

Curvature in Special Base Conformal Warped Products

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Abstract We introduce the concept of a base conformal warped product of two pseudo-Riemannian manifolds. We also define a subclass of this structure called as a special base conformal warped product. After, we explicitly mention many of the relevant fields where metrics of these forms and also considerations about their curvature related properties play important rolls. Among others, we cite general relativity, extra-dimension, string and supergravity theories as physical subjects and also the study of the spectrum of Laplace-Beltrami operators on p -forms in global analysis. Then, we give expressions for the Ricci tensor and scalar curvature of a base conformal warped product in terms of Ricci tensors and scalar curvatures of its base and fiber, respectively. Furthermore, we introduce specific identities verified by particular families of, either scalar or tensorial, nonlinear differential operators on pseudo-Riemannian manifolds. The latter allow us to obtain new interesting expressions for the Ricci tensor and scalar curvature of a special base conformal warped product and it turns out that not only the expressions but also the analytical approach used are interesting from the physical, geometrical and analytical point of view. Finally, we analyze, investigate and characterize possible solutions for the conformal and warping factors of a special base conformal warped product, which guarantee that the corresponding product is Einstein. Besides all, we apply these results to a generalization of the Schwarzschild metric.

Keywords Warped products · Conformal metrics · Ricci curvature · Scalar curvature · Laplace-Beltrami operator · Hessian · Semilinear equations · Positive solutions · Kaluza-Klein theory · String theory

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1 Introduction

The main concern of the present paper is so called base conformal warped products (for brevity, we call a product of this class as a *bcwp*) and their interesting curvature related geometric properties. One can consider *bcwp*'s as a generalization of the classical singly warped products. Before we mention physical motivations and applications of *bcwp*'s, we will explicitly define warped products and briefly mention their different types of extensions. This is the first of a series of articles where we deal with the study of curvature questions in *bcwp*'s, the latter also give rise to interesting problems in nonlinear analysis.

Let $B = (B_m, g_B)$ and $F = (F_k, g_F)$ be two pseudo-Riemannian manifolds of dimensions $m \geq 1$ and $k \geq 0$, respectively and also let $B \times F$ be the usual product manifold of B and F . Given a smooth function $w \in C_{>0}^\infty(B) = \{v \in C^\infty(B) : v > 0\}$, the *warped product* $B \times_w F = ((B \times_w F)_{m+k}, g = g_B + w^2 g_F)$ was first defined by Bishop and O'Neill in [21] in order to study manifolds of negative curvature. Moreover, they obtained expressions for the sectional, Ricci and scalar curvatures of a warped product in terms of sectional, Ricci and scalar curvatures of its base and fiber, respectively (see also [15–18, 83] and for other developments about warped products see for instance [28, 33, 41, 81, 111–113]).

From now on, we will use the Einstein summation convention over repeated indices and consider only connected manifolds. Furthermore, we will denote the Laplace-Beltrami operator on (B, g_B) by $\Delta_B(\cdot)$, i.e.,

$$\Delta_B(\cdot) = \nabla^{B^i} \nabla_{B_i}(\cdot) = \frac{1}{\sqrt{|g_B|}} \partial_i (\sqrt{|g_B|} g_B^{ij} \partial_j(\cdot)).$$

Note that Δ_B is elliptic if (B, g_B) is Riemannian and it is hyperbolic when (B, g_B) is Lorentzian. If (B, g_B) is neither Riemannian nor Lorentzian, then the operator is called as ultra-hyperbolic (see [24]).

In [88], Ponge and Reckziegel generalized the notion of warped product to twisted and doubly-twisted products, i.e., a *doubly-twisted product* $B \times_{(\psi_0; \psi_1)} F$ can be defined as the usual product $B \times F$ equipped with the pseudo-Riemannian metric $\psi_0^2 g_B + \psi_1^2 g_F$ where $\psi_0, \psi_1 \in C_{>0}^\infty(B \times F)$. In the case of $\psi_0 \equiv 1$, the corresponding *doubly-twisted product* is called as a *twisted product* by B.-Y. Chen (see [20, 27]). Clearly, if ψ_1 only depends on the points of B , then $B \times_{(1; \psi_1)} F$ becomes a warped product. One can also find other interesting generalizations in [39, 67, 101–103].

We recall that a pseudo-Riemannian manifold (B_m, g_B) is conformal to the pseudo-Riemannian manifold (B_m, \tilde{g}_B) , if and only if there exists $\eta \in C^\infty(B)$ such that $\tilde{g}_B = e^\eta g_B$.

From now on, we will call a doubly twisted product as a *base conformal warped product* when the functions ψ_0 and ψ_1 only depend on the points of B . For a precise definition, see Sect. 3. In this article, we deal with *bcwp*'s, and especially with a subclass called as *special base conformal warped products*, briefly *sbcwp*, which can be thought as a mixed structure of a conformal change in the metric of the base and a warped product, where there is a specific type of relation between the conformal factor and the warping function. Precisely, a *special base conformal warped product* is the usual product manifold $B_m \times F_k$ equipped with pseudo-Riemannian metric of the form $\psi^{2\mu} g_B + \psi^2 g_F$ where $\psi \in C_{>0}^\infty(B)$ and a parameter $\mu \in \mathbb{R}$. In this case, the corresponding *special base conformal warped product* is denoted by (ψ, μ) -*bcwp*. Note that when $\mu = 0$, we have a usual warped product and when $k = 0$ we have a usual conformal change in the base (the fiber is reduced to a point) and if $\mu = 1$ we are in the presence of a conformal change in the metric of a usual product pseudo-Riemannian manifold.

We remark here that a *sbcwp* can be expressed as a special conformal metric in a particular warped product, i.e.

$$\psi_0^2 g_B + \psi_1^2 g_F = \psi_0^2 \left(g_B + \frac{\psi_1^2}{\psi_0^2} g_F \right),$$

where $\psi_0, \psi_1 \in C^\infty_{>0}(B)$.

Metrics of this type have many applications in several topics from the areas of differential geometry, cosmology, relativity, string theory, quantum-gravity, etc. Now, we want to mention some of the major ones.

(i) In the construction of a large class of non trivial static anti de Sitter vacuum spacetimes

- In the Schwarzschild solutions of the Einstein equations

$$ds^2 = - \left(1 - \frac{2M}{r} \right) dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{1.1}$$

(see [6, 18, 59, 83, 96, 99]).

- In the Riemannian Schwarzschild metric, namely

$$(\mathbb{R}^2 \times \mathbf{S}^2, g_{Schw}), \tag{1.2}$$

where

$$g_{Schw} = u^2 d\phi^2 + u^{-2} dr^2 + r^2 g_{S^2(1)} \tag{1.3}$$

and $u^2 = 1 + r^2 - \frac{2m}{r}$, $m > 0$ (see [6]).

- In the “generalized Riemannian anti de Sitter \mathbf{T}^2 black hole metrics” (see §3.2 of [6] for details).

Indeed, let (F_2, g_F) be a pseudo-Riemannian manifold and g be a pseudo-metric on $\mathbb{R}_+ \times \mathbb{R} \times F_2$ defined by

$$g = \frac{1}{u^2(r)} dr^2 \pm u^2(r) dt^2 + r^2 g_F. \tag{1.4}$$

After the change of variables $s = r^2$, $y = \frac{1}{2}t$ and hence $ds^2 = 4r^2 dr^2$ and $dy^2 = \frac{1}{4} dt^2$. Then (1.4) is equivalent to

$$\begin{aligned} g &= \frac{1}{\sqrt{s}} \left[\frac{1}{4\sqrt{s}u^2(\sqrt{s})} ds^2 \pm 4\sqrt{s}u^2(\sqrt{s}) dy^2 \right] + s g_F \\ &= (s^{\frac{1}{2}})^{2(-\frac{1}{2})} \left[(2s^{\frac{1}{4}}u(s^{\frac{1}{2}}))^{2(-1)} ds^2 \pm (2s^{\frac{1}{4}}u(s^{\frac{1}{2}}))^2 dy^2 \right] + (s^{\frac{1}{2}})^2 g_F. \end{aligned} \tag{1.5}$$

Note that roughly speaking, g is a nested application of two (ψ, μ) -bcwp’s. That is, on $\mathbb{R}_+ \times \mathbb{R}$ and taking

$$\psi_1(s) = 2s^{\frac{1}{4}}u(s^{\frac{1}{2}}) \quad \text{and} \quad \mu_1 = -1, \tag{1.6}$$

the metric inside the brackets in the last member of (1.5) is a (ψ_1, μ_1) -bcwp, while the metric g on $(\mathbb{R}_+ \times \mathbb{R}) \times F_2$ is a (ψ_2, μ_2) -bcwp with

$$\psi_2(s, y) = s^{\frac{1}{2}} \quad \text{and} \quad \mu_2 = -\frac{1}{2}. \tag{1.7}$$

- (ii) In the Bañados-Teitelboim-Zanelli (BTZ) and de Sitter (dS) black holes (see [1, 13, 14, 39, 63, 86] for details).
- (iii) In the study of the spectrum of Laplace-Beltrami operator for p -forms. For instance in (1.1) of [7], the author considers the structure that follows: let \overline{M} be an n -dimensional compact, Riemannian manifold with boundary, and let y be a boundary-defining function; she endows the interior M of \overline{M} with a Riemannian metric ds^2 such that in a small tubular neighborhood of ∂M in M , ds^2 takes the form

$$ds^2 = e^{-2(a+1)t} dt^2 + e^{-2bt} d\theta_{\partial M}^2, \quad (1.8)$$

where $t := -\log y \in (c, +\infty)$ and $d\theta_{\partial M}^2$ is the Riemannian metric on ∂M (see [7, 80] and references therein for details).

- (iv) In the Kaluza-Klein theory (see [105, §7.6, Particle Physics and Geometry] and [84]) and in the Randall-Sundrum theory [47, 56, 89–91, 97] with μ as a free parameter. For example in [64] the following metric is considered

$$e^{2A(y)} g_{ij} dx^i dx^j + e^{2B(y)} dy^2, \quad (1.9)$$

with the notation $\{x^i\}$, $i = 0, 1, 2, 3$ for the coordinates in the 4-dimensional space-time and $x^5 = y$ for the fifth coordinate on an extra dimension. In particular, it takes the ansatz

$$B = \alpha A, \quad (1.10)$$

which corresponds exactly to our *sbcwp* metrics, considering $g_B = dy^2$, $g_F = g_{ij} dx^i dx^j$, $\psi(y) = e^{\frac{B(y)}{\alpha}} = e^{A(y)}$ and $\mu = \alpha$.

- (v) In String and Supergravity theories, for instance, in the Maldacena conjecture about the duality between compactifications of M/string theory on various Anti-de Sitter space-times and various conformal field theories (see [78, 79, 87]) and in warped compactifications (see [56, 98] and references therein). Besides these, there are also frequent occurrences of this type of metrics in string topics (see [50–54, 75, 85, 97] and also [1, 8, 86, 93] for some reviews about these topics).
- (vi) In the discussion of Birkhoff-type theorems (generally speaking these are the theorems in which the gravitational vacuum solutions admit more symmetry than the inserted metric ansatz, (see [59, p. 372] and [16, Chap. 3]) for rigorous statements), especially in (6.1) of [92] where, H-J. Schmidt considers a special form of a *bcwp* and basically shows that if a *bcwp* of this form is Einstein, then it admits one Killing vector more than the fiber has. In order to achieve that, the author considers for a specific value of μ , namely $\mu = (1 - k)/2$, the following problem:
- Does there exist a smooth function $\psi \in C_{>0}^\infty(B)$ such that the corresponding (ψ, μ) -bcwp $(B_2 \times F_k, \psi^{2\mu} g_B + \psi^2 g_F)$ is an Einstein manifold?* (see also (Pb-Eins.) below.)
- (vii) In questions of equivariant isometric embeddings (see [55]).
- (viii) In the study of bi-conformal transformations, bi-conformal vector fields and their applications (see [49, Remark in Sect. 7] and [48, Sects. 7 and 8]).

In order to study the curvature of (ψ, μ) -*bcwp*'s we organized the paper as follows:

In Sect. 2, we study a specific type of homogeneous non-linear second order partial differential operator closely related to those with terms including $\|\nabla^B(\cdot)\|_B^2 = g_B(\nabla^B(\cdot), \nabla^B(\cdot))$ and a generalization where the Hessian tensor is involved. Operators with this structure

are frequent in physics, differential geometry and analysis (see [10, 11, 37, 39, 58, 73, 110]).

In Sect. 3, we define precisely the base conformal warped products, compute their covariant derivatives and Riemann curvature tensor, Ricci tensor and scalar curvature.

In Sect. 4, applying the results of Sect. 2 we find a useful formula for the relation among the Ricci tensors (respectively the scalar curvatures) in a (ψ, μ) -bcwp. The principal results of this section are Theorem 4.1, about the Ricci tensor, and the theorem that follows about the scalar curvature.

Theorem 1.1 *Let $B = (B_m, g_B)$ and $F = (F_k, g_F)$ be two pseudo-Riemannian manifolds with dimensions $m \geq 2$ and $k \geq 0$, respectively. Suppose that S_B and S_F denote the scalar curvatures of $B = (B_m, g_B)$ and $F = (F_k, g_F)$, respectively. If $\mu \in \mathbb{R}$ is a parameter and $\psi \in C^\infty_{>0}(B)$ is a smooth function then, the scalar curvature S of the base conformal warped product $(B \times F, g = \psi^{2\mu}g_B + \psi^2g_F)$ verifies,*

(i) *If $\mu \neq -\frac{k}{m-1}$, then*

$$-\beta \Delta_B u + S_B u = S u^{2\mu\alpha+1} - S_F u^{2(\mu-1)\alpha+1} \tag{1.11}$$

where

$$\alpha = \frac{2[k + (m - 1)\mu]}{\{[k + (m - 1)\mu] + (1 - \mu)\}k + (m - 2)\mu[k + (m - 1)\mu]}, \tag{1.12}$$

$$\beta = \alpha 2[k + (m - 1)\mu] > 0 \tag{1.13}$$

and $\psi = u^\alpha > 0$.

(ii) *If $\mu = -\frac{k}{m-1}$, then*

$$-\left[-k^2 \frac{m-2}{m-1} + k(k+1)\right] \frac{|\nabla^B \psi|_B^2}{\psi^2} = \psi^{-2\frac{k}{m-1}} S - S_B - S_F \psi^{-2(\frac{k}{m-1}+1)}. \tag{1.14}$$

For the case of $m = 1$ see Remark 4.5.

The relation among the scalar curvatures in a warped product $B \times_w F$ is given by

$$S = -2k \frac{\Delta_B w}{w} - k(k-1) \frac{g_B(\nabla^B w, \nabla^B w)}{w^2} + S_B + \frac{S_F}{w^2}, \tag{1.15}$$

where Δ_B is the Laplace-Beltrami operator on (B, g_B) and S_B, S_F and S are the scalar curvatures of B, F and $B \times_w F$, respectively.

In the articles [36, 37] the authors transformed equation (1.15) into

$$-\frac{4k}{k+1} \Delta_B u + S_B u + S_F u^{1-\frac{4}{k+1}} = S u, \tag{1.16}$$

where $w = u^{\frac{2}{k+1}}$ and $u \in C^\infty_{>0}(B)$. Note that this result corresponds to the case of $\mu = 0$ in Theorem 1.1.

On the other hand, under a conformal change on the metric of a pseudo-Riemannian manifold $B = (B_m, g_B)$, i.e., $\tilde{g}_B = e^\eta g_B$ with $\eta \in C^\infty(B)$, the scalar curvature \tilde{S}_B associated to the metric \tilde{g}_B is related with the scalar curvature S_B by the equation

$$e^\eta \tilde{S}_B = S_B - (m-1) \Delta_B \eta - (m-1) \frac{m-2}{4} g_B(\nabla^B \eta, \nabla^B \eta). \tag{1.17}$$

When $m \geq 3$, the previous equation becomes

$$-4 \frac{m-1}{m-2} \Delta_B \varphi + S_B \varphi = \tilde{S}_B \varphi^{1+\frac{4}{m-2}}, \quad (1.18)$$

where $\tilde{g}_B = \varphi^{\frac{4}{m-2}} g_B$ and $\varphi \in C_{>0}^\infty(B)$.

There is an extensive number of publications about (1.18) (see [10, 11, 24, 25, 29, 30, 45, 46, 57, 60–62, 68, 69, 72, 94, 95]), especially due to its close relation with the so called Yamabe problem (see the original Yamabe's article [107] and the related questions posed by Trüdinger [100]), namely

(Ya) [107] Does there exist a smooth function $\varphi \in C_{>0}^\infty(B)$ such that $(B, \varphi^{\frac{4}{m-2}} g_B)$ has constant scalar curvature?

Analogously, in several articles the following problem has been studied (see [12, 26, 36–38, 42–44, 74, 108] among others).

(cscwp) Is there a smooth function $w \in C_{>0}^\infty(B)$ such that the warped product $B \times_w F$ (or equivalently $B \times_{(1,w)} F$) has constant scalar curvature?

The Yamabe problem needs the study of the existence of positive solutions of (1.18) with a constant $\lambda \in \mathbb{R}$ instead of \tilde{S}_B . On the other hand, the constant scalar curvature problem in warped products brings to the study of the existence of positive solutions of (1.16) with a parameter $\lambda \in \mathbb{R}$ instead of S .

Inspired by these, we propose a mixed problem between (Ya) and (cscwp), namely:

(Pb-sc) Given $\mu \in \mathbb{R}$, does there exist $\psi \in C_{>0}^\infty(B)$ such that the (ψ, μ) -bcwp $((B \times F)_{m+k}, \psi^{2\mu} g_B + \psi^2 g_F)$ has constant scalar curvature?

Note that when $\mu = 0$, (Pb-sc) corresponds to the problem (cscwp), whereas when the dimension of the fiber $k = 0$ and $\mu = 1$, then (Pb-sc) corresponds to (Ya) for the base manifold. Finally (Pb-sc) corresponds to (Ya) for the usual product metric with a conformal factor in $C_{>0}^\infty(B)$ when $\mu = 1$.

Under the hypothesis of Theorem 1.1(i), the analysis of the problem (Pb-sc) brings to the study of the existence and multiplicity of positive solutions $u \in C_{>0}^\infty(B)$ of

$$-\beta \Delta_B u + S_B u = \lambda u^{2\mu\alpha+1} - S_F u^{2(\mu-1)\alpha+1}, \quad (1.19)$$

where all the components of the equation are like in Theorem 1.1(i) and λ (the conjectured constant scalar curvature of the corresponding *sbcwp*) is a real parameter. We observe that an easy argument of separation of variables, like in [32, Sect. 2] and [37], shows that there exists a positive solution of (1.19) only if the scalar curvature of the fiber S_H is constant. Thus this will be a natural assumption in the study of (Pb-sc).

Furthermore note that the involved nonlinearities in the right hand side of (1.19) dramatically change with the choice of the parameters, the analysis of these changes is the subject of Sect. 5.

By taking into account the above considerations and the scalar curvature results obtained in this article, we will consider the study of (Pb-sc) and in particular, the questions mentioned above which are related to the existence and multiplicity of solution of (1.19) in our forthcoming articles (see [40]). Let us mention here that there are several partial results about semi-linear elliptic equations like (1.19) with different boundary conditions, see for instance [2–5, 26, 31, 34, 35, 106, 109].

In Sect. 6, we study particular problems related to Einstein manifolds. Deep studies about Einstein manifolds can be found in the books [18, 71] and the reviews [23, 68, 69, 109]. Besides, in [18] there is an approach to the existence of Einstein warped products (see also [70]).

Here, we consider suitable conditions that allow us to deal with some particular cases of the problem

(Pb-Eins.) Given $\mu \in \mathbb{R}$, does there exist $\psi \in C^\infty_{>0}(B)$ such that the corresponding (ψ, μ) -bcwp is an Einstein manifold?

More precisely, when B is an interval in \mathbb{R} (eventually \mathbb{R}) we reduce the problem to a single ordinary differential equation that can be solved by applying special functions. We give a more complete description if $B = (B_m, g_B)$ is a compact scalar flat manifold, in particular when $m = 1$. Furthermore we characterize Einstein manifolds with a precise type of metric of 2-dimensional base, generalizing (1.5). The latter result is very close to the work of H.-J. Schmidt in [92].

In the Appendix, we give a group of useful results about the behavior of the Laplace-Beltrami operator under a conformal change in the metric and we present the sketch of an alternative proof of Theorem 1.1 by applying a conformal change metric technique like in [37].

2 Some Families of Differential Operators

Throughout this section, $N = (N_n, h)$ is assumed to be a pseudo-Riemannian manifold of dimension n , $|\nabla(\cdot)|^2 = |\nabla^N(\cdot)|^2_N = h(\nabla^N(\cdot), \nabla^N(\cdot))$ and $\Delta_h = \Delta_N$.

Lemma 2.1 *Let L_h be the differential operator on $C^\infty_{>0}(N)$ defined by*

$$L_h v = \sum r_i \frac{\Delta_h v^{a_i}}{v^{a_i}}, \tag{2.1}$$

where any $r_i, a_i \in \mathbb{R}$, $\zeta := \sum r_i a_i \neq 0$, $\eta := \sum r_i a_i^2 \neq 0$ and the indices extend from 1 to $l \in \mathbb{N}$. Then for $\alpha = \frac{\zeta}{\eta}$ and $\beta = \frac{\zeta^2}{\eta}$ there results

$$L_h v = \beta \frac{\Delta_h v^{\frac{1}{\alpha}}}{v^{\frac{1}{\alpha}}}. \tag{2.2}$$

Proof In general, for a given real value t ,

$$\begin{aligned} \nabla v^t &= t v^{t-1} \nabla v, \\ \Delta_h v^t &= t[(t-1)v^{t-2}|\nabla v|^2 + v^{t-1}\Delta_h v] \quad \text{and} \\ \frac{\Delta_h v^t}{v^t} &= t \left[(t-1) \frac{|\nabla v|^2}{v^2} + \frac{\Delta_h v}{v} \right]. \end{aligned} \tag{2.3}$$

Thus, the right hand side of (2.2)

$$\begin{aligned} \beta \frac{\Delta_h v^{\frac{1}{\alpha}}}{v^{\frac{1}{\alpha}}} &= \beta \frac{1}{\alpha} \left[\left(\frac{1}{\alpha} - 1 \right) \frac{|\nabla v|^2}{v^2} + \frac{\Delta_h v}{v} \right] \\ &= \sum r_i a_i \left[\left(\frac{\sum r_i a_i^2}{\sum r_i a_i} - 1 \right) \frac{|\nabla v|^2}{v^2} + \frac{\Delta_h v}{v} \right] \\ &= \frac{|\nabla v|^2}{v^2} \sum_{i=1}^l r_i a_i (a_i - 1) + \frac{\Delta_h v}{v} \sum_{i=1}^l r_i a_i. \end{aligned}$$

And, again by (2.3), the left hand side of (2.2)

$$L_h v = \frac{|\nabla v|^2}{v^2} \sum_{i=1}^l r_i a_i (a_i - 1) + \frac{\Delta_h v}{v} \sum_{i=1}^l r_i a_i. \tag{2.4}$$

□

Remark 2.2 Note that (2.4) is independent of the hypothesis $\zeta := \sum r_i a_i \neq 0$ and $\eta := \sum r_i a_i^2 \neq 0$, it only depends on the structure of the operator L . Thus, the following expression is always satisfied

$$L_h v = (\eta - \zeta) \frac{|\nabla v|^2}{v^2} + \zeta \frac{\Delta_h v}{v}. \tag{2.5}$$

Corollary 2.3 *Let L_h be a differential operator defined by*

$$L_h v = r_1 \frac{\Delta_h v^{a_1}}{v^{a_1}} + r_2 \frac{\Delta_h v^{a_2}}{v^{a_2}} \quad \text{for } v \in C_{>0}^\infty(N), \tag{2.6}$$

where $r_1 a_1 + r_2 a_2 \neq 0$ and $r_1 a_1^2 + r_2 a_2^2 \neq 0$. Then, by changing the variables $v = u^\alpha$ with $0 < u \in C^\infty(N)$, $\alpha = \frac{r_1 a_1 + r_2 a_2}{r_1 a_1^2 + r_2 a_2^2}$ and $\beta = \frac{(r_1 a_1 + r_2 a_2)^2}{r_1 a_1^2 + r_2 a_2^2} = \alpha(r_1 a_1 + r_2 a_2)$ there results

$$L_h v = \beta \frac{\Delta_h u}{u}. \tag{2.7}$$

Remark 2.4 To the best of our knowledge, the only reference of an application of the identity in the form of (2.7) is an article where J. Lelong-Ferrand completed the solution given in another paper of her about a conjecture of A. Lichnerowicz concerning the conformal group of diffeomorphisms of a compact C^∞ Riemannian manifold, namely if such a manifold has the group of conformal transformations, then the manifold is globally conformal to the standard sphere of the same dimension. Her application corresponds to the values $r_1 = 1/(n - 1)$, $r_2 = -1/(n + 2)$, $a_1 = n - 1$ and $a_2 = n$ (see [73, p. 94 Proposition 2.2]).

Remark 2.5 By the change of variables as in Corollary 2.3 equations of the type

$$L_h v = r_1 \frac{\Delta_h v^{a_1}}{v^{a_1}} + r_2 \frac{\Delta_h v^{a_2}}{v^{a_2}} = H(v, x, s), \tag{2.8}$$

transform into

$$\beta \Delta_h u = u H(u^\alpha, x, s). \tag{2.9}$$

We will apply this argument several times throughout the paper.

Example 2.6 As it was mentioned in Sect. 1, the relation connecting the scalar curvatures of the base and the fiber in a warped product (see [15, 16, 18, 83]) is

$$S = -2k \frac{\Delta_{g_B} w}{w} - k(k-1) \frac{|\nabla^B w|_B^2}{w^2} + S_{g_B} + \frac{S_{g_F}}{w^2}. \tag{2.10}$$

By applying (2.3) with $t = k$ and $h = g_B$, it results the following

$$k \frac{\Delta_{g_B} w}{w} + \frac{\Delta_{g_B} w^k}{w^k} = -S + S_{g_B} + \frac{S_{g_F}}{w^2}. \tag{2.11}$$

Thus, by Remark 2.5 with $\alpha = \frac{2}{k+1}$, $\beta = \frac{4k}{k+1}$ and $w = u^\alpha$, we transform (2.10) into

$$\frac{4k}{k+1} \Delta_{g_B} u = u \left(-S + S_{g_B} + \frac{S_{g_F}}{u^{\frac{4}{k+1}}} \right), \tag{2.12}$$

which is equivalent to (1.16) introduced in [36, 37].

Remark 2.7 We have already mentioned that operators like L_h are present in different fields in Sect. 1. For instance, a similar situation to Example 2.6 can be found in the study of special cases of the Grad-Shafranov equation with a flow in plasma physics, see [58, 110].

Now, we consider H_h^v the Hessian of a function $v \in C^\infty(N)$, so that its second covariant differential $H_h^v = \nabla(\nabla v)$. Recall that it is the symmetric (0, 2) tensor field such that for any X, Y smooth vector fields on N ,

$$H_h^v(X, Y) = XYv - (\nabla_X Y)v = h(\nabla_X(\text{grad } v), Y). \tag{2.13}$$

Hence, for any $v \in C_{>0}^\infty(N)$ and for all $t \in \mathbb{R}$

$$H_h^{v^t} = t[(t-1)v^{t-2}dv \otimes dv + v^{t-1}H_h^v], \tag{2.14}$$

or equivalently

$$\frac{1}{v^t} H_h^{v^t} = t \left[(t-1) \frac{1}{v^2} dv \otimes dv + \frac{1}{v} H_h^v \right], \tag{2.15}$$

where \otimes is the usual tensorial product. Note the analogy of the latter expressions with (2.3) (for deeper information about the Hessian, see p. 86 of [83]).

Thus, by using the same technique applied in the proof of Lemma 2.1 and Remark 2.2, there results

Lemma 2.8 *Let \mathcal{H}_h be a differential operator on $C_{>0}^\infty(N)$ defined by*

$$\mathcal{H}_h v = \sum r_i \frac{H_h^{v^{a_i}}}{v^{a_i}}, \tag{2.16}$$

$\zeta := \sum r_i a_i$ and $\eta := \sum r_i a_i^2$, where the indices extend from 1 to $l \in \mathbb{N}$ and any $r_i, a_i \in \mathbb{R}$. Hence,

$$\mathcal{H}_h v = (\eta - \zeta) \frac{1}{v^2} dv \otimes dv + \zeta \frac{1}{v} H_h^v. \tag{2.17}$$

If furthermore, $\zeta \neq 0$ and $\eta \neq 0$, then

$$\mathcal{H}_h v = \beta \frac{H_h v^{\frac{1}{\alpha}}}{v^{\frac{1}{\alpha}}}, \tag{2.18}$$

where $\alpha = \frac{\zeta}{\eta}$ and $\beta = \frac{\zeta^2}{\eta}$.

3 About Base Conformal Warped Products

In this section, we define precisely *base conformal warped products* and compute covariant derivatives and curvatures of *base conformal warped products*. Several proofs contain standard but long computations, and hence will be omitted.

Let (B, g_B) and (F, g_F) be m and k dimensional *pseudo-Riemannian manifolds*, respectively. Then $M = B \times F$ is an $(m + k)$ -dimensional *pseudo-Riemannian manifold* with $\pi: B \times F \rightarrow B$ and $\sigma: B \times F \rightarrow F$ the usual projection maps.

Throughout this paper we use the *natural product coordinate system* on the *product manifold* $B \times F$, namely. Let (p_0, q_0) be a point in M and *coordinate charts* (U, x) and (V, y) on B and F , respectively such that $p_0 \in B$ and $q_0 \in F$. Then we can define a *coordinate chart* (W, z) on M such that W is an open subset in M contained in $U \times V$, $(p_0, q_0) \in W$ and for all (p, q) in W , $z(p, q) = (x(p), y(q))$, where $x = (x^1, \dots, x^m)$ and $y = (y^{m+1}, \dots, y^{m+k})$.

Clearly, the set of all (W, z) defines an atlas on $B \times F$. Here, for our convenience, we call the j -th *component* of y as y^{m+j} for all $j \in \{1, \dots, k\}$.

Let $\phi: B \rightarrow \mathbb{R} \in C^\infty(B)$ then the lift of ϕ to $B \times F$ is $\tilde{\phi} = \phi \circ \pi \in C^\infty(B \times F)$, where $C^\infty(B)$ is the set of all smooth real-valued functions on B .

Moreover, one can define *lifts of tangent vectors* as: Let $X_p \in T_p(B)$ and $q \in F$ then the lift $\tilde{X}_{(p,q)}$ of X_p is the unique tangent vector in $T_{(p,q)}(B \times \{q\})$ such that $d\pi_{(p,q)}(\tilde{X}_{(p,q)}) = X_p$ and $d\sigma_{(p,q)}(\tilde{X}_{(p,q)}) = 0$. We will denote the set of all lifts of all tangent vectors of B by $L_{(p,q)}(B)$.

Similarly, we can define *lifts of vector fields*. Let $X \in \mathfrak{X}(B)$ then the lift of X to $B \times F$ is the vector field $\tilde{X} \in \mathfrak{X}(B \times F)$ whose value at each (p, q) is the lift of X_p to (p, q) . We will denote the set of all lifts of all vector fields of B by $\mathfrak{L}(B)$.

Definition 3.1 Let (B, g_B) and (F, g_F) be *pseudo-Riemannian manifolds* and also let $w: B \rightarrow (0, \infty)$ and $c: B \rightarrow (0, \infty)$ be smooth functions. The *base conformal warped product* (briefly *bcwp*) is the *product manifold* $B \times F$ furnished with the metric tensor $g = c^2 g_B \oplus w^2 g_F$ defined by

$$g = (c \circ \pi)^2 \pi^*(g_B) \oplus (w \circ \pi)^2 \sigma^*(g_F). \tag{3.1}$$

By analogy with [88] we will denote this structure by $B \times_{(c;w)} F$. The function $w: B \rightarrow (0, \infty)$ is called the *warping function* and the function $c: B \rightarrow (0, \infty)$ is said to be the *conformal factor*.

If $c \equiv 1$ and w is not identically 1, then we obtain a *singly warped product*. If both $w \equiv 1$ and $c \equiv 1$, then we have a *product manifold*. If neither w nor c is constant, then we have a *nontrivial bcwp*.

If (B, g_B) and (F, g_F) are both *Riemannian manifolds*, then $B \times_{(c;w)} F$ is also a *Riemannian manifold*. We call $B \times_{(c;w)} F$ as a *Lorentzian base conformal warped product* if (F, g_F) is *Riemannian* and either (B, g_B) is *Lorentzian* or else (B, g_B) is a one-dimensional manifold with a *negative definite metric* $-dt^2$.

Notation 3.2 From now on, we will identify the operators defined on the base (respectively, fiber) of a *bcwp* with the name of the base (respectively, fiber) as a sub or super index. Unlike, the operators defined on the whole *bcwp* will not have labels. For instance, the Riemann curvature tensor of the base (B, g_B) will be denoted by R_B and likewise R_F denotes for that of the fiber (F, g_F) . Thus, the Riemann curvature tensor of $B \times_{(c;w)} F$ is denoted by R .

3.1 Covariant Derivatives

We state the covariant derivative formulas and the geodesic equation for a base conformal warped product manifold $B \times_{(c;w)} F$.

The gradient operator of smooth functions on $B \times_{(c;w)} F$ is denoted by ∇ and ∇^B and ∇^F denote the gradients of (B, g_B) and (F, g_F) , respectively (see Notation 3.2).

Proposition 3.3 *Let $\phi \in C^\infty(B)$ and $\psi \in C^\infty(F)$. Then*

$$\nabla\phi = \frac{1}{c^2}\nabla^B\phi \quad \text{and} \quad \nabla\psi = \frac{1}{w^2}\nabla^F\psi.$$

Also, we express the *covariant derivative* on $B \times F$ in terms of the *covariant derivatives* on B and F by using the Kozsul formula, which takes the following form on a base conformal warped product as above: Let $X, Y, Z \in \mathcal{L}(B)$ and $V, W, U \in \mathcal{L}(F)$, then

$$\begin{aligned} 2g(\nabla_{X+V}(Y+W), Z+U) &= (X+V)g(Y+W, Z+U) \\ &\quad + (Y+W)g(X+V, Z+U) \\ &\quad - (Z+U)g(X+V, Y+W) \\ &\quad + g([X+V, Y+W], Z+U) \\ &\quad - g([X+V, Z+U], Y+W) \\ &\quad - g([Y+W, Z+U], X+V), \end{aligned}$$

where $[\cdot, \cdot]$ denotes the Lie bracket.

Theorem 3.4 *Let $X, Y \in \mathcal{L}(B)$ and $V, W \in \mathcal{L}(F)$. Then*

- (1) $\nabla_X Y = \nabla_X^B Y + \frac{X(c)}{c}Y + \frac{Y(c)}{c}X - \frac{g_B(X, Y)}{c}\nabla^B c,$
- (2) $\nabla_X V = \nabla_V X = \frac{X(w)}{w}V,$
- (3) $\nabla_V W = \nabla_V^F W - \frac{w}{c^2}g_F(V, W)\nabla^B w.$

Remark 3.5 Let $X, Y \in \mathcal{L}(B)$ and $V, W \in \mathcal{L}(F)$. If $[\cdot, \cdot]$ denotes for the Lie bracket on $B \times_{(c;w)} F$, then $[X, Y] = [X, Y]_B$, $[X, V] = 0$ and $[V, W] = [V, W]_F$.

Proposition 3.6 *Let $(p, q) \in B \times_{(c;w)} F$. Then*

- (1) *The leaf $B \times \{q\}$ and the fiber $\{p\} \times F$ are totally umbilic.*
- (2) *The leaf $B \times \{q\}$ is totally geodesic.*
- (3) *The fiber $\{p\} \times F$ is totally geodesic when $(\nabla^B w)(p) = 0$.*

Now, we will establish the *geodesic equations* for *base conformal warped products*. The version for *singly warped products* is well known (compare p. 207 of [83]).

Proposition 3.7 *Let $\gamma = (\alpha, \beta): I \rightarrow B \times_{(c;w)} F$ be a (smooth) curve where $I \subseteq \mathbb{R}$. Then $\gamma = (\alpha, \beta)$ is a geodesic in $B \times_{(c;w)} F$ if and only if for any $t \in I$,*

$$\begin{aligned} (1) \quad \alpha'' &= -2 \frac{\alpha'(c)}{c} \alpha' + \frac{g_B(\alpha', \alpha')}{c} \nabla^B c + \frac{w g_F(\beta', \beta')}{c^2} \nabla^B w, \\ (2) \quad \beta'' &= -2 \frac{\alpha'(w)}{w} \beta'. \end{aligned}$$

Remark 3.8 If $\gamma = (\alpha, \beta): I \rightarrow B \times_{(c;w)} F$ is a geodesic in $B \times_{(c;w)} F$, then $\beta: I \rightarrow F$ is a pre-geodesic in (F, g_F) .

3.2 Riemannian Curvatures

From now on, we use the definition and the sign convention for the *curvature* as in [16, p. 16–25] (note the difference with [83]), namely. For an arbitrary n -dimensional pseudo-Riemannian manifold (N, h) , letting $X, Y, Z \in \mathcal{L}(N)$, we take the Riemann curvature tensor

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

Furthermore, for each $p \in N$, the Ricci curvature tensor is given by

$$\text{Ric}(X, Y) = \sum_{i=1}^n h(E_i, E_i) h(R(E_i, Y)X, E_i),$$

where $\{E_1, \dots, E_n\}$ is an orthonormal basis for $T_p N$.

Now, we give the Riemannian curvature formulas for a base conformal warped product. But first we state the Hessian tensor denoted by H (see Sect. 2) on this class of warped products.

Proposition 3.9 *Let $X, Y \in \mathcal{L}(B)$ and $V, W \in \mathcal{L}(F)$ and also let $\phi \in C^\infty(B)$ and $\psi \in C^\infty(F)$. Then, the Hessian H of $B \times_{(c;w)} F$ satisfies*

$$\begin{aligned} (1) \quad H^\phi(X, Y) &= H_B^\phi(X, Y) + \frac{g_B(X, Y)}{c} g_B(\nabla^B \phi, \nabla^B c) \\ &\quad - \frac{X(c)Y(\phi)}{c} - \frac{Y(c)X(\phi)}{c}, \\ (2) \quad H^\psi(X, Y) &= 0, \\ (3) \quad H^\phi(X, V) &= 0, \\ (4) \quad H^\psi(X, V) &= -\frac{X(w)V(\psi)}{w}, \\ (5) \quad H^\phi(V, W) &= \frac{w}{c^2} g_F(V, W) g_B(\nabla^B w, \nabla^B \phi), \\ (6) \quad H^\psi(V, W) &= H_F^\psi(V, W). \end{aligned}$$

Theorem 3.10 *Let $X, Y, Z \in \mathcal{L}(B)$ and $V, W, U \in \mathcal{L}(F)$. Then, the curvature Riemann tensor R of $B \times_{(c;w)} F$ satisfies*

- (1)
$$\begin{aligned} R(X, Y)Z &= R_B(X, Y)Z - \frac{H^c(Y, Z)}{c}X + \frac{H^c(X, Z)}{c}Y \\ &\quad + 2\frac{X(c)}{c^2}g_B(Y, Z)\nabla^B c - 2\frac{Y(c)}{c^2}g_B(X, Z)\nabla^B c \\ &\quad + \frac{g_B(X, Z)}{c}\nabla_Y^B \nabla^B c - \frac{g_B(Y, Z)}{c}\nabla_X^B \nabla^B c, \end{aligned}$$
- (2)
$$R(X, V)Y = \frac{H^w(X, Y)}{w}V,$$
- (3)
$$R(X, Y)V = 0,$$
- (4)
$$R(V, W)X = 0,$$
- (5)
$$R(V, X)W = wg_F(V, W)h^w(X),$$
- (6)
$$\begin{aligned} R(V, W)U &= R_F(V, W)U \\ &\quad + \frac{g_B(\nabla^B w, \nabla^B w)}{c^2}(g_F(V, U)W - g_F(W, U)V), \end{aligned}$$

where $h^w(X)$ is given in the remark that follows.

Remark 3.11 Note that $h^w(X) = \nabla_X \nabla w$ and $\nabla w = \frac{1}{c^2} \nabla^B w$. Hence,

$$\begin{aligned} h^w(X) &= -2\frac{X(c)}{c^3} \nabla^B w + \frac{1}{c^2} \left(\nabla_X^B \nabla^B w + \frac{X(c)}{c} \nabla^B w \right. \\ &\quad \left. + \frac{g_B(\nabla^B w, \nabla^B c)}{c} X - \frac{X(w)}{c} \nabla^B c \right). \end{aligned}$$

3.3 Ricci Curvatures

We compute Ricci curvatures of the base conformal warped product applying that if $\{E_1, \dots, E_m\}$ is a g_B -orthonormal frame field on an open set $U \subseteq B$ and $\{\tilde{E}_{m+1}, \dots, \tilde{E}_{m+k}\}$ is a g_F -orthonormal frame field on an open set $V \subseteq F$, then

$$\{c^{-1}E_1, \dots, c^{-1}E_m, w^{-1}\tilde{E}_{m+1}, \dots, w^{-1}\tilde{E}_{m+k}\}$$

is a g -orthonormal frame field on an open set $W \subseteq U \times V \subseteq B \times F$.

Proposition 3.12 *Let $\phi \in C^\infty(B)$ and $\psi \in C^\infty(F)$. Then, the Laplace-Beltrami operator Δ of $B \times_{(c;w)} F$ satisfies*

- (1)
$$\Delta\phi = \frac{\Delta_B \phi}{c^2} + \frac{m-2}{c^3} g_B(\nabla^B \phi, \nabla^B c) + \frac{1}{c^2} \frac{k}{w} g_B(\nabla^B w, \nabla^B \phi),$$
- (2)
$$\Delta\psi = \frac{\Delta_F \psi}{w^2}.$$

Theorem 3.13 *Let $X, Y \in \mathcal{L}(B)$ and $V, W \in \mathcal{L}(F)$. Then, the Ricci tensor Ric of $B \times_{(c;w)} F$ satisfies*

- (1) $\text{Ric}(X, Y) = \text{Ric}_B(X, Y)$

$$\begin{aligned}
 & - (m - 2) \frac{1}{c} H_B^c(X, Y) + 2(m - 2) \frac{1}{c^2} X(c)Y(c) \\
 & - \left[(m - 3) \frac{g_B(\nabla^B c, \nabla^B c)}{c^2} + \frac{\Delta_B c}{c} \right] g_B(X, Y) \\
 & - k \frac{1}{w} H_B^w(X, Y) - k \frac{g_B(\nabla^B w, \nabla^B c)}{wc} g_B(X, Y) \\
 & + k \frac{X(c)}{c} \frac{Y(w)}{w} + k \frac{Y(c)}{c} \frac{X(w)}{w},
 \end{aligned}$$
- (2) $\text{Ric}(X, V) = 0,$
- (3) $\text{Ric}(V, W) = \text{Ric}_F(V, W)$

$$\begin{aligned}
 & - \frac{w^2}{c^2} g_F(V, W) \left[(m - 2) \frac{g_B(\nabla^B w, \nabla^B c)}{wc} + \frac{\Delta_B w}{w} \right. \\
 & \left. + (k - 1) \frac{g_B(\nabla^B w, \nabla^B w)}{w^2} \right].
 \end{aligned}$$

An equivalent formulation of Theorem 3.13 is

Theorem 3.14 *The Ricci tensor Ric of $B \times_{(c;w)} F$ satisfies*

- (1) $\text{Ric} = \text{Ric}_B - \left[(m - 2) \frac{1}{c} H_B^c + k \frac{1}{w} H_B^w \right]$

$$\begin{aligned}
 & + 2(m - 2) \frac{1}{c^2} dc \otimes dc + k \frac{1}{wc} [dc \otimes dw + dw \otimes dc] \\
 & - \left[(m - 3) \frac{g_B(\nabla^B c, \nabla^B c)}{c^2} + \frac{\Delta_B c}{c} + k \frac{g_B(\nabla^B w, \nabla^B c)}{wc} \right] g_B
 \end{aligned}$$

on $\mathcal{L}(B) \times \mathcal{L}(B),$
- (2) $\text{Ric} = 0$ on $\mathcal{L}(B) \times \mathcal{L}(F),$
- (3) $\text{Ric} = \text{Ric}_F - \frac{w^2}{c^2} \left[(m - 2) \frac{g_B(\nabla^B w, \nabla^B c)}{wc} + \frac{\Delta_B w}{w} \right.$

$$\left. + (k - 1) \frac{g_B(\nabla^B w, \nabla^B w)}{w^2} \right] g_F$$
 on $\mathcal{L}(F) \times \mathcal{L}(F).$

Remark 3.15 If $m \neq 2$ and $k \neq 1$, applying (2.3), the expression of the Ricci tensor of $B \times_{(c;w)} F$ in Theorem 3.13 may be written as

- (1) $\text{Ric}(X, Y) = \text{Ric}_B(X, Y)$

$$- (m - 2) \frac{1}{c} H_B^c(X, Y) + 2(m - 2) \frac{1}{c^2} X(c)Y(c)$$

$$\begin{aligned}
 & -\frac{1}{m-2} \frac{\Delta_B c^{m-2}}{c^{m-2}} g_B(X, Y) \\
 & -k \frac{1}{w} H_B^w(X, Y) - k \frac{1}{wc} g_B(\nabla^B w, \nabla^B c) g_B(X, Y) \\
 & + k \frac{X(c)}{c} \frac{Y(w)}{w} + k \frac{Y(c)}{c} \frac{X(w)}{w},
 \end{aligned}$$

(2) $\text{Ric}(X, V) = 0,$

(3) $\text{Ric}(V, W) = \text{Ric}_F(V, W)$

$$-\frac{w^2}{c^2} g_F(V, W) \left[(m-2) \frac{1}{wc} g_B(\nabla^B w, \nabla^B c) + \frac{1}{k} \frac{\Delta_B w^k}{w^k} \right].$$

3.4 Scalar Curvature

By using the orthonormal frame introduced above, one can obtain the following result after a standard computation.

Theorem 3.16 *The scalar curvature S of $B \times_{(c;w)} F$ is given by*

$$\begin{aligned}
 c^2 S &= S_B + S_F \frac{c^2}{w^2} - 2(m-1) \frac{\Delta_B c}{c} - 2k \frac{\Delta_B w}{w} \\
 & - (m-4)(m-1) \frac{g_B(\nabla^B c, \nabla^B c)}{c^2} \\
 & - 2k(m-2) \frac{g_B(\nabla^B w, \nabla^B c)}{wc} \\
 & - k(k-1) \frac{g_B(\nabla^B w, \nabla^B w)}{w^2}.
 \end{aligned}$$

4 Curvature of $((B \times F)_{m+k}, \psi^{2\mu} g_B + \psi^2 g_F)$

From now on, we will deal with (ψ, μ) -bcwp's, i.e. $B \times_{(\psi^\mu; \psi)} F$, and specifically concentrate on its Ricci tensor and scalar curvature.

4.1 Ricci Tensor

Theorem 4.1 *Let $B = (B_m, g_B)$ and $F = (F_k, g_F)$ be two pseudo-Riemannian manifolds with dimensions $m \geq 3$ and $k \geq 1$, $\mu \in \mathbb{R} \setminus \{0, 1, \bar{\mu}, \bar{\mu}_\pm\}$ with $\bar{\mu} := -\frac{k}{m-2}$ and $\bar{\mu}_\pm := \bar{\mu} \pm \sqrt{\bar{\mu}^2 - \bar{\mu}}$ and $\psi \in C^\infty_{>0}(B)$. Then, the Ricci curvature tensor Ric of the base conformal warped product $B \times_{(\psi^\mu; \psi)} F$ verifies the relation*

$$\begin{aligned}
 \text{Ric} &= \text{Ric}_B + \beta^H \frac{1}{\psi^{\frac{1}{\alpha^H}}} H_B^{\psi^{\frac{1}{\alpha^H}}} - \beta^\Delta \frac{1}{\psi^{\frac{1}{\alpha^\Delta}}} \Delta_B \psi^{\frac{1}{\alpha^\Delta}} g_B \quad \text{on } \mathcal{L}(B) \times \mathcal{L}(B), \\
 \text{Ric} &= 0 \quad \text{on } \mathcal{L}(B) \times \mathcal{L}(F), \\
 \text{Ric} &= \text{Ric}_F - \frac{1}{\psi^{2(\mu-1)}} \frac{\beta^\Delta}{\mu} \frac{1}{\psi^{\frac{1}{\alpha^\Delta}}} \Delta_B \psi^{\frac{1}{\alpha^\Delta}} g_F \quad \text{on } \mathcal{L}(F) \times \mathcal{L}(F),
 \end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
 \alpha^\Delta &= \frac{1}{(m-2)\mu+k}, \\
 \beta^\Delta &= \frac{\mu}{(m-2)\mu+k}, \\
 \alpha^H &= \frac{-[(m-2)\mu+k]}{\mu[(m-2)\mu+k]+k(\mu-1)}, \\
 \beta^H &= \frac{[(m-2)\mu+k]^2}{\mu[(m-2)\mu+k]+k(\mu-1)}.
 \end{aligned}
 \tag{4.2}$$

Proof Applying Theorem 3.14 with $c = \psi^\mu$ and $w = \psi$, we obtain

$$\begin{aligned}
 \text{Ric} &= \text{Ric}_B - \left[(m-2) \frac{1}{\psi^\mu} H_B^{\psi^\mu} + k \frac{1}{\psi} H_B^\psi \right] \\
 &\quad + 2\mu[(m-2)\mu+k] \frac{1}{\psi^2} d\psi \otimes d\psi \\
 &\quad - \left[((m-3)\mu^2 + k\mu) \frac{g_B(\nabla^B \psi, \nabla^B \psi)}{\psi^2} + \frac{\Delta_B \psi^\mu}{\psi^\mu} \right] g_B \quad \text{on } \mathcal{L}(B) \times \mathcal{L}(B),
 \end{aligned}
 \tag{4.3}$$

$$\text{Ric} = 0 \quad \text{on } \mathcal{L}(B) \times \mathcal{L}(F),$$

$$\begin{aligned}
 \text{Ric} &= \text{Ric}_F - \frac{1}{\psi^{2(\mu-1)}} \left[((m-2)\mu+k-1) \frac{g_B(\nabla^B \psi, \nabla^B \psi)}{\psi^2} + \frac{\Delta_B \psi}{\psi} \right] g_F \\
 &\quad \text{on } \mathcal{L}(F) \times \mathcal{L}(F).
 \end{aligned}$$

So by (2.15) and (2.3), with $t = \mu \neq 0, 1$, there results

$$\begin{aligned}
 \text{Ric} &= \text{Ric}_B + \left[r_1^H \frac{1}{\psi^\mu} H_B^{\psi^\mu} + r_2^H \frac{1}{\psi} H_B^\psi \right] \\
 &\quad - \left[r_1^\Delta \frac{\Delta_B \psi^\mu}{\psi^\mu} + r_2^\Delta \frac{\Delta_B \psi}{\psi} \right] g_B \quad \text{on } \mathcal{L}(B) \times \mathcal{L}(B), \\
 \text{Ric} &= 0 \quad \text{on } \mathcal{L}(B) \times \mathcal{L}(F),
 \end{aligned}
 \tag{4.4}$$

$$\begin{aligned}
 \text{Ric} &= \text{Ric}_F - \frac{1}{\psi^{2(\mu-1)}} \left[((m-2)\mu+k-1) \frac{g_B(\nabla^B \psi, \nabla^B \psi)}{\psi^2} + \frac{\Delta_B \psi}{\psi} \right] g_F \\
 &\quad \text{on } \mathcal{L}(F) \times \mathcal{L}(F),
 \end{aligned}$$

where

$$\begin{aligned}
 (\mu-1)r_1^H &= (m-2)\mu+m-2+2k, \\
 (\mu-1)r_2^H &= -(m-2)2\mu^2-k(3\mu-1), \\
 (\mu-1)r_1^\Delta &= (m-2)\mu+k-1, \\
 (\mu-1)r_2^\Delta &= -\mu((m-2)\mu+k-\mu).
 \end{aligned}$$

Hence, using the notation introduced in Lemmas 2.8 and 2.1 and Remark 2.2,

$$\begin{aligned}
 \text{Ric} &= \text{Ric}_B + (\eta^H - \zeta^H) \frac{1}{\psi^2} d\psi \otimes d\psi + \zeta^H \frac{1}{\psi} \mathbf{H}_B^\psi \\
 &\quad - \left[(\eta^\Delta - \zeta^\Delta) \frac{g_B(\nabla^B \psi, \nabla^B \psi)}{\psi^2} + \zeta^\Delta \frac{\Delta_B \psi}{\psi} \right] g_B \quad \text{on } \mathcal{L}(B) \times \mathcal{L}(B), \\
 \text{Ric} &= 0 \quad \text{on } \mathcal{L}(B) \times \mathcal{L}(F), \\
 \text{Ric} &= \text{Ric}_F - \frac{1}{\psi^{2(\mu-1)}} \left[\left(\frac{\eta^\Delta}{\mu} - \frac{\zeta^\Delta}{\mu} \right) \frac{g_B(\nabla^B \psi, \nabla^B \psi)}{\psi^2} + \frac{\zeta^\Delta}{\mu} \frac{\Delta_B \psi}{\psi} \right] g_F \\
 &\quad \text{on } \mathcal{L}(F) \times \mathcal{L}(F),
 \end{aligned} \tag{4.5}$$

where

$$\begin{aligned}
 \zeta^H &= r_1^H \mu + r_2^H = -(m-2)\mu + k, \\
 \eta^H &= r_1^H \mu^2 + r_2^H = \mu[(m-2)\mu + k] + k(\mu-1), \\
 \zeta^\Delta &= r_1^\Delta \mu + r_2^\Delta = \mu, \\
 \eta^\Delta &= r_1^\Delta \mu^2 + r_2^\Delta = \mu[(m-2)\mu + k].
 \end{aligned} \tag{4.6}$$

Note that

$$\begin{aligned}
 \zeta^H = 0 &\iff \mu = \bar{\mu} := -\frac{k}{m-2}, \\
 \eta^H = 0 &\iff \mu = \bar{\mu}_\pm := \bar{\mu} \pm \sqrt{\bar{\mu}^2 - \bar{\mu}}, \\
 \zeta^\Delta = 0 &\iff \mu = 0, \\
 \eta^\Delta = 0 &\iff \mu = 0, -\frac{k}{m-2}.
 \end{aligned} \tag{4.7}$$

So, if $\mu \in \mathbb{R} \setminus \{0, 1, \bar{\mu}, \bar{\mu}_\pm\}$ and considering

$$\begin{aligned}
 \alpha^\Delta &= \frac{\zeta^\Delta}{\eta^\Delta}, \\
 \beta^\Delta &= \frac{(\zeta^\Delta)^2}{\eta^\Delta}, \\
 \alpha^H &= \frac{\zeta^H}{\eta^H}, \\
 \beta^H &= \frac{(\zeta^H)^2}{\eta^H},
 \end{aligned} \tag{4.8}$$

along with Lemmas 2.8 and 2.1 results the thesis. □

Remark 4.2 We will make some comments about the previous results and compare the above formulas with Ricci tensor formulas in the case of a conformal manifold and a warped product.

- (i) Note that the system (4.3) remains valid without conditions on $\mu, m \geq 1$ and $k \geq 0$.

Table 1 Einstein equations, $m \geq 3$, μ -exceptional cases in Theorem 4.1

μ	m	k	ζ^H	η^H	ζ^Δ	η^Δ	Genuine system	Formal system
0	≥ 3	≥ 1	$-k$	$-k$	0	0	(4.3)	(4.5)
1	≥ 3	≥ 1	$-[m - 2 + k]$	$m - 2 + k$	1	$m - 2 + k$	(4.3)	(4.5)
$\bar{\mu}$	≥ 3	≥ 1	0	$k(\bar{\mu} - 1)$	$\bar{\mu}$	0	(4.5)	-
$\bar{\mu}_\pm$	≥ 3	≥ 1	$k \frac{\bar{\mu}_\pm - 1}{\bar{\mu}_\pm}$	0	$\bar{\mu}_\pm$	$-k(\bar{\mu}_\pm - 1)$	(4.5)	-

- (ii) The system (4.3) with $\mu = 1, m \geq 1$ and $k = 0$ give the expression of the Ricci tensor under a conformal change in the base given by $\tilde{g}_B = \psi^2 g_B$, where $\psi \in C^\infty_{>0}(B)$ (see [11, 18]).
- (iii) For $\mu = 0, m \geq 1$ and $k \geq 1$ the system (4.3) reproduces the expressions of the Ricci tensor for a singly warped product [16, 18, 83].

The Table 1 is a synthesis of the μ -exceptional cases in the Theorem 4.1. In that table $\zeta^H, \eta^H, \zeta^\Delta$ and η^Δ are computed with the final expressions of (4.6). This is the reason to include the column titled “formal system”, and hence the systems written in that column are justified a posteriori.

Remark 4.3 Here, we consider the cases $m = 1$ and $m = 2$, with $k \geq 1$. The results and the proof are essentially the same as Theorem 4.1, but the conditions (4.7) take the following form.

$m = 1$:

$$\begin{aligned}
 \zeta^H = 0 &\iff \mu = k, \\
 \eta^H = 0 &\iff \mu = \bar{\mu}_\pm := k \mp \sqrt{k^2 - k}, \\
 \zeta^\Delta = 0 &\iff \mu = 0, \\
 \eta^\Delta = 0 &\iff \mu = 0, k.
 \end{aligned}
 \tag{4.9}$$

Thus the μ -exceptional cases are 0, 1, $k, \bar{\mu}_\pm$ (compare with [64]).

$m = 2$: Note that $k \geq 1$

$$\begin{aligned}
 \zeta^H = 0 &\text{ never,} \\
 \eta^H = 0 &\iff \mu = \frac{1}{2}, \\
 \zeta^\Delta = 0 &\iff \mu = 0, \\
 \eta^\Delta = 0 &\iff \mu = 0.
 \end{aligned}
 \tag{4.10}$$

Thus the μ -exceptional cases are 0, 1, $\frac{1}{2}$.

Hence like for Table 1, we can establish the Table 2.

Table 2 Einstein equations, $m = 1, 2$, μ -exceptional cases in Theorem 4.1

μ	m	k	ζ^H	η^H	ζ^Δ	η^Δ	Genuine system	Formal system
0	1	≥ 1	$-k$	$-k$	0	0	(4.3)	(4.5)
1	1	≥ 1	$-[-1+k]$	$-1+k$	1	$-1+k$	(4.3)	(4.5)
k	1	> 1	0	$k(k-1)$	k	0	(4.5)	-
$\bar{\mu}_\pm$	1	> 1	$k\frac{\bar{\mu}_\pm-1}{\bar{\mu}_\pm}$	0	$\bar{\mu}_\pm$	$-k(\bar{\mu}_\pm-1)$	(4.5)	-
0	2	≥ 1	$-k$	$-k$	0	0	(4.3)	(4.5)
1	2	≥ 1	$-k$	k	1	k	(4.3)	(4.5)
$\frac{1}{2}$	2	≥ 1	$-k$	0	$\frac{1}{2}$	$\frac{k}{2}$	(4.5)	-

4.2 Scalar Curvature

Theorem 4.4 Let $B = (B_m, g_B)$ and $F = (F_k, g_F)$ be two pseudo-Riemannian manifolds with $m \geq 2$ and $k \geq 1$, $\mu \in \mathbb{R} \setminus \{0, 1, -\frac{k}{m-1}\}$ and $\psi \in C^\infty_{>0}(B)$. Then, the scalar curvature S of the base conformal warped product $B \times_{(\psi^\mu; \psi)} F$ verifies the relation

$$-\beta \Delta_B u + S_B u = S u^{2\mu\alpha+1} - S_F u^{2(\mu-1)\alpha+1}, \tag{4.11}$$

where

$$\alpha = \frac{2[k + (m-1)\mu]}{\{[k + (m-1)\mu] + (1-\mu)k + (m-2)\mu[k + (m-1)\mu]\}}, \tag{4.12}$$

$$\beta = \alpha 2[k + (m-1)\mu] \tag{4.13}$$

and $\psi = u^\alpha > 0$.

Proof Applying Theorem 3.16 with $c = \psi^\mu$ and $w = \psi$, we obtain

$$\begin{aligned} \psi^{2\mu} S &= S_B + S_F \psi^{2(\mu-1)} - \left[2(m-1) \frac{\Delta_B \psi^\mu}{\psi^\mu} + 2k \frac{\Delta_B \psi}{\psi} \right] \\ &\quad - [(m-4)(m-1)\mu^2 + 2k(m-2)\mu + k(k-1)] \frac{g_B(\nabla^B \psi, \nabla^B \psi)}{\psi^2}. \end{aligned} \tag{4.14}$$

So by (2.3), with $t = \mu \neq 0, 1$, there results

$$\begin{aligned} \psi^{2\mu} S &= S_B + S_F \psi^{2(\mu-1)} \\ &\quad - \left\{ \left[2(m-1) + \frac{S}{\mu(\mu-1)} \right] \frac{\Delta_B \psi^\mu}{\psi^\mu} + \left[2k - \frac{S}{\mu-1} \right] \frac{\Delta_B \psi}{\psi} \right\}, \end{aligned}$$

where $\zeta = (m - 4)(m - 1)\mu^2 + 2k(m - 2)\mu + k(k - 1)$. Hence, by Lemma 2.1 and Remark 2.2 with

$$r_1 = 2(m - 1) + \frac{\zeta}{\mu(\mu - 1)},$$

$$r_2 = 2k - \frac{\zeta}{\mu - 1}$$

and

$$\begin{aligned} \zeta &= r_1\mu + r_2 = 2[k + (m - 1)\mu], \\ \eta &= r_1\mu^2 + r_2 = \{[k + (m - 1)\mu] + (1 - \mu)\}k + (m - 2)\mu[k + (m - 1)\mu] \\ &= \left\{ \frac{\zeta}{2} + (1 - \mu) \right\}k + (m - 2)\mu \frac{\zeta}{2}, \end{aligned} \tag{4.15}$$

we find

$$\begin{aligned} \psi^{2\mu} S &= S_B + S_F \psi^{2(\mu-1)} \\ &\quad - \left[(\eta - \zeta) \frac{g_B(\nabla^B \psi, \nabla^B \psi)}{\psi^2} + \zeta \frac{\Delta_B \psi}{\psi} \right]. \end{aligned} \tag{4.16}$$

Notice that (see also (A.18) in the Appendix)

$$\eta = (m - 1)(m - 2)\mu^2 + 2(m - 2)k\mu + (k + 1)k > 0 \quad \text{for all } \mu \in \mathbb{R}. \tag{4.17}$$

On the other hand, $\zeta = 0$ if and only if $\mu = -\frac{k}{m-1}$. Then the thesis follows by Lemma 2.1 and taking

$$\alpha = \frac{\zeta}{\eta} \quad \text{and} \quad \beta = \alpha\zeta. \tag{4.18}$$

□

The Table 3 is a synthesis of the cases not included in the Theorem 4.4. In that table ζ and η are computed with the expressions (4.15) instead of the originals in Remark 2.2. As above, this is the reason to include the column titled “formal equation”, and hence the equations written in that column are justified a posteriori.

All the other cases are covered in Theorem 4.4.

Remark 4.5 We want to make some comments about the results in the Table 3 where we have three important cases:

($\mu = 0$): As it was mentioned in Sect. 1, this case corresponds exactly to standard warped products. The relation (4.11) is well defined and reproduced in (1.16).

($\mu = 1, k = 0, m \geq 3$): This situation corresponds to a conformal change in the base. Again (4.11) is well defined and now reproduces (A.11) with $r = 2$, and hence (1.18) too.

($\mu = 1, k, m \geq 1, k + m \geq 3$): (i.e., rows 5 or 8) We have a conformal change in the usual product, more explicitly, $(B \times F, g = \psi^2(g_B + g_F))$. In this case (4.11) is well defined also, and reproduce with $\alpha = \frac{2}{m+k-2}$ and $\beta = 4\frac{m+k-1}{m+k-2}$, the equation

$$-4\frac{m+k-1}{m+k-2} \Delta_{g_B} u + (S_{g_B} + S_{g_F})u = Su^{1+\frac{4}{m+k-2}}, \tag{4.19}$$

where $g = u^{\frac{4}{m+k-2}}(g_B + g_F)$, $u \in C_{>0}^\infty(B)$, $\psi = u^{\frac{2}{m+k-2}}$ and $c_{m+k} = \beta$.

Table 3 Scalar curvature equation, μ -exceptional cases in Theorem 4.4

μ	m	k	ζ	η	Genuine equation	Formal equation	Equivalent equation	Geometrical meaning
0	≥ 1	0	0	0	(4.14)	(4.16)	$S = S_B$	-
0	≥ 1	≥ 1	$2k$	$(k+1)k$	(4.14) = (1.15)	(4.11)	(1.16)	singly warped
1	1	0	0	0	(4.14)	(4.16)	$S \equiv 0$	-
1	1	1	2	0	(4.14)	(4.16)	(A.14), $r = 2$	conformal product
1	1	≥ 2	$2k$	$k(k-1)$	(4.14)	(4.16)	(4.11) = (4.19)	conformal product
1	2	0	2	0	(4.14)	(4.16)	(A.14), $r = 2$	base conformal Nirenberg pb. type
1	≥ 3	0	$2(m-1)$	$(m-1)(m-2)$	(4.14)	(4.16)	(A.11), $r = 2$	base conformal Yamabe eq. type
1	≥ 2	≥ 1	$2[k+m-1]$	$\frac{\zeta}{2}(\frac{\zeta}{2}-1)$	(4.14)	(4.16)	(4.11) = (4.19)	conformal product
$-\frac{k}{m-1}$	≥ 2	≥ 1	0	> 0	(4.16)	-	(4.22)	-
$\neq \frac{k+1}{2}, 0, 1$	1	≥ 1	$2k$	$k(k+1-2\mu)$	(4.11)	-	-	-
$\frac{k+1}{2}$	1	> 1	$2k$	0	(4.16)	-	-	-

Now we will analyze the cases included neither in the previous items nor in Theorem 4.4.

($m = 1$): Let $k \geq 1$. It is clear that the involved differential equations are ordinary and $S_B \equiv 0$. If

- ($\mu \neq 0, 1, \frac{k+1}{2}$) By the same proof of Theorem 4.4, (4.11) is valid.
- ($\mu = 1, k \geq 2$) It is a particular case of the above item ($\mu = 1, k, m \geq 1, k + m \geq 3$), so (4.11) is true again.
- ($\mu = \frac{k+1}{2}, k \neq 1$) It is possible to apply (4.16) so

$$\psi^{k+1} S = 2k \left(-\frac{\Delta_B \psi}{\psi} + \frac{|\nabla^B \psi|_B^2}{\psi^2} \right) + S_F \psi^{k-1}. \tag{4.20}$$

- ($\mu = \frac{k+1}{2}, k = 1$) Clearly $\mu = 1$, hence (4.14) results by applying (A.14) with $r = 2$, i.e.

$$\psi^2 S = 2 \left(-\frac{\Delta_B \psi}{\psi} + \frac{|\nabla^B \psi|_B^2}{\psi^2} \right). \tag{4.21}$$

Confront with the precedent case.

($m \geq 2, \mu = -\frac{k}{m-1}$): In this case by (4.16) the relation among the scalar curvatures is

$$-k \left[1 + \frac{k}{m-1} \right] \frac{|\nabla^B \psi|_B^2}{\psi^2} = \psi^{-2\frac{k}{m-1}} S - S_B - S_F \psi^{-2(1+\frac{k}{m-1})}. \tag{4.22}$$

Remark 4.6 Note that $\beta > 0$ in Theorem 4.4, while this is not always true if $m = 1$.

Proof of Theorem 1.1 It is an immediate consequence of the above results of this section. \square

5 The Nonlinearities in a (ψ, μ) -bcwp Scalar Curvature Relations

In this section, we will mainly consider some general properties of the nonlinear partial differential equation in (4.11), regarding especially the type of nonlinearities. The main aim of this study is to deal with the question of existence and multiplicity of solutions for problem (Pb-sc). The corresponding results will be presented in forthcoming articles (see [40]).

From now on, we will denote by $discr(\cdot)$, the discriminant of a quadratic polynomial in one variable.

5.1 Base B_m with Dimension $m \geq 2$

Remark 5.1 Under the hypothesis of Theorem 4.4. In order to classify the type of non-linearities involved in (4.11), we will analyze the exponents as a function of the parameter μ and the dimensions of the base $m \geq 2$ and of the fiber $k \geq 1$ (see Table 4 below).

Note that by (4.17), $\alpha > 0$ if and only if $\mu > -\frac{k}{m-1}$ and by the hypothesis $\mu \neq -\frac{k}{m-1}$ in Theorem 4.4, results $\alpha \neq 0$.

We now introduce the following notation:

$$\begin{aligned}
 p &= p(m, k, \mu) = 2\mu\alpha + 1 \quad \text{and} \\
 q &= q(m, k, \mu) = 2(\mu - 1)\alpha + 1 = p - 2\alpha,
 \end{aligned}
 \tag{5.1}$$

where α is defined by (4.12).

Thus, for all m, k, μ as above, $p > 0$. Indeed, by (4.17), $p > 0$ if and only if $\varpi > 0$, where

$$\begin{aligned}
 \varpi &:= \varpi(m, k, \mu) \\
 &:= 4\mu[k + (m - 1)\mu] + (m - 1)(m - 2)\mu^2 + 2(m - 2)k\mu + (k + 1)k \\
 &= (m - 1)(m + 2)\mu^2 + 2mk\mu + (k + 1)k.
 \end{aligned}$$

But $\text{discr}(\varpi) \leq -4km^2 \leq -16$ and $m > 1$, so $\varpi > 0$.

Unlike p, q changes sign depending on m and k . Furthermore, it is important to determine the position of p and q with respect to 1 as a function of m and k . In order to do that, we define

$$D := \{(m, k) \in \mathbb{N}_{\geq 2} \times \mathbb{N}_{\geq 1} : \text{discr}(\varrho(m, k, \cdot)) < 0\},
 \tag{5.2}$$

where $\mathbb{N}_{\geq l} := \{j \in \mathbb{N} : j \geq l\}$,

$$\begin{aligned}
 \varrho &:= \varrho(m, k, \mu) \\
 &:= 4(\mu - 1)[k + (m - 1)\mu] + (m - 1)(m - 2)\mu^2 + 2(m - 2)k\mu + (k + 1)k \\
 &= (m - 1)(m + 2)\mu^2 + 2(mk - 2(m - 1))\mu + (k - 3)k
 \end{aligned}$$

and the discriminant of $\varrho(m, k, \cdot)$ is

$$\text{discr}(\varrho(m, k, \cdot)) = -4((m - 2)k - 4(m - 1))(k + m - 1).$$

Note that by (4.17), $q > 0$ if and only if $\varrho > 0$. Furthermore $q = 0$ if and only if $\varrho = 0$. But here $\text{discr}(\varrho(m, k, \cdot))$ changes its sign as a function of m and k .

In Table 4 below, we denote $\mathcal{CD} = (\mathbb{N}_{\geq 2} \times \mathbb{N}_{\geq 1}) \setminus D$ if $D \subseteq \mathbb{N}_{\geq 2} \times \mathbb{N}_{\geq 1}$ and $\mathcal{CI} = \mathbb{R} \setminus I$ if $I \subseteq \mathbb{R}$. If $(m, k) \in \mathcal{CD}$, let μ_- and μ_+ the two (eventually one, see Remark 5.3 below) roots of $q, \mu_- \leq \mu_+$. Besides, if $\text{discr}(\varrho(m, k, \cdot)) > 0$, then $\mu_- < 0$; unlike μ_+ can take any sign.

We remark that all the rows in Table 4 are *nonempty*, this means that the conditions established in each row are verified for a suitable choice of the parameters and manifolds. On the other hand, we observe that β is always positive as it was mentioned in Remark 4.6. Note that for any row in Table 4, the corresponding type of nonlinearity suggested by the exponents is modified by the scalar curvature of the fiber, S_F and by the function S .

Furthermore, depending on whether the base is Riemannian or not, then the linear part is elliptic or not, respectively.

Notation 5.2 In the last right hand side columns of Tables 4, 5, 6, 7 and 8, we will use the notation explained below:

- *super-lin* means that the corresponding exponent > 1 , roughly speaking *super-linear*
- *sub-lin* means that the corresponding exponent > 0 and < 1 , roughly speaking *sub-linear*
- *non-hom* means that the corresponding exponent $= 0$, roughly speaking *non-homogeneous*

Table 4 Nonlinearities in scalar curvature equation type (4.11) for $m \geq 2$, see Notation 5.2

$(m, k) \in$	$\mu \in$	α	p, q	Type of p, q non-linearity
$\mathbb{N}_{\geq 2} \times \mathbb{N}_{\geq 1}$	$(-\infty, -\frac{k}{m-1})$	< 0	$1 < p < q$	<i>super-lin</i>
D	$(-\frac{k}{m-1}, 0)$	$0 <$	$0 < q < p < 1$	<i>sub-lin</i>
CD	$(-\frac{k}{m-1}, 0) \cap (\mu_-, \mu_+)$	$0 <$	$q < 0 < p < 1$	$\begin{cases} \textit{sub-lin} \\ \textit{sing} \end{cases}$
CD	$(-\frac{k}{m-1}, 0) \cap \mathcal{C}[\mu_-, \mu_+]$	$0 <$	$0 < q < p < 1$	<i>sub-lin</i>
CD	$(-\frac{k}{m-1}, 0) \cap \{\mu_-, \mu_+\}$	$0 <$	$q = 0 < p = 2\alpha < 1$	$\begin{cases} \textit{sub-lin} \\ \textit{non-hom} \end{cases}$
D	$(0, 1)$	$0 <$	$0 < q < 1 < p$	$\begin{cases} \textit{super-lin} \\ \textit{sub-lin} \end{cases}$
CD	$(0, 1) \cap (\mu_-, \mu_+)$	$0 <$	$q < 0 < 1 < p$	$\begin{cases} \textit{super-lin} \\ \textit{sing} \end{cases}$
CD	$(0, 1) \cap \mathcal{C}[\mu_-, \mu_+]$	$0 <$	$0 < q < 1 < p$	$\begin{cases} \textit{super-lin} \\ \textit{sub-lin} \end{cases}$
CD	$(0, 1) \cap \{\mu_-, \mu_+\}$	$0 <$	$q = 0 < 1 < p = 2\alpha$	$\begin{cases} \textit{super-lin} \\ \textit{non-hom} \end{cases}$
$\mathbb{N}_{\geq 2} \times \mathbb{N}_{\geq 1}$	$(1, +\infty)$	$0 <$	$1 < q < p$	<i>super-lin</i>

• *sing* means that the corresponding exponent < 0 , roughly speaking *singular*.

However, all these conditions depend strongly on the corresponding coefficients in the whole specific non-linearity. More clearly, we can say that the right columns of the tables mentioned above are exact when S and S_F are strictly positive constants.

Remark 5.3 Note that when we consider $discr(\varrho) = 0$, we look for solutions $(m, k) \in \mathbb{N}_{\geq 2} \times \mathbb{N}_{\geq 1}$, in particular ordered pairs with natural components. It is easy to see that

$$\begin{aligned}
 D_0 &= \{(m, k) \in \mathbb{N}_{\geq 2} \times \mathbb{N}_{\geq 1} : discr(\varrho(m, k)) = 0\} \\
 &= \left\{ (m, k) \in \mathbb{N}_{\geq 3} \times \mathbb{N}_{\geq 1} : k = 4 \frac{m-1}{m-2} \right\} \\
 &= \{(3, 8), (4, 6), (6, 5)\}.
 \end{aligned}
 \tag{5.3}$$

All the other solutions of $discr(\varrho) = 0$ in \mathbb{R}^2 have no natural components.

Then, for $(m, k) = (3, 8) \in D_0$ ($(4, 6), (6, 5)$ respectively), $-\frac{k}{m-1}$ takes the value -4 ($-2, -1$ respectively) and $\mu_- = \mu_+ = -2$ ($-1, -\frac{1}{2}$ respectively). In such a case, when $\mu = \mu_- = \mu_+$, the fifth row in Table 4 establishes that $q = 0$, $p = \frac{1}{3}$ ($\frac{1}{2}, \frac{2}{3}$ respectively), $\alpha = \frac{1}{6}$ ($\frac{1}{4}, \frac{1}{3}$ respectively) and $\beta = \frac{4}{3}$ ($\frac{3}{2}, \frac{10}{3}$ respectively).

Note that for the elements in D_0 , the sum of the two components is either 11 or 10, both particularly interesting values in the physical applications. More precisely in the problems of the extra dimensions in cosmology, super-gravity and string theory (i.e. see [1, 8, 50–53, 89, 90]).

Notation 5.4 From now on, for $m \geq 3$ we will denote the Sobolev critical exponent by $2^* = \frac{2m}{m-2}$ and $p_Y = q_Y = \frac{4}{m-2} + 1 = \frac{m+2}{m-2} = 2^* - 1$.

Remark 5.5 Let $m \geq 3$. Now we will show that there exist particular values μ_{p_Y} and μ_{q_Y} such that the position of μ with respect to them, indicates that the corresponding p or q are sub-critical, critical or super-critical. The critical and super-critical cases will correspond to the conditions in the first row of Table 4. Indeed, by an easy but lengthy computation we have

$$p > p_Y: \text{ if and only if } \mu < \mu_{p_Y} = -\frac{k+1}{m-2}.$$

$$q > q_Y: \text{ if and only if } \mu < \mu_{q_Y} = -\frac{k}{m-2}.$$

Moreover,

$p = p_Y$: is verified if and only if $\mu = -\frac{k+1}{m-2}$; and consequently $\alpha = -\frac{2}{k+1}$, $\beta = 4\frac{m-1}{m-2} - 4\frac{k}{k+1} > 0$ and $q = p_Y + \frac{4}{k+1}$. Hence (4.11) takes the form

$$-\left(4\frac{m-1}{m-2} - 4\frac{k}{k+1}\right)\Delta_B u + S_B u = S u^{p_Y} - S_F u^{p_Y + \frac{4}{k+1}}. \tag{5.4}$$

$q = q_Y$: is verified if and only if $\mu = \mu_{q_Y} = -\frac{k}{m-2}$; and consequently

$$\alpha = -\frac{2}{k+m-2}, \quad \beta = \frac{4k}{(k+m-2)(m-2)} > 0 \quad \text{and}$$

$$p = q_Y - \frac{2}{k+m-2}.$$

Hence the equation (4.11) takes the form

$$-\frac{4k}{(k+m-2)(m-2)}\Delta_B u + S_B u = S u^{q_Y - \frac{2}{k+m-2}} - S_F u^{q_Y}. \tag{5.5}$$

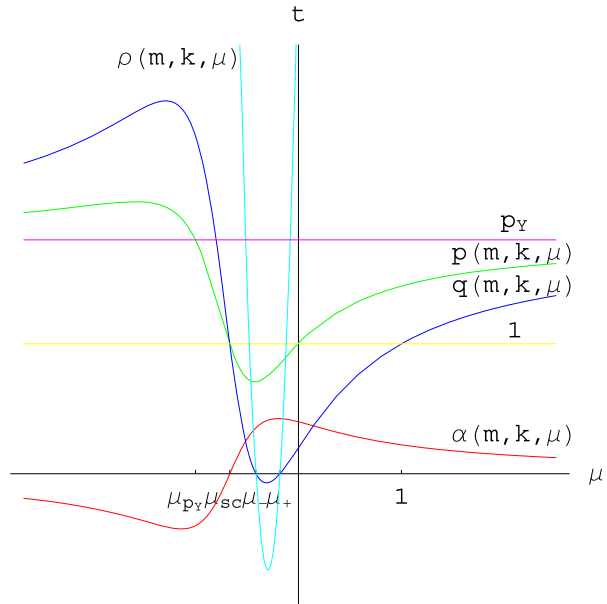
Note that μ_{q_Y} is the exceptional value $\bar{\mu}$ in Theorem 4.1 (see Table 1).

We observe also that $\mu_{p_Y} < \mu_{q_Y} < -\frac{k}{m-1}$, so that at least one of the two exponents is no sub-critical only if we stay in the conditions of the first row in the Table 4.

Remark 5.6 Let $m \geq 3$. Now, we will study the behavior of (4.11), when $\mu \rightarrow \pm\infty$. Consider $\mu \rightarrow \pm\infty$, then by (4.12) we have (see Table 4)

$$\alpha = \frac{2}{\left\{1 + \frac{1}{\frac{\mu-1}{\mu+m-1}}\right\}k + (m-2)\mu} \rightarrow \pm 0 \tag{5.6}$$

Fig. 1 Example:
 $(m, k) = (6, 4) \in CD$



and

$$\alpha\mu \rightarrow \frac{2}{m-2}. \tag{5.7}$$

Hence,

$$\begin{aligned} \beta &= \alpha 2[k + (m - 1)\mu] = \alpha 2k + 2(m - 1)\alpha\mu \rightarrow \beta_Y = 4\frac{m - 1}{m - 2}, \\ p &= 2\mu\alpha + 1 \rightarrow p_Y = \frac{4}{m - 2} + 1, \\ q &= 2(\mu - 1)\alpha + 1 = p - 2\alpha \rightarrow q_Y = \frac{4}{m - 2} + 1, \end{aligned} \tag{5.8}$$

with $q_Y = p_Y$. Thus, roughly speaking the limit equation of (4.11) for $\mu \rightarrow \pm\infty$ results

$$-4\frac{m-1}{m-2}\Delta_B u + S_B u = (S - S_F)u^{\frac{4}{m-2}+1}, \tag{5.9}$$

by “a suitable definition of S ”. Notice the similarity of this equation with the Yamabe type equation associated to a conformal change in the base (see (1.18)). Furthermore, by the last part of Remark 5.5, the approximation is by super-critical problems when $\mu \rightarrow -\infty$ and by sub-critical problems when $\mu \rightarrow +\infty$.

5.2 Base B_m with Dimension $m = 1$

Remark 5.7 As in the case of Remark 5.1, we will classify the type of non linearities involved in (4.11), obviously when this equation is verified (see Remark 4.5 and cases either $k \geq 1$ and $\mu \neq 0, 1, \frac{k+1}{2}$ or $k \geq 2$ and $\mu = 1$ there). Furthermore $m = 1$ implies that the

equations in (4.11) are ordinary differential equations and that the curvature tensor of the base is 0, and consequently $S_B \equiv 0$. Analogously, $S_F \equiv 0$ if $k = 1$. Hence, we will analyze the exponents as a function of the parameter μ and the dimension of the fiber $k \geq 1$.

Similar to the case of $m \geq 2$, for any row in the Tables 5, 6, 7, 8, the corresponding type of nonlinearity is modified by the scalar curvature of the fiber S_F and by the function S .

The problem (Pb-sc) for $m = 1$ and the corresponding nonlinear ordinary differential equations for low values of k are particularly interesting in physical applications (see [64–66], Kaluza-Klein theory and Randall-Sundrum theory).

By these hypothesis, we have

$$0 \neq \alpha = \frac{2}{-2\mu + k + 1} = \frac{1}{-\mu + k_1} \tag{5.10}$$

and

$$0 \neq \beta = \frac{4k}{-2\mu + k + 1} = \frac{2k}{-\mu + k_1}, \tag{5.11}$$

where

$$1 \leq k_1 := \frac{k + 1}{2}. \tag{5.12}$$

Note that by (5.10), we have that $\alpha > 0$ if and only if $\mu < k_1$. By (5.11), we also have that $\beta > 0$ if and only if $\mu < k_1$.

Furthermore by the same notation introduced in (5.1), we have

$$p = p(1, k, \mu) = 2\mu\alpha + 1 = \frac{\mu + \frac{k+1}{2}}{-\mu + \frac{k+1}{2}} = \frac{\mu + k_1}{-\mu + k_1} \tag{5.13}$$

and

$$q = q(1, k, \mu) = 2(\mu - 1)\alpha + 1 = p - 2\alpha = \frac{\mu + \frac{k-3}{2}}{-\mu + \frac{k+1}{2}} = \frac{\mu + k_1 - 2}{-\mu + k_1}. \tag{5.14}$$

In particular,

- (i) $\mu > k_1$ if and only if $\alpha < 0$ if and only if $p < q$.
- (ii) $p < 1$ if and only if $\mu\alpha < 0$ and $q < 1$ if and only if $(\mu - 1)\alpha < 0$.
- (iii) $p > 0$ if and only if $\mu \in (-k_1, k_1)$.
- (iv) $q > 0$ if and only if $\mu \in (2 - k_1, k_1)$ or $\mu \in (k_1, 2 - k_1)$.
- (v) $2 - k_1 < 0$ if and only if $3 < k$.

Now we will separately analyze the cases $k \geq 4$, $k = 3$, $k = 2$ and $k = 1$ (see (v) above and the first paragraph of this section).

$k \geq 4$: then $2 - k_1 \leq -\frac{1}{2} < 0 < \frac{5}{2} \leq k_1$. Thus we obtain Table 5.

$k = 3$: this implies $2 - k_1 = 0 < k_1 = 2$. Hence we have Table 6.

$k = 2$: so $0 < 2 - k_1 = \frac{1}{2} < k_1 = \frac{3}{2}$. It follows that Table 7.

$k = 1$: in this case $0 < 2 - k_1 = k_1 = 1$. But since $S_F \equiv 0$, q is non-influent. Thus we obtain Table 8.

Table 5 Nonlinearities in scalar curvature equation type (4.11) for $m = 1$ and $k \geq 4$, see Notation 5.2

$\mu \in$	$\alpha \in$	p, q	Type of p, q non-linearity
$(-\infty, -k_1)$	$(0, \frac{1}{2k_1})$	$q < p < 0$	<i>sing</i>
$\{-k_1\}$	$\{\frac{1}{2k_1}\}$	$q < p = 0 < 1$	$\left\{ \begin{array}{l} \textit{non-hom} \\ \textit{sing} \end{array} \right.$
$(-k_1, 2 - k_1)$	$(\frac{1}{2k_1}, \frac{1}{2(k_1-1)})$	$q < 0 < p < 1$	$\left\{ \begin{array}{l} \textit{sub-lin} \\ \textit{sing} \end{array} \right.$
$\{2 - k_1\}$	$\{\frac{1}{2(k_1-1)}\}$	$q = 0 < p = \frac{1}{k_1-1} < 1$	$\left\{ \begin{array}{l} \textit{sub-lin} \\ \textit{non-hom} \end{array} \right.$
$(2 - k_1, 0)$	$(\frac{1}{2(k_1-1)}, \frac{1}{k_1})$	$0 < q < p < 1$	<i>sub-lin</i>
$(0, 1)$	$(\frac{1}{k_1}, \frac{1}{k_1-1})$	$0 < q < 1 < p$	$\left\{ \begin{array}{l} \textit{super-lin} \\ \textit{sub-lin} \end{array} \right.$
$\{1\}$	$\frac{1}{k_1-1}$	$q = 1 < p = \frac{k_1+1}{k_1-1}$	$\left\{ \begin{array}{l} \textit{super-lin} \\ \textit{lin} \end{array} \right.$
$(1, k_1)$	$(\frac{1}{k_1-1}, +\infty)$	$1 < q < p$	<i>super-lin</i>
$(k_1, +\infty)$	$(-\infty, 0)$	$p < q < 0$	<i>sing</i>

Table 6 Nonlinearities in scalar curvature equation type (4.11) for $m = 1$ and $k = 3$, see Notation 5.2

$\mu \in$	$\alpha \in$	p, q	Type of p, q non-linearity
$(-\infty, -2)$	$(0, \frac{1}{4})$	$q < p < 0$	<i>sing</i>
$\{-2\}$	$\{\frac{1}{4}\}$	$q = -\frac{1}{2} < p = 0$	<i>non-hom / sing</i>
$(-2, 0)$	$(\frac{1}{4}, \frac{1}{2})$	$q < 0 < p < 1$	<i>sub-lin / sing</i>
$(0, 1)$	$(\frac{1}{2}, 1)$	$0 < q < 1 < p$	<i>super-lin / sub-lin</i>
$\{1\}$	$\{1\}$	$q = 1 < p = 3$	<i>super-lin / lin</i>
$(1, 2)$	$(1, +\infty)$	$1 < q < p$	<i>super-lin</i>
$(2, +\infty)$	$(-\infty, 0)$	$p < q < 0$	<i>sing</i>

6 Some Examples and Final Remarks

We consider the usual definition of Einstein manifolds (see [10, 11, 16, 59, 77, 83]). For some other alternative but close definitions see [18]. For dimension ≥ 3 these definitions are coincident.

Definition 6.1 A pseudo-Riemannian manifold (N_n, h) is said to be an Einstein manifold with $\lambda \in C^\infty(N)$ if and only if $Ric_h = \lambda h$.

Thus, the followings hold by letting (N_n, h) be a pseudo-Riemannian manifold,

Table 7 Nonlinearities in scalar curvature equation type (4.11) for $m = 1$ and $k = 2$, see Notation 5.2

$\mu \in$	$\alpha \in$	p, q	Type of p, q non-linearity
$(-\infty, -\frac{3}{2})$	$(0, \frac{1}{3})$	$q < p < 0$	<i>sing</i>
$\{-\frac{3}{2}\}$	$\{\frac{1}{3}\}$	$q = -\frac{2}{3} < p = 0$	<i>non-hom / sing</i>
$(-\frac{3}{2}, 0)$	$(\frac{1}{3}, \frac{2}{3})$	$q < 0 < p < 1$	<i>sub-lin / sing</i>
$(0, \frac{1}{2})$	$(\frac{2}{3}, 1)$	$q < 0 < 1 < p$	<i>super-lin / sing</i>
$\{\frac{1}{2}\}$	$\{1\}$	$q = 0 < p = 2$	<i>super-lin / non-hom</i>
$(\frac{1}{2}, 1)$	$(1, 2)$	$0 < q < 1 < p$	<i>super-lin / sub-lin</i>
$\{1\}$	$\{2\}$	$q = 1 < p = 5$	<i>super-lin / lin</i>
$(1, \frac{3}{2})$	$(2, +\infty)$	$1 < q < p$	<i>super-lin</i>
$(\frac{3}{2}, +\infty)$	$(-\infty, 0)$	$p < q < 0$	<i>sing</i>

Table 8 Nonlinearities in scalar curvature equation type (4.11) for $m = 1$ and $k = 1$, see Notation 5.2

$\mu \in$	$\alpha \in$	p	Type of p, q non-linearity
$(-\infty, -1)$	$(0, \frac{1}{2})$	$p < 0$	<i>sing</i>
$\{-1\}$	$\{\frac{1}{2}\}$	$p = 0$	<i>non-hom</i>
$(-1, 0)$	$(\frac{1}{2}, 1)$	$0 < p < 1$	<i>sub-lin</i>
$(0, 1)$	$(1, +\infty)$	$1 < p$	<i>super-lin</i>
$(1, +\infty)$	$(-\infty, 0)$	$p < 0$	<i>sing</i>

- (i) if (N_n, h) is Einstein with λ and $n \geq 3$, then λ is constant and $\lambda = S_N/n$, where S_N is the scalar curvature of (N_n, h) .
- (ii) if (N_n, h) is Einstein with λ and $n = 2$, then λ is not necessarily constant.

Remark 6.2 Let $M = B_m \times_{(\psi^\mu; \psi)} F_k$ be a (ψ, μ) -bcwp such that the Ricci curvature tensor Ric is given by (4.1). So, M is an Einstein manifold with λ if and only if (F, g_F) is Einstein with ν constant (note that when $k = 2$, ν is constant by the equations and not by the above item (i)) and the system that follows is verified

$$\begin{aligned}
 \lambda \psi^{2\mu} g_B &= Ric_B + \beta^H \frac{1}{\psi^{\frac{1}{\alpha^H}}} H_B^{\psi^{\frac{1}{\alpha^H}}} - \beta^\Delta \frac{1}{\psi^{\frac{1}{\alpha^\Delta}}} \Delta_B \psi^{\frac{1}{\alpha^\Delta}} g_B \quad \text{on } \mathcal{L}(B) \times \mathcal{L}(B), \\
 \lambda \psi^2 &= \nu - \frac{1}{\psi^{2(\mu-1)}} \frac{\beta^\Delta}{\mu} \frac{1}{\psi^{\frac{1}{\alpha^\Delta}}} \Delta_B \psi^{\frac{1}{\alpha^\Delta}},
 \end{aligned}
 \tag{6.1}$$

where the coefficients are given by (4.8). Compare this system with the well known results for an arbitrary warped product in [18, 70, 83].

So taking the g_B -trace of the first equation in (6.1) results

$$\begin{aligned} \lambda m \psi^{2\mu} &= S_B + \beta^H \frac{1}{\psi^{\frac{1}{\alpha^H}}} \Delta_B \psi^{\frac{1}{\alpha^H}} - m \beta^\Delta \frac{1}{\psi^{\frac{1}{\alpha^\Delta}}} \Delta_B \psi^{\frac{1}{\alpha^\Delta}}, \\ \lambda \psi^2 &= \nu - \frac{1}{\psi^{2(\mu-1)}} \frac{\beta^\Delta}{\mu} \frac{1}{\psi^{\frac{1}{\alpha^\Delta}}} \Delta_B \psi^{\frac{1}{\alpha^\Delta}}. \end{aligned} \tag{6.2}$$

At this point we observe that we meet all the hypothesis to apply Lemma 2.1, thus (6.2) is equivalent to

$$\begin{aligned} \lambda m \psi^{2\mu} &= S_B + \beta_{\text{tr}} \frac{1}{\psi^{\frac{1}{\alpha_{\text{tr}}}}} \Delta_B \psi^{\frac{1}{\alpha_{\text{tr}}}}, \\ \lambda \psi^2 &= \nu - \frac{1}{\psi^{2(\mu-1)}} \frac{\beta^\Delta}{\mu} \frac{1}{\psi^{\frac{1}{\alpha^\Delta}}} \Delta_B \psi^{\frac{1}{\alpha^\Delta}}, \end{aligned} \tag{6.3}$$

where

$$\begin{aligned} \alpha_{\text{tr}} &= \frac{\zeta_{\text{tr}}}{\eta_{\text{tr}}}, \\ \beta_{\text{tr}} &= \frac{\zeta_{\text{tr}}^2}{\eta_{\text{tr}}}, \end{aligned} \tag{6.4}$$

with

$$\begin{aligned} \zeta_{\text{tr}} &= \frac{\beta^H}{\alpha^H} - m \frac{\beta^\Delta}{\alpha^\Delta} = \zeta^H - m \zeta^\Delta = -2(m-1)\mu - k, \\ \eta_{\text{tr}} &= \frac{\beta^H}{(\alpha^H)^2} - m \frac{\beta^\Delta}{(\alpha^\Delta)^2} = \eta^H - m \eta^\Delta \\ &= -(m-1)\mu[(m-2)\mu + k] + k(\mu-1). \end{aligned} \tag{6.5}$$

Note that for $m = 1$, we have $S_B \equiv 0$, thus the system (6.2) and hence (6.3) are equivalent to the Einstein condition with λ . In this case, the coefficients take the form

$$\begin{aligned} \alpha_{\text{tr}} &= \frac{-1}{\mu-1}, \\ \beta_{\text{tr}} &= \frac{k}{\mu-1}, \\ \alpha^\Delta &= \frac{1}{-\mu+k}, \\ \beta^\Delta &= \frac{\mu}{-\mu+k}. \end{aligned} \tag{6.6}$$

Example 6.3 First of all, note that the interesting solutions of the involved ordinary differential equations must be nonnegative and moreover positive for us. So along this example, when we speak of solutions, it should be understood that we consider only positive solutions, unless explicitly mentioned otherwise. We now consider Remark 6.2 with B as a real

interval (i.e. $m = \dim B = 1$) equipped with the usual metric $\pm dr^2$ and (F^k, g_F) is an Einstein manifold with v . We immediately observe that in $(B, \pm dr^2)$, we have the following expressions:

$$\begin{aligned} \nabla^B(\cdot) &= \pm(\cdot)', \\ |\nabla^B(\cdot)|_B^2 &= \pm|(\cdot)'|^2, \\ H_B^{(\cdot)}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) &= (\cdot)'', \\ \Delta_B(\cdot) &= \pm(\cdot)'', \end{aligned} \tag{6.7}$$

where $(\cdot)'$ means the usual derivative with respect to r . Thus, by (6.3), if $\mu \in \mathbb{R} \setminus \{0, 1, k, \overline{\mu}_\pm\}$ ¹ (see Table 2) the corresponding (ψ, μ) -bcwp is an Einstein manifold with λ if and only if (λ, ψ) verifies the system

$$\begin{aligned} \lambda\psi^{2\mu} &= \pm \frac{k}{\mu - 1} \frac{1}{\psi^{1-\mu}} (\psi^{1-\mu})'', \\ \lambda\psi^2 &= v \mp \frac{1}{\psi^{2(\mu-1)}} \frac{1}{-\mu + k} \frac{1}{\psi^{-\mu+k}} (\psi^{-\mu+k})'', \end{aligned} \tag{6.8}$$

or still by changing variables $v = \psi^{1-\mu}$, if and only if (λ, v) verifies the system

$$\begin{aligned} \text{(a)} \quad \lambda v^{\frac{2\mu}{1-\mu}} &= \pm \frac{k}{\mu - 1} \frac{1}{v} v'', \\ \text{(b)} \quad \lambda v^{\frac{2}{1-\mu}} &= v \mp v^2 \frac{1}{-\mu + k} \frac{1}{v^{\frac{-\mu+k}{1-\mu}}} (v^{\frac{-\mu+k}{1-\mu}})''. \end{aligned} \tag{6.9}$$

So, applying (2.3) to the right hand side of (6.9) (b) results that a solution (λ, v) of (6.9) (a) is solution of (6.9) (b) if and only if it is a solution to the first order ordinary differential equation

$$\lambda v^{\frac{2}{1-\mu}} = v \mp \frac{k - 1}{(1 - \mu)^2} (v')^2 + \frac{\lambda}{k} v^{\frac{2}{1-\mu}} \tag{6.10}$$

or equivalently to

$$(k - 1) \left(\pm \frac{1}{(1 - \mu)^2} (v')^2 + \frac{\lambda}{k} v^{\frac{2}{1-\mu}} \right) = v. \tag{6.11}$$

We divide the study in two cases, namely.

$k \geq 2$: In this case, (6.11) is central, taking its derivative we obtain that any regular solution of this verifies

$$2 \frac{k - 1}{1 - \mu} v' \left(\pm \frac{1}{1 - \mu} v'' + \frac{\lambda}{k} v^{\frac{2}{1-\mu}-1} \right) = 0. \tag{6.12}$$

Hence, we have the following result:

¹The signs \pm in μ_\pm are not relative to the signs in the metric $\pm dr^2$, these are relative only with Table 2.

If (λ, v) is a solution of (6.9), then it is a solution of (6.11). Moreover, if (λ, v) is a solution of (6.11) then v is constant or is a solution of (6.9).

Thus, we have proved:

If a (ψ, μ) -bcwp is Einstein with λ , then $0 < v = \psi^{1-\mu}$ satisfies (6.11), where λ is necessarily constant (indeed $m + k \geq 3$). Furthermore, if $0 < v = \psi^{1-\mu}$ is a non-constant solution of (6.11), then the corresponding (ψ, μ) -bcwp is Einstein with λ . Furthermore, if $0 < \psi$ is a constant, then a (ψ, μ) -bcwp is Einstein if and only if $\lambda = 0 = v$.

We observe that (6.11) may be solved by the method of separation of variables

$$\frac{dv}{dr} = v' = \sqrt{\pm(1 - \mu)^2 \left(\frac{v}{k-1} - \frac{\lambda}{k} v^{\frac{2}{1-\mu}} \right)}. \tag{6.13}$$

Thus, its solutions are given by

$$\int^v \frac{1}{\sqrt{\pm(1 - \mu)^2 \left(\frac{v}{k-1} - \frac{\lambda}{k} w^{\frac{2}{1-\mu}} \right)}} dw = r. \tag{6.14}$$

For suitable values of the parameters, the latter integral may be solved by applying special functions (more specifically, hypergeometric functions called also Gauss-Kummer series and elliptic functions, see for example [19, 104] or apply Mathematica, Maple etc.). As we mentioned in Sect. 1, metrics of this type are considered in Randal-Sundrum theory [64] and in super-gravity theories.

One particular simpler case of the above results corresponds to $\mu = -1$, namely, $(\psi, -1)$ -bcwp with $k \geq 2$: In this case, (6.13) reduces to

$$\frac{dv}{dr} = v' = \sqrt{\pm 4 \left(\frac{v}{k-1} - \frac{\lambda}{k} v \right)} \tag{6.15}$$

and (6.14) to

$$r = \int^v \frac{1}{\sqrt{\pm 4 \left(\frac{v}{k-1} - \frac{\lambda}{k} w \right)}} dw = \mp \frac{k}{\lambda} \sqrt{\frac{v\lambda - kv\lambda + kv}{\mp k \pm k^2}} + \gamma, \tag{6.16}$$

with a real constant γ . Hence,

$$v(r) = \mp \frac{\lambda}{k} (r + \gamma)^2 + \frac{v}{\lambda} \frac{k}{k-1}. \tag{6.17}$$

$k = 1$: First of all, $v = 0$ and $\lambda \in C^\infty(B)$. Hence, unlike to the case of $k \geq 2$, (6.11) gives no information and (6.9) (a) and (6.9) (b) coincide. Thus, we proved that:

A (ψ, μ) -bcwp is Einstein with $\lambda \in C^\infty(B)$ if and only if $0 < v = \psi^{1-\mu}$ satisfies (6.9) (a) with $k = 1$.

For the completeness of the exposition, we will write a few lines about possibly the easiest case that follows.

$(\psi, -1)$ -bcwp with $k = 1$: Here, (6.9) (a) takes the trivial form

$$v'' = \mp 2\lambda, \tag{6.18}$$

where $\lambda \in C^\infty(B)$. So

$$v(r) = \mp 2 \int \int^\omega \lambda(\tau) d\tau d\omega. \tag{6.19}$$

In particular, if λ is constant then this results $\psi^2(r) = v(r) = \mp 2\lambda r^2 + ar + b$, with a and b real constants such that $\mp 2\lambda r^2 + ar + b$ is positive. It is clear that the latter condition depends on the base interval B_1 and the parameter λ .

Example 6.4 Like in Example 6.3, we consider only positive solutions, unless otherwise explicitly mentioned. By applying Remark 6.2 when B is a compact Riemannian manifold of $\dim B = m = 1$ with metric g_B and (F^k, g_F) is an Einstein manifold with v , we have that if $\mu \in \mathbb{R} \setminus \{0, 1, k, \overline{\mu}_\pm\}$ (see Table 2) the corresponding (ψ, k) -bcwp is an Einstein manifold with λ if and only if (λ, ψ) verifies the system

$$\begin{aligned} \lambda \psi^{2\mu} &= \frac{k}{\mu - 1} \frac{1}{\psi^{1-\mu}} \Delta_B(\psi^{1-\mu}), \\ \lambda \psi^2 &= v - \frac{1}{\psi^{2(\mu-1)}} \frac{1}{-\mu + k} \frac{1}{\psi^{-\mu+k}} \Delta_B(\psi^{-\mu+k}), \end{aligned} \tag{6.20}$$

Thus, by integrating on B and applying the compactness of B and also considering the positivity of ψ we conclude that $\lambda = v = 0$ and ψ is a positive constant. So, we proved that:

Let B be a compact Riemannian manifold of $\dim B = m = 1$ with metric g_B and (F^k, g_F) be an Einstein manifold with v where $\mu \in \mathbb{R} \setminus \{0, 1, k, \overline{\mu}_\pm\}$. A (ψ, μ) -bcwp is Einstein with λ if and only if $\lambda = v = 0$ and ψ is a positive constant (in particular, a trivial product).

Remark 6.5 The same order of ideas of Example 6.4 and considering especially (6.3) and (6.1), allow us to prove the following:

Let (B_m, g_B) be a scalar flat compact Riemannian manifold and (F_k, g_F) be a pseudo-Riemannian manifold. Furthermore, suppose that $\mu \in \mathbb{R} \setminus \{0, 1, k, \overline{\mu}_\pm\}$. A (ψ, μ) -bcwp is Einstein with a constant λ if and only if (F_k, g_F) is Einstein with $v = 0, \lambda = 0$ and ψ is a positive constant (in particular a usual product) and (B_m, g_B) is Ricci-flat.

Remark 6.6 Let $k \geq 2$ be and let us assume the hypothesis of Remark 6.2.

(i) It is easy to verify that in (6.3),

$$\alpha_{tr} = \alpha^\Delta \tag{6.21}$$

if and only if

$$(m - 1)(m - 2)\mu^2 + 2(m - 1)k\mu + k(k - 1) = 0. \tag{6.22}$$

Note that the latter (6.22) is also equivalent to

$$\alpha^\Delta = \alpha^H. \tag{6.23}$$

Since $m, k \in \mathbb{N}$ and $k \geq 1$, for any $m > 2$ (6.22) has two real solutions, namely

$$\tilde{\mu}_{\pm} = \frac{-(m-1)k \pm \sqrt{(m-1)k(k+m-2)}}{(m-1)(m-2)}, \tag{6.24}$$

while for $m = 2$ has only one solution

$$\tilde{\mu} = \frac{1-k}{2}. \tag{6.25}$$

We remark here that the latter is exactly the value of the parameter considered by H.-J. Schmidt in his studies about Birkhoff’s theorems in [92] (see (vi) in Sect. 1).

(ii) If (6.21) is satisfied for some μ , then (6.3) implies the functional equation

$$\lambda m \psi^{2\mu} = S_B + \beta_{\text{tr}}(v - \lambda \psi^2) \psi^{2(\mu-1)} \frac{\mu}{\beta^\Delta}, \tag{6.26}$$

or equivalently, by (4.8), (4.6), (6.4) and (6.5)

$$\lambda m \psi^{2\mu} = S_B - [2(m-1)\mu + k](v - \lambda \psi^2) \psi^{2(\mu-1)}, \tag{6.27}$$

or still

$$[m - 2(m-1)\mu - k]\lambda \psi^{2\mu} + [2(m-1)\mu + k]v \psi^{2(\mu-1)} = S_B. \tag{6.28}$$

We observe that if μ is such that (6.21) is satisfied, we reobtained Remark 6.5 for this specific value of μ without the hypothesis of compactness of the base, as a consequence of (6.28) and (6.1).

(iii) When $m = 2$ and μ is like in (6.25), then (6.28) takes the form

$$\lambda \psi^{1-k} + v \psi^{-(k+1)} = S_B. \tag{6.29}$$

Example 6.7 Now, we consider an interesting application of (6.3) with $m = 2$ and $k \geq 2$, containing as particular case the Schwarzschild type metrics considered in (i) of Sect. 1. Along the development of this example, we will prove the statement that follows:

Let (F_k, g_F) be Einstein with constant Ricci curvature v and dimension $k \geq 2$. Then, $\mathbb{R}_+ \times \mathbb{R} \times F_k$ furnished with a metric

$$g = s^{\frac{1}{k}-1} \left[\frac{1}{4\sqrt{su^2}(\sqrt{s})} ds^2 \pm 4\sqrt{su^2}(\sqrt{s}) dy^2 \right] + s^{\frac{2}{k}} g_F, \tag{6.30}$$

is Einstein with constant Ricci curvature λ where $(s, y) \in \mathbb{R}_+ \times \mathbb{R}$ if and only if u^2 is given by (6.50), where λ and C are such that the right hand side of (6.50) results positive.

Let (F_k, g_F) be Einstein with v and $(B_2, g_B) = (\mathbb{R}_+ \times \mathbb{R}, g_B)$ be a pseudo-Riemannian manifold endowed with the metric

$$g_B = (\psi_1(s))^{2(-1)} ds^2 \pm (\psi_1(s))^2 dy^2, \tag{6.31}$$

where ψ_1 is defined as $\psi_1(s) = 2s^{\frac{1}{4}}u(s^{\frac{1}{2}})$, like in (1.6). So by applying the second row of Table 8 we have,

$$S_B(s) = -\Delta_{ds^2} \psi_1^2(s) = S_B u^2 \Big|_{r=s^{\frac{1}{2}}}, \tag{6.32}$$

where S_B is the linear second order ordinary differential operator defined by

$$S_B f(r) = r^{-3} f(r) - r^{-2} \frac{d}{dr} f \Big|_r - r^{-1} \frac{d^2}{dr^2} f \Big|_r, \quad f \in C^\infty(\mathbb{R}_+). \tag{6.33}$$

We now consider $B_2 \times F_k = \mathbb{R}_+ \times \mathbb{R} \times F_k$ endowed with the metric

$$g = (\psi_2(s, y))^{2\mu_2} g_B + (\psi_2(s, y))^2 g_F, \tag{6.34}$$

where $\psi_2(s, y) = s^{\frac{1}{k}}$ and $\mu_2 = \frac{1-k}{2}$ (compare with (1.7) when $k = 2$).

Hence, since $(B_2 \times F_k, g)$ satisfies the hypothesis of Remark 6.6 (see Remark 4.3), if $(B_2 \times F_k, g)$ is Einstein with λ , then ψ_2 satisfies (6.3). Furthermore, the relation (6.21) is verified with $\mu_2 = \tilde{\mu}$ (see (6.25)). Consequently, ψ_2 must verify (6.29), precisely

$$\lambda \psi_2^{1-k} + \nu \psi_2^{-(k+1)} = S_B. \tag{6.35}$$

Therefore, by (6.32) and the definition of ψ_2

$$\lambda r^{\frac{2}{k}-2} + \nu r^{-2-\frac{2}{k}} = S_B u^2 \Big|_r, \tag{6.36}$$

or equivalently

$$\lambda r^{1+\frac{2}{k}} + \nu r^{1-\frac{2}{k}} = r^3 S_B u^2 \Big|_r, \tag{6.37}$$

where $r = s^{\frac{1}{2}}$.

Note that the latter is an Euler (also called equidimensional) equation. It is easy to show that for any real constants ν and λ , the general solution of (6.37) has the form

$$u^2(r) = \lambda \left(1 - \left(1 + \frac{2}{k} \right)^2 \right)^{-1} r^{1+\frac{2}{k}} + \nu \left(1 - \left(1 - \frac{2}{k} \right)^2 \right)^{-1} r^{1-\frac{2}{k}} + v_h(r), \tag{6.38}$$

where v_h is a solution of the homogeneous equation

$$0 = u^2 - r \frac{d}{dr} u^2 \Big|_r - r^2 \frac{d^2}{dr^2} u^2 \Big|_r, \tag{6.39}$$

namely a linear combination of r and r^{-1} .

It is clear that the choices of ν , λ and v_h will be such that the function u^2 be nonnegative.

Furthermore, we observe that among all the solutions of (6.36) there are spurious solutions of (6.1), the reason is that (6.2) is only a necessary condition of (6.1). Indeed, (6.38) is a solution of (6.1) if and only if $v_h(r) = C \frac{1}{r}$, where C is an arbitrary constant. In order to prove this, we note the following facts about (B_2, g_B) which is assumed as above:

- (i) Since $m = 2$,

$$\text{Ric}_B = \frac{1}{2} S_B g_B. \tag{6.40}$$

- (ii) By Proposition 3.9,

$$H_B^s = \left(\frac{1}{2} \Delta_{g_B^s} \right) g_B. \tag{6.41}$$

(iii) By Proposition 3.12 and the definition of ψ_1 ,

$$\Delta_{g_B} s = 2\psi_1 \frac{d}{ds} \psi_1 = \frac{d}{ds} \psi_1^2 = \mathcal{L}\left(r, \frac{d}{dr}\right) u^2 \Big|_{s^{\frac{1}{2}}}, \tag{6.42}$$

where

$$\mathcal{L}\left(r, \frac{d}{dr}\right) f \Big|_r = 2\left[r^{-1} + \frac{d}{dr}\right] f \Big|_r. \tag{6.43}$$

For $(B_2 \times F_k, g)$, since the coefficients given by (4.8) verify (6.21), they take the values:

$$\begin{aligned} \alpha^\Delta &= \alpha^H = \frac{1}{k}, \\ \beta^\Delta &= \frac{\tilde{\mu}}{k} = \frac{1-k}{2k}, \\ \beta^H &= \frac{k}{2\tilde{\mu}-1} = -1, \end{aligned} \tag{6.44}$$

hence, (6.1) takes the form

$$\begin{aligned} \lambda \psi_2^{1-k} g_B &= \text{Ric}_B - \frac{1}{\psi_2^k} \left(H_B^{\psi_2^k} + \frac{1-k}{2k} \Delta_B \psi_2^k g_B \right) \quad \text{on } \mathcal{L}(B) \times \mathcal{L}(B), \\ \lambda \psi_2^2 &= \nu - \psi_2^{k+1} \frac{1}{k} \frac{1}{\psi_2^k} \Delta_B \psi_2^k. \end{aligned} \tag{6.45}$$

So by the definition of ψ_2 , (6.40) and (6.41), (6.45) results equivalent to

$$\begin{aligned} \lambda \psi_2^{1-k} &= \frac{1}{2} S_B - \frac{1}{2} \frac{1}{k} \frac{1}{\psi_2^k} \Delta_B \psi_2^k, \\ \lambda \psi_2^{1-k} - \nu \psi_2^{-(k+1)} &= -\frac{1}{k} \frac{1}{\psi_2^k} \Delta_B \psi_2^k, \end{aligned} \tag{6.46}$$

or moreover, by easy computations, to

$$\begin{aligned} \lambda \psi_2^{1-k} + \nu \psi_2^{-(k+1)} &= S_B, \\ \lambda \psi_2 - \nu \psi_2^{-1} &= -\frac{1}{k} \Delta_B \psi_2^k. \end{aligned} \tag{6.47}$$

Note in the above steps the reduction from 4 to 2 equations. Furthermore the first equation of (6.47) is exactly (6.35). Recalling again that $\psi_2(s, y) = s^{\frac{1}{k}}$, (6.47) takes the form

$$\begin{aligned} \lambda s^{\frac{1}{k}-1} + \nu s^{-\frac{1}{k}-1} &= S_B, \\ \lambda s^{\frac{1}{k}} - \nu s^{-\frac{1}{k}} &= -\frac{1}{k} \Delta_B s, \end{aligned} \tag{6.48}$$

and since $s^{\frac{1}{2}} = r$, by (6.32) and (6.42),

$$\begin{aligned} \text{(a)} \quad & \lambda r^{\frac{2}{k}-2} + \nu r^{-\frac{2}{k}-2} = \mathcal{S}_B u^2|_r, \\ \text{(b)} \quad & \lambda r^{\frac{2}{k}} - \nu r^{-\frac{2}{k}} = -\frac{1}{k} \mathcal{L}\left(r, \frac{d}{dr}\right) u^2|_r. \end{aligned} \tag{6.49}$$

We observe that deriving the second equation of (6.49) and multiplying by r^{-1} , we obtain the first equation. So any regular solution of (b) is a solution of (a) in (6.49).

On the other hand it is easy to show that a general solution of (6.49) (b) is

$$u^2(r) = \lambda \left(1 - \left(1 + \frac{2}{k}\right)^2\right)^{-1} r^{1+\frac{2}{k}} + \nu \left(1 - \left(1 - \frac{2}{k}\right)^2\right)^{-1} r^{1-\frac{2}{k}} + C \frac{1}{r}, \tag{6.50}$$

where C is an arbitrary constant.

Thus, since (6.49) (a) coincides with (6.36), a solution (6.38) of the latter is a solution of (6.49) if and only if $v_h(r) = C \frac{1}{r}$, where C is an arbitrary constant Q.E.D.

As we mentioned in the first paragraph of this example, important solutions of the Einstein vacuum equations are included in the above discussion (namely, compare with Sect. 1 (i)). We will write explicitly some cases with $k = 2$ but the situation is more general. Let $B_2 \times F_2$ be endowed with a metric of the form

$$g = s^{-\frac{1}{2}} g_B + s g_{F_2}, \tag{6.51}$$

where (B_2, g_B) is like in (6.31), $(s, y) \in B_2 = \mathbb{R}_+ \times \mathbb{R}$ and (F_2, g_F) is a pseudo-Riemannian manifold of dimension $k = 2$.

Ricci flat: If $\lambda = 0$, then (6.37) takes the form

$$\nu = u^2 - r(u^2)' - r^2(u^2)'' . \tag{6.52}$$

It is easy to verify that $u^2(r) = \nu + C \frac{1}{r}$ is a solution of (6.52). In particular, when $C = -2M$, $M > 0$ and $\nu = 1$ we obtain the classical Schwarzschild solution (compare with (1.1) and (6.50)). While, the condition “ $C = 0$ and $\nu = 1$ ” arises the Minkowski metric of an empty space-time in spherical terms.

Riemman-Schwarzschild: If $\lambda = -3$ and $\nu = 1$, then (6.37) takes the form

$$-3r^2 + 1 = u^2 - r(u^2)' - r^2(u^2)'' . \tag{6.53}$$

It is easy to verify that for any positive M , $u^2(r) = 1 - \frac{2M}{r} + r^2$ is a solution of (6.53) (compare with (1.3) and (6.50)).

Thus, (6.37) contains a large family of important solutions of the Einstein equation. An analogous procedure can be applied to build the static BTZ (2 + 1)-black hole solution, we leave the computations to the reader (see [1, 13, 14, 39, 63] for details about BTZ).

Remark 6.8 Let $F = (F_k, g_F)$ be a pseudo-Riemannian Einstein manifold with constant ν and dimension $k \geq 1$.

We recall the principal result in [70] in the context of Riemannian manifolds, namely: an Einstein warped product with a non-positive scalar curvature and compact base is a trivial Riemannian product space, so that the warping function results constant. Thus, if $B =$

(B_m, g_B) is a compact Riemannian manifold with dimension $m \geq 3$ and $\mu \in \mathbb{R} \setminus \{0, 1, \bar{\mu}, \bar{\mu}_\pm\}$ (compare with Theorem 4.1), then our system (6.1) admits a non-constant positive solution only if $\lambda > 0$. But if we let F and μ be as above, then there exists a metric on B_m admitting no $\psi \in C^\infty_{>0}(B)$ such that the corresponding (ψ, μ) -bcwp is Einstein with $\lambda > 0$. Indeed, multiplying the first equation of (6.3) by $\psi^{\frac{1}{\alpha v}}$ and integrating on B respect to the measure dg_B there results

$$\lambda m \int_B \psi^{2\mu + \frac{1}{\alpha v}} dg_B = \int_B S_B \psi^{\frac{1}{\alpha v}} dg_B.$$

Now, we recall the Aubin result “any manifold of dimension ≥ 3 possesses a complete metric of constant negative scalar curvature” (see [9, 22, 76]). So if g_B is a such metric on our compact B_m , i.e. $S_B < 0$, then λ cannot be positive (contradiction).

In conclusion, let F and μ as above. On every compact manifold B of dimension ≥ 3 , there exists a Riemannian metric g_B such that a (ψ, μ) -bcwp with base (B, g_B) is Einstein with λ if and only if ψ is constant, (B, g_B) is Einstein with $\lambda m \psi^{2\mu}$ and $\lambda \psi^2 = v \leq 0$.

The case $\mu = 0$, i.e. singly warped product, was considered in [82]. The remaining values of μ (i.e. $1, \bar{\mu}, \bar{\mu}_\pm$) can be analyzed with an analogous approach with suitable changes, yet by applying (4.5) and (2.3).

A particular example of the latter results (i.e. $\mu = -1$) is the following interesting application of them: Let (F_k, g_F) be a pseudo-Riemannian Einstein manifold of dimension $k \geq 1$. Then on any compact manifold B_m of dimension ≥ 3 there exists a metric g_B such that there is no $\psi \in C^\infty_{>0}(B)$ such that $(B \times F, \psi^{-2}g_B + \psi^2g_F)$ is a non trivial (i.e ψ non constant) Einstein manifold.

7 Conclusions and Future Directions

Now, we would like to summarize the content of the paper and to propose our future plans on this topic.

In brief, we introduced and studied curvature properties of a type of product of two pseudo-Riemannian manifolds called *base conformal warped product* by us, roughly speaking the metric of a such product is a mixture of a conformal metric on the base and a warped metric. As we mentioned in Sect. 1, these kind of metrics and considerations about their curvatures are very frequent in different physical areas, for instance relativity, extra-dimension theories (Kaluza-Klein, Randall-Sundrum), string and super-gravity theories; also in global analysis for example in the study of the spectrum of Laplace-Beltrami operators on p -forms, etc.

In Sect. 2, we started our discussion by considering particular families of either scalar or tensorial nonlinear partial differential operators on pseudo-Riemannian manifolds and studied useful identities verified by them. The latter allowed us to find reduced expressions of the Ricci tensor and scalar curvature used not only in Sect. 4 and Sect. 5, but also in the study of multiply warped products in [39]. The operated reductions can be considered as generalizations of those used by Yamabe in [107], in order to obtain the famous expression (1.18) for the behavior of the scalar curvature under a conformal change and those used in [37] with the same aim but for a singly warped product (see also Remark 2.4 for other particular application).

In Sect. 3, we defined precisely *base conformal warped products* of pseudo-Riemannian manifolds and computed their Levi-Civita connection, Hessian, Laplace-Beltrami operator and Riemannian curvatures.

In Sect. 4 and from then on, we concentrated on a very commonly used physical ansatz, namely when the conformal factor acting on the metric of the base and the warping function acting on the metric of the fiber are related by an exponent, so that one is a power of the other (see the examples in Sect. 1). We called a product manifold furnished with a metric form like above as a special base conformal warped product. Then, we turned our attention to the structure of the relations that connect the different types of curvatures, especially Ricci and scalar. More explicitly, we obtained more approachable relations by applying the results of Sect. 2 but also some formulas even in some exceptional cases corresponding to the situations where the results of Sect. 2 are unapplicable.

In Sect. 5, we focused on a classification of the type of nonlinearities arose in the relation among the involved scalar curvatures of a *special base conformal warped product*, previously obtained in Sect. 4. Similar to the study made in the latter, we classified the nonlinearities according to the value of the exponent parameter μ , the dimensions of the base and the fiber and finally the scalar curvature of the fiber. The aim of this classification is to study in future works the problem of prescribing constant/nonconstant scalar curvature in *special base conformal warped products*, indeed in these problems, the type of nonlinearities, ellipticity/hyperbolicity of the linear part of differential equations connecting the involved scalar curvatures and compactness of the base play a very central role.

At this point, we would like to note that the previous problems as well as the study of the Einstein equation on *base conformal warped products*, *special base conformal warped products* and their *generalizations to multi-fiber cases*, give rise to a reach family of interesting problems not only in differential geometry and physics (see for instance, the several recent works of R. Argurio, J.P. Gauntlett, S. Kachru, M.O. Katanaev, J. Maldacena, H.-J. Schmidt, E. Silverstien, A. Strominger, P.S. Wesson among many others), but also in non linear analysis (see the different works of A. Ambrosetti, T. Aubin, Y. Choquet-Bruat, J.F. Escobar, E. Hebey, R. Schoen, S.-T. Yau among others), which will be the subject matter of future works (see [40]).

In Sect. 6, we analyzed, investigated and characterized possible solutions for the conformal and warping factors of a *special base conformal warped product* which guarantee that the corresponding product is Einstein. We apply the same order of ideas to a generalization of the Schwarzschild metrics also. Among the considered cases there are important metrics in questions of relativity, cosmology, hight energy physics, etc.

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Appendix

We first show some interesting properties about the behavior of the Laplace-Beltrami operator under a conformal change in the metric.

Let $N = (N_n, h)$ be a pseudo-Riemannian manifold of dimension n and let

$$\Delta_h(\cdot) = \frac{1}{\sqrt{|h|}} \partial_i (\sqrt{|h|} h^{ij} \partial_j(\cdot)), \quad (\text{A.1})$$

be the Laplace-Beltrami operator related to the metric h , where we denote the usual volume element by $\sqrt{|h|} := \sqrt{|\det h|}$.

Lemma A.1 *Let $u \in C_{>0}^\infty(N)$ and $r \in \mathbb{R}$. Then,*

$$u^r \Delta_{u^r h}(\cdot) = r \frac{n-2}{2} h \left(\frac{\nabla u}{u}, \nabla(\cdot) \right) + \Delta_h(\cdot). \tag{A.2}$$

Proof Denote $\tilde{h} = u^r h$, there results $\tilde{h}_{ij} = u^r h_{ij}$, $\tilde{h}^{ij} = u^{-r} h^{ij}$ and $\det \tilde{h} = u^{nr} \det h$. Thus,

$$\begin{aligned} \Delta_{\tilde{h}}(\cdot) &= \frac{1}{u^{\frac{n}{2}r} \sqrt{\tilde{h}}} \partial_i (u^{\frac{n}{2}r} \sqrt{\tilde{h}} u^{-r} h^{ij} \partial_j(\cdot)) \\ &= \frac{1}{u^{\frac{n}{2}r} \sqrt{\tilde{h}}} \left[\left(\frac{n}{2} - 1 \right) r u^{(\frac{n}{2}-1)r-1} \partial_i u \sqrt{\tilde{h}} h^{ij} \partial_j(\cdot) \right. \\ &\quad \left. + u^{(\frac{n}{2}-1)r} \partial_i (\sqrt{\tilde{h}} h^{ij} \partial_j(\cdot)) \right]. \end{aligned} \tag{A.3}$$

So multiplying by u^r ,

$$u^r \Delta_{\tilde{h}}(\cdot) = \left(\frac{n}{2} - 1 \right) r u^{-1} \partial_i u h^{ij} \partial_j(\cdot) + \Delta_h(\cdot). \tag{A.4}$$

□

Lemma A.2 *Let $u, w \in C_{>0}^\infty(N)$ and $r \in \mathbb{R}$. Then,*

$$u^r \frac{1}{w} \Delta_{u^r h} w = r \frac{n-2}{4} \frac{\Delta_h(uw)}{uw} - r \frac{n-2}{4} \frac{\Delta_h u}{u} + \left(1 - r \frac{n-2}{4} \right) \frac{\Delta_h w}{w}. \tag{A.5}$$

In particular, if $w = u$, then

$$u^r \frac{1}{u} \Delta_{u^r h} u = r \frac{n-2}{4} \frac{\Delta_h u^2}{u^2} + \left(1 - r \frac{n-2}{2} \right) \frac{\Delta_h u}{u} = \frac{1}{r \frac{n-2}{2} + 1} \frac{\Delta u^{r \frac{n-2}{2} + 1}}{u^{r \frac{n-2}{2} + 1}}, \tag{A.6}$$

where the latter equality is true when $r \neq -\frac{2}{n-2}$. Moreover, if $n \geq 3$ and $r = \frac{4}{n-2}$, then

$$u^{\frac{4}{n-2}} \frac{1}{u} \Delta_{u^{\frac{4}{n-2}} h} u = \frac{\Delta_h u^2}{u^2} - \frac{\Delta_h u}{u} = \frac{1}{3} \frac{\Delta_h u^3}{u^3}. \tag{A.7}$$

Proof First of all, we observe that

$$\frac{\Delta_h(uw)}{uw} = 2h \left(\frac{\nabla u}{u}, \frac{\nabla w}{w} \right) + \frac{\Delta_h u}{u} + \frac{\Delta_h w}{w}, \tag{A.8}$$

hence

$$h \left(\frac{\nabla u}{u}, \frac{\nabla w}{w} \right) = \frac{1}{2} \frac{\Delta_h(uw)}{uw} - \frac{1}{2} \left(\frac{\Delta_h u}{u} + \frac{\Delta_h w}{w} \right). \tag{A.9}$$

On the other hand, by Lemma A.1,

$$\begin{aligned}
 u^r \frac{1}{w} \Delta_{u^r h} w &= r \frac{n-2}{2} h \left(\frac{\nabla u}{u}, \frac{\nabla w}{w} \right) + \frac{\Delta_h w}{w} \\
 &= r \frac{n-2}{4} \left[\frac{\Delta_h(uw)}{uw} - \frac{\Delta_h u}{u} - \frac{\Delta_h w}{w} \right] + \frac{\Delta_h w}{w} \\
 &= r \frac{n-2}{4} \frac{\Delta_h(uw)}{uw} - r \frac{n-2}{4} \frac{\Delta_h u}{u} + \left(1 - r \frac{n-2}{4} \right) \frac{\Delta_h w}{w}. \tag{A.10}
 \end{aligned}$$

In (A.6), the first equality is immediate by taking $w = u$ in (A.5). In order to obtain the second equality of (A.6) it is sufficient to apply Remark 2.5 with $\alpha = \beta = \frac{1}{r \frac{n-2}{2} + 1}$. Finally, (A.7) is an obvious consequence of (A.6). □

Remark A.3 Now we compute the useful relation between the scalar curvatures under a conformal change in the metric h when the conformal metric is written in the form $\tilde{h} = v^r h$, $h \in C^\infty_{>0}(N)$ instead of an exponential form like in (1.17). Consider $v^r = e^\eta$, so that $\eta = r \log v$ and applying (1.17) and (2.3) (note that $t \neq 0, 1$) we obtain

$$\begin{aligned}
 v^r S_{\tilde{h}} &= S_h - (n-1)r \left[\Delta_h \log v + \frac{n-2}{4} |\nabla \log v|^2 \right] \\
 &= S_h - (n-1)r \left[\left(-1 + \frac{n-2}{4} r \right) \frac{|\nabla v|^2}{v^2} + \frac{\Delta_h v}{v} \right] \\
 &= S_h - (n-1)r \left[\left(-1 + \frac{n-2}{4} r \right) \frac{1}{(t-1)t} \frac{\Delta_h v^t}{v^t} \right. \\
 &\quad \left. + \left(1 - \left(-1 + \frac{n-2}{4} r \right) \frac{1}{t-1} \right) \frac{\Delta_h v}{v} \right]. \tag{A.11}
 \end{aligned}$$

Without lose of generality we assume r is nonzero, it is clear that $S_{\tilde{h}} = S_h$ when $r = 0$. At this point, we have two cases:

$(n \geq 3)$: By Remark 2.5 with $\alpha = \beta = \frac{4}{n-2} \frac{1}{r}$,

$$v^r S_{\tilde{h}} = S_h - (n-1) \frac{4}{n-2} \frac{\Delta_h v^{\frac{n-2}{4} r}}{v^{\frac{n-2}{4} r}}, \tag{A.12}$$

which contents as a particular case (1.18) when $r = \frac{4}{n-2}$.

$(n = 2)$: In this case (A.11) says

$$v^r S_{\tilde{h}} = S_h - \frac{r}{t-1} \left[-\frac{1}{t} \frac{\Delta_h v^t}{v^t} + t \frac{\Delta_h v}{v} \right]. \tag{A.13}$$

Moreover, if we apply (2.4) the latter equation becomes

$$v^r S_{\tilde{h}} = S_h + r \left(\frac{|\nabla v|^2}{v^2} - \frac{\Delta v}{v} \right). \tag{A.14}$$

Note that in (A.13) it is not possible to apply Remark 2.5.

Remark A.4 Now, as we mentioned in Sect. 1, we will outline an alternative proof of Theorem 1.1 by applying a conformal change metric technique like in [37]. We will concentrate in Theorem 4.4 when $m \geq 3$. The same order of ideas may be used for the case $m = 2$.

Proof (of Theorem 4.4 when $m \geq 3$, $\mu \neq -\frac{1}{m-2}$) Since $g = \tilde{g}_B + \psi^{2\mu} g_F$ with $\tilde{g}_B = \psi^{2\mu} g_B$, an application of (1.15) to ψ results

$$S = -2k \frac{\Delta_{\psi^{2\mu} g_B} \psi}{\psi} - k(k-1) \frac{\psi^{2\mu} g_B (\psi^{-2\mu} \nabla \psi, \psi^{-2\mu} \nabla \psi)}{\psi^2} + S_{\psi^{2\mu} g_B} + \frac{S_{g_F}}{\psi^2}. \tag{A.15}$$

So by multiplying by $\psi^{2\mu}$ and applying (A.6) (note that $\mu \neq -\frac{1}{m-2}$), (2.3) (with $t \neq 0, 1$) and equation (A.12) we obtain

$$\begin{aligned} \psi^{2\mu} S &= -\frac{2k}{\mu(m-2)+1} \frac{\Delta_{g_B} \psi^{\mu(m-2)+1}}{\psi^{\mu(m-2)+1}} \\ &\quad - k(k-1) \frac{1}{t-1} \left[\frac{1}{t} \frac{\Delta_{g_B} \psi^t}{\psi^t} - \frac{\Delta_{g_B} \psi}{\psi} \right] \\ &\quad + S_{g_B} - (m-1) \frac{4}{m-2} \frac{\Delta_h \psi^{\frac{m-2}{4} 2\mu}}{\psi^{\frac{m-2}{4} 2\mu}} + S_{g_F} \psi^{2(\mu-1)} \\ &= -\left[\frac{2k}{\mu(m-2)+1} \frac{\Delta_{g_B} \psi^{\mu(m-2)+1}}{\psi^{\mu(m-2)+1}} + \frac{k(k-1)}{t(t-1)} \frac{\Delta_{g_B} \psi^t}{\psi^t} \right. \\ &\quad \left. - \frac{k(k-1)}{(t-1)} \frac{\Delta_{g_B} \psi}{\psi} \right. \\ &\quad \left. + (m-1) \frac{4}{m-2} \frac{\Delta_h \psi^{\frac{m-2}{4} 2\mu}}{\psi^{\frac{m-2}{4} 2\mu}} \right] + S_{g_B} + S_{g_F} \psi^{2(\mu-1)}. \end{aligned} \tag{A.16}$$

The hypothesis of Lemma 2.1 is verified, indeed: since $\mu \neq -\frac{k}{m-1}$,

$$2(k + (m-1)\mu) \neq 0 \tag{A.17}$$

and

$$\begin{aligned} &2k(\mu(m-2)+1) + k(k-1) + (m-1)(m-2)\mu^2 \\ &= \{[k + (m-1)\mu] + (1-\mu)\}k + (m-2)\mu[k + (m-1)\mu] \\ &= [k + (m-1)\mu](k + (m-2)\mu) + (1-\mu)k \\ &= (m-1)(m-2)\mu^2 + 2(m-2)k\mu + (k+1)k \\ &> 0. \end{aligned} \tag{A.18}$$

Thus, by applying Lemma 2.1 with

$$\alpha = \frac{2[k + (m-1)\mu]}{\{[k + (m-1)\mu] + (1-\mu)\}k + (m-2)\mu[k + (m-1)\mu]}$$

and also $\beta = \alpha 2[k + (m-1)\mu]$ (thus $\beta > 0$) and $u = \psi^{\frac{1}{\alpha}}$, we obtain that:

$$-\beta \frac{\Delta_{g_B} u}{u} = u^{2\mu\alpha} S - S_{g_B} - S_{g_F} u^{2(\mu-1)\alpha}. \tag{A.19}$$

□

Remark A.5 By using the latter technique, the case of $\mu = -\frac{k}{m-2}$ must be analyzed separately. However, it is possible to prove (4.11) in a similar way too.

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