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# STABILITY ANALYSIS OF SWITCHED TIME DELAY SYSTEMS*

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**Abstract.** This paper addresses the asymptotic stability of switched time delay systems with heterogeneous time invariant time delays. Piecewise Lyapunov–Razumikhin functions are introduced for the switching candidate systems to investigate the stability in the presence of an infinite number of switchings. We provide sufficient conditions in terms of the minimum dwell time to guarantee asymptotic stability under the assumptions that each switching candidate is delay-independently or delay-dependently stable. Conservatism analysis is also provided by comparing with the dwell time conditions for switched delay-free systems. Finally, a numerical example is given to validate the results.

**Key words.** asymptotic stability, switched systems, time delay, dwell time

**AMS subject classifications.** 93D05, 93D20, 93C05, 93C23

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## 1. Introduction.

Switching control offers a new look into the design of complex control systems (e.g., nonlinear systems, parameter varying systems, and uncertain systems) [1, 8, 9, 19, 21, 27]. Unlike the conventional adaptive control techniques that rely on continuous tuning, the switching control method updates the controller parameters in a discrete fashion based on the switching logic. The resulting closed-loop systems have hybrid behaviors (e.g., continuous dynamics, discrete time dynamics, and jump phenomena). One of the most challenging issues in the area of hybrid systems is the stability analysis in the presence of control switching. We refer to [9] for a general review on switching control methods.

In particular, we are interested in the stability analysis of switched time delay systems. In fact, time delay systems are ubiquitous in chemical processes, aerodynamics, and communication networks [3, 14]. To further complicate the situation, the time delays are usually time varying and uncertain [24, 26]. It has been shown that robust $\mathcal{H}^\infty$ controllers can be designed for such infinite-dimensional plants, where robustness can be guaranteed within some uncertainty bounds [4]. In order to incorporate a larger operating range or better robustness, controller switching can be introduced, which results in switched closed-loop systems with time delays. For delay-free systems, stability analysis and design methodology have been investigated recently in the framework of hybrid dynamical systems [1, 2, 8, 11, 19, 21, 25]. In particular, [21] provided sufficient conditions on the stability of the switching control systems based on Filippov solutions to discontinuous differential equations and Lyapunov functionals; [19] proposed a dwell-time-based switching control, where a sufficiently large dwell time can guarantee system stability. A more flexible result was obtained in [10], where the average dwell time was introduced for switching control. In [25] the results of [10] were extended to linear parameter varying (LPV) systems. LaSalle's invariance

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principle was extended to a class of switched linear systems for stability analysis [8]. Despite the variety and significance of the many results on hybrid system stability, stability of switched time delay systems hasn't been adequately addressed due to the general difficulty of infinite-dimensional systems [7].

Two important approaches in the stability analysis of time delay systems are the (1) Lyapunov–Krasovskii method and (2) Lyapunov–Razumikhin method [6, 20]. Various sufficient conditions with respect to the stability of time delay systems have been given using Riccati-type inequalities or linear matrix inequalities (LMIs) [3, 12, 14, 24]. Meanwhile, stability analysis in the presence of switching has been discussed in some recent works [16, 18, 22]. In [18] stability and stabilizability were discussed for discrete time switched time delay systems; [16] considered a similar stability problem in a continuous time domain. Note that [18] and [16] produce *trajectory-dependent* results without taking admissible switching signals into consideration.

The main contribution of this paper is a collection of results on the *trajectory-independent* stability of continuous time switched time delay systems using piecewise Lyapunov–Razumikhin functions. The dwell time of the switching signals is constructively given, which guarantees asymptotic stability for the delay-independent case and the delay-dependent case, respectively. Note that the asymptotic stability of finite-dimensional linear systems indicates exponential stability, while this is not the case for infinite-dimensional systems [7, 15]. This poses the key challenge in the analysis of switched time delay systems, where we do not assume exponential convergence of the switching candidates, as opposed to most of the results in the literature [8, 10, 17, 19].

The paper is organized as follows. The problem is defined in section 2. In section 3, the main results on the stability of switched time delay systems are presented in terms of the dwell time of the switching signals. Conservatism analysis is provided by comparing with the dwell-time conditions for switching delay-free systems in section 4. The results are illustrated with a numerical example in section 5, followed by concluding remarks in section 6.

**2. Problem definition.** For convenience, we would like to employ the following notation. The general retarded functional differential equations (RFDEs) with time delay $r$ can be described as

$$\dot{x}(t) = f(t, x_t) \tag{2.1}$$

with initial condition $\phi(\cdot) \in C([-r, 0], \mathbb{R}^n)$, where $x_t$ denotes the state defined by $x_t(\theta) = x(t+\theta)$, $-r \le \theta \le 0$. We use $\|\cdot\|$ to denote the Euclidean norm of a vector in $\mathbb{R}^n$, and $|f|_{[t-r,t]}$ for the $\infty$-norm of $f$, i.e.,

$$|f|_{[t-r,t]} := \sup_{t-r\le\theta\le t} \|f(\theta)\|,$$

where $f$ is an element of the Banach space $C([t-r,t], \mathbb{R}^n)$.

Consider the following switched time delay systems:

$$\Sigma_t : \begin{cases} \dot{x}(t) = A_{q(t)}x(t) + \bar{A}_{q(t)}x(t-\tau_{q(t)}), \quad t \ge 0, \\ x_0(\theta) = \phi(\theta) \ \ \forall \theta \in [-\tau_{max}, 0], \end{cases} \tag{2.2}$$

where $x(t) \in \mathbb{R}^n$ and $q(t)$ is a piecewise switching signal taking values on the set $\mathcal{F} := \{1, 2, \ldots, l\}$, i.e., $q(t) = k_j$, $k_j \in \mathcal{F}$ $\forall t \in [t_j, t_{j+1})$, where $t_j$, $j \in \mathbb{Z}^+ \cup \{0\}$, is the $j$th switching time instant. It is clear that the trajectory of $\Sigma_t$ in any arbitrary switching interval $t \in [t_j, t_{j+1})$ obeys

$$\Sigma_{k_j}: \begin{cases} \dot{x}(t) = A_{k_j}x(t) + \bar{A}_{k_j}x(t-\tau_{k_j}), & t \in [t_j,\ t_{j+1}), \\ x_{t_j}(\theta) = \phi_j(\theta) \ \ \forall \theta \in [-\tau_{k_j}, 0], \end{cases} \tag{2.3}$$

where $\phi_j(\theta)$ is defined as

$$\phi_j(\theta) = \begin{cases} x(t_j+\theta), & -\tau_{k_j} \le \theta < 0, \\ \lim_{h\to 0^-} x(t_j+h), & \theta = 0. \end{cases} \tag{2.4}$$

We introduce the triplet $\Sigma_i := (A_i, \bar{A}_i, \tau_i) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^+$ to describe the $i$th candidate system of (2.2). Thus $\forall t \ge 0$, we have $\Sigma_t \in \mathcal{A} := \{\Sigma_i : i \in \mathcal{F}\}$, where $\mathcal{A}$ is the family of candidate systems of (2.2). In (2.2), $\phi(\cdot) : [-\tau_{max}, 0] \to \mathbb{R}^n$ is a continuous and bounded vector-valued function, where $\tau_{max} = \max_{i\in\mathcal{F}}\{\tau_i\}$ is the maximal time delay of the candidate systems in $\mathcal{A}$.

Similar to [8], we say that the switched time delay system $\Sigma_t$ described by (2.2) is *stable* if there exists a function $\bar{\alpha}$ of class $\mathcal{K}$ [1] such that

$$\|x(t)\| \le \bar{\alpha}(|x|_{[t_0-\tau_{max}, t_0]}) \quad \forall t \ge t_0 \ge 0 \tag{2.5}$$

along the trajectory of (2.2). Furthermore, $\Sigma_t$ is *asymptotically stable* when $\Sigma_t$ is stable and $\lim_{t\to+\infty} x(t) = 0$.

LEMMA 2.1 (see [3, 14]). *Suppose for a given triplet $\Sigma_i \in \mathcal{A}$, $i \in \mathcal{F}$, there exists symmetric and positive-definite $P_i \in \mathbb{R}^{n\times n}$, such that the following LMI with respect to $P_i$ is satisfied for some $p_i > 1$ and $\alpha_i > 0$:*

$$\begin{bmatrix} P_iA_i + A_i^TP_i + p_i\alpha_iP_i & P_i\bar{A}_i \\ \bar{A}_i^TP_i & -\alpha_iP_i \end{bmatrix} < 0. \tag{2.6}$$

*Then $\Sigma_i$ is asymptotically stable independent of delay.*

If all candidate systems of (2.2), $\Sigma_i \in \mathcal{A}$, are delay-independently asymptotically stable satisfying (2.6), we denote $\mathcal{A}$ by $\tilde{\mathcal{A}}$.

LEMMA 2.2 (see [3, 14]). *Suppose for a given triplet $\Sigma_i \in \mathcal{A}$, $i \in \mathcal{F}$, there exists symmetric and positive-definite $P_i \in \mathbb{R}^{n\times n}$, and a scalar $p_i > 1$, such that*

$$\begin{bmatrix} \tau_i^{-1}\Omega_i & P_i\bar{A}_iM_i \\ M_i^T\bar{A}_i^TP_i & -R_i \end{bmatrix} < 0, \tag{2.7}$$

*where*

$$\begin{aligned} \Omega_i &= (A_i+\bar{A}_i)^TP_i + P_i(A_i+\bar{A}_i) + \tau_ip_i(\alpha_i+\beta_i)P_i, \\ M_i &= [A_i \ \ \bar{A}_i], \\ R_i &= \mathrm{diag}(\alpha_iP_i, \beta_iP_i), \end{aligned}$$

*and $\alpha_i > 0$, $\beta_i > 0$ are scalars. Then $\Sigma_i$ is asymptotically stable dependent on delay.*

Similarly we denote $\mathcal{A}$ by $\tilde{\mathcal{A}}_d$ if all candidate systems of (2.2) are delay-dependently asymptotically stable satisfying (2.7).

[1] A continuous function $\bar{\alpha}(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is a class $\mathcal{K}$ function if it is strictly increasing and $\bar{\alpha}(0) = 0$.

In what follows, we will establish sufficient conditions to guarantee stability of switched system (2.2) for the delay-independent case and the delay-dependent case. Therefore, we will assume that $\mathcal{A} = \tilde{\mathcal{A}}$ and $\mathcal{A} = \tilde{\mathcal{A}}_d$, respectively, in the corresponding sections in this paper. An important method in stability analysis of switched systems is based on the construction of the common Lyapunov function (CLF), which allows for arbitrary switching. However, this method is too conservative from the perspective of controller design because it is usually difficult to find the CLF for all the candidate systems, particularly for time delay systems whose stability criteria are only sufficient in most of the circumstances. A recent paper [28] explored the CLF method for switched time delay systems with three very strong assumptions: (i) each candidate system has the same time delay $\tau$; (ii) each candidate is assumed to be delay-independently stable; (iii) the $A$-matrix is always symmetric and the $\bar{A}$-matrix is always in the form of $\delta I$. In the present paper, we consider an alternative method using piecewise Lyapunov–Razumikhin functions for a general class of systems (2.2) and obtain stability conditions in terms of the dwell time of the switching signal. This method can be used for the case with delay-independent criterion (2.6) and the case with delay-dependent criterion (2.7).

**3. Main results on dwell-time-based switching.** For a given positive constant $\tau_D$, the switching signal set based on the dwell time $\tau_D$ is denoted by $S[\tau_D]$, where for any switching signal $q(t) \in S[\tau_D]$, the distance between any consecutive discontinuities of $q(t)$, $t_{j+1} - t_j$, $j \in \mathbb{Z}^+ \cup \{0\}$, is larger than $\tau_D$ [10, 19]. A sufficient condition on the minimum dwell time to guarantee the stable switching will be given using piecewise Lyapunov–Razumikhin functions. Note that the dwell-time-based switching is trajectory independent [8].

Before presenting the main result of this paper, we recall the following lemma [7] for general RFDEs (2.1).

LEMMA 3.1 (see [7]). *Suppose $u, v, w, p : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous, nondecreasing functions, $u(0) = v(0) = 0$, $u(s)$, $v(s)$, $w(s)$, $p(s)$ positive for $s > 0$, $p(s) > s$, and $v(s)$ strictly increasing. If there is a continuous function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that*

$$u(\|x(t)\|) \le V(t,x) \le v(\|x(t)\|), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n, \tag{3.1}$$

*and*

$$\dot{V}(t, x(t)) \le -w(\|x(t)\|) \tag{3.2}$$

*if*

$$V(t+\theta, x(t+\theta)) < p(V(t, x(t))) \quad \forall \theta \in [-r, 0], \tag{3.3}$$

*then the solution $x = 0$ of the RFDE is uniformly asymptotically stable.*

A particular case of (2.1) is a linear time delay system $\Sigma_i$, $i \in \mathcal{F}$, where we can construct the corresponding Lyapunov–Razumikhin function in the quadratic form

$$V_i(t,x) = x^T(t) P_i x(t), \quad P_i = P_i^T > 0. \tag{3.4}$$

Apparently $V_i$ can be bounded by

$$u_i(\|x(t)\|) \le V_i(t,x) \le v_i(\|x(t)\|) \ \forall x \in \mathbb{R}^n, \tag{3.5}$$

where

$$u_i(s) := \kappa_i s^2, \quad v_i(s) := \bar{\kappa}_i s^2, \tag{3.6}$$

in which $\kappa_i := \sigma_{min}[P_i] > 0$ denotes the smallest singular value of $P_i$ and $\bar{\kappa}_i := \sigma_{max}[P_i] > 0$ the largest singular value of $P_i$.

PROPOSITION 3.2. *For each time delay system $\Sigma_i$ with Lyapunov–Razumikhin function defined by* (3.4)*, assume that* (3.2) *and* (3.3) *are satisfied for some $w_i(s)$. Then we have*

$$|x|_{[t_m-\tau_i,t_m]} \le \left(\frac{\bar{\kappa}_i}{\kappa_i}\right)^{1/2} |x|_{[t_n-\tau_i,t_n]} \quad \forall t_m \ge t_n \ge 0. \tag{3.7}$$

*Proof.* Define

$$\bar{V}_i(t,x) := \sup_{-\tau_i \le \theta \le 0} V_i(t+\theta, x(t+\theta)) \tag{3.8}$$

for $t \ge 0$. We have

$$\kappa_i(|x|_{[t-\tau_i,t]})^2 \le \bar{V}_i(t,x) \le \bar{\kappa_i}(|x|_{[t-\tau_i,t]})^2, \quad t \ge 0. \tag{3.9}$$

The definition of $\bar{V}_i(t,x)$ implies $\exists \theta_0 \in [-\tau_i, 0]$, such that $\bar{V}_i(t,x) = V(t+\theta_0, x(t+\theta_0))$. Introduce the upper right-hand derivative of $\bar{V}_i(t,x)$ as

$$\dot{\bar{V}}_i^+ = \limsup_{h\to 0^+} \frac{1}{h}[\bar{V}_i(t+h, x(t+h)) - \bar{V}_i(t, x(t))].$$

We have the following:

(i) If $\theta_0 = 0$, i.e., $V_i(t+\theta, x(t+\theta)) \le V_i(t, x(t)) < p(V_i(t, x(t)))$, we have $\dot{V}_i(t,x) < 0$ by (3.2). Therefore $\dot{\bar{V}}_i^+ \le 0$.

(ii) If $-\tau_i < \theta_0 < 0$, we have $\bar{V}_i(t+h, x(t+h)) = \bar{V}_i(t,x)$ for $h > 0$ sufficiently small, which results in $\dot{\bar{V}}_i^+ = 0$.

(iii) If $\theta_0 = -\tau_i$, the continuity of $V_i(t,x)$ implies $\dot{\bar{V}}_i^+ \le 0$.

The above analysis shows that

$$\bar{V}_i(t_m) \le \bar{V}_i(t_n) \quad \forall t_m \ge t_n \ge 0. \tag{3.10}$$

Recalling (3.9), we have

$$\kappa_i(|x|_{[t_m-\tau_i,t_m]})^2 \le \bar{V}_i(t_m) \le \bar{V}_i(t_n) \le \bar{\kappa}_i(|x|_{[t_n-\tau_i,t_n]})^2 \tag{3.11}$$

for any $t_m \ge t_n \ge 0$. This implies (3.7) and proves the result. ∎

Suppose all of the conditions of Lemma 3.1 are satisfied for general RFDEs (2.1). We also have the following result.

LEMMA 3.3 (see [7]). *Suppose $|\phi|_{[t_0-r,t_0]} \le \bar{\delta}_1$, $\bar{\delta}_1 > 0$, and $\bar{\delta}_2 > 0$ such that $v(\bar{\delta}_1) = u(\bar{\delta}_2)$. For all $\eta$ satisfying $0 < \eta \le \bar{\delta}_2$, we have*

$$V(t,x) \le u(\eta) \quad \forall t \ge t_0 + T. \tag{3.12}$$

*Here*

$$T = \frac{Nv(\bar{\delta}_1)}{\gamma} \tag{3.13}$$

*is defined by $\gamma = \inf_{v^{-1}(u(\eta)) \le s \le \bar{\delta}_2} w(s)$ and $N = \lceil (v(\bar{\delta}_1) - u(\eta))/a \rceil$, where $\lceil \cdot \rceil$ is the ceiling integer function and $a > 0$ satisfies $p(s) - s > a$ for $u(\eta) \le s \le v(\bar{\delta}_1)$.*

**3.1. The case with delay-independent criterion.** Consider the switched time delay systems $\Sigma_t$ defined by (2.2) and assume each candidate system $\Sigma_i$, $i \in \mathcal{F}$, delay-independently asymptotically stable satisfying (2.6) (i.e., $\mathcal{A} = \tilde{\mathcal{A}}$). A sufficient condition on the minimum dwell time to guarantee the asymptotic stability can be derived using multiple piecewise Lyapunov–Razumikhin functions. In order to state the main result we make some preliminary definitions.

For the switched delay systems (2.2), first assume $\tau_D > \tau_{max}$. Consider an arbitrary switching interval $[t_j, t_{j+1})$ of the piecewise switching signal $q(t) \in S[\tau_D]$, where $q(t) = k_j$, $k_j \in \mathcal{F}$ $\forall t \in [t_j, t_{j+1})$ and $t_j$ is the $j$th switching time instant for $j \in \mathbb{Z}^+ \cup \{0\}$ and $t_0 = 0$. The state variable $x_j(t)$ defined on this interval obeys (2.3). For the convenience of using "sup", we define $x_j(t_{j+1}) = \lim_{h\to 0^-} x_j(t_{j+1}+h) = x_{j+1}(t_{j+1})$ based on the fact that $x(t)$ is continuous for $t \geq 0$. Therefore $x_j(t)$ is now defined on a compact set $[t_j, t_{j+1}]$. Recall (2.4); the initial condition $\phi_j(t)$ of $\Sigma_{k_j}$ is $\phi_j(t) = x(t) = x_{j-1}(t), t \in [t_j - \tau_{k_j}, t_j]$ for $j \in \mathbb{Z}^+$, which is true because $\tau_D > \tau_{max}$.

Construct the Lyapunov–Razumikhin function

$$V_{k_j}(x_j, t) = x_j^T(t) P_{k_j} x_j(t), \quad t \in [t_j, t_{j+1}], \tag{3.14}$$

for (2.3). Then we have

$$\kappa_{k_j} \|x_j(t)\|^2 \leq V_{k_j}(t, x_j) \leq \bar{\kappa}_{k_j} \|x_j(t)\|^2 \quad \forall x_j \in \mathbb{R}^n. \tag{3.15}$$

A straightforward calculation gives the time derivative of $V_{k_j}(t, x_j(t))$ along the trajectory of (2.3),

$$\dot{V}_{k_j}(t, x_j) = x_j^T (A_{k_j}^T P_{k_j} + P_{k_j} A_{k_j}) x_j + 2x_j^T(t) P_{k_j} \bar{A}_{k_j} x_j(t - \tau_{k_j}), \tag{3.16}$$

where

$$\begin{aligned} & 2x_j^T(t) P_{k_j} \bar{A}_{k_j} x_j(t - \tau_{k_j}) \\ \leq\ & \alpha_{k_j} x_j^T(t - \tau_{k_j}) P_{k_j} x_j(t - \tau_{k_j}) \\ & + \alpha_{k_j}^{-1} x_j^T(t) P_{k_j} \bar{A}_{k_j} P_{k_j}^{-1} \bar{A}_{k_j}^T P_{k_j} x_j(t) \quad \forall \alpha_{k_j} > 0. \end{aligned}$$

Applying the Razumikhin condition with $p(s) = p_{k_j} s$, $p_{k_j} > 1$, we obtain

$$x_j^T(t - \tau_{k_j}) P_{k_j} x_j(t - \tau_{k_j}) \leq p_{k_j} x_j^T(t) P_{k_j} x_j(t) \tag{3.17}$$

for

$$V_{k_j}(t + \theta, x_j(t + \theta)) < p_{k_j} V_{k_j}(t, x_j(t)) \quad \forall \theta \in [-\tau_{k_j}, 0].$$

Let

$$S_{k_j} := -(A_{k_j}^T P_{k_j} + P_{k_j} A_{k_j} + p_{k_j} \alpha_{k_j} P_{k_j} + \alpha_{k_j}^{-1} P_{k_j} \bar{A}_{k_j} P_{k_j}^{-1} \bar{A}_{k_j}^T P_{k_j}). \tag{3.18}$$

We have

$$\dot{V}_{k_j}(t, x_j) \leq -x_j^T(t) S_{k_j} x_j(t). \tag{3.19}$$

Because $\Sigma_t \in \tilde{\mathcal{A}}$, we have $S_{k_j} > 0$ from Lemma 2.1. Furthermore we can select $w(s) = w_{k_j} s^2$ in Lemma 3.1, such that (3.2) is satisfied, where $w_{k_j} := \sigma_{min}[S_{k_j}] > 0$.

Define

$$\lambda := \max_{i\in\mathcal{F}} \frac{\bar{\kappa}_i}{\kappa_i} \tag{3.20}$$

and

$$\mu := \max_{i\in\mathcal{F}} \frac{\bar{\kappa}_i}{w_i}. \tag{3.21}$$

Now we are ready to state the main result.

THEOREM 3.4. *Let the dwell time be defined by* $\tau_D := T^* + \tau_{max}$, *where*

$$T^* := \lambda\mu \left\lfloor \frac{\lambda - 1}{\bar{p} - 1} + 1 \right\rfloor, \tag{3.22}$$

*with* $\bar{p} := \min_{i\in\mathcal{F}}\{p_i\} > 1$, *and* $\lfloor\cdot\rfloor$ *being the floor integer function. Then the system* (2.2) *with* $\Sigma_t \in \tilde{\mathcal{A}}$ *is asymptotically stable for any switching rule* $q(t) \in S[\tau_D]$.

*Proof.* First we claim that $\forall\tau > \tau_D$, there exist $0 < \beta < 1$ and $0 < \alpha < 1$, such that $\tau \geq \bar{T} + \tau_{max}$, where

$$\bar{T} := \frac{\lambda\mu}{\alpha^2} \left\lceil \frac{\lambda - \alpha^2}{\alpha^2\beta(\bar{p} - 1)} \right\rceil. \tag{3.23}$$

For a given $\tau$, to find such $\alpha$ and $\beta$ define $\tilde{T} + \tau_{max} := \tau > \tau_D = T^* + \tau_{max}$, and consider the two cases below.

(1) If $\lfloor(\lambda - 1)/(\bar{p} - 1)\rfloor =: k < (\lambda - 1)/(\bar{p} - 1) < k + 1$, then we can find $\Delta_1 > 0$ and $\Delta_2 > 0$ small enough, such that

$$\left\lceil \frac{\lambda - \alpha_1^2}{\alpha_1^2\beta(\bar{p} - 1)} \right\rceil = \left\lceil \frac{\lambda - 1}{\bar{p} - 1} \right\rceil = k + 1 = \left\lfloor \frac{\lambda - 1}{\bar{p} - 1} + 1 \right\rfloor$$

with $\alpha_1 = (1 + \Delta_1)^{-\frac{1}{2}} < 1$ and $\beta = (1 + \Delta_2)^{-\frac{1}{2}} < 1$. Let $\tilde{T} = T^* + \epsilon$, $\epsilon > 0$. It is easy to check that

$$\frac{\lambda\mu}{\alpha_2^2} \left\lceil \frac{\lambda - \alpha_1^2}{\alpha_1^2\beta(\bar{p} - 1)} \right\rceil = \frac{\lambda\mu}{\alpha_2^2}(k + 1) \leq (k + 1)\lambda\mu + \epsilon = \tilde{T}, \tag{3.24}$$

where $0 < \alpha_2 = (1 + \Delta_3)^{-\frac{1}{2}} < 1$ with $0 < \Delta_3 \leq \frac{\epsilon}{(k+1)\lambda\mu}$. Now choosing $0 < \alpha = \max\{\alpha_1, \alpha_2\} < 1$, we have $\bar{T} \leq \tilde{T}$, which is straightforward from (3.23) and (3.24).

(2) If $(\lambda - 1)/(\bar{p} - 1) = k > 0$ is an integer, then we can similarly find $0 < \alpha_1 < 1$ and $0 < \beta < 1$ such that

$$\left\lceil \frac{\lambda - \alpha_1^2}{\alpha_1^2\beta(\bar{p} - 1)} \right\rceil = \left\lceil \frac{\lambda - 1}{\bar{p} - 1} + 1 \right\rceil = k + 1 = \left\lfloor \frac{\lambda - 1}{\bar{p} - 1} + 1 \right\rfloor.$$

In the same fashion as (1), we can constructively have $0 < \alpha < 1$ and $0 < \beta < 1$ such that $\bar{T} \leq \tilde{T}$.

This proves the first claim.

The second claim we make is that $\|x_j(t)\| \leq \alpha\delta_j$ for any $t \geq t_j + \bar{T}, t \in [t_j, t_{j+1}]$, where we assume $|\phi_j(t)|_{[t_j - \tau_{k_j}, t_j]} \leq \delta_j$. To show this fact, we can choose $\bar{\delta}_1 = \delta_j$,

$\bar{\delta}_2 = \bar{\delta}_1 \sqrt{\bar{\kappa}_{k_j}/\kappa_{k_j}} \geq \bar{\delta}_1$, and select $\eta = \alpha\bar{\delta}_1$ in Lemma 3.3. It is straightforward that $0 < \eta < \bar{\delta}_1 \leq \bar{\delta}_2$. Recalling (3.12) and (3.13), we have

$$V_{k_j}(t, x_j) \leq \kappa_{k_j}\eta^2 \quad \text{for} \quad t \geq t_j + T, \tag{3.25}$$

where

$$\begin{aligned} T &= \frac{Nv(\bar{\delta}_1)}{\gamma} \frac{\lceil (v(\bar{\delta}_1) - u(\eta))/a \rceil v(\bar{\delta}_1)}{\inf_{v^{-1}(u(\eta)) \leq s \leq \bar{\delta}_2} w(s)} \\ &= \frac{\bar{\kappa}_{k_j}^2 \lceil (v(\bar{\delta}_1) - u(\eta))/a \rceil}{\alpha^2 w_{k_j} \kappa_{k_j}}. \end{aligned} \tag{3.26}$$

Combining (3.15) and (3.25) yields

$$\|x_j(t)\| \leq \alpha\delta_j \quad \text{for} \quad t \geq t_j + T. \tag{3.27}$$

Now choosing $a = \beta(p_{k_j} - 1)\kappa_{k_j}\eta^2$, we have

$$T = \frac{\bar{\kappa}_{k_j}^2 \left\lceil \frac{(\bar{\kappa}_{k_j}/\kappa_{k_j}) - \alpha^2}{\alpha^2 \beta(p_{k_j} - 1)} \right\rceil}{\alpha^2 w_{k_j} \kappa_{k_j}} \leq \bar{T}. \tag{3.28}$$

Therefore from (3.27) and (3.28) we have

$$|x_j|_{[t_j + \bar{T}, t_{j+1}]} \leq \alpha\delta_j, \tag{3.29}$$

as claimed.

Now recall that $t_{j+1} - t_j > \tau_D$. Therefore $t_{j+1} - t_j \geq \bar{T} + \tau_{max} \geq \bar{T} + \tau_{k_{j+1}}$. Also notice that $\phi_{j+1}(t) = x_j(t), t \in [t_{j+1} - \tau_{k_{j+1}}, t_{j+1}]$. We have

$$\begin{aligned} |\phi_{j+1}|_{[t_{j+1} - \tau_{k_{j+1}}, t_{j+1}]} &= |x_j|_{[t_{j+1} - \tau_{k_{j+1}}, t_{j+1}]} \\ &\leq |x_j|_{[t_j + \bar{T}, t_{j+1}]} \leq \alpha\delta_j := \delta_{j+1} \end{aligned} \tag{3.30}$$

and $\delta_0$ is defined as $\delta_0 := |\phi|_{[-\tau_{max}, 0]} \geq |\phi|_{[-\tau_{k_0}, 0]}$. Therefore we obtain a convergent sequence $\{\delta_i\}, i = 0, 1, 2, \ldots$, where $\delta_i = \alpha^i \delta_0$.

Meanwhile, (3.7) implies

$$|x_j|_{[t - \tau_{k_j}, t]} \leq \sqrt{\frac{\bar{\kappa}_{k_j}}{\kappa_{k_j}}} |x_j|_{[t_j - \tau_{k_j}, t_j]} \quad \forall t \in [t_j, t_{j+1}]. \tag{3.31}$$

Hence

$$\begin{aligned} &\sup_{t \in [t_j, t_{j+1}]} \|x_j(t)\| \\ &\leq \sup_{t \in [t_j, t_{j+1}]} |x_j|_{[t - \tau_{k_j}, t]} \leq \sqrt{\lambda} |x_j|_{[t_j - \tau_{k_j}, t_j]} \\ &\leq \sqrt{\lambda}\delta_j = \alpha^j \sqrt{\lambda}\delta_0, \end{aligned} \tag{3.32}$$

which implies the asymptotic stability of the switched time delay system $\Sigma_t$ with the switching signal $q(t) \in S_{[\tau_D]}$. $\square$

**3.2. The case with delay-dependent criterion.** In a similar fashion, we can investigate the stability of the switched time delay system $\Sigma_t$ of (2.2) under the assumption that $\Sigma_t \in \bar{\mathcal{A}}_d$. Hence each candidate system $\Sigma_i$, $i \in \mathcal{F}$, is delay-dependently asymptotically stable satisfying (2.7). We assume $\tau_D^d > 2\tau_{max}$ in this scenario. Similar to the proof of Theorem 3.4, we consider an arbitrary switching interval $[t_j, t_{j+1})$ of the piecewise switching signal $q(t) \in S[\tau_D^d]$ , where the state variable $x_j(t)$ defined on this interval obeys (2.3). The first order model transformation [7] of (2.3) results in

$$\begin{aligned} \dot{x}_j(t) &= (A_{k_j} + \bar{A}_{k_j})x_j(t) \\ &\quad - \bar{A}_{k_j} \int_{-\tau_{k_j}}^{0} [A_{k_j} x_j(t+\theta) + \bar{A}_{k_j} x(t+\theta-\tau_{k_j})] d\theta, \end{aligned} \tag{3.33}$$

where the initial condition $\psi_j(t)$ is defined as $\psi_j(t) = x_{j-1}(t)$, $t \in [t_j - 2\tau_{k_j}, t_j]$, for $j \in \mathbb{Z}^+$, and $\psi_0(t)$ defined by

$$\psi_0(t) = \begin{cases} \phi(t), & t \in [-\tau_{max}, 0], \\ \phi(-\tau_{max}), & t \in [-2\tau_{max}, -\tau_{max}). \end{cases}$$

By using the Lyapunov–Razumikhin function (3.14), we obtain the time derivative of $V_{k_j}(t, x_j(t))$ along the trajectory of (3.33),

$$\begin{aligned} \dot{V}_{k_j}(t, x_j) &= x_j^T(t)[P_{k_j}(A_{k_j} + \bar{A}_{k_j}) + (A_{k_j} + \bar{A}_{k_j})^T P_{k_j}] x_j(t) \\ &\quad - \int_{-\tau_{k_j}}^{0} [2x_j^T(t) P_{k_j} \bar{A}_{k_j} (A_{k_j} x_j(t+\theta) + \bar{A}_{k_j} x_j(t+\theta-\tau_{k_j})] d\theta. \end{aligned}$$

Assume $V_{k_j}(t+\theta, x_j(t+\theta)) < p(V_{k_j}(t, x_j(t)))$ $\forall \theta \in [-2\tau_{k_j}, 0]$, where $p(s) = p_{k_j} s$, $p_{k_j} > 1$. We have [3, 14]

$$\dot{V}_{k_j}(t, x_j) \leq -x_j^T(t) S_{k_j}^d x_j(t), \tag{3.34}$$

where

$$\begin{aligned} S_{k_j}^d := -\Big\{ & P_{k_j}(A_{k_j} + \bar{A}_{k_j}) + (A_{k_j} + \bar{A}_{k_j})^T P_{k_j} \\ & + \tau_{k_j} \Big[ \alpha_{k_j}^{-1} P_{k_j} \bar{A}_{k_j} A_{k_j} P_{k_j}^{-1} \bar{A}_{k_j}^T A_{k_j}^T P_{k_j} \\ & + \beta_i^{-1} P_{k_j} (\bar{A}_{k_j})^2 P_{k_j}^{-1} (\bar{A}_{k_j}^T)^2 P_{k_j} + p_{k_j}(\alpha_{k_j} + \beta_{k_j}) P_{k_j} \Big] \Big\}. \end{aligned} \tag{3.35}$$

Because $\Sigma_t \in \tilde{\mathcal{A}}_d$, we have $S_{k_j}^d > 0$ from Lemma 2.2. Therefore we can select $w(s) = w_{k_j}^d s^2$ in Lemma 3.1, such that (3.2) holds, where $w_{k_j}^d := \sigma_{min}[S_{k_j}^d] > 0$.

THEOREM 3.5. *Let the dwell time be* $\tau_D^d := T_d^* + 2\tau_{max}$, *where*

$$T_d^* := \lambda \mu_d \left\lfloor \frac{\lambda - 1}{\bar{p} - 1} + 1 \right\rfloor, \tag{3.36}$$

*with*

$$\mu_d := \max_{i \in \mathcal{F}} \frac{\bar{\kappa}_i}{w_i^d} \tag{3.37}$$

*and the other parameters are the same as those defined in Theorem* 3.4*. Then system* (2.2) *with* $\Sigma_t \in \tilde{\mathcal{A}}_d$ *is asymptotically stable for any switching rule* $q(t) \in S[\tau_D^d]$*.*

*Proof.* We can apply arguments similar to those used in the proof of Theorem 3.4 to obtain the following inequality:

$$\sup_{t\in[t_j,t_{j+1}]} \|x_j(t)\| \leq \sqrt{\lambda}\delta_j^d, \tag{3.38}$$

where $|\psi_j(t)|_{[t_j-2\tau_{k_j},t_j]} \leq \delta_j^d$ and $\delta_{j+1}^d = \alpha\delta_j^d$. Note that $\delta_0^d$ can be selected as

$$\delta_0^d := |\psi|_{[-2\tau_{max},0]} = |\phi|_{[-\tau_{max},0]} = \delta_0.$$

It is clear that $|\psi|_{[-2\tau_{k_0},0]} \leq \delta_0^d$, which further implies $\delta_j^d = \delta_j$, $j \in \mathbb{Z}^+ \cup \{0\}$. The upper bound of the state variable $x(t)$ of the switched time delay systems $\Sigma_t$ is bounded by a decreasing sequence $\{\delta_i\}, i = 0, 1, 2, \ldots,$ converging to zero, which implies the asymptotic stability and proves this theorem. ∎

The dwell-time-based stability analysis proposed in this paper is general in the sense that it can be used for other stability results based on Razumikhin theorems as long as the correspondingly Lyapunov functions are in quadratic forms. Particularly, Theorem 3.5 can be extended easily to the case where $\Sigma_t$ has time-varying time delays and parameter uncertainties, which has important applications such as TCP (transmission control protocol) congestion control of computer networks [13, 26].

*Remark* 3.6. Note that the Lyapunov–Krasovskii method has been used to analyze the stability of time delay systems, with which some less conservative stability conditions have been provided [5]. However, it is difficult to employ piecewise Lyapunov–Krasovskii functionals for dwell-time-based analysis similar to Theorems 3.4 and 3.5. Recall the general form of Lyapunov–Krasovskii functional $V(t, x_t)$ [20] for delay system (2.1), such that

$$u(\|x(t)\|) \leq V(t, x_t) \leq v(|x|_{[t-\tau,t]}), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n,$$

and

$$\dot{V}(t, x(t)) \leq -w(\|x(t)\|).$$

The upper bound of $V(t, x_t)$ is dependent on the $\infty$-norm of the trajectory, while other bounds on $V(t, x_t)$ and $\dot{V}(t, x_t)$ are on the Euclidean norm of the trajectory, which poses the technical challenge of estimating the trajectory bound and decaying rate for the switched delay systems (2.2).

**4. Conservatism analysis.** The dwell-time-based stability results had been obtained for switched linear systems free of delays [10, 19]. It is interesting to compare the conservatism of the results presented in this paper with those for delay-free systems.

In fact, one extreme case of the switched system $\Sigma_t$ is $\tau_i = 0$ and $\bar{A}_i = 0$ for $i \in \mathcal{A}$, which corresponds to the delay-free scenario. For each candidate system $\dot{x} = A_i x$, a sufficient and necessary condition to guarantee asymptotic stability is $\exists P_i = P_i^T > 0$, such that $Q_i := -(A_i^T P_i + P_i A_i) > 0$. Correspondingly a dwell-time-based stability for such a switched delay-free system is $q(t) \in S_{[\tilde{\tau}_D]}$, where

$$\tilde{\tau}_D = \tilde{\mu} \ln \lambda, \tag{4.1}$$

where $\lambda$ is defined by (3.20) and

$$\tilde{\mu} := \max_{i\in\mathcal{F}} \frac{\bar{\kappa}_i}{\tilde{w}_i}, \tag{4.2}$$

where $\tilde{w}_i := \sigma_{min}[Q_i] > 0$.

On the other hand, in our case for $\tau_i = 0$ and $\bar{A}_i = 0$, we observe that

$$\lim_{\alpha_i\to 0^+} S_i = \lim_{\alpha_i,\beta_i\to 0^+} S_i^d = Q_i, \quad i \in \mathcal{F}, \tag{4.3}$$

from (3.18) and (3.35), which indicates $\mu = \mu_d = \tilde{\mu}$ by (3.21), (3.37), and (4.2). Accordingly we can select $p_i > 1$, $i \in \mathcal{F}$, sufficiently large such that $\lfloor \frac{\lambda-1}{\bar{p}-1} + 1 \rfloor = 1$ in (3.22) and (3.36) and obtain

$$\tau_D = T^* = \lambda\mu = \lambda\mu_d = T_d^* = \tau_D^d. \tag{4.4}$$

Therefore

$$\tau_D = \tau_D^d = \lambda\tilde{\mu} > \tilde{\mu}\ln\lambda = \tilde{\tau}_D. \tag{4.5}$$

The dwell times derived for switched time delay systems are proportional to $\lambda$, in contrast to the logarithm of $\lambda$ for switched delay-free systems. This gap is due to the fact that asymptotic stability for linear delay-free systems implies exponential stability. However, for time delay systems, the sufficient stability conditions based on the Lyapunov–Razumikhin theorem do not guarantee exponential stability. As a matter of fact, the exponential estimates for time delay systems require additional assumptions besides asymptotic stability [15].

It should be noted that stability conditions for switched time delay systems are also considered in [22, 23], where the authors give a sufficient condition to guarantee *uniform* stability (see Theorem 6.1 of [22] for notation and details): $\Gamma e^{L(\Lambda+h)} \le 1$. Apparently, this condition does not hold for the switched system (2.2) because in our case $\Gamma = 1$, and hence

$$\Gamma e^{L(\Lambda+h)} = e^{L(\Lambda+h)} > 1 \quad \forall\Lambda > 0,\ L > 0,\ h > 0.$$

**5. Numerical example.** In this section, we use an illustrative example to demonstrate the results in section 3.

*Example.* Consider the following switched time delay system with 2 candidates:

$$\Sigma_t : \begin{cases} \quad \dot{x} = A_{q(t)}x(t) + \bar{A}_{q(t)}x(t - \tau_{q(t)}), \quad t \ge 0, \\ x(t) = \phi(t) \ \ \forall t \in [-\tau_{max}, 0], \\ q(t) \in \{1, 2\}, \end{cases} \tag{5.1}$$

where the switching candidate systems $\Sigma_1 := (A_1, \bar{A}_1, \tau_1)$ and $\Sigma_2 := (A_2, \bar{A}_2, \tau_2)$ are determined by

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix} -1 & 0 \\ -0.5 & -1 \end{bmatrix};$$

$$A_2 = \begin{bmatrix} -1 & 0.5 \\ 0 & -1 \end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix} -1 & 0 \\ 0.1 & -1 \end{bmatrix};$$

TABLE 5.1
*Parameters calculated with respect to switched time delay system $\Sigma_t$ and switched delay-free system $\bar{\Sigma}_t$.*

| Parameters | $\Sigma_t$ (with delay) | $\bar{\Sigma}_t$ (delay free) |
|---|---|---|
| $\lambda$ | 1.7224 | 1.7224 |
| $\mu_d$ | 1.5216 | N/A |
| $\tilde{\mu}_d$ | N/A | 0.7056 |
| $\bar{p}$ | 1.4 | N/A |
| $T_d^*$ | 5.3147 | N/A |
| Dwell time | $\tau_D^d = 6.5147$ | $\tilde{\tau}_D = 0.3836$ |

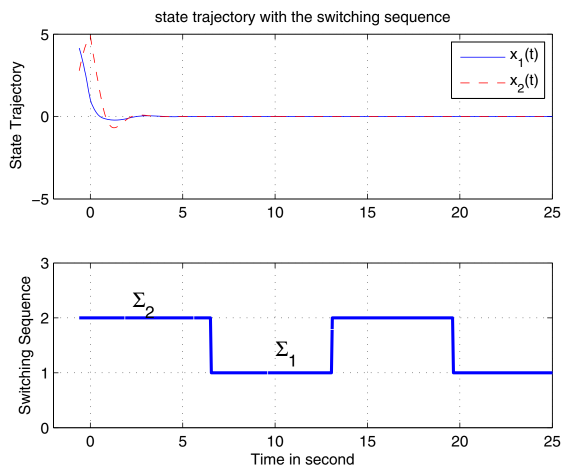


FIG. 5.1. *The state trajectory of $\Sigma_t$ in the presence of switching.*

and $\tau_1 = 0.3, \tau_2 = 0.6$. The initial condition of (5.1) is chosen as

$$\phi(t) = \begin{bmatrix} 5\cos(\frac{\pi}{2.4}t + \frac{\pi}{6}) \\ 5\sin(\frac{\pi}{2.4}t + \frac{\pi}{6}) \end{bmatrix} \quad \forall t \in [-0.6, 0].$$

It is clear that $\Sigma_1$ and $\Sigma_2$ are delay-dependently stable, which can be verified by Lemma 2.2. Applying Theorem 3.5 gives the dwell time $\tau_D^d = 6.52$, which guarantees the asymptotic stability of the switched time delay system (5.1). For the purpose of comparison, we also calculate the dwell time $\tilde{\tau}_D$ of the delay-free system $\bar{\Sigma}_t$ : $\dot{x} = A_{q(t)}x(t)$, $q(t) \in \{1, 2\}$. The results are shown in Table 5.1.

The switched time delay system $\Sigma_t$ described by (5.1) is simulated in MATLAB, where we start with $\Sigma_2$ and perform switching every $\tau_D^d$ seconds. The state trajectory is depicted in Figure 5.1, where we clearly see the asymptotic convergence in the presence of switching. Also, we provide the phase portrait in Figure 5.2 with respect to $x_1(t)$ and $x_2(t)$, which better illustrates the switching and the stability of the switched system.

It is also interesting to investigate the relation between time delays of (2.2) and the corresponding dwell time $\tau_D^d$. For this purpose, we took $\tau_1 = \tau_2 = \tau$ in (5.1) with $\tau$ varying from 0.1 to 0.7. The results are shown in Table 5.2. We should also indicate that the free parameters $\alpha_i$, $\beta_i$, and $p_i$ can be further optimized to reduce the values of $\tau_D^d$ given in the table. However, this is an open problem deserving a

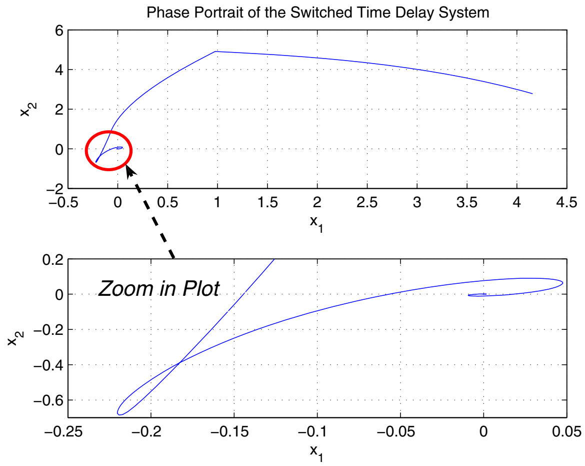


FIG. 5.2. *Phase portrait of the switched time delay system* $\Sigma_t$.

TABLE 5.2
*Dwell time values versus time delays of* $\Sigma_t$.

| $\tau$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 |
|---|---|---|---|---|---|---|---|
| $\tau_D^d$ | 0.93 | 1.49 | 3.36 | 4.83 | 9.14 | 106.23 | 950.58 |

separate study. Nevertheless, the results given in the table suggest an exponentially increasing behavior of $\tau_D^d$ with the delay. Similar behavior is observed for the $\mathcal{H}^\infty$ optimal cost in weighted sensitivity minimization for systems with delays [4].

**6. Concluding remarks.** We provided stability analysis for switched linear systems with time delays, where each candidate system is assumed to be delay-independently or delay-dependently asymptotically stable. We showed the existence of a dwell time of the switching signal, such that the switched time delay system is asymptotically stable independent of the trajectory. The dwell time values for both scenarios are constructively given. The results are compared with the dwell-time conditions for switched delay-free systems. Optimization of the minimum dwell times that we have derived, in terms of the free parameters appearing in the LMI conditions, is an interesting open problem. An interesting extension of this work is to investigate stability and controller synthesis for switched interval time delay systems, which will potentially offer a hybrid control method for large time delay systems and time varying delay systems.

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