Journal of Number Theory 129 (2009) 1366–1378

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On the representations of integers by the sextenary quadratic form $x^2 + y^2 + z^2 + 7s^2 + 7t^2 + 7u^2$ and 7-cores ☆

Alexander Berkovich [a,*], Hamza Yesilyurt [b]

[a] Department of Mathematics, University of Florida, 358 Little Hall, Gainesville, FL 32611, USA
[b] Department of Mathematics, Bilkent University, 06800 Bilkent/Ankara, Turkey

ARTICLE INFO

Article history:
Received 12 April 2008
Revised 4 July 2008
Available online 22 October 2008
Communicated by Matthias Beck

MSC:
primary 05A20, 11F27
secondary 05A19, 11P82

Keywords:
7-cores
Sextenary forms
Modular equations

ABSTRACT

In this paper we derive an explicit formula for the number of representations of an integer by the sextenary form $x^2 + y^2 + z^2 + 7s^2 + 7t^2 + 7u^2$. We establish the following intriguing inequalities

$$2\omega(n+2) \geqslant a_7(n) \geqslant \omega(n+2) \quad \text{for } n \neq 0, 2, 6, 16.$$

Here $a_7(n)$ is the number of partitions of n that are 7-cores and $\omega(n)$ is the number of representations of n by the sextenary form $(x^2 + y^2 + z^2 + 7s^2 + 7t^2 + 7u^2)/8$ with x, y, z, s, t and u being odd positive integers.

1. Introduction

Recall that a partition is called a t-core if it has no rim hooks of length t [10]. Let $a_t(n)$ be the number of t-core partitions of n. It is well known that [8,11]

$$\sum_{n\geqslant 0} a_t(n)q^n = \sum_{\vec{n}\in\mathbb{Z}^t,\ \vec{n}.\vec{1}_t=0} q^{\frac{t}{2}\|\vec{n}\|^2+\vec{b}_t.\vec{n}} = \frac{E^t(q^t)}{E(q)}, \tag{1.1}$$

where

☆ Research was supported in part by NSA grant H98230-07-01-0011.
* Corresponding author.
E-mail addresses: alexb@math.ufl.edu (A. Berkovich), hamza@fen.bilkent.edu.tr (H. Yesilyurt).

0022-314X/$ – see front matter
doi:10.1016/j.jnt.2008.09.001

$$\vec{b_t} := (0, 1, 2, \dots, t-1), \qquad \vec{1_t} := (1, 1, \dots, 1),$$
$$E(q) := \prod_{n=1}^{\infty}\left(1 - q^n\right). \tag{1.2}$$

Let

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \qquad \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}. \tag{1.3}$$

Throughout the paper we assume that q is a complex number with $|q| < 1$. For convenience, the coefficient of q^n in the expansion of $H(q)$ will be denoted as $[q^n]H(q)$. For a partition π, BG-rank(π) is defined as an alternating sum of parities of parts of π [2,3]. In [4], the authors found positive *eta*-quotient representations for the 7-core generating functions $\sum_{n\geqslant 0} a_{7,j}(n)q^n$, where $a_{7,j}(n)$ denotes the number of 7-cores of n with BG-rank $= j$ and established a number of inequalities for $a_{7,j}(n)$ with $j = -1, 0, 1, 2$ and $a_7(n)$. In this paper, we prove lower and upper bounds for $a_7(n)$, namely

Theorem 1.1.

$$\left[q^n\right]\left(1 + 5q^6 + q^{16} + 2q\psi^3(q)\psi^3\left(q^7\right)\right) \geqslant \left[q^n\right]\frac{E^7(q^7)}{E(q)}, \tag{1.4}$$

where the inequality is strict if $n \neq 0, 6$ or 16.

Theorem 1.2.

$$\left[q^n\right]\left(\frac{E^7(q^7)}{E(q)} + q^2\right) \geqslant \left[q^n\right]\left(q\psi^3(q)\psi^3\left(q^7\right)\right). \tag{1.5}$$

We should remark that the inequality in (1.5) is only strict as it can be seen in the proof of Theorem 1.2 when n is even and $n \neq 2$. Theorem 1.1 and Theorem 1.2 are proved in Sections 5 and 6.

It is well known that every integer can be written as sum of three triangular numbers, that is $[q^n]\psi^3(q) > 0$ for all $n \geqslant 0$. This together with (1.5) implies that

$$\left[q^n\right]\frac{E^7(q^7)}{E(q)} > 0 \quad \text{for all } n \geqslant 3, \text{ and hence for all } n \geqslant 0.$$

In fact, Granville and Ono showed that [9] if $t \geqslant 4$, then

$$\left[q^n\right]\frac{E^t(q^t)}{E(q)} > 0 \quad \text{for all } n \geqslant 0.$$

The lower bound given by (1.5) improves the Granville–Ono result when $t = 7$.

Essential to our proofs are the following theta function identities which we prove in Section 4 by employing the theory of modular equations

Theorem 1.3.

$$7\varphi^3(-q)\varphi^3\left(-q^7\right) = -49\left(q^2\frac{E^7(q^7)}{E(q)} + qE^3(q)E^3\left(q^7\right)\right) + 56\left(7q^4\frac{E^7(q^{14})}{E(q^2)} + q^2E^3\left(q^2\right)E^3\left(q^{14}\right)\right)$$
$$-\frac{E^7(q)}{E(q^7)} + 8\frac{E^7(q^2)}{E(q^{14})}, \tag{1.6}$$

and

$$56q^3\psi^3(q)\psi^3(q^7) = 49q^2\frac{E^7(q^7)}{E(q)} + 7qE^3(q)E^3(q^7) - 49\left(q^4\frac{E^7(q^{14})}{E(q^2)} + q^2E^3(q^2)E^3(q^{14})\right) + \frac{E^7(q)}{E(q^7)} - \frac{E^7(q^2)}{E(q^{14})}. \tag{1.7}$$

By employing Theorem 1.3, we derive explicit formulas for the number of representations of an integer by the sextenary forms $x^2+y^2+z^2+7s^2+7t^2+7u^2$, and $(x^2+y^2+z^2+7s^2+7t^2+7u^2)/8$ with x, y, z, s, t and u being odd for the later case. Before we state the formulas, we define for a prime p, $p \equiv 1, 2, 4 \pmod 7$,

$$F(p,r) := \frac{\beta^{2r+2} - \bar{\beta}^{2r+2}}{\beta^2 - \bar{\beta}^2},$$

with

$$\beta = x + \sqrt{-7}y, \qquad \bar{\beta} = x - \sqrt{-7}y,$$

where x and y are the positive unique integers satisfying $p = x^2 + 7y^2$ provided $p > 2$. If $p = 2$, then

$$\beta = (1+\sqrt{-7})/2, \qquad \bar{\beta} = (1-\sqrt{-7})/2.$$

Corollary 1.4. *Let $\nu(n)$ be the number of representations of n by the sextenary form $x^2+y^2+z^2+7s^2+7t^2+7u^2$. Suppose n has the prime factorization*

$$n = 7^c 2^d \prod_{i=1}^{r} p_i^{v_i} \prod_{j=1}^{s} q_j^{w_j},$$

with p_i odd $p_i \equiv 1, 2, 4 \pmod 7$, $q_j \equiv 3, 5, 6 \pmod 7$, and $b = \sum_{j=1}^{s} w_j$.
If n is odd, then

$$\nu(n) = \frac{1}{8}\left(7^{2c+1} - (-1)^b\right)\prod_{i=1}^{r}\frac{1-p_i^{2v_i+2}}{1-p_i^2}\prod_{j=1}^{s}\frac{(-1)^{w_j}+q_j^{2w_j+2}}{1+q_j^2} + \frac{21}{4}(-7)^c\prod_{i=1}^{r}F(p_i,v_i)\prod_{j=1}^{s}\frac{q_i^{w_j}(1+(-1)^{w_j})}{2}. \tag{1.8}$$

If n is even, then

$$\nu(n) = \frac{1}{24}\left(7^{2c+1} - (-1)^b\right)\left(4^{d+1}-7\right)\prod_{i=1}^{r}\frac{1-p_i^{2v_i+2}}{1-p_i^2}\prod_{j=1}^{s}\frac{(-1)^{w_j}+q_j^{2w_j+2}}{1+q_j^2} - \frac{3}{4}(-7)^c\left(7F(2,d)+8F(2,d-1)\right)\prod_{i=1}^{r}F(p_i,v_i)\prod_{j=1}^{s}\frac{q_i^{w_j}(1+(-1)^{w_j})}{2}. \tag{1.9}$$

Corollary 1.5. *Let $\omega(n)$ be the number of representations of n by the sextenary form $(x^2+y^2+z^2+7s^2+7t^2+7u^2)/8$ with x, y, z, s, t and u being odd positive integers. Suppose n has the prime factorization*

$$n = 7^c 2^d \prod_{i=1}^{r} p_i^{v_i} \prod_{j=1}^{s} q_j^{w_j},$$

with p_i odd $p_i \equiv 1, 2, 4 \pmod 7$, $q_j \equiv 3, 5, 6 \pmod 7$, and $b = \sum_{j=1}^{s} w_j$.

If n is odd, then

$$\omega(n) = \frac{1}{64}\bigl(7^{2c+1} - (-1)^b\bigr) \prod_{i=1}^{r} \frac{1 - p_i^{2v_i+2}}{1 - p_i^2} \prod_{j=1}^{s} \frac{(-1)^{w_j} + q_j^{2w_j+2}}{1 + q_j^2} - \frac{3}{32}(-7)^c \prod_{i=1}^{r} F(p_i, v_i) \prod_{j=1}^{s} \frac{q_i^{w_j}(1 + (-1)^{w_j})}{2}. \tag{1.10}$$

If n is even, then

$$\omega(n) = \frac{1}{64}4^d\bigl(7^{2c+1} - (-1)^b\bigr) \prod_{i=1}^{r} \frac{1 - p_i^{2v_i+2}}{1 - p_i^2} \prod_{j=1}^{s} \frac{(-1)^{w_j} + q_j^{2w_j+2}}{1 + q_j^2} - \frac{3}{32}(-7)^c\bigl(F(2,d) + 7F(2,d-1)\bigr) \prod_{i=1}^{r} F(p_i, v_i) \prod_{j=1}^{s} \frac{q_i^{w_j}(1 + (-1)^{w_j})}{2}. \tag{1.11}$$

The rest of the paper is organized as follows. In the next section, we recall two Lambert series identities of Ramanujan which we extensively use in our proofs. In Section 3, we give a brief introduction to modular equations. Then, we prove Theorem 1.3 and from it we derive Corollary 1.4 and Corollary 1.5. In Sections 5 and 6, Theorem 1.1 and Theorem 1.2 are proven.

2. Two Lambert series identities of Ramanujan

We start with two Lambert series identities of Ramanujan [6] which we will employ in our proofs.

$$L(q) := \frac{8}{7}\left(1 - \frac{E^7(q)}{E(q^7)}\right) - 7qE^3(q)E^3\bigl(q^7\bigr) = \sum_{n=1}^{\infty}\left(\frac{n}{7}\right)\frac{n^2q^n}{1-q^n} \tag{2.1}$$

and

$$K(q) := 8q^2\frac{E^7(q^7)}{E(q)} + qE^3(q)E^3\bigl(q^7\bigr) = \sum_{n=1}^{\infty}\left(\frac{n}{7}\right)\frac{q^n(1+q^n)}{(1-q^n)^3}. \tag{2.2}$$

We should remark that (2.1) and (2.2) are equivalent under the imaginary transformation [7]. It is easy to see that

$$L(q) = \sum_{n=1}^{\infty}\left(\sum_{d|n} d^2\left(\frac{d}{7}\right)\right)q^n \quad \text{and} \quad K(q) = \sum_{n=1}^{\infty}\left(\sum_{d|n} d^2\left(\frac{n/d}{7}\right)\right)q^n. \tag{2.3}$$

The coefficients of $L(q)$ and $K(q)$ are clearly multiplicative. The reader may wish to consult [1] for background on multiplicative functions, convolution of multiplicative functions and Legendre's symbol. Using multiplicity it is easy to conclude from (2.3) that

$$\left[q^n\right]L[q] = (-1)^b \prod_{i=1}^{r} \frac{1-p_i^{2v_i+2}}{1-p_i^2} \prod_{j=1}^{s} \frac{(-1)^{w_j}+q_j^{2w_j+2}}{1+q_j^2}, \tag{2.4}$$

$$\left[q^n\right]K[q] = 7^{2c} \prod_{i=1}^{r} \frac{1-p_i^{2v_i+2}}{1-p_i^2} \prod_{j=1}^{s} \frac{(-1)^{w_j}+q_j^{2w_j+2}}{1+q_j^2}, \tag{2.5}$$

where n has the prime factorization

$$n = 7^c \prod_{i=1}^{r} p_i^{v_i} \prod_{j=1}^{s} q_j^{w_j},$$

with $p_i \equiv 1, 2, 4 \pmod 7$, $q_j \equiv 3, 5, 6 \pmod 7$, and $b = \sum_{j=1}^{s} w_j$. We note that (2.5) was stated as Lemma 1 in [8].

Next, let

$$M(q) := qE^3(q)E^3\left(q^7\right). \tag{2.6}$$

From [8, p. 11, Lemma 2], we have

$$\left[q^n\right]M(q) = \begin{cases} (-7)^c \prod_{i=1}^{r} F(p_i, v_i) \prod_{j=1}^{s} q_i^{w_j} & \text{if each } w_j \text{ is even,} \\ 0 & \text{otherwise,} \end{cases} \tag{2.7}$$

where the prime factorization of n is defined as above and

$$F(p, r) := \frac{\beta^{2r+2} - \bar{\beta}^{2r+2}}{\beta^2 - \bar{\beta}^2}, \tag{2.8}$$

with

$$\beta = x + \sqrt{-7}y, \qquad \bar{\beta} = x - \sqrt{-7}y,$$

where x and y are the positive unique integers satisfying $p = x^2 + 7y^2$ provided $p \equiv 1, 2, 4 \pmod 7$ and $p > 2$. If $p = 2$, then

$$\beta = (1 + \sqrt{-7})/2, \qquad \bar{\beta} = (1 - \sqrt{-7})/2.$$

Next, we give background information on modular equations.

3. Modular equations

For $0 < k < 1$, the complete elliptic integral of the first kind $K(k)$, associated with the modulus k, is defined by

$$K(k) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

The number $k' := \sqrt{1-k^2}$ is called the *complementary modulus*. Let K, K', L, and L' denote complete elliptic integrals of the first kind associated with the moduli k, k', ℓ, and ℓ', respectively. Suppose that

$$n\frac{K'}{K} = \frac{L'}{L} \tag{3.1}$$

for some positive rational integer n. A relation between k and ℓ induced by (3.1) is called a *modular equation of degree n*. There are several definitions of a modular equation in the literature. For example, see the books by R.A. Rankin [12, p. 76] and B. Schoeneberg [13, pp. 141–142]. Following Ramanujan, set

$$\alpha = k^2 \quad \text{and} \quad \beta = \ell^2.$$

We often say that β has degree n over α. If

$$q = \exp(-\pi K'/K), \tag{3.2}$$

two of the most fundamental relations in the theory of elliptic functions are given by the formulas [5, pp. 101–102],

$$\varphi^2(q) = \frac{2}{\pi}K(k) \quad \text{and} \quad \alpha = k^2 = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}. \tag{3.3}$$

Eq. (3.3) and elementary theta function identities make it possible to write each modular equation as a theta function identity. Ramanujan derived an extensive "catalogue" of formulas [5, pp. 122–124] giving the "evaluations" of $E(q)$, $\varphi(q)$, $\psi(q)$, and $\chi(q)$ at various powers of the arguments in terms of

$$z := z_1 := \frac{2}{\pi}K(k), \quad \alpha, \quad \text{and} \quad q.$$

The evaluations that will be needed in this paper are as follows:

$$\phi(-q) = \sqrt{z}\{(1-\alpha)\}^{1/4}, \tag{3.4}$$

$$\psi(-q) = q^{-1/8}\sqrt{\frac{1}{2}z}\{\alpha(1-\alpha)\}^{1/8}, \tag{3.5}$$

$$E(-q) = 2^{-1/6}q^{-1/24}\sqrt{z}\{\alpha(1-\alpha)\}^{1/24}, \tag{3.6}$$

$$E(q^2) = 2^{-1/3}q^{-1/12}\sqrt{z}\{\alpha(1-\alpha)\}^{1/12}, \tag{3.7}$$

$$E(q^4) = 4^{-1/3}q^{-1/6}\sqrt{z}\alpha^{1/6}(1-\alpha)^{1/24}. \tag{3.8}$$

We should remark that in the notation of [5], $E(q)=f(-q)$. If q is replaced by q^n, then the evaluations are given in terms of

$$z_n := \frac{2}{\pi}K(l), \quad \beta, \quad \text{and} \quad q^n,$$

where β has degree n over α.

Lastly, the multiplier m of degree n is defined by

$$m=\frac{\varphi^2(q)}{\varphi^2(q^n)}=\frac{z}{z_n}. \tag{3.9}$$

The proofs of the following modular equations of degree 7 can be found in [5, p. 314, Entry 19(i), (iii), (viii)]

$$(\alpha\beta)^{1/8}+\big\{(1-\alpha)(1-\beta)\big\}^{1/8}=1, \tag{3.10}$$

$$m=\frac{1-4(\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)})^{1/24}}{\{(1-\alpha)(1-\beta)\}^{1/8}-(\alpha\beta)^{1/8}}, \qquad \frac{7}{m}=-\frac{1-4(\frac{\alpha^7(1-\alpha)^7}{\beta(1-\beta)})^{1/24}}{\{(1-\alpha)(1-\beta)\}^{1/8}-(\alpha\beta)^{1/8}}, \tag{3.11}$$

$$m-7/m=2\big((\alpha\beta)^{1/8}-\big\{(1-\alpha)(1-\beta)\big\}^{1/8}\big)\big(2+(\alpha\beta)^{1/4}+\big\{(1-\alpha)(1-\beta)\big\}^{1/4}\big). \tag{3.12}$$

4. Proof of Theorem 1.3

In the language of modular equations the identities (1.6) and (1.7) are reciprocals of each other [5, p. 216, Entry 24(v)] and so we only prove (1.7). In (1.7), we replace q by $-q$ and use the evaluations given in (3.5)–(3.7), we find that

$$\begin{aligned}
&-7\sqrt{z^3}\sqrt{z_7^3}\big\{\alpha\beta(1-\alpha)(1-\beta)\big\}^{3/8}\\
&=\frac{49}{2}\frac{\sqrt{z_7^7}}{\sqrt{z}}\left(\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\right)^{1/24}-\frac{7}{2}\sqrt{z^3}\sqrt{z_7^3}\big\{\alpha\beta(1-\alpha)(1-\beta)\big\}^{1/8}\\
&\quad-\frac{49}{4}\frac{\sqrt{z_7^7}}{\sqrt{z}}\left(\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\right)^{1/12}-\frac{49}{4}\sqrt{z^3}\sqrt{z_7^3}\big\{\alpha\beta(1-\alpha)(1-\beta)\big\}^{1/4}\\
&\quad+\frac{1}{2}\frac{\sqrt{z^7}}{\sqrt{z_7}}\left(\frac{\alpha^7(1-\alpha)^7}{\beta(1-\beta)}\right)^{1/24}-\frac{1}{4}\frac{\sqrt{z^7}}{\sqrt{z_7}}\left(\frac{\alpha^7(1-\alpha)^7}{\beta(1-\beta)}\right)^{1/12}.
\end{aligned} \tag{4.1}$$

We divide both sides of (4.1) by $\sqrt{z^3}\sqrt{z_7^3}$ and use (3.9) and conclude that (4.1) is equivalent to

$$\begin{aligned}
&\frac{49}{m^2}\left\{1-\left(1-\left(\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\right)^{1/24}\right)^2\right\}+m^2\left\{1-\left(1-\left(\frac{\alpha^7(1-\alpha)^7}{\beta(1-\beta)}\right)^{1/24}\right)^2\right\}\\
&\quad+7\big\{4\big\{\alpha\beta(1-\alpha)(1-\beta)\big\}^{3/8}-2\big\{\alpha\beta(1-\alpha)(1-\beta)\big\}^{1/8}-7\big\{\alpha\beta(1-\alpha)(1-\beta)\big\}^{1/4}\big\}\\
&=0.
\end{aligned} \tag{4.2}$$

We prove (4.2).

Set $t := (\alpha\beta)^{1/8}$. Then, by (3.10), we have

$$\{(1-\alpha)(1-\beta)\}^{1/8} = 1-t. \tag{4.3}$$

Let $x := 1-2t$. From (3.11) we have

$$\left(\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\right)^{1/24} = \frac{1-xm}{4} \quad \text{and} \quad \left(\frac{\alpha^7(1-\alpha)^7}{\beta(1-\beta)}\right)^{1/24} = \frac{1+7x/m}{4}. \tag{4.4}$$

Similarly, (3.12) is equivalent to

$$m - 7/m = 2(2t-1)\left(2t^2-2t+3\right). \tag{4.5}$$

Now using (4.4) and (4.5), we find after some algebra that

$$\begin{aligned}
&\frac{49}{m^2}\left\{1-\left(1-\left(\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\right)^{1/24}\right)^2\right\} + m^2\left\{1-\left(1-\left(\frac{\alpha^7(1-\alpha)^7}{\beta(1-\beta)}\right)^{1/24}\right)^2\right\} \\
&\quad = \frac{7}{16}\left((m-7/m)^2 + 6x(m-7/m) - 14x^2 + 14\right) \\
&\quad = 7\left(4t^6 - 12t^5 + 19t^4 - 18t^3 + 5t^2 + 2t\right).
\end{aligned} \tag{4.6}$$

Moreover,

$$\begin{aligned}
&7\left\{4\{\alpha\beta(1-\alpha)(1-\beta)\}^{3/8} - 2\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} - 7\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4}\right\} \\
&\quad = 7\left(4t^3(1-t)^3 - 2t(1-t) - 7t^2(1-t)^2\right) \\
&\quad = -7\left(4t^6 - 12t^5 + 19t^4 - 18t^3 + 5t^2 + 2t\right).
\end{aligned} \tag{4.7}$$

This completes the proof of (4.2). Hence the proof of Theorem 1.3 is complete. Next, we prove Corollary 1.4. The proof of Corollary 1.5 is very similar to that of Corollary 1.4 and we forgo its proof.

Proof of Corollary 1.4. From (1.6) with q replaced by $-q$, and the definitions (2.1), (2.2), and (2.6), we have that

$$8\varphi^3(q)\varphi^3\left(q^7\right) = 8 + L(-q) - 7K(-q) - 8L\left(q^2\right) + 56K\left(q^2\right) - 42M(-q) - 48M\left(q^2\right). \tag{4.8}$$

Therefore,

$$8\nu(2n+1) = 8\left[q^{2n+1}\right]\left(\varphi^3(q)\varphi^3\left(q^7\right)\right) = \left[q^{2n+1}\right]\left(7K(q) - L(q) + 42M(q)\right) \tag{4.9}$$

and

$$\begin{aligned}
8\nu(2n) &= 8\left[q^{2n}\right]\left(\varphi^3(q)\varphi^3\left(q^7\right)\right) \\
&= \left[q^{2n}\right]\left(8 + L(q) - 8L\left(q^2\right) - 7K(q) + 56K\left(q^2\right) - 42M(q) - 48M\left(q^2\right)\right).
\end{aligned} \tag{4.10}$$

These two equations together with (2.4), (2.5) and (2.7) imply (1.8) and (1.9). □

5. Proof of Theorem 1.1

From (1.7) and the definitions (2.1), (2.2), and (2.6), we have that

$$32q^2\left(2q\psi^3(q)\psi^3\left(q^7\right)-\frac{E^7(q^7)}{E(q)}\right)=3K(q)-7K\left(q^2\right)-L(q)-2M(q)+L\left(q^2\right)-42M\left(q^2\right). \tag{5.1}$$

Explicit check shows that (1.4) is valid for $n=0$, 6 or $n=16$ and so we assume in this section that $n\neq 0$, 6 or 16. From (5.1), we see that

$$32\left[q^{2n-1}\right]\left(2q\psi^3(q)\psi^3\left(q^7\right)-\frac{E^7(q^7)}{E(q)}\right)=\left[q^{2n+1}\right]\left(3K(q)-L(q)-2M(q)\right). \tag{5.2}$$

Let $r(n):=[q^n](3K(q)-L(q)-2M(q))$. Instead of proving that (5.2) is nonnegative, we will prove the stronger statement that if $n>1$, then

$$r(n)>0. \tag{5.3}$$

If $[q^n]M(q)=0$, then by (2.4) and (2.5), we have that

$$r(n)=\left[q^n\right]\left(3K(q)-L(q)\right)=\left(3.7^{2c}-(-1)^b\right)\prod_{i=1}^{r}\frac{1-p_i^{2v_i+2}}{1-p_i^2}\prod_{j=1}^{s}\frac{(-1)^{w_j}+q_j^{2w_j+2}}{1+q_j^2}>0, \tag{5.4}$$

where n has the prime factorization

$$n=7^c\prod_{i=1}^{r}p_i^{v_i}\prod_{j=1}^{s}q_j^{w_j},$$

with $p_i\equiv 1,2,4 \pmod 7$, $q_j\equiv 3,5,6 \pmod 7$, and $b=\sum_{j=1}^{s}w_j$.

Let $s(n):=[q^n]M(q)$, assuming now that $s(n)\neq 0$, we have by (2.4), (2.5), (2.7),

$$r(n)=\left|s(n)\right|7^{-c}\left\{\left(3.7^{2c}-1\right)\prod_{i=1}^{r}\frac{1-p_i^{2v_i+2}}{(1-p_i^2)|F(p_i,v_i)|}\prod_{j=1}^{s}\frac{1+q_j^{2w_j+2}}{q_j^{w_j}(1+q_j^2)}-2.7^c.\frac{s(n)}{|s(n)|}\right\}. \tag{5.5}$$

From (2.8), we observe that

$$F(p,r)=\frac{\beta^{2r+2}-\bar\beta^{2r+2}}{\beta^2-\bar\beta^2}=\beta^{2r}+\beta^{2r-2}\bar\beta^2+\cdots+\bar\beta^{2r},$$

where $p=\beta\bar\beta$. Therefore,

$$\left|F(p,r)\right|\leqslant (r+1)p^r. \tag{5.6}$$

It is easy to show that if p and q as above and w is even, then

$$\frac{1-p^{2v+2}}{|F(p,v)|(1-p^2)}\geqslant\begin{cases}11 & \text{if } p>2 \text{ or } v>2,\\ 5/3 & \text{if } p=2,\ v=1,\\ 21/5 & \text{if } p=2,\ v=2\end{cases} \tag{5.7}$$

and

$$\frac{q^{2w+2}+1}{q^w(1+q^2)} \geqslant \begin{cases} 11 & \text{if } q>3 \text{ or } w>3, \\ 8 & \text{if } q=3,\ w=2. \end{cases} \tag{5.8}$$

Using (5.7) and (5.8) in (5.5), we conclude that

$$r(n) \geqslant 7^{-c}\left(\left(3.7^{2c}-1\right).\frac{5}{3}-2.7^c\right) > 7^{-c}\left(\left(3.7^{2c}-1\right)-2.7^c\right) \geqslant 0. \tag{5.9}$$

Next, we look at even-indexed coefficients. From (5.1), we find that

$$\begin{aligned} &32\left[q^{2n}\right]\left(2q\psi^3(q)\psi^3\left(q^7\right)-\frac{E^7(q^7)}{E(q)}\right) \\ &\quad=\left[q^{2n+2}\right]\left(3K(q)-7K\left(q^2\right)-L(q)-2M(q)+L\left(q^2\right)-42M\left(q^2\right)\right). \end{aligned} \tag{5.10}$$

Therefore, it remains to prove

$$\left[q^n\right]\left(3K(q)-7K\left(q^2\right)-L(q)-2M(q)+L\left(q^2\right)-42M\left(q^2\right)\right)>0, \tag{5.11}$$

where n is an even integer, $n \neq 0+2=2$, $6+2=8$ or $16+2=18$. Suppose n has the prime factorization

$$n=7^c 2^d \prod_{i=1}^{r} p_i^{v_i} \prod_{j=1}^{s} q_j^{w_j},$$

where p_i odd $p_i \equiv 1,2,4 \pmod 7$, $q_j \equiv 3,5,6 \pmod 7$, $b=\sum_{j=1}^{s} w_j$ and $d>0$. Employing (2.4) and (2.5), we find that

$$\begin{aligned} &\left[q^n\right]\left(3K(q)-7K\left(q^2\right)-L(q)+L\left(q^2\right)\right) \\ &\quad=\frac{1}{3}\left(7^{2c}\left(5.4^d+4\right)-(-1)^b.3.4^d\right)\prod_{i=1}^{r}\frac{1-p_i^{2v_i+2}}{1-p_i^2}\prod_{j=1}^{s}\frac{(-1)^{w_j}+q_j^{2w_j+2}}{1+q_j^2}>0, \end{aligned}$$

which proves (5.11) if $[q^n]M(q)=0$. Thus, we assume now that $[q^n]M(q)\neq 0$ that is w_i and hence b are all even, by (2.7), we find that

$$\begin{aligned} &\left[q^n\right]\left(3K(q)-7K\left(q^2\right)-L(q)-2M(q)+L\left(q^2\right)-42M\left(q^2\right)\right) \\ &\quad=\prod_{j=1}^{s} q_j^{w_j}\prod_{i=1}^{r}\left|F(p_i,v_i)\right|\left\{\frac{1}{3}\left(7^{2c}\left(5.4^d+4\right)-3.4^d\right)\prod_{i=1}^{r}\frac{1-p_i^{2v_i+2}}{(1-p_i^2)|F(p_i,v_i)|}\prod_{j=1}^{s}\frac{1+q_j^{2w_j+2}}{(1+q_j^2)q_j^{w_j}}\right. \\ &\qquad\left.-2.(-7)^c\left(F(2,d)+21F(2,d-1)\right)\prod_{i=1}^{r}\frac{F(p_i,v_i)}{|F(p_i,v_i)|}\right\} \\ &\quad\geqslant\frac{1}{3}\left(7^{2c}\left(5.4^d+4\right)-3.4^d\right)\prod_{i=1}^{r}\frac{1-p_i^{2v_i+2}}{(1-p_i^2)|F(p_i,v_i)|}\prod_{j=1}^{s}\frac{1+q_j^{2w_j+2}}{(1+q_j^2)q_j^{w_j}} \\ &\qquad-2.7^c\left|F(2,d)+21F(2,d-1)\right|. \end{aligned} \tag{5.12}$$

Let

$$S_1 := \frac{1}{3}\big(7^{2c}\big(5.4^d+4\big)-3.4^d\big)A(n)-2.7^c\big|F(2,d)+21F(2,d-1)\big|, \tag{5.13}$$

where

$$A(n) := \prod_{i=1}^{r}\frac{1-p_i^{2v_i+2}}{(1-p_i^2)|F(p_i,v_i)|}\prod_{j=1}^{s}\frac{1+q_j^{2w_j+2}}{(1+q_j^2)q_j^{w_j}}. \tag{5.14}$$

From (5.7), (5.8), and (5.6), we find that

$$S_1 \geqslant S_2 := \frac{1}{3}\big(7^{2c}\big(5.4^d+4\big)-3.4^d\big)-2.7^c\big|F(2,d)+21F(2,d-1)\big| \tag{5.15}$$

$$> S_3 := \frac{1}{3}\big(7^{2c}\big(5.4^d+4\big)-3.4^d\big)-2.7^c\big((d+1)2^d+21d2^{d-1}\big) \tag{5.16}$$

$$= \frac{7^c}{3}\big(7^c\big(5.4^d+4\big)-3.4^d-3(23d+2)2^d\big). \tag{5.17}$$

It is easy to show that $S_3>0$ if $c\geqslant 1$ except for $c=1$ and $d=2$ but $S_2>0$ for $c=1$ and $d=2$. Observe that $S_3>0$ if $d>8$ and $c=0$. Direct evaluation shows that $S_2>0$ if $c=0$, $d=4$, 5, 6, 7 or 8. For the remaining cases, $c=0$, $d=1$, 2, or 3, by (5.7) and (5.8), we have that $A(n)\geqslant 11$ unless $n=2^d3^w$ with $0\leqslant w\leqslant 2$. The validity of (5.11) can easily be checked for $n=4$, 12, 36 ($n=2$, 8 or 18 and $w=1$ are already excluded). Assuming $A(n)\geqslant 11$, direct computation show that $S_1>0$ if $c=0$, $d=1$, 2 or 3. Hence, the proof of Theorem 1.1 is complete.

6. Proof of Theorem 1.2

From (1.7), and the definitions (2.1), (2.2), and (2.6), we find that

$$64q^2\left(\frac{E^7(q^7)}{E(q)}-q\psi^3(q)\psi^3(q^7)\right)$$
$$= L(q)-L\big(q^2\big)+K(q)-K\big(q^2\big)-2M(q)+6\big(K\big(q^2\big)+7M\big(q^2\big)\big)+2K\big(q^2\big). \tag{6.1}$$

It is clear from (2.5) that $[q^n]K(q^2)>0$. Below we assume that $n\neq 2$. Validity of (1.5) for the corresponding value of $n=2$ can easily be checked. Therefore, it suffices to prove that if $n\neq 2$, then

$$\big[q^n\big]\big(K(q)+7M(q)\big)>0 \tag{6.2}$$

and

$$\big[q^n\big]\big(L(q)-L\big(q^2\big)+K(q)-K\big(q^2\big)-2M(q)\big)\geqslant 0. \tag{6.3}$$

We start with (6.2). From (2.5),

$$\big[q^n\big]\big(K(q)+7M(q)\big)\geqslant 0 \quad \text{if } s(n) := \big[q^n\big]M(q)=0. \tag{6.4}$$

Assuming that $s(n)\neq 0$, by (2.5), (2.7), (5.7), and by (5.8), we find that

$$\frac{[q^n]K(q)}{|s(n)|} = 7^c \prod_{i=1}^{r} \frac{1-p_i^{2v_i+2}}{(1-p_i^2)|F(p_i,v_i)|} \prod_{j=1}^{s} \frac{1+q_j^{2w_j+2}}{q_j^{w_j}(1+q_j^2)} > 7, \tag{6.5}$$

provided $n \neq 2$ or 4. However, $s(4) > 0$ and so we conclude that

$$[q^n]\big(K(q)+7M(q)\big) = \big|s(n)\big|\left(\frac{[q^n]K(q)}{|s(n)|} + 7\frac{s(n)}{|s(n)|}\right) > 0. \tag{6.6}$$

Next, we prove (6.3). Assume as before that n has the prime factorization,

$$n = 7^c 2^d \prod_{i=1}^{r} p_i^{v_i} \prod_{j=1}^{s} q_j^{w_j},$$

with p_i odd $p_i \equiv 1,2,4 \pmod 7$, $q_j \equiv 3,5,6 \pmod 7$, $b = \sum_{j=1}^{s} w_j$.

From (2.4) and (2.5), we find that

$$[q^n]\big(L(q)-L\big(q^2\big)+K(q)-K\big(q^2\big)\big) = \big(7^{2c}+(-1)^b\big)2^{2d} \prod_{i=1}^{r} \frac{1-p_i^{2v_i+2}}{(1-p_i^2)} \prod_{j=1}^{s} \frac{(-1)^{w_j}+q_j^{2w_j+2}}{(1+q_j^2)} \geqslant 0,$$

which proves (6.3) if $s(n) = 0$. Next assume that $s(n) \neq 0$. Then, w_j and b are all even and by employing (2.7), (5.7), and (5.8), we find that

$$[q^n]\big(L(q)-L\big(q^2\big)+K(q)-K\big(q^2\big)-2M(q)\big) \tag{6.7}$$

$$= \big|s(n)\big|\left\{\frac{(7^c+7^{-c})2^{2d}}{|F(2,d)|} \prod_{i=1}^{r} \frac{1-p_i^{2v_i+2}}{(1-p_i^2)|F(p_i,v_i)|} \prod_{j=1}^{s} \frac{1+q_j^{2w_j+2}}{q_j^{w_j}(1+q_j^2)} - 2\frac{s(n)}{|s(n)|}\right\} \geqslant 0, \tag{6.8}$$

since $|F(2,d)| \leqslant (d+1)2^d \leqslant 2^{2d}$ by (5.6).

7. Concluding remarks

We would like to point out that another upper bound for the coefficients of 7-cores is given by the inequality

$$[q^n]\big(\varphi^3(q)\varphi^3\big(q^7\big)\big) > 5[q^n]\big(q^3\psi^3(q)\psi^3\big(q^7\big)\big) > [q^n]\left(q^2\frac{E^7(q^7)}{E(q)}\right) \quad \text{for } n \neq 2,4,7,14,22,29,58. \tag{7.1}$$

The proof of the first part of this inequality is similar to that of Theorem 1.1 and is omitted for space considerations. The second part of this inequality follows from Theorem 1.1. It would be interesting to prove all these inequalities for 7-cores in a completely elementary manner. It is natural to ask if our inequalities extend to general t-cores. We offer the following inequality as a conjecture:

$$[q^n]\big(\psi(q)\psi\big(q^t\big)\big)^{(t-1)/2} \geqslant [q^n]\left(\frac{E^t(q^t)}{E(q)}\right),$$

valid for all n, provided that t is an odd integer greater or equal to 11.

Acknowledgments

We would like to thank George Andrews, Frank Garvan, Michael Somos and James Sellers for their interest and helpful comments.

References

[1] T.M. Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York, 1976.
[2] A. Berkovich, F.G. Garvan, On the Andrews–Stanley refinement of Ramanujan's congruence modulo 5 and generalization, Trans. Amer. Math. Soc. 358 (2006) 703–726.
[3] A. Berkovich, F.G. Garvan, The BG-rank of a partition and its applications, Adv. in Appl. Math. 40 (3) (2008) 377–400.
[4] A. Berkovich, H. Yesilyurt, New Identities for 7-cores with prescribed BG-rank, Discrete Math. 308 (22) (2008) 5246–5259.
[5] B.C. Berndt, Ramanujan's Notebooks, Part III, Springer-Verlag, New York, 1991.
[6] H.H. Chan, New proofs of Ramanujan's partition identities for moduli 5 and 7, J. Number Theory 53 (1995) 144–159.
[7] H.H. Chan, On the equivalence of Ramanujan's partition identities and a connection with the Rogers–Ramanujan continued fraction, J. Math. Anal. Appl. 198 (1) (1996) 111–120.
[8] F. Garvan, D. Kim, D. Stanton, Cranks and t-cores, Invent. Math. 101 (1990) 1–17.
[9] A. Granville, K. Ono, Defect zero p-blocks for finite simple groups, Trans. Amer. Math. Soc. 348 (1) (1996) 331–347.
[10] G. James, A. Kerber, The Representation Theory of the Symmetric Group, Addison–Wesley, Reading, MA, 1981.
[11] A.A. Klyachko, Modular forms and representations of symmetric groups, J. Soviet. Math. 26 (1984) 1879–1887.
[12] R.A. Rankin, Modular Forms and Functions, Cambridge University Press, Cambridge, 1977.
[13] B. Schoeneberg, Elliptic Modular Functions, Springer-Verlag, New York, 1974.