STABILIZING FIRST-ORDER CONTROLLERS WITH DESIRED STABILITY REGION

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Abstract

In this paper, we determine the set of all stabilizing first-order controllers that place the poles of the closed-loop system in a desired stability region. The solution is based on a generalization of the Hermite–Biehler theorem applicable to polynomials with complex coefficients and the application of a modified stabilizing gain algorithm to three subsidiary plants. The method given is also applicable to PID controllers.

Key Words

Hermite–Biehler theorem, stabilization, first-order controllers, regional pole placement

1. Introduction

In many applications, stability of the closed-loop system is not enough, and it is usually required that the poles of the closed-loop system lie in more restrictive stability regions. It is known that time domain specifications for a closed-loop system can be translated into desired closed-loop poles locations in the frequency domain. These are specified in terms of the damping ratio and damped natural frequency of the closed-loop poles [1]. A desired stability region S in the complex plane is shown in Fig. 1, [2]. The region S is the intersection of three regions $S_{-\lambda}$, S_θ, and $S_{-\theta}$ where $S_{-\lambda} := \{s : s \in \mathbf{C}, \; Re[s] < -\lambda\}$ is a shifted Hurwitz stability region, $S_\theta := \{s : s \in \mathbf{C}, \; Re[se^{-j\theta}] < 0\}$ and $S_{-\theta} := \{s : s \in \mathbf{C}, \; Re[se^{j\theta}] < 0\}$ are rotated Hurwitz stability regions. In [3], it is stated that if all the poles of the closed-loop system lie in the region S, then the step response of the compensated system exhibits a settling time of no more than $4/\lambda$ and a maximum overshoot corresponding to the angle θ. In [4], the region S is approximated by a circular region and a design procedure that combines linear-quadratic optimal control with

regional pole placement is given. See also [5–11] for different methods of solving this problem. Recently, a method for determining the set of all proportional controllers that place the closed-loop poles in the region S was given in [2].

The quest for an analytic design method for first-order controllers has been around for decades. Recently several computational methods have been proposed to determine the set of all stabilizing first-order controllers. In [12, 13], stabilizing first-order controllers for continuous and discrete-time systems were determined using boundary crossing theorem to identify boundaries of stability region of two parameters, by sweeping over the third parameter the complete set can be determined. Using these results, it has been shown that it is possible to obtain H_∞ optimal design with first-order controllers [14, 15]. Alternative methods have been used to determine the total set of controllers' parameters that stabilize a given system. An exact solution to stabilizing discrete-time systems by first-order controllers was given in [16]. Using extensions of the Hermite–Biehler theorem the set of all stabilizing first-order controllers were determined in [17, 18]. In this paper, we give a method to determine the set of all first-order controllers that place the poles of the closed-loop system in the region S. Once this set is determined, it is more convenient to search, among such controllers, those that satisfy other performance criteria imposed on the unit step response.

The paper is organized as follows. In Section 2, a generalization of the Hermite–Biehler theorem applicable to polynomials with complex coefficients is stated. This theorem is then used to convert the problem of determining gains such that a plant have a certain number of real roots to an equivalent problem of signature determination. In Section 3, we give an algorithm that solves the problem of determining all stabilizing first-order controllers that place the poles of the closed-loop system in a desired stability region. Section 4 contains some concluding remarks.

2. A Generalization of the Hermite–Biehler Theorem

In this section, a generalization of the Hermite–Biehler theorem to polynomials with complex coefficients [19] is

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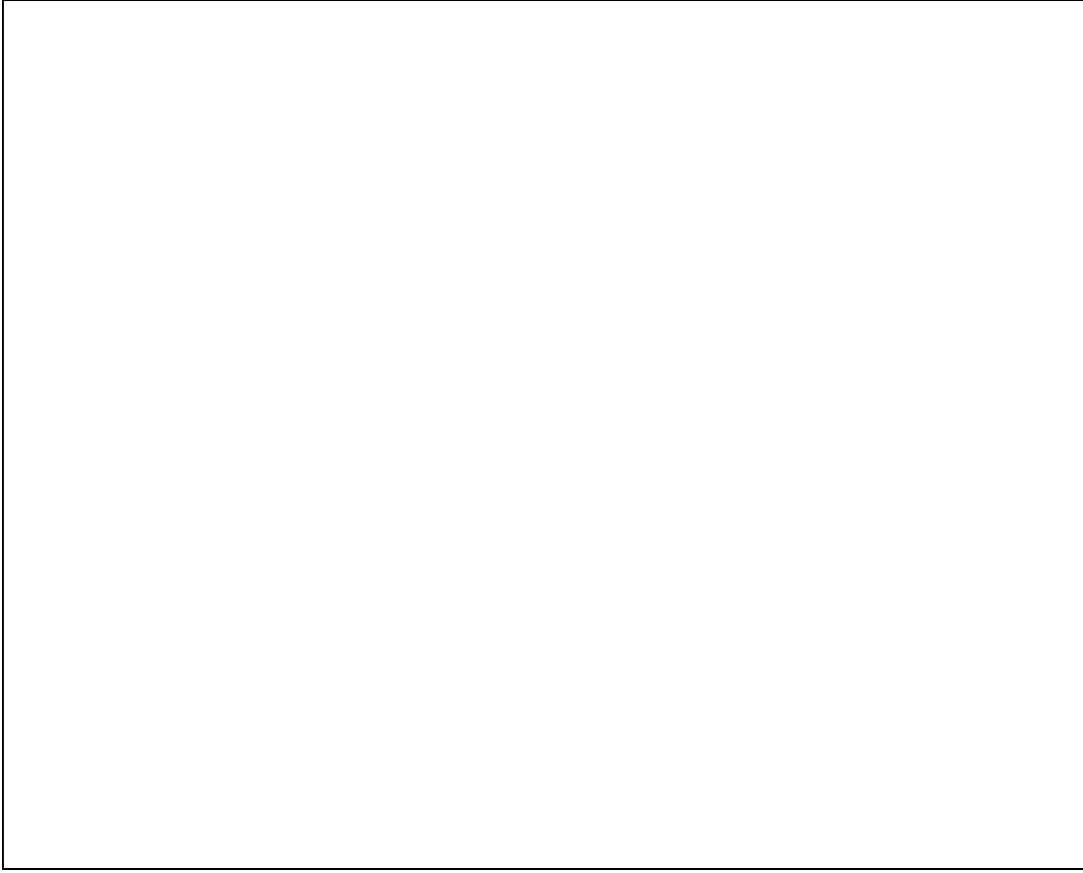
Figure 1. Stability region S.

presented. Before proceeding any further, let us fix the notation used in this paper. Let $\mathbf{R}$ denotes the set of real numbers and $\mathbf{C}$ denotes the set of complex numbers and let $\mathbf{C}_-$, $\mathbf{C}_0$, $\mathbf{C}_+$ denote the points in the open left half, $j\omega$-axis, and the open right half of the complex plane, respectively. Given a set of polynomials $\psi_1, \ldots, \psi_k \in \mathbf{R}[s]$ not all zero and $k > 1$, their *greatest common divisor* (with highest coefficient 1) is unique and it is denoted by $gcd\{\psi_1, \ldots, \psi_k\}$. If $gcd\{\psi_1, \ldots, \psi_k\}, = 1$, then we say $(\psi_1, \ldots, \psi_k)$ is *coprime*. The derivative of ψ is denoted by ψ'. The set $\mathcal{H}$ of Hurwitz stable polynomials are $\mathcal{H} = \{\psi(s) \in \mathbf{C}[s] : \psi(s) = 0 \Rightarrow s \in \mathbf{C}_-\}$. The *signature* $\sigma(\psi)$ *of a polynomial* $\psi \in \mathbf{C}[s]$ is the difference between the number of its $\mathbf{C}_-$ roots and $\mathbf{C}_+$ roots. Given $\psi \in \mathbf{C}[s]$, the real and imaginary parts (a, b) of $\psi(s)$ are the unique polynomials a, $b \in \mathbf{R}[\omega]$ such that $\psi(j\omega) = a(\omega) + jb(\omega)$. Finally, let us define the *signum function* $\mathcal{S} : \mathbf{R} \to \{-1, 0, 1\}$ by

$$\mathcal{S}r = \begin{cases} -1 & \text{if } r < 0 \\ 0 & \text{if } r = 0 \\ 1 & \text{if } r > 0 \end{cases}$$

Theorem 1. [19] Let a non-zero polynomial $\psi \in \mathbf{C}[s]$ of *degree* n have the real-imaginary parts (a, b). Let $\omega_1 < \omega_2 < \cdots < \omega_k$ be the real, distinct finite roots of b

with odd multiplicities. Also let $\omega_0 = -\infty$, $\omega_{k+1} = \infty$, and ξ_n be the leading coefficient of $\psi(s)$. Then

$$\sigma(\psi) = \begin{cases} \frac{1}{2}\big\{\mathcal{S}a(\omega_0)(-1)^k + 2\sum_{i=1}^k \mathcal{S}a(\omega_i)(-1)^{k-i} \\ \quad - \mathcal{S}a(\omega_{k+1})\big\}\mathcal{S}b(\infty) \\ \text{if } n \text{ is even and } \xi_n \text{is purely real,} \\ \text{or } n \text{ is odd and } \xi_n \text{is purely imaginary.} \\ \\ \frac{1}{2}\big\{2\sum_{i=1}^k \mathcal{S}a(\omega_i)(-1)^{k-i}\big\}\mathcal{S}b(\infty) \\ \text{if } n \text{ is even and } \xi_n \text{is not purely real,} \\ \text{or } n \text{ is odd and } \xi_n \text{is not purely imaginary.} \end{cases}$$
(1)

Proof: See [2, 19].

The following extension of Lemma 1 in [17] transforms the problem of determining the number of real roots of a real polynomial to an equivalent problem of finding the signature of a complex polynomial.

Lemma 1. A non-zero polynomial $\psi \in \mathbf{R}[u]$, has r real roots without counting the multiplicities if and only if the signature of the complex polynomial $\overline{\psi}(j\omega) = \psi(w) + j\psi'(w)$ is $-r$.

Proof: We first assume that (ψ, ψ') is coprime. If $deg\ \psi = n$, then $deg\ \psi' = n - 1$, $deg\ \overline{\psi} = n$, and the highest

coefficient $\overline{\xi}_n$ of $\overline{\psi}(s)$ depends only on the highest coefficient ξ_n of $\psi(\omega)$. If n is even, then $(j\omega)^n$ is real. As $\xi_n = (j\omega)^n \overline{\xi}_n$ is real, it follows that $\overline{\xi}_n$ is real. If n is odd, then $(j\omega)^n$ is imaginary and using similar arguments it follows that $\overline{\xi}_n$ is imaginary. In both cases, n even or odd, we use the first equation of (1) in Theorem 1 to calculate the signature of $\overline{\psi}(s)$. Let $\psi(\omega)$ have r real distinct roots $\omega_1 < \omega_2 < \cdots < \omega_r$. Since $\psi'(w)$ is the derivative of $\psi(w)$, it follows that between any two consecutive real roots ω_i and ω_{i+1} of $\psi(\omega)$, there is an odd number of real roots of $\psi'(\omega)$: $v_{i1} < v_{i2} < \cdots < v_{ij}$, where j is an odd integer. Since $\mathcal{S}\psi(v_{i1}) = \mathcal{S}\psi(v_{i2}) = \cdots = \mathcal{S}\psi(v_{ij})$, it follows that $2\mathcal{S}\psi(v_{i1}) - 2\mathcal{S}\psi(v_{i2}) + \cdots + (-1)^j 2\mathcal{S}\psi(v_{ij}) = 2\mathcal{S}\psi(v_{i1})$. In the interval $(-\infty, \omega_1)$ or (ω_r, ∞), $\psi'(\omega)$ has an even number of real roots which do not affect the signature as the sign of ψ is constant throughout the interval. Finally note that $\mathcal{S}\psi(\infty)\mathcal{S}\psi'(\infty) = 1, \ldots, \mathcal{S}\psi(v_{01})\mathcal{S}\psi'(\infty) = (-1)^{r-1}$, $\mathcal{S}\psi(-\infty)\mathcal{S}\psi'(\infty) = (-1)^r$. Using these facts in (1) of Theorem 1, we get $\sigma(\overline{\psi}) = \frac{1}{2}\{\mathcal{S}\psi(-\infty)(-1)^{r-1} + 2\mathcal{S}\psi(v_{01}) (-1)^{r-2} + \cdots - \mathcal{S}\psi(\infty)\}\mathcal{S}\psi'(\infty) = -r$. Therefore, by Theorem 1, the signature of $\overline{\psi}(s)$ is $-r$. Conversely, let the signature of $\overline{\psi}(s)$ be $-r$. Using the first equation of (1) in Theorem 1, it follows that $\psi(\omega)$ changes sign exactly r times. Hence, $\psi(\omega)$ has r real roots. For non-coprime pair (ψ, ψ'), repeating similar arguments it is easy to prove that $\psi(\omega)$ has r real roots without counting the multiplicities if and only if the signature of $\overline{\psi}(s)$ is $-r$. ∎

We now briefly describe a method to determine constant stabilizing gains for complex polynomials. This method will be used in the next section. Given a proper plant $g(s) = p(s)/q(s)$, where $p, q \in \mathbf{C}[s]$ are coprime, the set $A_r(p,q) := \{\alpha \in \mathbf{R} \,:\, \sigma[\phi(s, \alpha)] = \sigma[q(s) + \alpha p(s)] = r\}$ is the set of all real α such that $\phi(s, \alpha)$ has signature equal to r. Let (h, g) and (f, e) be the real-imaginary parts of q and p, respectively, so that $q(j\omega) = h(\omega) + jg(\omega)$, $p(j\omega) = f(\omega) + je(\omega)$. Let $d := \gcd\{f, e\}$ so that $f = d\overline{f}$, $e = d\overline{e}$, for coprime polynomials $\overline{f}, \overline{e} \in \mathbf{R}[\omega]$. Then, the polynomial $\overline{p}(s)$ such that $\overline{p}(j\omega) := \overline{f}(\omega) + j\overline{e}(\omega)$ is free of $\mathbf{C}_0$ roots. Let $m = \deg \overline{p}$ less than or equal to $n = \deg q$ and let (H, G) be the real-imaginary parts of $q(s)\overline{p}^*(s)$ where $\overline{p}^*(j\omega) := \overline{f}(\omega) - j\overline{e}(j\omega)$. Also let $F(\omega) := p(s)\overline{p}^*(s)$. By a simple computation, it follows that,

$$
\begin{aligned}
H(\omega) &= h(\omega)\overline{f}(\omega) + g(\omega)\overline{e}(\omega) \\
G(\omega) &= g(\omega)\overline{f}(\omega) - h(\omega)\overline{e}(\omega) \\
F(\omega) &= f(\omega)\overline{f}(\omega) + e(\omega)\overline{e}(\omega)
\end{aligned}
\tag{2}
$$

If $G \not\equiv 0$ and if they exist, let the *real roots with odd multiplicities of* $G(\omega)$ be $\{\omega_1, \ldots, \omega_k\}$ with the ordering $\omega_1 < \omega_2 < \cdots < \omega_k$, with $\omega_0 := -\infty$ and $\omega_{k+1} := \infty$ for notational convenience and let ξ be the leading coefficient of $[q(s) + \alpha p(s)]\overline{p}^*(s)$. The following algorithm determines whether $A_r(p,q)$ is empty or not and outputs its elements when it is not empty:

Algorithm 1.

1. Calculate

$$
\overline{\alpha}_j = \begin{cases}
-\frac{H}{F}(\omega_j), j = 1, \ldots, k \ \& \ F(\omega_j) \neq 0, \\
\quad \textit{if } n + m \textit{ is even and } \xi \textit{ is not purely real,} \\
\quad \textit{or } n + m \textit{ is odd and } \xi \textit{ is not} \\
\quad \textit{ purely imaginary.} \\
\\
-\frac{H}{F}(\omega_j), j = 0, \ldots, k+1 \ \& \ F(\omega_j) \neq 0, \\
\quad \textit{if } n + m \textit{ is even and } \xi \textit{ is purely real,} \\
\quad \textit{or } n + m \textit{ is odd and } \xi \textit{ is purely imaginary.}
\end{cases}
$$

and sort the distinct $\overline{\alpha}_j$'s in ascending order

$$
\alpha_0 < \alpha_1 < \cdots < \alpha_{k+2} < \alpha_{k+3}
$$

where $\alpha_0 = -\infty$ and $\alpha_{k+3} = \infty$.

2. Identify all the sequences of signums

$$
\mathcal{I} = \begin{cases}
\{i_1, \ldots, i_k\} & \textit{if } n + m \textit{ is even and } \xi \textit{ is} \\
& \textit{not purely real,} \\
& \textit{or } n + m \textit{ is odd and } \xi \textit{ is} \\
& \textit{not purely imaginary.} \\
\{i_0, i_1, \ldots, i_{k+1}\} & \textit{if } n + m \textit{ is even and } \xi \textit{ is} \\
& \textit{purely real,} \\
& \textit{or } n + m \textit{ is odd and } \xi \textit{ is} \\
& \textit{purely imaginary.}
\end{cases}
$$

where $i_j \in \{-1, 1\}$ for $j = 0, 1, \ldots, k+1$, that correspond to the intervals (α_j, α_{j+1}) for $j = 0, \ldots, k+2$.

3. For each signum sequence $\mathcal{I}_j$ from step 2, if

$$
r + \sigma(p^*) = \begin{cases}
\{(-1)^{k-1}i_0 + \cdots + i_{k-2} - i_{k-1} + i_k\} \\
\quad \mathcal{S}G(\infty) \\
\quad \textit{if } n + m \textit{ is even and } \xi \textit{ is not} \\
\quad \textit{ purely real,} \\
\quad \textit{or } n + m \textit{ is odd and } \xi \textit{ is not} \\
\quad \textit{ purely imaginary.} \\
\\
\frac{1}{2}\{(-1)^k i_0 + \cdots - 2i_{k-1} + 2i_k - i_{k+1}\} \\
\quad \mathcal{S}G(\infty) \\
\quad \textit{if } n + m \textit{ is even and } \xi \textit{ is} \\
\quad \textit{ purely real,} \\
\quad \textit{or } n + m \textit{ is odd and } \xi \textit{ is} \\
\quad \textit{ purely imaginary.}
\end{cases}
$$

holds, then $(\alpha_j, \alpha_{j+1}) \in A_r(p, q)$

Remark 1. By step 3 of Algorithm 1, a necessary condition for the existence of an $\alpha \in A_r(p,q)$ is that the imaginary part of $[q(s) + \alpha p(s)]\overline{p}^*(s)$ has at least $|r + \sigma(\overline{p}^*)|$ real roots with odd multiplicities if $n + m$ is even and ξ is not purely real, or $n + m$ is odd and ξ is not purely imaginary, and $|r + \sigma(\overline{p}^*) - 1|$ real roots with odd multiplicities if $n + m$ is even and ξ is purely real, or $n + m$ is odd and ξ is purely imaginary.

3. First-Order Controllers

Given a plant $g(s) = p(s)/q(s)$ and a first-order controller $c(s) = \alpha_2 s + \alpha_3/s + \alpha_1$, our objective is to find all values of $(\alpha_1, \alpha_2, \alpha_3)$ such that the closed-loop characteristic polynomial $\phi(s, \alpha_1, \alpha_2, \alpha_3) = (s + \alpha_1)q(s) + (\alpha_2 s + \alpha_3)p(s)$ has all its roots in the region S given in Fig. 1. This is equivalent to solving three subproblems using the stability regions $S_{-\lambda}$, S_θ, and $S_{-\theta}$ and finding the intersection of the solution sets.

Let us first solve the problem for the stability region S_θ. Let us replace s by $e^{j\theta}s$, then $\phi_\theta(s, \alpha_1, \alpha_2, \alpha_3) = (e^{j\theta}s + \alpha_1)q(e^{j\theta}s) + (\alpha_2 e^{j\theta}s + \alpha_3)p(e^{j\theta}s)$. Since θ is constant, we have $e^{j\theta} = \beta + j\gamma$ and $p, q \in C[s]$. The new characteristic polynomial is given by

$$\begin{aligned}
\phi_\theta^0(s, \alpha_1, \alpha_2, \alpha_3) &= [(\beta + j\gamma)s + \alpha_1]q(s) \\
&\quad + [\alpha_2(\beta + j\gamma)s + \alpha_3]p(s) \\
&= q_0(s) + \alpha_3 p_0(s)
\end{aligned}$$

where $q_0(s) = [(\beta + j\gamma)s + \alpha_1]q(s) + [\alpha_2(\beta + j\gamma)s]p(s)$, $p_0(s) = p(s)$. Roots of $\phi(s, \alpha_1, \alpha_2, \alpha_3)$ in stability region S_θ is equivalent to roots of $\phi_\theta^0(s, \alpha_1, \alpha_2, \alpha_3)$ in the open left half complex plane. Using the generalized Hermite–Biehler theorem applicable to complex polynomials and Lemma 1, we describe in what follows a method to compute all values of $(\alpha_1, \alpha_2, \alpha_3)$ such that $\phi_\theta^0(s, \alpha_1, \alpha_2, \alpha_3)$ is Hurwitz stable. Recall that $q(j\omega) = h(\omega) + jg(\omega)$, $p(j\omega) = f(\omega) + je(\omega)$, $\overline{p}(j\omega) = \overline{f}(\omega) + j\overline{e}(\omega)$, and $q(j\omega)\overline{p}^*(j\omega) = H(\omega) + jG(\omega)$, $p(j\omega)\overline{p}^*(j\omega) = F(\omega)$, where H, G, and F are given by (2). Multiplying $\phi_\theta^0(j\omega, \alpha_1, \alpha_2, \alpha_3)$ by $\overline{p}_0^*(j\omega)$ we obtain

$$\begin{aligned}
\psi_\theta^1(j\omega, \alpha_1, \alpha_2, \alpha_3) &= [-\omega(\gamma H(\omega) + \beta G(\omega)) + \alpha_1 H(\omega) \\
&\quad - \alpha_2 \omega \gamma F(\omega) + \alpha_3 F(\omega)] \\
&\quad + j[\omega(\beta H(\omega) - \gamma G(\omega)) \\
&\quad + \alpha_1 G(\omega) + \alpha_2 \omega \beta F(\omega)]
\end{aligned}$$

Note that only two parameters (α_1, α_2) appear in the imaginary part of $\psi_\theta^1(s)$. Suitable ranges of (α_1, α_2) can be determined using Remark 1 and Lemma 1 as described below. The reasoning behind the algorithm which determines the set of parameters α_1, α_2, α_3 of a stabilizing first-order controller can be explained as follows: suppose $\phi_\theta^0(s)$ is Hurwitz stable for some α_1, α_2, $\alpha_3 \in$ **R**. By Remark 1, it follows that the imaginary part $\omega[\beta H(\omega) - \gamma G(\omega)] + \alpha_1 G(\omega) + \alpha_2 \omega \beta F(\omega)$ of $\psi_\theta^1(s)$ has at least $r_1 = |n + 1 + \sigma(\overline{p}^*)|$ real roots with odd multiplicities. Suppose the imaginary part of $\psi_\theta^1(s)$ has r_1 real roots with odd multiplicities. By Lemma 1, $\sigma[\phi_\theta^1(s)] = -r_1$, where

$$\begin{aligned}
\phi_\theta^1(j\omega) &= H_1(j\omega) + \alpha_1 G_1(j\omega) + \alpha_2 F_1(j\omega) \\
&= q_1(j\omega) + \alpha_2 p_1(j\omega) \quad\quad (3)
\end{aligned}$$

and $H_0(\omega) = \omega[\beta H(\omega) - \gamma G(\omega)]$, $F_0(\omega) = \omega\beta F(\omega)$, $G_0(\omega) = G(\omega)$, $H_1(j\omega) = H_0(\omega) + jH_0'(\omega)$, $F_1(j\omega) = F_0(\omega) + jF_0'(\omega)$, $G_1(j\omega) = G_0(\omega) + jG_0'(\omega)$, $q_1(s) = H_1(s) + \alpha_1 G_1(s)$, and $p_1(s) = F_1(s)$. To find the suitable ranges of α_1 and α_2, we modify $\phi_\theta^1(s)$ as follows: let $B := gcd\{F_0, F_0'\}$ so that $F_0 = B\overline{F}_0, F_0' = B\overline{F}_0'$ for coprime polynomials $\overline{F}_0, \overline{F}_0' \in \mathbf{R}[w]$. Also let $\overline{p}_1(j\omega) := \overline{F}_0(\omega) + j\overline{F}_0'(\omega)$. By a simple computation, it follows that,

$$\begin{aligned}
\psi_\theta^2(j\omega, \alpha_1, \alpha_2) &= \phi_\theta^1(j\omega, \alpha_1, \alpha_2)\overline{p}_1^*(j\omega) \\
&= H_{2r}(\omega) + \alpha_1 G_{2r}(\omega) + \alpha_2 F_{2r}(\omega) \\
&\quad + j[H_{2i}(\omega) + \alpha_1 G_{2i}(\omega)]
\end{aligned}$$

where

$$\begin{aligned}
H_{2r}(\omega) &= H_0(\omega)\overline{F}_0(\omega) + H_0'(\omega)\overline{F}_0'(\omega), \\
F_{2r}(\omega) &= F_0(\omega)\overline{F}_0(\omega) + F_0'(\omega)\overline{F}_0'(\omega), \\
G_{2r}(\omega) &= G_0(\omega)\overline{F}_0(\omega) + G_0'(\omega)\overline{F}_0'(\omega), \\
H_{2i}(\omega) &= H_0'(\omega)\overline{F}(\omega) - H_0(\omega)\overline{F}_0'(\omega), \\
G_{2i}(\omega) &= G_0'(\omega)\overline{F}_0(\omega) - G_0(\omega)\overline{F}_0'(\omega).
\end{aligned}$$

Now only one parameter α_1 appears in the imaginary part of $\psi_\theta^2(s)$. Once more by Remark 1, since $\sigma[\phi_\theta^1(s)\overline{p}_1^*(s)] = -r_1 + \sigma[\overline{p}_1^*(s)]$ the imaginary part of $\phi_\theta^1(s)\overline{p}_1^*(s)$ should have at least $r_2 = |-r_1 + \sigma(\overline{p}_1^*)|$ real roots with odd multiplicities . Now the set of $\alpha_1 \in \mathbf{R}$ which achieves r_2 real roots with odd multiplicities in $H_{2i}(\omega) + \alpha_1 G_{2i}(\omega)$ can be determined by applying Algorithm 1 to $q_2(j\omega) = H_2(j\omega) = H_{2i}(\omega) + jH_{2i}'(\omega)$ and $p_2(j\omega) = G_2(j\omega) = G_{2i}(\omega) + jG_{2i}'(\omega)$. In each step, we eliminate one of the controller's parameters and determine conditions to find the remaining ones. The algorithm below traces the above steps backwards by repetition of the following steps (i)–(iii):

(i) Pick a value of α_1 such that the number of real roots with odd multiplicities of $H_{2i}(\omega) + \alpha_1 G_{2i}(\omega)$ is r_2 or greater.

(ii) Determine using Algorithm 1 all $\alpha_2 \in \mathbf{R}$ such that $\sigma[\phi_\theta^1(s)] = -r_1$. This is equivalent to determining values of α_2 such that $H_0(\omega) + \alpha_1 G_0(\omega) + \alpha_2 F_0(\omega)$ has r_1 real roots with odd multiplicities.

(iii) For every α_2 determined, find using Algorithm 1 again, all α_3 such that $\phi_\theta^0(s)$ is Hurwitz stable.

Algorithm 2.
1. *Partition the real axis into intervals (or union of intervals) such that the number of real roots with odd multiplicities of $H_{2i}(\omega) + \alpha_1 G_{2i}(\omega)$ is constant in each interval.*
2. *Fix $r_1 = |n + \sigma(\overline{p}_0^*) + 1|$.*
   (a) Find admissible ranges of α_1 from the intervals found in the first step.

i. *Fix an α_1 in the admissible range.*

ii. *Apply Algorithm 1 to $q_1(s)$ and $p_1(s)$. (This calculates admissible values of α_2 such that $H_0(\omega) + \alpha_1 G_0(\omega) + \alpha_2 F_0(\omega)$ has r_1 real roots with odd multiplicities.)*

   A. *Fix an α_2 from the range determined in 2.a.ii.*

   B. *Apply Algorithm 1 to $q_0(s)$ and $p_0(s)$. (This calculates all admissible values of α_3 such that ϕ_θ^0 is in $\mathcal{H}$.)*

   C. *Increment α_2 and go to step 2.a.ii.B.*

iii. *Increment α_1 and go to step 2.a.ii.*

(b) *Increment r_1 and go to step 2.a.*

For the stability region $S_{-\theta}$, it was shown in [2] for the case of proportional controllers, that $S_{-\theta}$ and S_θ have exactly the same set of stabilizing controllers. This conclusion holds for first-order controllers. To see this, suppose that for a given triplet $(\overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}_3)$, s_0 is a root of $\phi(s, \alpha_1, \alpha_2, \alpha_3)$, then $(e^{j\theta} s_0 + \overline{\alpha}_1) q(e^{j\theta} s_0) + (\overline{\alpha}_2 e^{j\theta} s_0 + \overline{\alpha}_3) p(e^{j\theta} s_0) = 0$. As $q(s)$ and $p(s)$ are real polynomials, it follows that $(e^{-j\theta} s_0^* + \overline{\alpha}_1) q(e^{-j\theta} s_0^*) + (\overline{\alpha}_2 e^{-j\theta} s_0^* + \overline{\alpha}_3) p(e^{-j\theta} s_0^*) = 0$ where s_0^* is the complex conjugate of s_0. Since s_0^* and s_0 have the same real part, it follows that $(\overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}_3)$ is stabilizing triplet for the stability region $S_{-\theta}$ if and only if it is stabilizing triplet for the stability region S_θ.

Now let us consider the problem of determining the stabilizing values of $(\alpha_1, \alpha_2, \alpha_3)$ for the shifted Hurwitz stability region $S_{-\lambda}$. Let us replace s by $s - \lambda$ and make the corresponding changes. We now solve the usual stabilization problem for the new characteristic polynomial $\phi_\lambda(s, \alpha_1, \alpha_2, \alpha_3)$. As we are using a dynamic controller, the new characteristic polynomial is given by

$$\phi_\lambda(s, \alpha_1, \alpha_2, \alpha_3) = (s + \alpha_1 - \lambda) q(s) + (\alpha_2 s + \alpha_3 - \alpha_2 \lambda) p(s).$$

Multiplying $\phi_\lambda(s, \alpha_1, \alpha_2, \alpha_3)$ by $\overline{p}(-s)$ we obtain

$$\begin{aligned}
\psi_\lambda(s, \alpha_1, \alpha_2, \alpha_3) = \; & s^2 G(s^2) - \lambda H(s^2) + \alpha_1 H(s^2) \\
& - \alpha_2 \lambda F(s^2) + \alpha_3 F(s^2) + s[H(s^2) \\
& - \lambda G(s^2) + \alpha_1 G(s^2) + \alpha_2 F(s^2)]
\end{aligned}$$

We can use the method described above to find stabilizing values of $(\alpha_1, \alpha_2, \alpha_3)$. In [18], an alternative method that take advantage of the fact that $\psi_\lambda(s)$ is a real polynomial was given.

Example 1. Consider a first-order controller to stabilize the unstable plant $g(s) = p(s)/q(s)$ where $q(s) = s^5 + 3s^4 + 29s^3 + 15s^2 - 3s + 60$, $p(s) = s^3 - 6s^2 + 2s - 1$, and the stability region S is the intersection of two rotated stability regions $S_{\pi/18}$ and $S_{-\pi/18}$. Let us replace s by $e^{j\pi/18} s$, then $q(s) = (0.6428 + j0.7660)s^5 + (2.2981 + j1.9284)s^4 + (25.1147 + j14.5000)s^3 + (14.0954 + j5.1303)s^2 - (2.9544 + j0.5209)s + 60$, $p(s) = (0.8660 + j0.5000)s^3 - (5.6382 + j2.0521)s^2 + (1.9696 + j0.3473)s - 1$. Using Algorithm 2, the stabilizing values of $(\alpha_1, \alpha_2, \alpha_3)$ are obtained as shown in Fig. 2.   •

Remark 2. This method can be applied to PID controllers. Let $c(s) = (\alpha_1 s^2 + \alpha_2 s + \alpha_3)/s$, then we obtain

$$\begin{aligned}
\psi_\lambda(s, \alpha_1, \alpha_2, \alpha_3) = \; & \big[s^2 G(s^2) - \lambda H(s^2) + \alpha_1 s^2 F(s^2) \\
& + \alpha_1 \lambda^2 F(s^2) - \alpha_2 \lambda F(s^2) + \alpha_3 F(s^2) \big] \\
& + s[H(s^2) - \lambda G(s^2) - \alpha_1 2\lambda F(s^2) \\
& + \alpha_2 F(s^2)]
\end{aligned}$$

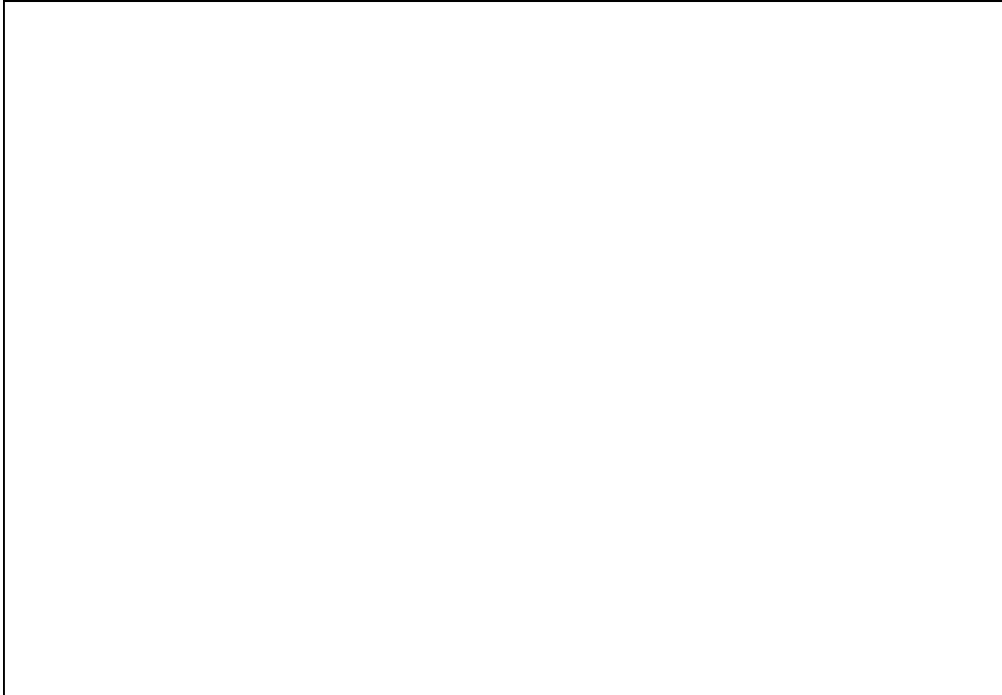
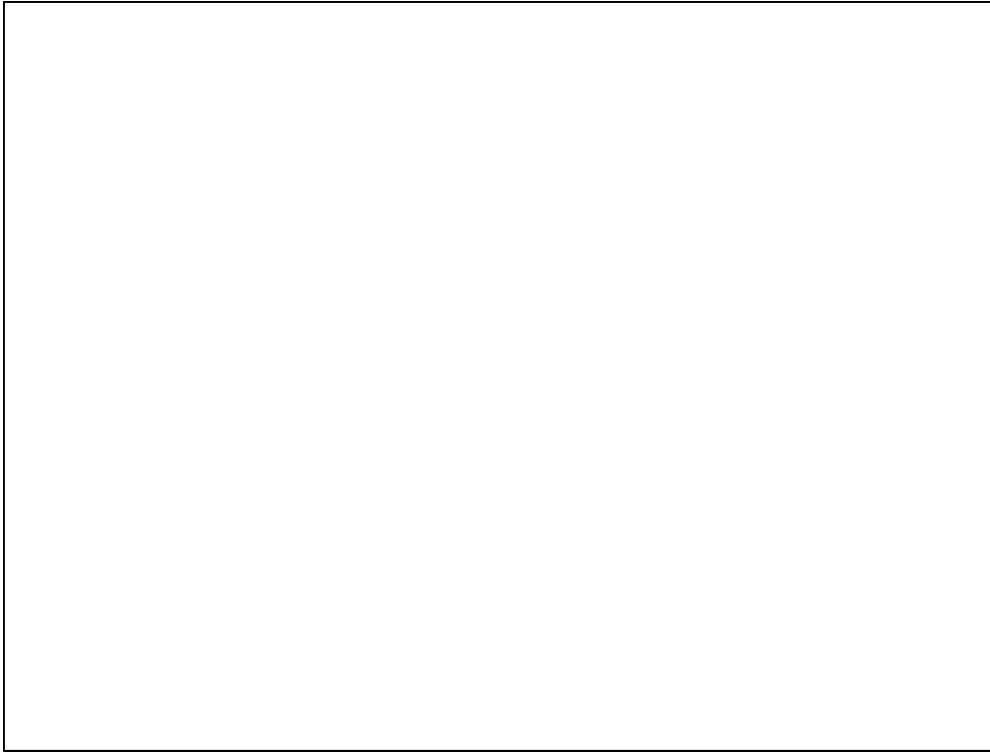Figure 2. Stabilizing values $(\alpha_1, \alpha_2, \alpha_3)$.
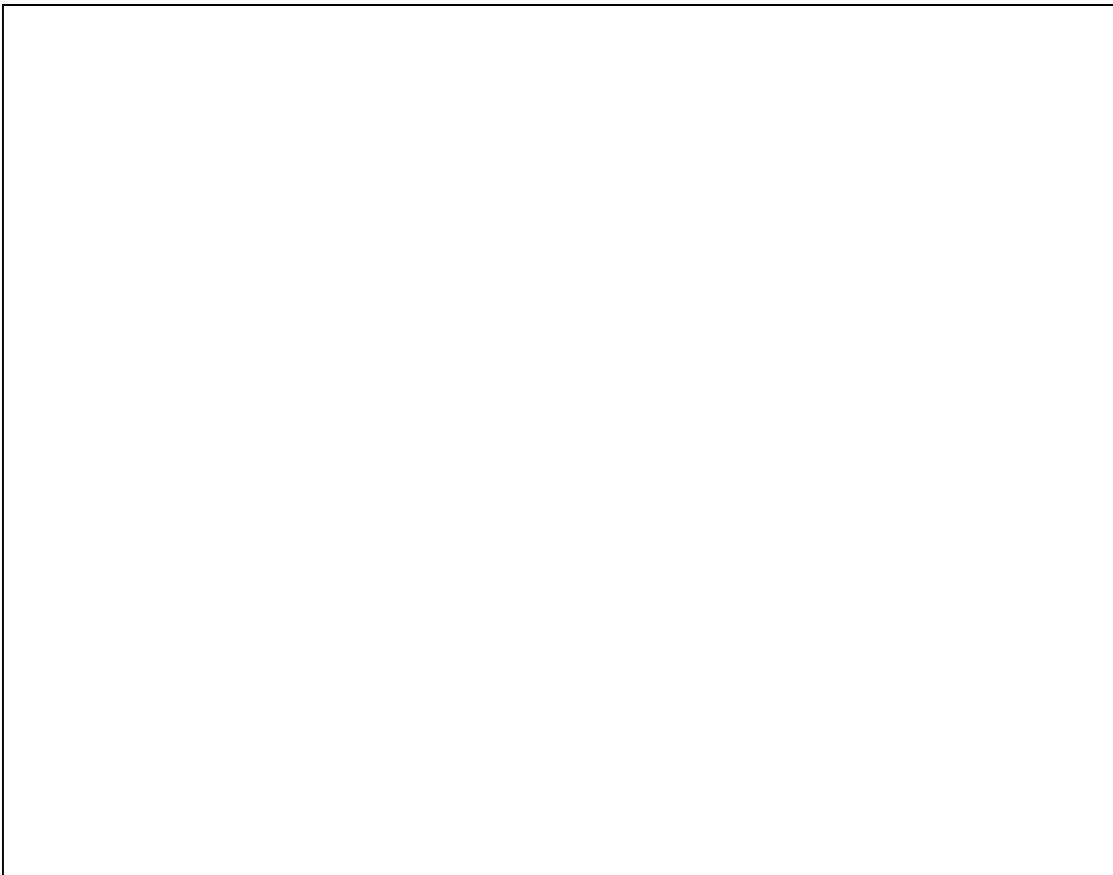
Figure 3. Stabilizing values (α_1, α_2).

Figure 4. Attainable roots with respect to region S for $\alpha_1 = -0.7599$.

and

$$\psi_\theta(j\omega, \alpha_1, \alpha_2, \alpha_3) = \big[-\omega(\gamma H_0(\omega) + \beta G_0(\omega)) - \alpha_1\omega^2$$
$$(\beta^2 - \gamma^2)F_0(\omega) - \alpha_2\omega\gamma F_0(\omega)$$
$$+ \alpha_3 F_0(\omega)\big] + j\big[\omega(\beta H_0(\omega) - \gamma G_0(\omega))$$
$$- \alpha_1\omega^2 2\beta\gamma F_0(\omega) + \alpha_2\omega\beta F_0(\omega)\big]$$

As two parameters (α_1, α_2) appear in the odd part of $\psi_\lambda(s, \alpha_1, \alpha_2, \alpha_3)$, imaginary part of $\psi_\theta(s, \alpha_1, \alpha_2, \alpha_3)$, we can directly apply the method developed for first-order controllers.

Example 2. Consider a PI controller $c(s) = (\alpha_1 s + \alpha_2)/s$ to stabilize the plant $g(s) = p(s)/q(s)$ given in [2], where $q(s) = s^3 + 3s^2 + 4s$, $p(s) = s^2 + 2s - 2$. The stability region S is given in Fig. 1 and specified by the parameters $\gamma = 0.5$ and $\theta = \pi/6$. For the rotated Hurwitz stability regions S_θ and $S_{-\theta}$, replacing s by $se^{j\pi/6}$, we get $q(s) = js^3 + (1.5 + 2.5981j)s^2 + (3.4641 + 2j)s$, $p(s) = (0.5 + 0.866j)s^2 + (1.7321 + j)s - 2$. For the shifted Hurwitz stability region $S_{-\gamma}$, replacing s by $s - 0.5$ we get $q(s) = s^3 + 1.5s^2 + 1.75s - 1.375$, $p(s) = s^2 + s - 2.75$. Using these new polynomials and the method described in this section, we obtain the stabilizing values of (α_1, α_2) as shown in Fig. 3. Let $\alpha_1 = -0.7599$, then $(-0.1489, -0.13)$ is the stabilizing interval for α_2. To check the results obtained, the root-locus for the values of α_2 in this interval is shown in Fig. 4 and clearly these roots belong to the stability region S.

4. Conclusions

This paper gives a computational method to determine the set of all first-order controllers that place the poles of the closed-loop system in a sector of the left-half plane. The computation is based on a generalization of the Hermite–Biehler theorem applicable to complex polynomials. Since this method is based on eliminating one of the controller's parameters, at each step, and determining conditions to find the remaining ones, extension of this method to controllers with more than three parameters is possible.

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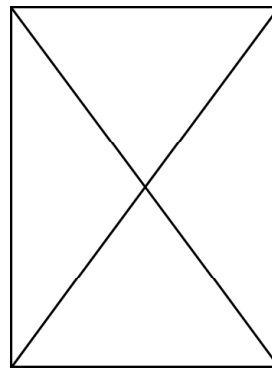
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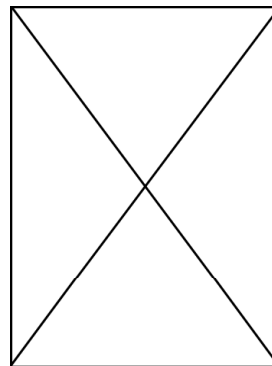
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