The Noether Map I

Article *in* Proceedings of the American Mathematical Society · November 2005

DOI: 10.1515/FORUM.2009.028

Mara D Neusel
Texas Tech University

The Noether Map

Mara D. Neusel

DEPT. OF MATH. AND STATS.
TEXAS TECH UNIVERSITY
MS 1042
LUBBOCK, TX 79409
USA

MARA.D.NEUSEL@TTU.EDU

Müfit Sezer

DEPARTMENT OF MATHEMATICS
BOĞAZICI ÜNIVERSITESI
BEBEK
ISTANBUL
TURKEY

MUFIT.SEZER@BOUN.EDU.TR

October 23rd 2005

AMS CODE: 13A50 Invariant Theory, 20J06 Group Cohomology
KEYWORDS: Invariant Theory of Finite Groups, Integral Closure, Noether Map, Modular Invariant Theory, Orbit Chern Classes, Transfer, Projective $\mathbb{F}G$-Modules, Tate Cohomology, Degree Bounds, Cohen-Macaulay Defect
The first author is partially supported by NSA Grant No. H98230-05-1-0026

Typeset by LS TEX

SUMMARY : *Let* $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ *be a faithful representation of a finite group* G. *In this paper we study the image of the associated Noether map*

$$\eta_G^G : \mathbb{F}[V(G)]^G \longrightarrow \mathbb{F}[V]^G.$$

It turns out that the image of the Noether map characterizes the ring of invariants in the sense that its integral closure $\overline{\mathrm{Im}(\eta_G^G)} = \mathbb{F}[V]^G$. *This is true without any restrictions on the group, representation, or ground field. Furthermore, we show that the Noether map is surjective, i.e., its image integrally closed, if* $V = \mathbb{F}^n$ *is a projective* $\mathbb{F}G$*-module. Moreover, we show that the converse of this statement is true if* G *is a* p*-group and* $\mathbb{F}$ *has characteristic* p, *or if* ϱ *is a permutation representation. We apply these results and obtain upper bounds on the Noether number and the Cohen-Macaulay defect of* $\mathbb{F}[V]^G$. *We illustrate our results with several examples.*

Let $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a faithful representation of a finite group G over a field $\mathbb{F}$. The representation ϱ induces naturally an action of G on the vector space $V = \mathbb{F}^n$ of dimension n and hence on the ring of polynomial functions $\mathbb{F}[V] = \mathbb{F}[x_1, \ldots, x_n]$. Our interest is focused on the subring of invariants

$$\mathbb{F}[V]^G = \{f \in \mathbb{F}[V]^G \mid gf = f \ \forall \ g \in G\},$$

which is a graded connected Noetherian commutative algebra.

In the first section of this paper we introduce the Noether map and show that its image characterizes the ring of invariants. In Section 2 we consider projective $\mathbb{F}G$-modules V, and show that the Noether map is surjective in this case. The next section deals with the converse: In Section 3 we show that the Noether map is surjective if and only if V is $\mathbb{F}G$-projective in the cases of p-groups and of permutation representations. In Section 4 we derive some results about degree bounds and the Cohen-Macaulay defect of $\mathbb{F}[V]^G$. Furthermore we present some examples.

§1. The Noether Map

Let $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a representation of a group G of order d. Let $\mathbb{F}[V]$ be the symmetric algebra on V^*. Denote by $\mathbb{F}G$ the group algebra. Let

$$V(G) = \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}G, V) \cong \mathbb{F}G \otimes V$$

be the coinduced module $\mathrm{coind}_1^G(V)$. The group G acts on $V(G)$ by left multiplication on the first component. We obtain a G-equivariant surjection

$$(\bigstar) \qquad V(G) \longrightarrow V, (g, v) \longmapsto gv.$$

Let us choose a basis $e_1, \ldots, e_n$ for V. Let $x_1, \ldots, x_n$ be the standard dual basis for V^*, and set $G = \{g_1, \ldots, g_d\}$. Then $V(G)$ can be written as

$$V(G) = \mathrm{span}_{\mathbb{F}}\{e_{ij} \mid i = 1, \ldots, n, \ j = 1, \ldots, d\},$$

and the map $(\bigstar)$ translates into

$$V(G) \longrightarrow V, \ e_{ij} \longmapsto g_j e_i.$$

Similarly, we have

$$V(G)^* = \mathrm{span}_{\mathbb{F}}\{x_{ij} \mid i = 1, \ldots, n, \ j = 1, \ldots, d\}$$

with

$$V(G)^* \longrightarrow V^*, \ x_{ij} \longmapsto g_j x_i.$$

We obtain a surjective G-equivariant map between the rings of polynomial functions

$$\eta_G : \mathbb{F}[V(G)] \longrightarrow \mathbb{F}[V].$$

The group G acts on $\mathbb{F}[V(G)]$ by permuting the basis elements x_{ij}. By restriction to the induced ring of invariants, we obtain the classical Noether map, cf. Section 4.2 in [11],

$$\eta_G^G : \mathbb{F}[V(G)]^G \longrightarrow \mathbb{F}[V]^G.$$

We note that $V(G)$ is the n-fold regular representation of G. Thus $\mathbb{F}[V(G)]^G$ are the n-fold vector invariants of the regular representation of G.

In the classical nonmodular case, where $p \nmid d$, the map η_G^G is surjective, see Proposition 4.2.2 in [11]. This does not remain true in the modular case as we illustrate in the next example.

EXAMPLE 1: Let $\varrho : \mathbb{Z}/2 \hookrightarrow \mathrm{GL}(3, \mathbb{F}_2)$ be the 3-dimensional representation of $\mathbb{Z}/2$ over the field with two elements afforded by the matrix

$$\varrho(g) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$\mathbb{F}[x_1,\ x_2,\ x_3]^{\mathbb{Z}/2} = \mathbb{F}[x_1 + x_2, x_1x_2, x_3]$$

and

$$\begin{aligned} &\mathbb{F}[x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}]^{\mathbb{Z}/2} \\ &\quad = \mathbb{F}[x_{i1} + x_{i2}, x_{i1}x_{i2}, x_{i1}x_{i+1,2} + x_{i2}x_{i+1,1}, x_{11}x_{21}x_{31} + x_{12}x_{22}x_{32}], \end{aligned}$$

where $i \in \mathbb{Z}/3$, cf. Example 2 in Section 2.3, [11] or Example 1 in Section 3.2, loc.cit. We obtain

$$\mathrm{Im}(\eta_{\mathbb{Z}/2}^{\mathbb{Z}/2}) = \mathbb{F}[x_1 + x_2, x_1x_2, x_3^2, (x_1 + x_2)x_3].$$

Thus the Noether map is no longer surjective, because the invariant x_3 is not in its image. However, note that the integral closure of the image of the Noether map is the ring of invariants $\mathbb{F}[V]^G$. This is always true as we see in this section.

Recall the **transfer map**

$$\mathrm{Tr}^G : \mathbb{F}[V] \longrightarrow \mathbb{F}[V]^G;\ f \longmapsto \sum_{g \in G} gf,$$

see, e.g., Section 2.2. in [11]. By construction the transfer is an $\mathbb{F}[V]^G$-module homomorphism. We denote by

$$\mathbb{F}[\mathrm{Im}(\mathrm{Tr}^G)] \subseteq \mathbb{F}[V]^G$$

the subalgeba generated by the image of the transfer.

We observe that any element

$$\frac{f_1}{f_2} \in \mathbb{F}(V)$$

can be written as the quotient of some polynomial by an invariant polynomial in the following way

$$\frac{f_1}{f_2} = \frac{f_1 \frac{\mathbf{N}(f_2)}{f_2}}{\mathbf{N}(f_2)},$$

where $\mathbf{N}(f) = \prod_{g \in G} gf$ denotes the **Norm** of f. This allows us to extend the transfer to a map of $\mathbb{F}(V)^G$-modules between the respective fields of fractions

$$\mathrm{Tr}^G : \mathbb{F}(V) \longrightarrow \mathbb{F}(V)^G;\ \frac{f_1}{f_2} \longmapsto \frac{\sum_{g \in G} gf_1}{f_2},$$

where we assume that $f_2 \in \mathbb{F}[V]^G$.

PROPOSITION 1.1: *We have that*

$$\mathbb{F}(\mathrm{Tr}^G(\mathbb{F}(V))) = \mathit{IFIF}(\mathbb{F}[\mathrm{Im}(\mathrm{Tr}^G)]) = \mathbb{F}(V)^G,$$

where $\mathit{IFIF}(-)$ *denotes the field of fractions functor.*

PROOF: Let $\frac{\mathrm{Tr}^G(f_1)}{\mathrm{Tr}^G(f_2)} \in \mathit{IFIF}(\mathbb{F}[\mathrm{Im}(\mathrm{Tr}^G)])$. Then

$$\frac{\mathrm{Tr}^G(f_1)}{\mathrm{Tr}^G(f_2)} = \mathrm{Tr}^G\left(\frac{f_1}{\mathrm{Tr}^G(f_2)}\right) \in \mathrm{Tr}^G(\mathbb{F}(V)).$$

To prove the reverse inclusion take an element

$$\mathrm{Tr}^G(\frac{f_1}{f_2}) \in \mathrm{Tr}^G(\mathbb{F}(V)),$$

where $f_2 \in \mathbb{F}[V]^G$. Choose a polynomial $f \in \mathbb{F}[V]$ such that $\mathrm{Tr}^G(f) \neq 0$. (Recall that the transfer map is never zero by Propositon 2.2.4 in [11].) Then we have

$$\mathrm{Tr}^G(\frac{f_1}{f_2}) = \frac{\mathrm{Tr}^G(f_1)}{f_2} = \frac{\mathrm{Tr}^G(f_1)\mathrm{Tr}^G(f)}{f_2\mathrm{Tr}^G(f)} = \frac{\mathrm{Tr}^G(f_1)\mathrm{Tr}^G(f)}{\mathrm{Tr}^G(ff_2)} \in \mathit{IFIF}(\mathbb{F}[\mathrm{Im}(\mathrm{Tr}^G)]).$$

We come to the second equality. Since $\mathbb{F}[\mathrm{Im}(\mathrm{Tr}^G)] \subseteq \mathbb{F}[V]^G$ we have that

$$\mathit{IFIF}(\mathbb{F}[\mathrm{Im}(\mathrm{Tr}^G)]) \subseteq \mathbb{F}(V)^G.$$

To prove the reverse inclusion, let $\frac{f_1}{f_2} \in \mathbb{F}(V)^G$ where without loss of generality $f_1,\ f_2 \in \mathbb{F}[V]^G$. Let $\mathrm{Tr}^G(f) \neq 0$ for some suitable $f \in \mathbb{F}[V]$. Thus

$$\frac{f_1}{f_2} = \frac{\mathrm{Tr}^G(f)f_1}{\mathrm{Tr}^G(f)f_2} = \frac{\mathrm{Tr}^G(f f_1)}{\mathrm{Tr}^G(f f_2)} \in \mathbb{FF}(\mathbb{F}[\mathrm{Im}(\mathrm{Tr}^G)])$$

as desired. □

PROPOSITION 1.2: *The integral closure of the image of the Noether map is the ring of invariants*

$$\overline{\mathrm{Im}(\eta_G^G)} = \mathbb{F}[V]^G.$$

PROOF: By Proposition 1.1 and Lemma 4.2.1 in [11] we have the following commutative diagram:

$$\begin{array}{ccccccc}
\mathbb{F}[\mathrm{Im}(\mathrm{Tr}^G)] & \subseteq & \mathrm{Im}(\eta_G^G) & \subseteq & \mathbb{F}[V]^G & \subseteq & \mathbb{F}[V] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{FF}(\mathbb{F}[\mathrm{Im}(\mathrm{Tr}^G)]) & = & \mathbb{FF}(\mathrm{Im}(\eta_G^G)) & = & \mathbb{F}(V)^G & \subseteq & \mathbb{F}(V).
\end{array}$$

Let $x_1, \ldots, x_n \in V^*$ be a basis. Then the coefficients of the polynomials

$$F_i(X) = \prod_{g\in G} (X - g x_i),$$

are the orbit chern classes of x_i counted with multiplicities

$$\sigma_1(x_i) = \mathrm{Tr}^G(x_i), \cdots, \sigma_d(x_i) = \mathbf{N}(x_i).$$

Thus they are in the image of η_G^G. Denote by A the $\mathbb{F}$-algebra generated by these coefficients. By construction A is finitely generated, thus noetherian. Furthermore $\mathbb{F}[V]$ is finitely generated as an A-module, thus as an $\mathrm{Im}(\eta_G^G)$-module since $A \subseteq \mathrm{Im}(\eta_G^G)$. Therefore the extension

$$\mathrm{Im}(\eta_G^G) \subseteq \mathbb{F}[V]$$

is finite, and

$$\overline{\mathrm{Im}(\eta_G^G)} = \mathbb{F}[V]^G$$

as desired. □

We close this section with an immediate corollary of the preceding result:

COROLLARY 1.3: *The Krull dimension of the image of the Noether map coincides with the Krull dimension of the ring of invariants, which in turn is equal to* $n = \dim_{\mathbb{F}} V$. □

ADDENDUM: Define a map $E : \mathbb{F}[V] \longrightarrow \mathbb{F}[V(G)]^G,\ x_i \longmapsto \sum_{j=1}^{d} x_{ij}$. Then we obtain a commutative triangle as follows:

$$\begin{array}{ccc} \mathbb{F}[V(G)]^G & \xrightarrow{\eta_G^G} & \mathbb{F}[V]^G \\ \uparrow E & \nearrow \mathrm{Tr}^G & \\ \mathbb{F}[V] & & \end{array}$$

If $p \nmid d$, then the preceding diagram proves that the Noether map is surjective, since the transfer is surjective, see Lemma 4.2.1 in [11]. We want to add the following observation:

PROPOSITION 1.4: *The algebra generated by the image of the transfer map is equal to the image of the Noether map if and only if V is a nonmodular $\mathbb{F}G$-module.*

PROOF: By Lemma 4.2.1 in [11] the image of the transfer is always contained in the image of the Noether map. Thus if $p \nmid |G|$, then the transfer is surjective, and hence the Noether map. If $p \mid |G|$, then the transfer is no longer surjective. Indeed, the height of the image of the transfer is at most $n-1$, see Theorem 6.4.7 in [11]. Thus the Krull dimension of $\mathbb{F}[\mathrm{Im}(\mathrm{Tr}^G)]$ is strictly less than n. On the other hand the Krull dimension of the image of the Noether map is n by Proposition 1.2. Thus they cannot be equal. □

§2. Projective Modules

In this section we want to study the question when the Noether map is surjective.

We note that the $\mathbb{F}G$ module V is projective if and only if its dual vector space V^* is injective which in turn is equivalent to projective because G is a finite group. We will make frequently use of this fact in what follows.

PROPOSITION 2.1: *If V is a projective $\mathbb{F}G$-module, then the Noether map is surjective.*

PROOF: By construction we have a short exact sequence of $\mathbb{F}G$-modules as follows

$$0 \longrightarrow W^* \longrightarrow V(G)^* \longrightarrow V^* \longrightarrow 0.$$

Since V^* is projective, this sequence splits and

$$V(G)^* \overset{\varphi}{\cong} V^* \oplus W^* \xrightarrow{\mathrm{pr}} V^*$$

as $\mathbb{F}G$-modules. Taking invariants we obtain a commutative diagram

$$\begin{array}{ccc} \mathbb{F}[V(G)]^G & \xrightarrow{\varphi^*} & \mathbb{F}[V \oplus W]^G \\ & \searrow \eta_G^G & \downarrow \mathrm{pr}^* \\ & & \mathbb{F}[V]^G \end{array}$$

Thus η_G^G is surjective because φ^* as well as pr^* are. □

REMARK: Since nonmodular $\mathbb{F}G$-modules are always projective we recover the classical result that η_G^G is surjective for every nonmodular representation of G.

COROLLARY 2.2: *Let $\rho : G \hookrightarrow \mathrm{GL}(p, \mathbb{F})$ be a permutation representation of the finite group G over a field $\mathbb{F}$ of characteristic p. Then η_G^G is surjective.*

PROOF: Let $\psi : \Sigma_p \hookrightarrow \mathrm{GL}(p, \mathbb{F})$ be the defining representation of the symmetric group in p letters. Since ρ is a permutation representation we have that

$$\rho(G) \leq \psi(\Sigma_p) \leq \mathrm{GL}(p, \mathbb{F}).$$

Since $V = \mathbb{F}^p$ is a projective Σ_p-module it is projective as a $\mathbb{F}G$-module. Thus by Proposition 2.1 the Noether map η_G^G is surjective. □

EXAMPLE 1: If $\psi : \Sigma_n \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ is the defining representation of the symmetric group in n letter over a field of charactersitic p, where $p < n$, then neither V is projective as a module over Σ_n nor is $\eta_{\Sigma_n}^{\Sigma_n}$ surjective. The latter is true because in degree one [1] we have

$$\mathbb{F}[V(\Sigma_n)]_{(1)}^{\Sigma_n} = \mathrm{span}_{\mathbb{F}}\{\sum_{j=1}^{n!} x_{ij} \mid i = 1, \ldots, n\}$$

and thus

$$\eta_{\Sigma_n}^{\Sigma_n}(\sum_{j=1}^{n!} x_{ij}) = (n-1)! \sum_{i=1}^{n} x_i \equiv 0 \bmod p.$$

Therefore the first elementary symmetric function $e_1 = x_1 + \cdots + x_n \in \mathbb{F}[V]^{\Sigma_n}$ is not hit. Therefore, V is not $\mathbb{F}\Sigma_n$-projective. This is not a new result: For the defining representation $\psi : \Sigma_n \hookrightarrow \mathrm{GL}(n, \mathbb{F})$, $V = \mathbb{F}^n$ is a projective $\mathbb{F}\Sigma_n$-module if and only if $p \geq n$. This follows from Corollary 7 on Page 33 of [1]. See Theorem 3.5 in Section 3 for a generalization of this.

EXAMPLE 2: Let $\psi : A_n \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be the defining representation of the alternating group in n letters over a field of characteristic p. By Corollary 2.2 the Noether map $\eta_{A_n}^{A_n}$ is surjective if $n \leq p$. We want to check what happens if $n > p$.

We start by considering the Noether map

$$\eta_{A_n}^{A_n} : \mathbb{F}[V(A_n)]^{A_n} \longrightarrow \mathbb{F}[V]^{A_n}$$

[1] For a graded object A we denote the homogeneous degree i-part by $A_{(i)}$.

in degree one. We have

$$\mathbb{F}[V(A_n)]^{A_n}|_{(1)} = \mathrm{span}_{\mathbb{F}}\{\sum_{j=1}^{|A_n|} x_{ij} \mid i = 1, \ldots, n\}$$

and

$$\mathbb{F}[V]^{A_n}|_{(1)} = \mathrm{span}_{\mathbb{F}}\{e_1 = x_1 + \cdots + x_n\}.$$

Thus we have

$$\eta_{A_n}^{A_n}(\sum_{j=1}^{|A_n|} x_{ij}) = |\mathrm{Stab}_{A_n}(x_i)|e_1 = |A_{n-1}|e_1 = \frac{(n-1)!}{2}e_1.$$

Thus the elementary symmetric function e_1 is in the image of the Noether map if and only if

$$\frac{(n-1)!}{2} \in \mathbb{F}^\times.$$

This in turn happens exactly when

(1) p is odd and $p \geq n$,
(2) $p = 2$ and $n \leq 4$.

We know already that the Noether map is surjective in the first case. If p is even and $n \leq 3$ we are in the nonmodular case, so the Noether map is again surjective. Thus the only case that we have to check by hand is the defining representation of A_4 over a field of characteristic 2.

We note that the 2-Sylow subgroup of A_4 is the Klein-Four-Group $\mathbb{Z}/2 \times \mathbb{Z}/2$. When we restrict $\psi|_{\mathbb{Z}/2\times\mathbb{Z}/2}$ we obtain the regular representation of $\mathbb{Z}/2 \times \mathbb{Z}/2$. Thus V is $\mathbb{F}(\mathbb{Z}/2 \times \mathbb{Z}/2)$-projective. Therefore, V is $\mathbb{F}A_4$-projective. Hence the Noether map is surjective. Indeed, a short calculation shows that

$$\begin{aligned}
\eta_{A_4}^{A_4}(o(x_{11})) &= 3e_1 = e_1,\\
\eta_{A_4}^{A_4}(o(x_{11}x_{12})) &= e_2,\\
\eta_{A_4}^{A_4}(o(x_{11}x_{21}x_{31})) &= 3e_3 = e_3,\\
\eta_{A_4}^{A_4}(o(x_{11}x_{12}x_{13}x_{14})) &= 3e_4 = e_4,\\
\eta_{A_4}^{A_4}(o(x_{11}^3x_{21}^2x_{31})) &= o(x_1^3x_2^2x_3),
\end{aligned}$$

where $o(-)$ denotes the orbit sum of $-$, and $g_1 = (1)$, $g_2 = (12)(34)$, $g_3 = (13)(24)$, and $g_4 = (14)(23)$.

§3. P-Groups and Permutation Representations

For nonmodular representations the Noether map is always surjective and V is always projective. Therefore, we restrict ourselves to modular representations in what follows.

In this section we want to show that the converse Proposition 2.1 is true in the case of p-groups P and in the case of permutation representations. The next two results settle the case of $P \cong \mathbb{Z}/p$.

LEMMA 3.1: *Let P be a cyclic p-group, and let $\mathbb{F}$ have characteristic p. Then*

$$\operatorname{Im}(\operatorname{Tr}^P)_{(1)} \subsetneqq \mathbb{F}[V]^P_{(1)}$$

unless V is the k-fold regular representation of P for some $k \in \mathbb{N}$.

PROOF: Since the transfer is additive it suffices to consider indecomposable modules only.

Let the order of the group be p^s. Then up to isomorphism there are exactly p^s indecomposable $\mathbb{F}P$-modules $V_1, \ldots, V_{p^s}$ with $\dim_{\mathbb{F}} V_i = i$. The action of P on V_i is afforded by the matrix consisting of one Jordan block with 1's on the diagonal and superdiagonal. Note that $V_i^P = V_1$ for all i.

Set $\Delta = g - 1$ where $g \in P$ is a generator. Then

$$\Delta(V_i^*) = \begin{cases} V_{i-1}^* & \text{for } i = 2, \ldots, p^s \\ 0 & \text{for } i = 1. \end{cases}$$

Since, $\operatorname{Tr}^P = \Delta^{p^s-1}$, we obtain

$$\operatorname{Tr}^P(V_i^*) = \Delta^{p^s-1}(V_i^*) = \begin{cases} 0 & \text{for } i = 1, \ldots, p^s - 1 \\ V_1^* & \text{for } i = p^s \end{cases}$$

as desired. □

In Theorem 3.2 [8] (and the following remark) a more precise version of the preceding result is shown: the transfer is surjective in degrees prime to the characteristic in the case of k-fold regular representations. We obtain the following corollary that we note here for later reference.

COROLLARY 3.2: *Let $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a faithful representation of a finite group. Let $i \in \mathbb{F}^\times$. Then*

$$\operatorname{Im}(\eta^G_G|_{(i)}) = \operatorname{Im}(\operatorname{Tr}^G|_{(i)}).$$

PROOF: By construction we obtain a commutative diagram as follows

$$\begin{array}{ccc} \mathbb{F}[V(G)]^G|_{(i)} & \xrightarrow{\eta_G^G|_{(i)}} & \mathbb{F}[V]^G|_{(i)} \\ \uparrow{\scriptstyle \mathrm{Tr}^G|_{(i)}} & & \uparrow{\scriptstyle \mathrm{Tr}^G|_{(i)}} \\ \mathbb{F}[V(G)]|_{(i)} & \xrightarrow{\eta_G|_{(i)}} & \mathbb{F}[V]|_{(i)}. \end{array}$$

By Theorem 3.2 [8] and the remark following it the transfer map on the left

$$\mathrm{Tr}^G|_{(i)} : \mathbb{F}[V(G)]|_{(i)} \longrightarrow \mathbb{F}[V(G)]^G|_{(i)}$$

is surjective. By construction the lower map $\eta_G|_{(i)}$ is surjective. Thus the result follows. □

Even though Proposition 3.4 contains the following result as a special case, we want to leave the proof in, because it is so simple and uses just some linear algebra, cf. Lemma 3.2 in [6].

PROPOSITION 3.3: *Let $G = P$ a cyclic p-group. Then the following are equivalent*

(1) *The Noether map is surjective.*
(2) *The Noether map is surjective in degree one.*
(3) *V is a projective $\mathbb{F}P$-module.*

PROOF: The implication (1) ⇒ (2) is trivial. The implication (3) ⇒ (1) was proven in Proposition 2.1. Thus we need to show that V is projective if $\eta_P^P|_{(1)}$ is surjective.

By Corollary 3.2 we have that $\mathrm{Im}(\eta_G^G|_{(i)}) = \mathrm{Im}(\mathrm{Tr}^G|_{(i)})$. Since the transfer is surjective in degree one exactly when V is a k-fold regular representation by Lemma 3.1, we have that V is the k-fold regular representation and hence projective. □

THEOREM 3.4: *Let $\varrho : P \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a representation of a p-group over a field $\mathbb{F}$ of characteristic p. Then the following are equivalent:*

(1) *The Noether map is surjective.*
(2) *The Noether map is surjective in degree one.*
(3) *V is a projective $\mathbb{F}P$-module.*

PROOF: The implication (1) ⇒ (2) is trivial. The implication (3) ⇒ (1) was proven in Proposition 2.1. Thus we need to show that V is projective if $\eta_P^P|_{(1)}$ is surjective.

Consider the short exact sequence of $\mathbb{F}P$-modules

$$(*) \qquad 0 \longrightarrow K^* \longrightarrow V(P)^* \xrightarrow{\eta_P|_{(1)}} V^* \longrightarrow 0.$$

The module $V(P)$ is free and therefore cohomologically trivial. Thus the long exact cohomology sequence breaks up into

$$0 \longrightarrow (K^*)^P \longrightarrow (V(P)^*)^P \stackrel{\eta_P^P|_{(1)}}{\longrightarrow} (V^*)^P \longrightarrow \mathrm{H}^1(P, K^*) \longrightarrow 0$$

and

$$\mathrm{H}^i(P, V^*) \cong \mathrm{H}^{i+1}(P, K^*) \quad \forall\ i \geq 1.$$

Since $\eta_P^P|_{(1)}$ is surjective by assumption, we obtain

$$\mathrm{H}^1(P, K^*) = 0.$$

Thus

$$\widehat{\mathrm{H}}^1(P, K^*) = \mathrm{H}^1(P, K^*) = 0,$$

where $\widehat{\mathrm{H}}^*(-,-)$ denotes the Tate cohomology. Thus K^* is a projective $\mathbb{F}P$-module by Theorem 8.5, Chapter VI in [2]. Since P is finite and K^* finitely generated, this implies that K^* is injective, see Corollary 2.7 in [3]. Thus the sequence $(*)$ splits and V^* is projective as desired. □

We illustrate this result with an example.

EXAMPLE 1: Let $\mathbb{F}$ be the field with q elements of characteristic p. Let $P \leq \mathrm{GL}(n, \mathbb{F})$ be a p-Sylow subgroup of the general linear group. With assume without loss of generality that P consists of upper triangular matrices with 1's on the diagonal. Then

$$\mathbb{F}[V(P)]^P_{(1)} = \mathrm{span}_{\mathbb{F}}\{o(x_{i1}) = \sum_{j=1}^{|P|} x_{ij} \mid i = 1, \ldots, n\}.$$

Thus

$$\begin{aligned}
\eta_P^P(o(x_{i1})) &= \sum_{j=1}^{|P|} g_j x_i \\
&= \sum_{(a_{i+1},\ldots,a_n)\in\mathbb{F}^{n-i}} (x_i + a_{i+1}x_{i+1} + \cdots + a_n x_n) \\
&= q^{\frac{n(n-1)}{2}-(n-i)}\Big(q^{n-i}x_i + q^{n-i-1}\Big(\sum_{a_{i+1}\in\mathbb{F}} a_{i+1}x_{i+1} + \cdots + \sum_{a_n\in\mathbb{F}} a_n x_n\Big)\Big). \\
&= q^{\frac{n(n-1)}{2}} x_i + q^{\frac{n(n-1)}{2}-1}\Big(\sum_{a_{i+1}\in\mathbb{F}} a_{i+1}x_{i+1} + \cdots + \sum_{a_n\in\mathbb{F}} a_n x_n\Big)\Big).
\end{aligned}$$

The factor $q^{\frac{n(n-1)}{2}}$ is nonzero if and only if $n = 0$ or $n = 1$. Since we are considering the modular case this cannot happen.

The factor $q^{\frac{n(n-1)}{2}-1}$ is nonzero if and only if $n = 2$.

Thus we proceed by having a closer look at the two-dimensional case: We have by the above calculations

$$\begin{aligned} \eta_P^P(o(x_{11})) &= \sum_{j=1}^{|P|} g_j x_1 = \sum_{a_2 \in \mathbb{F}} (x_1 + a_2 x_2) = \Big(\sum_{a_2 \in \mathbb{F}} a_2\Big) x_2, \\ \eta_P^P(o(x_{21})) &= \sum_{j=1}^{|P|} g_j x_2 = 0 \end{aligned}$$

If p is odd then for every nonzero $a_2 \in \mathbb{F}$ there exists a negative $-a_2 \neq a_2$. Therefore

$$\sum_{a_2 \in \mathbb{F}} a_2 = 0.$$

If $p = 2$ then

$$\Big(\sum_{a_2 \in \mathbb{F}} a_2\Big) x_2 = \begin{cases} x_2 & \text{if } q = 2 \\ 0 & \text{if } q > 2. \end{cases}$$

Thus we have that the Noether map is surjective if and only if $n = 2 = p = q$. Explicitely we find

$$\eta_P^P(o(x_{11})) = x_2 \quad \text{and} \quad \eta_P^P(o(x_{11}x_{12})) = x_1^2 + x_1 x_2.$$

Note that in this case

$$\mathrm{Syl}_2(\mathrm{GL}(2,\ \mathbb{F}_2)) \cong \mathbb{Z}/2$$

and our representation is projective.

THEOREM 3.5: *Let $\rho : G \hookrightarrow \mathrm{GL}(n,\ \mathbb{F})$ be a permutation representation of a finite group of order d. Then the Noether map η_G^G is surjective if and only if $V = \mathbb{F}^n$ is projective.*

PROOF: By Proposition 2.1 we know that η_G^G is surjective if V is projective as $\mathbb{F}G$-module.

We show that the converse is also true as follows:

Let η_G^G be surjective, then its restriction to degree one, $\eta_G^G|_{(1)}$, is also surjective:

$$\eta_G^G|_{(1)} : (V(G)^*)^G \longrightarrow (V^*)^G.$$

We note that $(V(G)^*)^G$ has an $\mathbb{F}$-basis consisting of

$$o(x_{ij}) = \sum_{j=1}^{d} x_{ij} \quad \text{for } i = 1, \ldots,\ n.$$

Therefore, the image under the Noether map is spanned by

$$\eta_G^G\left(\sum_{j=1}^{d} x_{ij}\right) = k_i o(x_i) \quad \text{for } i = 1, \dots, n,$$

where

$$k_i = \left|\mathrm{Stab}_G(x_i)\right|$$

is the order of the stabilizer of x_i in G. Since ϱ is a permutation representation, $(V^*)^G$ is spanned by the orbit sums of $x_1, \dots, x_n$. It follows that k_i's are not zero, since the Noether map is surjective. Hence

$$\left|\mathrm{Stab}_G(x_i)\right| \not\equiv 0 \bmod p.$$

In other words, no element in a p-Sylow subgroup P of G fixes x_i, $i = 1, \dots, n$. Therefore

$$(\maltese) \qquad o^P(x_i) = \mathrm{Tr}^P(x_i) = \eta_P^P|_{(1)}(x_{i1}),$$

where $o^P(-)$ denotes the orbit sum under the action of P, and g_1 is the identity element. Since $(V^*)^P$ is also spanned by the orbit sums of the x_i's, we found in ($\maltese$) that $\eta_P^P|_{(1)}$ is surjective. Therefore, η_P^P is surjective by Proposition 3.4. Hence V^* is a projective $\mathbb{F}P$-module, by the same Propositon 3.4. Since P is a p-Sylow subgroup of G, the module V^* is projective as a $\mathbb{F}G$-module, see Corollary 3 on Page 66 of [1]. □

§4. Applications and Examples

Let $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a faithful representation of a finite group of order d. Set $V = \mathbb{F}^n$. Recall that $\beta(\mathbb{F}[V]^G)$ is the maximal degree of an $\mathbb{F}$-algebra generator of $\mathbb{F}[V]^G$ in a minimal generating set, the so-called Noether number.

PROPOSITION 4.1: *If V is a projective $\mathbb{F}G$-module then*

$$\beta(\mathbb{F}[V]^G) \le \max\{d, \ n\binom{d}{2}\}.$$

PROOF: If V is $\mathbb{F}G$-projective then the Noether map η_G^G is surjective by Proposition 2.1. Thus, since η_G^G is an $\mathbb{F}$-algebra map, a set of generators of $\mathbb{F}[V(G)]^G$ is mapped onto a set of generators of $\mathbb{F}[V]^G$. Since $V(G)$ is a permutation module with n transitive components each of which has degree d,

it is generated by elements of degree at most $\max\{d, \ n\binom{d}{2}\}$, by Corollary 3.10.9 in [5] and the result follows. □

REMARK: Let $\rho : G \hookrightarrow \mathrm{GL}(n,\ \mathbb{F})$ be a representation of a finite group G of order d. Assume that the characteristic of $\mathbb{F}$ is zero or strictly larger than d. (This is the strongly nonmodular case.) Then

$$\beta(\mathbb{F}[V]^G) \leq \beta(\mathbb{F}[W]^G)$$

where W is the regular $\mathbb{F}G$-module, see Theorem 4.1.4 in [11]. Thus our Proposition 4.1 is a characteristic-free generalization: for *projective* $\mathbb{F}G$-modules V of dimension n, the upper bound for $\beta(\mathbb{F}[V]^G)$ is given by $\beta(\mathbb{F}[W]^G)$ where W is $\oplus_n \mathbb{F}G$.

The degree bound given above is sharp as we illustrate with the following example.

EXAMPLE 1: Let A_3 be the alternating group in three letters. Let $\mathbb{F}$ be a field containing a primitive 3rd root of unity $\omega \in \mathbb{F}$. Then we obtain a faithful representation

$$\rho : A_3 \hookrightarrow \mathrm{GL}(1,\ \mathbb{F}), (123) \longmapsto \omega.$$

We have

$$\mathbb{F}[x]^{A_3} = \mathbb{F}[x^3], \quad \text{and} \quad \mathbb{F}[x_{11}, x_{12}, x_{13}]^{A_3} = \mathbb{F}[e_1, e_2, e_3, o(x_{11}^2 x_{12})],$$

where the e_i's are the elementary symmetric functions in the x_{1j}'s. Thus

$$\beta(\mathbb{F}[x]^{A_3}) = 3 = \beta(\mathbb{F}[x_{11}, x_{12}, x_{13}]^{A_3}) = \max\{3, \binom{3}{2}\}.$$

Before we proceed we want to compare the degree bound given in Proposition 4.1 with the known general bounds, see [9] for an overview of this topic.

(1) In the nonmodular case, we have that $\beta(\mathbb{F}[V]^G) \leq |G|$ by Theorem 2.3.3 in [11]. This bound is better since

$$|G| \leq \max\{n|G|, n\binom{|G|}{2}\}.$$

(2) The general degree bound given in Theorem 3.8.11 in [5] is

$$\beta(\mathbb{F}[V]^G) \leq n(|G|-1) + |G|^{n2^{n-1}} n^{2^{n-1}+1}.$$

A short calculation shows that

$$\max\{n|G|, n\binom{|G|}{2}\} \leq n(|G|-1) + |G|^{n2^{n-1}} n^{2^{n-1}+1}.$$

Thus the bound given in Proposition 4.1 is always better (where it applies).

(3) If the ground field $\mathbb{F}$ is finite of order q, we have another general

degree bound given by:

$$\beta(\mathbb{F}[V]^G) \leq \begin{cases} \frac{q^n-1}{q-1}(nq-n-1) & \text{if } n \geq 3, \\ 2q^2-q-2 & \text{if } n=2, \end{cases}$$

see Theorem 16.4 in [7]. This bound behaves worse than the one of Proposition 4.1 if $q > |G|$.

(4) Finally in [4] a bound of a completely different flavor is proven. In particular it depends on a choice of a homogeneous system of parameters. In our Example 1 we found that the bound of Proposition 4.1 is sharp. If we apply Theorem 2.3 in [4] to this example we obtain

$$\beta(\mathbb{F}[x]^{A_3}) \leq \text{degree}(f),$$

where $f \in \mathbb{F}[x]^{A_3}$ is a system of parameters. If we make the unlucky choice of $f = x^9$ the bound given in [4] is no longer sharp.

We denote by CMdefect(–) the Cohen-Macaulay defect. The following result tells us that the Cohen-Macaulay defect of the ring of invariants of n copies of the regular representation of a finite group G is an upper bound for the Cohen-Macaulay defect of the ring of invariants $\mathbb{F}[V]^G$ in the case where V is projective.

PROPOSITION 4.2: *If V is $\mathbb{F}G$-projective then*

$$\text{CMdefect}(\mathbb{F}[V]^G) \leq \text{CMdefect}(\mathbb{F}[V(G)]^G).$$

PROOF: Since V is $\mathbb{F}G$-projective, we have the $\mathbb{F}G$-module decomposition

$$V(G) = V \oplus K.$$

Thus the result follows from [10]. □

REMARK: The inequality in the preceding result is sharp since the Cohen-Macaulay defect of any nonmodular representation is zero.

References

[1] J. L. Alperin, *Local Representation Theory*, Cambridge Studies in Advanced Mathematics 11, Cambridge University Press, Cambridge 1986.

[2] Kenneth S. Brown, *Cohomology of Groups*, Graduate Texts in Mathematics 87, Springer-Verlag, New York NY 1982.

[3] Jon F. Carlson, *Modules and Group Algebras*, Lectures in Mathematics ETH Zürich , Birkhäuser Verlag, Basel-Boston-Berlin 1996.

[4] Jianjun Chuai, A new Degree Bound for Invariant Rings, *Proceedings of the AMS* 133 (2005), 1325-1333.

[5] Harm Derken and Gregor Kemper, *Computational Invariant Theory*, Encylopaedia of Mathematical Sciences, Springer-Verlag, Heidelberg 2002.

[6] Peter Fleischmann, Gregor Kemper, and R. James Shank, On the Depth of Cohomology Modules, *Quaterly J. of Math* (2004), 167-184.

[7] Dikran B. Karagueuzian and Peter Symonds, The Module Structure of a Group Action on a Polynomial Ring: A Finiteness Theorem, preprint 2002.

[8] Mara D. Neusel, The Transfer in the Invariant Theory of Modular Permutation Representations, *Pacific J. of Math.* 199 (2001) 121-136.

[9] Mara D. Neusel, Degree Bounds - An Invitation to Postmodern Invariant Theory -, *Topology and its Applications*, to appear.

[10] Mara D. Neusel, Comparing the Depths of Rings of Invariants, pp 189-192 in: *CRM Conference Proceedings* Volume 35, edited by Eddy Campbell and David Wehlau, AMS, Providence RI 2004.

[11] Mara D. Neusel and Larry Smith, *Invariant Theory of Finite Groups*, Mathematical Surveys and Monographs Vol.94, AMS, Providence RI 2002.

Mara D. Neusel
Department of Mathematics and Statistics
Mail Stop 1042
Texas Tech University
Lubbock TX 79409
USA
mara.d.neusel@ttu.edu

Müfit Sezer
Department of Mathematics
Boğazici Üniversitesi
Bebek
Istanbul
Turkey
mufit.sezer@boun.edu.tr