphism $\beta^{H,V} : \mathrm{ind}^{\Gamma}_{\nabla}\,\mathcal{S}^{\nabla}_{H,V} \to \mathrm{coind}^{\Gamma}_{\Delta}\,\mathcal{S}^{\Delta}_{H,V}$ is an isomorphism.*

Proof. First, suppose that β is an isomorphism. We are to show that the simple biset functor $S^{\Gamma}_{H,V}$ is both injective and projective. By the exact sequence in Proposition 6.2, we get

$$\mathrm{ind}^{\Gamma}_{\Delta}\,\mathcal{S}^{\nabla}_{H,V} \cong S^{\Gamma}_{H,V} \cong \mathrm{coind}^{\Gamma}_{\nabla}\,\mathcal{S}^{\Delta}_{H,V}.$$

Now consider any short exact sequence

$$0 \longrightarrow K \longrightarrow F \xrightarrow{\ \phi\ } S^{\Gamma}_{H,V} \longrightarrow 0$$

of biset functors. By the second isomorphism above we get $\phi \in \mathrm{Hom}_{\Gamma}(F, \mathrm{coind}^{\Gamma}_{\nabla}\,\mathcal{S}^{\Delta}_{H,V})$. (Note that since ϕ is a morphism of biset functors, preimage of any $v \in S^{\Gamma}_{H,V}(X)$ is in the kernel of destriction maps to the isomorphism class of H.) By adjointness of coinduction and restriction we get a morphism $\bar{\phi} \in \mathrm{Hom}_\nabla(\mathrm{res}^{\Gamma}_{\nabla} F, \mathcal{S}^{\Delta}_{H,V})$. Since V is projective as an $R\mathrm{Out}(H)$-module, the morphism $\bar{\phi}_H : F(H) \to V$ of $R\mathrm{Out}(H)$-modules splits. That is to saying that V is a direct summand of the $R\mathrm{Out}(H)$-module $F(H)$. Therefore, there is a morphism $\bar{\psi} : \mathcal{S}^{\nabla}_{H,V} \to \mathrm{res}^{\Gamma}_{\Delta} F$ of Δ-modules given by sending V to the direct summand of $F(H)$ which is isomorphic to V. Now by adjointness of induction and restriction the map $\bar{\psi}$ induces a map $\psi \in \mathrm{Hom}_\Gamma(\mathrm{ind}^{\Gamma}_{\Delta} \mathcal{S}^{\nabla}_{H,V}, F)$. Clearly the composition $\phi \circ \psi$ is the mark homomorphism, which is an isomorphism by our assumption. Hence the composition $\psi \circ \beta^{-1}$ is the required splitting for the above exact sequence.

Similarly it can be shown that any short exact sequence of biset functors

$$0 \longrightarrow S^{\Gamma}_{H,V} \xrightarrow{\alpha} F \longrightarrow C \longrightarrow 0$$

splits, that is, $S^{\Gamma}_{H,V}$ is injective.

Conversely, suppose the simple biset functor $S^{\Gamma}_{H,V}$ is both injective and surjective. Then both of the sequences

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathrm{ind}^{\Gamma}_{\Delta} \mathcal{S}^{\nabla}_{H,V} \longrightarrow S^{\Gamma}_{H,V} \longrightarrow 0$$

and

$$0 \longrightarrow S^{\Gamma}_{H,V} \longrightarrow \mathrm{coind}^{\Gamma}_{\nabla} \mathcal{S}^{\Delta}_{H,V} \longrightarrow \mathcal{C} \longrightarrow 0$$

splits. But $\mathrm{ind}^{\Gamma}_{\Delta} \mathcal{S}^{\nabla}_{H,V}$ and $\mathrm{coind}^{\Gamma}_{\nabla} \mathcal{S}^{\Delta}_{H,V}$ are indecomposable. Hence the result follows. □

Now the equivalence of the statements (i) and (iii) of Theorem 6.7 is clear. Incidentally, the approach of the proof of this theorem, applied in the context of Mackey functors, taken together with [2, Proposition 2.4], can be used to give a proof of the following corollary.

Corollary 6.9. *(See [15].) The Mackey algebra $\mu_R(G)$ is semisimple if and only if $|G|$ is invertible in R.*

Indeed in [12], it is shown that the simple Mackey functors can be constructed using the methods of the present paper. In particular, we have a mark morphism associated to each simple Mackey functor $\mathcal{S}^{G}_{H,V}$ where H is a subgroup of G and V is a simple $RN_G(H)/H$-module. Now one can modify the above proof to get the above corollary.

However, the mark morphism for the alchemic algebra for G over a field R of characteristic zero is not an isomorphism, in general. In the following we find necessary and sufficient condition for the mark morphism to be an isomorphism, completing the proof of Theorem 6.7. Note that the result is inspired by the semisimplicity theorem proved by Barker and Bouc, independently.

Theorem 6.10. *Let R be a field of characteristic zero. The mark morphism $\beta^{H,V} : \mathrm{ind}^{\Gamma}_{\nabla} \mathcal{S}^{\nabla}_{H,V} \to \mathrm{coind}^{\Gamma}_{\Delta} \mathcal{S}^{\Delta}_{H,V}$ is an isomorphism for any simple pair (H,V) for G if and only if the group G is cyclic.*

Proof. First, suppose G is not cyclic. We are to show that the mark morphism is not an isomorphism for some pair (H,V). But we have $\mathrm{ind}^{\Gamma}_{\nabla} \mathcal{S}^{\nabla}_{1,1} \cong B^G$ by Lemma 6.4 and it is well known by a theorem of Ritter and Segal that $\mathcal{R}_{\mathbb{Q}} \cong S_{1,1}$ is not isomorphic to B^G. So the mark morphism for the simple pair $(1,1)$ is not an isomorphism.

Conversely, suppose that the group G is cyclic. Then we are to show that the mark morphism is an isomorphism. But this is equivalent to showing that $\mathrm{coind}_{\Delta}^{\Gamma} \mathcal{S}_{H,V}^{\Delta}$ and $\mathrm{ind}_{\nabla}^{\Gamma} \mathcal{S}_{H,V}^{\nabla}$ are simple, that is, to showing that the subfunctor $\mathcal{I}_{\mathrm{coind}_{\Delta}^{\Gamma} \mathcal{S}_{H,V}^{\Delta},H}$ of $\mathrm{coind}_{\Delta}^{\Gamma} \mathcal{S}_{H,V}^{\Delta}$ is not proper and the subfunctor $\mathcal{K}_{\mathrm{ind}_{\nabla}^{\Gamma} \mathcal{S}_{H,V}^{\nabla},H}$ is zero. We shall show the first claim. The second one follows from a similar argument. Without loss of generality, we shall prove the claim for the top coordinate module, that is we shall prove that $\mathrm{coind}_{\Delta}^{\Gamma} \mathcal{S}_{H,V}^{\Delta}(G) = \mathcal{I}_{\mathrm{coind}_{\Delta}^{\Gamma} \mathcal{S}_{H,V}^{\Delta},H}(G)$. For simplicity, write $\mathcal{I}(G)$ for $\mathcal{I}_{\mathrm{coind}_{\Delta}^{\Gamma} \mathcal{S}_{H,V}^{\Delta},H}(G)$. Now any element of $\mathcal{I}(G)$ is a sum of tinflated elements $\mathrm{tin}_I^G v_I$ for some $I \cong H$ and where we write v_I for the generator of the one-dimensional $R\mathrm{Out}(H)$-module $\mathcal{S}_{H,V}^{\Delta}(I)$. By Corollary 5.3, we have

$$\left(\mathrm{tin}_I^G v_I\right)_J = \sum \mathrm{c}_{J,I}^{y\lambda} v_I$$

where the sum is over the representatives of the double cosets $J^*\backslash G/I^*$ such that $I = I \sqcap J^y$ and $J = J \sqcap {}^yI$. But G is cyclic, hence $J^y = J$ for any $y \in G$. Therefore the Jth coordinate $(\mathrm{tin}_I^G v_I)_J$ of $\mathrm{tin}_I^G v_I$ is non-zero only if the equalities $I = I \sqcap J$ and $J = J \sqcap I$ hold. Moreover when the equalities hold we have

$$\left(\mathrm{tin}_I^G v_I\right)_J = \left|J^*\backslash G/I^*\right| \mathrm{c}_{J,I}^{\lambda} v_I.$$

Clearly $\mathrm{c}_{J,I}^{\lambda} v_I = v_J$ and hence

$$\left(\mathrm{tin}_I^G v_I\right)_J = \left|J^*\backslash G/I^*\right| v_J.$$

Finally we have

$$\mathrm{tin}_I^G v_I = \left(\left|J^*\backslash G/I^*\right| v_J\right)_{J \in S_I}$$

where $S_I = \{J \preccurlyeq G\colon\ J \cong H,\ I = I \sqcap J,\ J = J \sqcap I\}$. Now our aim is to show that the set $\mathcal{B} = \{\mathrm{tin}_I^G v_I\colon\ I \preccurlyeq G,\ I \cong H\}$ forms a basis for the coordinate module $\mathrm{coind}_{\Delta}^{\Gamma} \mathcal{S}_{H,V}^{\Delta}(G)$. In order to show this, we first determine the set S_I for any I and then show that the set $\mathcal{B}$ is linearly independent. Since the order of the set $\mathcal{B}$ is equal to the order of the canonical basis of $\mathrm{coind}_{\Delta}^{\Gamma} \mathcal{S}_{H,V}^{\Delta}(G)$, consisting of characteristic functions of subquotients of G, the proof would be established.

Now write $G = C_1 \times C_2 \times \cdots \times C_n$ as a direct product of (cyclic) Sylow subgroups. For simplicity, we shall prove our claim for the case that $H \preccurlyeq C_1$. The general case follows from the same argument.

Let $I \preccurlyeq G$ be a subquotient isomorphic to H. Then it is easy to see that $J \preccurlyeq G$ is in S_I if and only if the equalities $J^* = I^* \times Z$ and $J_* = I_* \times Z$ hold for some $Z \preccurlyeq G$ with $(|J^*|, |Z|) = 1$. Indeed the converse statement is trivial and the first implication follows from the equalities

$$\frac{J^*}{J_*} = \frac{(J^* \cap I^*)J_*}{(J^* \cap I_*)J_*} \quad \text{and} \quad \frac{I^*}{I_*} = \frac{(I^* \cap J^*)I_*}{(I^* \cap J_*)I_*}$$

using the fact that all groups involved are cyclic.

Hence we obtain a description of the set S_I. However we can make it more explicit. Let $J \in S_J$ be the subquotient with J^* has minimal order among all $J' \in S_I$. Then clearly $J \preccurlyeq C_1$ and $S_J = S_I$. Now we can identify the set S_J with the set of subgroups of $G_1 = G/C_1$. Note further that given $L, N \in S_J$ we have the equality

$$\left|L^*\backslash G/N^*\right| = \left|C_1/J^*\right|\left|L_1^*\backslash G_1/N_1^*\right|$$

where $L^* = J^* \times L_1^*$ and $N^* = J^* \times N_1^*$. Indeed we have

$$\left|L^*\backslash G/N^*\right| = \frac{|G|}{|L^*N^*|} = \frac{|G||L^* \cap N^*|}{|L^*||N^*|}.$$

Now both the denominator and the numerator of the last quotient is divisible by the square of the order of J^*. Canceling these, we get the above equality.

Hence it remains to show that the elements of the set $\mathcal{B}$ are linearly independent. It is clear that the sets S_I gives a partition of the set of subquotients of G isomorphic to H. Hence it suffices to show that the elements of the set $\mathcal{B}_J = \{\mathrm{tin}_I^G\, v_I\colon\ I \in S_J\}$ are linearly independent where J is the representative of the set S_J with minimal order of its numerator. But this is clear since for any $I \preccurlyeq J$, we have

$$\mathrm{tin}_I^G\, v_I = \left(\left|C_1/J^*\right|\left|J_1^*\backslash G_1/I_1^*\right| v_I\right)_{I \in S_J}$$

and any linear relation among these elements gives a linear relation between the columns of the double coset matrix $[|L\backslash G_1/N|]_{L,N \leqslant G_1}$ which is well known to be invertible for cyclic groups. Thus we have proved that $\mathrm{coind}_\Delta^\Gamma\, \mathcal{S}_{H,V}^\Delta(G) = \mathcal{I}(G)$ proving that $\mathrm{coind}_\Delta^\Gamma\, \mathcal{S}_{H,V}^\Delta$ is simple, as required. $\square$

Now by Corollary 5.3, we can describe the simple biset functors for a cyclic group G, explicitly.

Corollary 6.11. *Suppose the group G is cyclic. Let $H \preccurlyeq G$ and V be a simple $R\mathrm{Out}(H)$-module. Then*

$$S_{H,V}^\Gamma(K) = \bigoplus_{C \preccurlyeq K,\, \phi: C \cong H} {}^\phi V.$$

The isogation, destriction and tinflation maps are as in the Corollary 5.3.

We end with a generalization of Theorem 6.7 to the finite-dimensional algebras over a field. Let A be a finite-dimensional algebra over a field k. Let e be an idempotent of the algebra A. We denote by A_e the truncated subalgebra eAe.

Given a simple A-module V, it is well known that the restriction eV of V to the subalgebra A_e is either zero or a simple A_e-module. Conversely, given a simple A_e-module W, there exists unique, up to isomorphism, simple A-module $S_{e,W}$ such that $eS_{e,W} \cong W$. Indeed we can construct the simple module $S_{e,W}$ in two ways, as follows. See [13] for further details.

For the first construction, consider the A-module

$$F = \mathrm{ind}_{A_e}^A\, W := Ae \otimes_{A_e} W.$$

It is clear that the A-module F satisfies $eF \cong W$. However the A-module F is not necessarily simple. We let K be the sum of all submodules of F annihilated by the idempotent e. Then it is easy to show that the A-submodule K is the unique maximal submodule of the A-module F and it intersects eF trivially. Therefore the quotient F/K is simple satisfying $e(F/K) \cong W$. We leave it as an easy exercise to show that the simple module F/K is unique, up to isomorphism. Note further that the induced module F is indecomposable. To see this consider the adjunction isomorphism

$$\mathrm{Hom}_A(F, F) \cong \mathrm{Hom}_{A_e}(W, eF) \cong \mathrm{Hom}_{A_e}(W, W)$$

which is indeed an isomorphism of rings.

The second way of constructing the simple modules is to consider the dual construction. Let W be as above. Put

$$E = \mathrm{coind}_{A_e}^{A} W := \mathrm{Hom}_{A_e}(eA, W).$$

Again we have

$$eE = e\,\mathrm{Hom}_{A_e}(A, W) \cong \mathrm{Hom}_{A_e}(eAe, W) \cong W$$

since e is the identity element of A_e and we only consider the A_e-invariant maps. As above, the A-module E is not simple, in general. But it is indecomposable, that is its socle is simple with the property that $e\,\mathrm{soc}(E) \cong W$. Therefore by the uniqueness of the simple module $S_{e,W}$, we have $\mathrm{soc}(E) \cong S_{e,W}$.

Now we are ready to prove our generalization, Theorem 1.4.

Proof of Theorem 1.4. It is clear that (1) implies (2). We shall prove that (2) implies (1). Given a simple A-module V, we have $gV \neq 0$ for some $g \in \{e, f\}$. Then by construction, we have $S_{g,gV} \cong V$. But by the assumption we have

$$S_{g,gV} \cong \mathrm{ind}_{A_g}^{A} gV.$$

Moreover the simple module gV is projective since the algebra A_g is semisimple. Hence the A-module V is projective. Similarly V is also injective. But V was arbitrary, thus A is semisimple. □

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