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QUANTUM CANONICAL TRANSFORMATIONS IN WEYL–WIGNER–GROENEWOLD–MOYAL FORMALISM

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A conjecture in quantum mechanics states that any quantum canonical transformation can decompose into a sequence of three basic canonical transformations; gauge, point and interchange of coordinates and momenta. It is shown that if one attempts to construct the three basic transformations in star-product form, while gauge and point transformations are immediate in star-exponential form, interchange has no correspondent, but it is possible in an ordinary exponential form. As an alternative approach, it is shown that all three basic transformations can be constructed in the ordinary exponential form and that in some cases this approach provides more useful tools than the star-exponential form in finding the generating function for given canonical transformation or vice versa. It is also shown that transforms of c-number phase space functions under linear–nonlinear canonical transformations and intertwining method can be treated within this argument.

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1. Introduction

Weyl–Wigner–Groenewold–Moyal (WWGM) formalism provides us a quantization and dequantization scheme based on Weyl's correspondence[1] and Wigner's quasi-distribution function[2] between quantum mechanical operators and c-number phase space functions. The product rule of these functions is given by Groenewold–Moyal's

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so-called twisted or star-product ($\star$-product).[3,4] For a comprehensive treatment of the subject the reader may consult Refs. 5–7.

Since the playground of quantum canonical transformations (QCT's) is the quantum phase-space, it is natural to introduce connection between QCT's and their c-number phase-space picture. Starting with the pioneering works of B. Leaf,[8–10] various aspects of this subject have been studied in the literature much considering behavior of the Wigner function under CT's. An extensive list of references can be found in Ref. 7.

Surprisingly, two independent fundamental types of invertible phase-space maps in one variable were proposed as the elementary generators of the entire classical and QCT's, that is every CT can be decomposed as finite or infinite sequences of the elementary CT's.[11] These are linear and point CT's. Later elaborations of this conjecture in quantum mechanics led to a triplet as a wider class including gauge transformations, point transformations and finally interchange of coordinates and momenta.[12,13] As a crash problem, this statement has not been proven in a general framework yet. But, though it is not true for every CT it applies to a large and relevant class of CT's.

The present work deals mainly with the implementation of the conjecture stated above in WWGM formalism. First, we give a brief summary on the fundamental QCT's following Refs. 12 and 13. In Sec. 2, we see that if we use the algebra isomorphism between the Hilbert space operators and the c-number phase-space functions, gauge and point transformations appear immediately in $\star$-exponential form just as in the expected form, but the interchanging remains out of this isomorphism. Still, we will be able to construct the interchanging in an ordinary exponential form. By accepting this result as our main guide, Sec. 3 is devoted to show that other two fundamental CT's can also be set on an isomorphism independent background. Section 4 shows that the generators in the ordinary exponential form are compatible with the well-known behaviors of functions under both linear and nonlinear CT's. In Sec. 5, after construction of the intertwining method in terms of $\star$-product we emphasize that the intertwining equation may be used to determine the relation between nonintertwined potentials well. Finally, Sec. 6 contains a short summary and conclusions.

A QCT is defined as

$$\hat{F}(\hat{q},\hat{p})\hat{q}\hat{F}^{-1}(\hat{q},\hat{p}) = \hat{Q}(\hat{q},\hat{p})\,, \quad \hat{F}(\hat{q},\hat{p})\hat{p}\hat{F}^{-1}(\hat{q},\hat{p}) = \hat{P}(\hat{q},\hat{p})\,, \tag{1}$$

where $[\hat{Q},\hat{P}] = \hat{Q}\hat{P} - \hat{P}\hat{Q} = i\hbar$ and $\hat{F}(\hat{q},\hat{p})$ is the generating function (GF) which is an arbitrary complex function (unitary or nonunitary), and $\hat{F}^{-1}$ is the algebraic inverse of $\hat{F}$. Action of the exponential version of the transformation on an arbitrary quantum phase-space function $\hat{u}(\hat{q},\hat{p})$ is given by the well-known form

$$e^{\lambda\hat{f}}\hat{u}(\hat{q},\hat{p})e^{-\lambda\hat{f}} = \hat{u} + \lambda[\hat{f},\hat{u}] + \frac{\lambda^2}{2!}[\hat{f},[\hat{f},\hat{u}]] + \frac{\lambda^3}{3!}[\hat{f},[\hat{f},[\hat{f},\hat{u}]]] + \cdots\,, \tag{2}$$

where λ is a pure imaginary number with a continuous parameter. The gauge transformation is generated, via (2), by complex function $f(\hat{q})$:

$$e^{\lambda f(\hat{q})}\hat{q}e^{-\lambda f(\hat{q})} = \hat{q}\,, \quad e^{\lambda f(\hat{q})}\hat{p}e^{-\lambda f(\hat{q})} = \hat{p} + i\hbar\lambda\partial_q f\,, \tag{3}$$

where $\partial_q \equiv \frac{\partial}{\partial q}$.

The point CT (change of variables) is given by

$$e^{\lambda f(\hat{q})\hat{p}}\hat{q}e^{-\lambda f(\hat{q})\hat{p}} = A(\hat{q})\,, \quad e^{\lambda f(\hat{q})\hat{p}}\hat{p}e^{-\lambda f(\hat{q})\hat{p}} = (\partial_q A)^{-1}\hat{p}\,, \tag{4}$$

where

$$A(\hat{q}) = e^{-i\hbar\lambda f(\hat{q})\partial_q}\hat{q}\,. \tag{5}$$

In (4), it is immediate to see that when the order in $f(\hat{q})\hat{p}$ is reversed, the order in $(\partial_q A)^{-1}\hat{p}$ is reversed.

Finally, the interchange of coordinates and momenta

$$\hat{I}\hat{q}\hat{I}^{-1} = \hat{p}\,, \quad \hat{I}\hat{p}\hat{I}^{-1} = -\hat{q} \tag{6}$$

is achieved by the Fourier transform operator $\hat{I}$ whose definition is given by the action

$$\hat{I}^{\pm 1} f(\hat{q}) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} f(\hat{q}')e^{\pm i\hat{q}\hat{q}'/\hbar}\,dq'\,. \tag{7}$$

As a special case of the linear CT, the interchanging can also be constructed by the composition of gauge transformations[13]

$$\hat{F}_I(\hat{q},\hat{p}) = e^{i\hat{q}^2/(2\hbar)}e^{i\hat{p}^2/(2\hbar)}e^{i\hat{q}^2/(2\hbar)}\,. \tag{8}$$

But because of the linearity property,

$$\hat{I}\hat{u}(\hat{q},\hat{p})\hat{I}^{-1} = \hat{u}(\hat{p},-\hat{q})\,, \tag{9}$$

the middle term in (8) cannot be accepted as an independent transformation within the class composed of gauge, point and interchange transformations. But note that, at the beginning it is possible to give the gauge (3) and point (4) transformations so as to be based on function $f(\hat{p})$ such as $e^{\lambda f(\hat{p})}$ and $e^{\lambda f(\hat{p})\hat{q}}$ respectively. Throughout the text our choice will be $f(\hat{q})$.

On the other hand, the linear CT itself can be decomposed into the form[13]

$$\begin{aligned}\hat{F}_L(\hat{q},\hat{p}) &= e^{\gamma\hat{q}\hat{p}}e^{\beta\hat{q}^2}e^{\alpha\hat{p}^2}\\ &= e^{\gamma\hat{q}\hat{p}}e^{\beta\hat{q}^2}\hat{I}e^{\alpha\hat{q}^2}\hat{I}^{-1}\,,\end{aligned} \tag{10}$$

where α, β, γ are the pure imaginary numbers compatible with a linear CT, therefore linear CT is not an element of the class defined above.

2. Implementations in WWGM Formalism

Given a c-number phase-space monomial $q^m p^n$ with nonnegative integers m, n, its image in the Hilbert space as a symmetrically ordered operator is determined by the Fourier transform

$$\hat{F}(\hat{q},\hat{p}) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} d\sigma\, d\tau\, dq\, dp\, F(q,p) e^{i[\sigma(\hat{q}-q)+\tau(\hat{p}-p)]/\hbar} \tag{11}$$

which serves as the quantization procedure called Weyl quantization.[1] Conversely, given operator $\hat{F}(\hat{q},\hat{p})$, the phase-space kernels $F(q,p)$ are specified simply by the correspondence

$$\hat{q} \to q\,, \qquad \hat{p} \to p \tag{12}$$

provided that $\hat{F}(\hat{q},\hat{p})$ is symmetrically ordered.

The associative (but non-Abelian in general) $\star$-product corresponding to the operator product in the Hilbert space is given by

$$\star = e^{\frac{i\hbar}{2}\left(\overleftarrow{\partial}_q \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_q\right)}\,, \tag{13}$$

where the arrows indicate the direction that the derivatives act. $\star$-product of c-number phase-space monomials and operator product of their images are in a complete algebra isomorphism given by the equation[3]

$$\hat{F}(\hat{q},\hat{p})\hat{G}(\hat{q},\hat{p}) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} d\sigma\, d\tau\, dq\, dp [F(q,p) \star G(q,p)] \times e^{i[\sigma(\hat{q}-q)+\tau(\hat{p}-p)]/\hbar}\,. \tag{14}$$

For example, while $q \star p^2$ is going to $\hat{q}\hat{p}^2$, $qp \star p$ goes to $\hat{q}\hat{p}^2 - i\hbar\hat{p}/2$ which is equivalent to the product of symmetrically ordered images of qp and p. By means of this isomorphism, it is possible to make practically some simple (de)quantization operations. For example, the operator $\hat{p}\hat{q}$ which is not symmetrically ordered is the quantized version of $p \star q = qp - i\hbar/2$. Conversely, quantized version of the c-number function $qp = (q \star p + p \star q)/2$ is $(\hat{q}\hat{p} + \hat{p}\hat{q})/2$. Therefore these examples induce that the Weyl quantization procedure of a c-number function is automatically reduced to write it in terms of the $\star$-product, meanwhile the dequantization procedure is easier obviously. Although this (de)quantization scheme as a unitary mapping is restricted to the monomials, it can be set for more general classes of functions and operators.[6,7] Now it has been recently understood that this correspondence can be achieved in terms of a kernel function.[14,15] In these general terms, for the sake of generality, we assume that there always exists a one to one correspondence between arbitrary $F(q,p)$ and $\hat{F}(\hat{q},\hat{p})$. If this general correspondence which can be summarized as

$$\begin{aligned} F(q,p) &\leftrightarrow \hat{F}(\hat{q},\hat{p})\,, \\ F(q,p) \star G(q,p) &\leftrightarrow \hat{F}(\hat{q},\hat{p})\hat{G}(\hat{q},\hat{p}) \end{aligned} \tag{15}$$

is used for the QCT (1), the corresponding transformation in the c-number phase-space can be written as[16]

$$\begin{aligned} F(q,p)\star q\star F^{-1}(q,p) &= Q(q,p)\,,\\ F(q,p)\star p\star F^{-1}(q,p) &= P(q,p) \end{aligned} \tag{16}$$

satisfying

$$\{Q,P\}^M = i\hbar \tag{17}$$

and $F\star F^{-1} = F^{-1}\star F = 1$, where $\{Q,P\}^M = Q\star P - P\star Q$ is the Moyal bracket and $F^{-1}(q,p)$ is the algebraic inverse of the GF $F(q,p)$.

If we employ the facts that $\{F,q\}^M = i\hbar\partial_p F$ and $\{F,p\}^M = -i\hbar\partial_q F$, we can write the definition (16) in a more useful form

$$Q(q,p) = q - i\hbar\partial_p F(q,p)\star F^{-1}(q,p)\,, \tag{18a}$$

$$P(q,p) = p + i\hbar\partial_q F(q,p)\star F^{-1}(q,p)\,. \tag{18b}$$

The correspondence (15) implies the gauge transformation (3) in WWGM formalism as

$$e_\star^{\lambda f(q)}\star q\star e_\star^{-\lambda f(q)} = q\,,\quad e_\star^{\lambda f(q)}\star p\star e_\star^{-\lambda f(q)} = p + i\hbar\lambda\partial_q f\,, \tag{19}$$

which gives the dequantized form of (3), where the $\star$-exponential is given by[17–20]

$$e_\star^{\lambda f(q,p)} = 1 + \lambda f(q,p) + \frac{\lambda^2}{2!}\, f(q,p)\star f(q,p) + \cdots\,. \tag{20}$$

When the property

$$\partial_\lambda e_\star^{\lambda f(q,p)} = f(q,p)\star e_\star^{\lambda f(q,p)} = e_\star^{\lambda f(q,p)}\star f(q,p) \tag{21}$$

is performed, one can obtain

$$\begin{aligned} e_\star^{\lambda f(q,p)}\star u(q,p)\star e_\star^{-\lambda f(q,p)} &= \sum_{n,r=0}^{\infty}\binom{n}{r}(-1)^r\frac{\lambda^{n+r}}{(n+r)!}(f\star)^n u(\star f)^r\\ &= u + \lambda\{f,u\}^M + \frac{\lambda^2}{2!}\{f,\{f,u\}^M\}^M\\ &\quad + \frac{\lambda^3}{3!}\{f,\{f,\{f,u\}^M\}^M\}^M + \cdots\,. \end{aligned} \tag{22}$$

Then the point CT (4) amounts to

$$e_\star^{\lambda f(q)\star p}\star q\star e_\star^{-\lambda f(q)\star p} = A(q)\,,\quad e_\star^{\lambda f(q)\star p}\star p\star e_\star^{-\lambda f(q)\star p} = (\partial_q A)^{-1}\star p\,, \tag{23}$$

where $A(q)$ is as in (5). In attempting to construct the interchange GF (7) it should be considered the fact that the c-number phase-space GF's are not operators. Therefore searching a function taking integral (or maybe taking derivative) is meaningless and such a function is not available in the $\star$-product argument within the isomorphism given above. Still, things can be put right by converting (16) into the system of partial differential equations[21]

$$F_I(q,p) \star q = p \star F_I(q,p)\,, \qquad F_I(q,p) \star p = -q \star F_I(q,p)\,. \tag{24}$$

The solution

$$F_I(q,p) = e^{i(q^2+p^2)/\hbar} \tag{25}$$

can be accepted as the GF searched for the last member of the set of fundamental transformations. Note that $F_I(q,p)$ appears in an ordinary exponential form not in the $\star$-exponential form like the others. Therefore it may be natural to ask whether each of the three basic GF's can be obtained in the ordinary exponential form. The proceeding section is devoted to discuss this approach.

3. Ordinary Exponential versus $\star$-exponential

Equation (16) is used for two main purposes; given GF, finding the CT, i.e. the pair $(Q(q,p), P(q,p))$ and vice versa. When the transformations are defined in terms of the $\star$-exponential form, given GF determination of $Q(q,p)$ and $P(q,p)$ is much easier, since the expansion (22) is a powerful tool. But conversely, given a CT, determination of the GF is generally a tedious matter. On the other hand, it is always possible to convert (16) into a system of partial differential equations for the GF's in the ordinary exponential form;

$$e^{\lambda f(q,p)} \star q = Q(q,p) \star e^{\lambda f(q,p)}\,, \quad e^{\lambda f(q,p)} \star p = P(q,p) \star e^{\lambda f(q,p)}\,, \tag{26}$$

which is a modified definition of the CT's that can be used to find both the GF and the pair $(Q(q,p), P(q,p))$. As a remarkable point note that, in operating (26) to find the CT, one does not need to know the inverse of the GF.

Alternatively, if the CT is defined simply by eliminating the star sign, (22) is deformed to strictly different expansion

$$e^{\lambda f(q,p)} \star u(q,p) \star e^{-\lambda f(q,p)} = \sum_{n,r=0}^{\infty} \binom{n}{r} (-1)^r \frac{\lambda^{n+r}}{(n+r)!} f^n \star u \star f^r\,. \tag{27}$$

This form behaves in a contrast way such that the right-hand side of (27) is not so easy to evaluate. In order to get rid of this problem, one may attempt to convert (27) into (26). This is always possible when $f(q,p) = f(q)$ or $f(q,p) = f(p)$ because with this condition the term $\exp[-\lambda f(q,p)]$ in (27) is always the inverse of $\exp[\lambda f(q,p)]$. Thus the definitions (26) and (27) becomes equivalent. Since the equations

$$e_\star^{\lambda f} = e^{\lambda f}\,, \qquad (f\star)^n = f^n\star \tag{28}$$

also hold, $e_\star^{\lambda f}$ and $e^{\lambda f}$ generate the same CT via (22) or (27) equivalently.

These remarks allow us to use the ordinary exponential form to generate the gauge transformation directly:

$$e^{\lambda f(q)} \star q \star e^{-\lambda f(q)} = q\,, \quad e^{\lambda f(q)} \star p \star e^{-\lambda f(q)} = p + i\hbar\lambda\partial_q f\,. \tag{29}$$

Conversely, given gauge transformation $Q = q$, $P = p + u(q)$, the GF

$$F_G(q,p) = e^{-\frac{i}{\hbar}\int u(q)dq} \tag{30}$$

appears as the solution to system of partial differential equations

$$-\partial_p F_G = \partial_p F_G\,, \quad i\hbar\partial_q F_G = u\left(q + \frac{i\hbar\partial_p}{2}\right)F_G \tag{31}$$

which is obtained from (26).

On the other hand, according to (16) and the canonicity condition (17), the most general form of the point transformation must satisfy the system of partial differential equations

$$F_P(q,p) \star q = Q(q) \star F_P(q,p)\,, \tag{32a}$$

$$F_P(q,p) \star p = [\tilde{Q}(q)p + \chi(q)] \star F_P(q,p)\,, \tag{32b}$$

where $\tilde{Q}(q) = [\partial_q Q(q)]^{-1}$ and $\chi(q)$, for the time being, is an arbitrary function. The system (32) may be solved by looking for solutions of the form

$$F_P(q,p) = e^{\lambda[f(q)p+g(q)]}\,. \tag{33}$$

Indeed, consider the facts that

$$F(q) \star e^{G(q)p+H(q)} = F\left[q + \frac{i\hbar G(q)}{2}\right]e^{G(q)p+H(q)}\,, \tag{34}$$

$$\frac{\partial}{\partial q}F\big[\underline{q} + G(q)\big] = \frac{\partial G}{\partial q}\frac{\partial F}{\partial q}[q + G(q)]\,, \tag{35}$$

where F, G, H are arbitrary functions and $\underline{q}$ is considered as constant under the operation ∂_q, which is originated from a crucial property of the $\star$-product. Then (32a) requires

$$Q(\upsilon) = q - \frac{i\hbar\lambda}{2}\,f\,, \tag{36}$$

where $\upsilon = q + i\hbar\lambda f/2$. The equality of the coefficients of p's with equal powers on both sides of (32b) requires

$$\tilde{Q}(\upsilon) = \left[\frac{\partial Q}{\partial q}(\upsilon)\right]^{-1} = \frac{2 + i\hbar\lambda\partial_q f}{2 - i\hbar\lambda\partial_q f}\,, \tag{37}$$

$$\chi(\upsilon) = \frac{i\hbar\lambda}{2}\left[1 + \tilde{Q}(\upsilon)\right]\frac{\partial g}{\partial q} - \frac{\lambda\hbar^2}{4}\frac{\partial\tilde{Q}}{\partial q}(\upsilon)\frac{\partial f}{\partial q}\,. \tag{38}$$

Given CT, i.e. $Q(q)$ and $\chi(q)$ or GF, i.e. $f(q)$ and $g(q)$, (36) and (38) provide analytical or numeric solutions. We now examine some important cases.

(i) $f = c_1$ and $g = g(q)$, where c_1 is any constant. Such a choice gives

$$Q(q) = q - i\hbar\lambda c_1\,, \tag{39}$$

$$P(q,p) = p + i\hbar\lambda\,\frac{\partial g}{\partial q}\left(q - \frac{i\hbar\lambda c_1}{2}\right) \tag{40}$$

due to (36) and (38) respectively. If $c_1 = 0$, then this is the gauge transformation (29). This result shows obviously that the gauge transformation is a special case of the point transformation and it makes the gauge transformation unnecessary as an independent fundamental transformation so long as the point transformation is defined by (32) with $\chi(q) \neq 0$. On the other hand, one may define the point transformation so as to be $\chi(q) = 0$ without destroying the canonicity condition (17). With this definition, the gauge transformation becomes a necessary member of the class of fundamental transformations. Now, (36) is still valid and (38) gives

$$g(q) = -\frac{i\hbar}{2}\int \frac{\frac{\partial \tilde{Q}}{\partial q}(\upsilon)}{1+\tilde{Q}(\upsilon)}\,\frac{\partial f}{\partial q}\,dq\,. \qquad (41)$$

If $f(q)$ and also $g(q)$ are chosen arbitrarily as nonconstant functions, according to (38) we see that the existence of $\chi(q)$ in $P(q,p)$ becomes generally inevitable. The following example may make these points more clear.

Consider the CT $Q = 1/q$, $P = -q^2 p$. Equations (36) and (38) (or (41)) give rise to

$$f(q) = \pm\frac{2}{\hbar\lambda}(1-q^2)^{1/2} \qquad (42)$$

and

$$g(q) = -\frac{1}{2\lambda}\ln(q^2-1) \qquad (43)$$

respectively, where $g(q)$ is evaluated for the $f(q)$ with positive sign. Conversely, given GF containing the same $f(q)$ with the positive sign in (42) and $g = 0$, we get the CT as

$$Q(q) = \frac{1}{q}\,, \qquad P(q,p) = -q^2 p - \frac{i\hbar}{q}\frac{1+q^2}{1-q^2}\,. \qquad (44)$$

(ii) $f = f(q)$ and $g = c_2$, where c_2 is any constant.

(a) $f = q$. (36) and (38) amount to the scaling transformation

$$Q(q) = kq\,, \quad P(q,p) = \frac{1}{k}\,p \qquad (45)$$

with $\chi = 0$, where $k = (2 - i\hbar\lambda)/(2 + i\hbar\lambda)$, $(\lambda \neq 2i/\hbar)$. Note that the scaling transformation (45) is compatible with (23) for a different $A(q)$. Since

$$e_\star^{\mu q\star p} \star q \star e_\star^{-\mu q\star p} = e^{-i\hbar\mu} q\,, \quad e_\star^{\mu q\star p} \star p \star e_\star^{-\mu q\star p} = e^{i\hbar\mu} \star p = e^{i\hbar\mu} p\,, \quad (46)$$

where μ is a pure imaginary number, (45) can be generated by the $\star$-exponential function

$$e_\star^{\frac{i}{\hbar}(\ln k) q\star p} \qquad (47)$$

corresponding to the quantized form of $e^{\lambda qp}$. This result, i.e. $e^{i(\ln k)\hat{q}\hat{p}/\hbar}$, is not immediate if one attempts to quantize $e^{\lambda qp}$ using the Weyl correspondence (11).

(b) $f = q^2$ generates the CT

$$\begin{aligned} Q(q) &= -q + \frac{2i}{\hbar\lambda}(1 \pm \eta)\,, \\ P(q,p) &= -\frac{\eta}{(\eta+2)}p - \lambda\hbar^2\frac{\eta+1}{\eta(\eta+2)^2}\,, \end{aligned} \tag{48}$$

where $\eta = (1+2i\hbar\lambda q)^{1/2}$ and $P(q,p)$ is evaluated for the $Q(q)$ with positive sign.

(iii) (a) $Q(q) = \ln q$, $P(q,p) = qp$ which is one of the three successive transformations in transforming the quantum Liouville Hamiltonian to a free particle. For this, one must solve $e^{q-i\hbar\lambda f(q)/2} = q + i\hbar\lambda f(q)/2$ numerically.[21]

(b) The inverse of (a) is $Q(q) = e^q$, $P(q,p) = e^{-q}p$. It is a typical example for a spectrum nonpreserving transformation that one encounters in the phase space representations of the radial dimension.[21,22] For this transformation one obtains $e^{q+i\hbar\lambda f(q)/2} = q - i\hbar\lambda f(q)/2$ for which a numerical solution is necessary.

(iv) Finally, one may choose

$$g(q) = \frac{i\hbar}{2}\,\partial_q f\,, \qquad \chi(q) = \frac{i\hbar}{2}\,\partial_q\tilde{Q} \tag{49}$$

so that the transformation becomes

$$e^{\lambda f\star p} \star q = Q(q) \star e^{\lambda f\star p}\,, \quad e^{\lambda f\star p} \star p = \tilde{Q} \star p \star e^{\lambda f\star p} \tag{50}$$

which is the ordinary exponential analogous of (23). But (38) shows that such a transformation induces the condition

$$\frac{\partial\tilde{Q}}{\partial q}(\upsilon) = \frac{\partial}{\partial q}\tilde{Q}(\upsilon) \tag{51}$$

which is not always possible. One possible case is the scaling transformation given above.

4. Transform of Functions

In this section we consider the behavior of phase space functions under the linear and nonlinear CT's in turn within the ordinary exponential form. While the linear case can be investigated in a general framework, the nonlinear case is given by a particular example. We show that both results are compatible with the ones in the literature.[23,24]

The linear CT's satisfy the equation

$$F(q,p) \star u(q,p) \star F^{-1}(q,p) = u(FqF^{-1}, FpF^{-1}) = u(Q,P)\,, \tag{52}$$

for any arbitrary phase space function $u(q,p)$, which is especially shown in the literature for the Wigner functions.[23,25] Now, we would like to show this covariance in a general compact way keeping ourselves in the ordinary exponential form. In doing so, we will use the Lie operator method which is very suitable for the treatment.

The Lie operator associated with the transformation (22) is defined by[26]

$$\hat{L}_M = f\star - \star f = f\left(q + \frac{i\hbar\partial_p}{2}, p - \frac{i\hbar\partial_q}{2}\right) - f\left(q - \frac{i\hbar\partial_p}{2}, p + \frac{i\hbar\partial_q}{2}\right). \quad (53)$$

$\hat{L}_M$ acts on $u(q,p)$ such as

$$\hat{L}_M u = \{f, u\}^M = f \star u - u \star f\,, \quad (54)$$

that the result is obviously a Moyal bracket. The powers of $\hat{L}$ are given by

$$\begin{gathered} \hat{L}_M^0 u = u\,, \qquad \hat{L}_M u = \{f, u\}^M\,, \\ \hat{L}_M^2 u = \hat{L}_M\{f, u\}^M = \{f, \{f, u\}^M\}^M\,, \end{gathered} \quad (55)$$

and so on. Thus the construction of the transformation (22) in terms of the Lie operator is straightforward, so that

$$e_\star^{\lambda f} \star u \star e_\star^{-\lambda f} = e^{\lambda \hat{L}_M} u\,. \quad (56)$$

The linear CT is given by

$$Q(q,p) = aq + bp\,, \qquad P(q,p) = cq + dp\,, \quad (57)$$

where a, b, c, d are real constants satisfying $ad - bc = 1$ and $a + d + 2 \neq 0$. The compact GF as the solution to (26) is given by

$$F(q,p) = e^{2iA[bp^2 - cq^2 + (a-d)qp]/\hbar}\,, \quad (58)$$

where $A = 1/(a+d+2)$. Since the Lie operator method is based on the $\star$-exponential, (58) is not suitable to generate the linear CT within the procedure (56). But first, let us consider the decomposed form of the linear CT

$$F_L'(q,p) = e_\star^{\gamma q \star p} \star e_\star^{\beta q^2} \star e_\star^{\alpha p^2} \quad (59)$$

which corresponds to (10). Second, if we go ahead one step more we reach, by (28) and (47), that

$$F_L(q,p) = e^{\lambda qp} \star e^{\beta q^2} \star e^{\alpha p^2} \quad (60)$$

which is a decomposition of (58) in terms of ordinary exponentials. The uniqueness principle of the GF's allows us to use (60) instead of (58) and therefore we reach the result

$$F_L \star u(q,p) \star F_L^{-1} = u(F_L \star q \star F_L^{-1}, F_L \star p \star F_L^{-1})\,. \quad (61)$$

Indeed, the first movement gives

$$e^{\alpha p^2} \star u(q,p) \star e^{-\alpha p^2} = e^{\alpha \hat{L}_{M_1}} u(q,p) = u(q - 2i\hbar\alpha p, p)\,, \quad (62)$$

where $\hat{L}_{M_1} = p^2 \star - \star\, p^2 = -2i\hbar p\partial_q$. The second one amounts to

$$\begin{aligned} e^{\beta q^2} \star u(q - 2i\hbar\alpha p, p) \star e^{-\beta q^2} &= e^{\beta \hat{L}_{M_2}} u(q - 2i\hbar\alpha p, p) \\ &= u\big((1 + 4\hbar^2\alpha\beta)q - 2i\hbar\alpha p, p + 2i\hbar\beta q\big)\,, \end{aligned} \tag{63}$$

where $\hat{L}_{M_2} = q^2 \star - \star\, q^2 = 2i\hbar q\partial_p$. And finally the third one can be achieved by the Lie operator

$$\hat{L}_{M_3} = -i\hbar q\partial_q + i\hbar p\partial_p \tag{64}$$

and it can generate the scaling transformation (45) with the choice $\gamma = i(\ln k)/\hbar$. Therefore if $u(q,p)$ is expanded in power series it is easy to see that

$$e^{\gamma \hat{L}_{M_3}} u(q,p) = u\left(kq, \frac{p}{k}\right), \tag{65}$$

where we used the facts that

$$e^{-i\hbar\gamma q\partial_q} q^n = (kq)^n\,, \quad e^{i\hbar\gamma p\partial_p} p^n = \left(\frac{p}{k}\right)^n. \tag{66}$$

Consequently,

$$F_L \star u(q,p) \star F_L^{-1} = u(aq + bp, cq + dp) \tag{67}$$

with

$$a = (1 + 4\hbar^2\alpha\beta)k\,, \quad b = -\frac{2i\hbar\alpha}{k}\,, \quad c = 2i\hbar\beta k\,, \quad d = \frac{1}{k}\,. \tag{68}$$

If the same procedure is run for the decomposition

$$F_I'(q,p) = e_\star^{iq^2/2\hbar} \star e_\star^{ip^2/2\hbar} \star e_\star^{iq^2/2\hbar} \tag{69}$$

corresponding to the interchanging (8), it can be concluded that

$$e^{i(q^2+p^2)/\hbar} \star u(q,p) \star \left[e^{i(q^2+p^2)/\hbar}\right]^{-1} = u(p, -q)\,, \tag{70}$$

which is just as expected from the fact that the interchanging (25) is a special case of the linear CT (58). This result relies on the uniqueness of the solution to (26), i.e. the uniqueness of GF. But one should be aware that the composition (69) is not a special case of (59). This situation is originated from the fact that because any finite CT can be achieved by many different basic transformation steps, the decomposition of the transformation is not unique.

Now this time consider the nonlinear CT

$$Q(q,p) = q\,, \quad P(q,p) = p + \nu q^2\,, \tag{71}$$

generated by

$$F(q,p) = e^{-\frac{i\nu}{3\hbar}q^3}\,, \tag{72}$$

where ν is a parameter. Thus the Lie operator is

$$\hat{L}_M = q^3 \star - \star\, q^3 = -\frac{1}{4}i\hbar^3\partial_p^3 + 3i\hbar q^2\partial_p\,. \tag{73}$$

Therefore $f(q,p)$ transforms as follows:

$$\begin{aligned} e^{-\frac{i\nu}{3\hbar}\hat{L}_M} f(q,p) &= e^{-\frac{\nu\hbar^2}{12}\partial_p^3+\nu q^2\partial_p} f(q,p) \\ &= e^{-\frac{\nu\hbar^2}{12}\partial_p^3} f(q,p+\nu q^2) \\ &= \left(1-\frac{\nu\hbar^2}{12}\partial_p^3+\cdots\right) f(q,p+\nu q^2) \\ &= f(q,p+\nu q^2)-\frac{\nu\hbar^2}{12}\partial_p^3 f(q,p+\nu q^2)+\cdots . \end{aligned} \tag{74}$$

As a special case it may be remarkable to point out that one can show easily that any gauge transformation $e^{\lambda f(q)}$ (linear or nonlinear) transforms p^2 and p^{-1} as $(p+i\hbar\lambda\partial_q f)^2$ and $(p+i\hbar\lambda\partial_q f)^{-1}$ respectively. Note that the inverses mean the algebraic inverses. If the interchange transformation is employed, the same situation appears for q^2 and q^{-1}. Therefore this fact and (67) say that any gauge transformation can be applied directly to the harmonic oscillator or Coulomb-type problems.

As a physical realization of the facts that have been stated so far on the transform of functions, we will consider the transformation of the $\star$-genvalue equation $H(q,p)\star W(q,p) = EW(q,p)$ by a suitable example, where $W(q,p)$ is the Wigner function of the system $H(q,p)$ and E is the correspondent eigenvalue.[27] A CT converts the $\star$-genvalue equation into another $\star$-genvalue equation $H'(q,p)\star W'(q,p) = EW'(q,p)$ where

$$\begin{aligned} H'(q,p) &= F(q,p)\star H(q,p)\star F^{-1}(q,p)\,, \\ W'(q,p) &= F(q,p)\star W(q,p)\star F^{-1}(q,p)\,. \end{aligned} \tag{75}$$

Now consider the system with linear potential $H(q,p) = p^2+q$. The Wigner function of the system satisfying the $\star$-genvalue differential equation is the Airy function

$$W(q,p) = \mathrm{Ai}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(t^3/3+\xi t)}dt\,, \tag{76}$$

where $\xi(q,p) = (2/\hbar)^{2/3}(q+p^2-E)$.[24] The CT

$$Q(q,p) = p-q^2\,, \qquad P(q,p) = -q \tag{77}$$

that can be constructed by means of a two-step GF

$$F(q,p) = e^{\frac{iq^3}{3\hbar}} \star e^{\frac{i}{\hbar}(q^2+p^2)} \tag{78}$$

should convert the Hamilton and Wigner functions of the linear potential system into that of the free particle system $H(q,p) = p$. Indeed, the transformation of the Hamilton function is immediate by (70) and (74). On the other hand, the first step of the GF transforms the Wigner function as $\mathrm{Ai}[\xi(p,-q)]$. The second step acts as

$$\begin{aligned} e^{\frac{i}{3\hbar}\hat{L}_M}\,\mathrm{Ai}[\xi(p,-q)] &= e^{\frac{\hbar^2}{12}\partial_p^3 - q^2\partial_p}\,\mathrm{Ai}[\xi(p,-q)] \\ &= e^{\frac{\hbar^2}{12}\partial_p^3}\,\mathrm{Ai}[\xi(p-q^2,-q)] \\ &= \left(\frac{\hbar}{2}\right)^{2/3}\delta(p-E)\,. \end{aligned} \tag{79}$$

This is the free particle Wigner function.

5. Intertwining

Suppose that there exists a c-number phase-space function $L(q,p)$ making a link between two Hamilton functions in potential form by means of the transformation

$$L(q,p)\star H_0(q,p)\star L^{-1}(q,p) = H_1(q,p)\,, \tag{80}$$

where $H_0 = p^2 + V_0(q)$ and $H_1 = p^2 + V_1(q)$. It is apparent that (80) is equivalent to

$$L(q,p)\star H_0(q,p) = H_1(q,p)\star L(q,p)\,, \tag{81}$$

and therefore $L(q,p)$ is an intertwining GF. Expansion of the $\star$-product reduces (81) to the differential equation relating the two potentials;

$$V_1\left(q+\frac{i\hbar\partial_p}{2}\right)L(q,p) = V_0\left(q-\frac{i\hbar\partial_p}{2}\right)L(q,p) + 2i\hbar p\partial_q L(q,p)\,. \tag{82}$$

The choice

$$L(q,p) = p - i\varphi(q) \tag{83}$$

leads to the well-known consistency conditions

$$V_1(q) + V_0(q) = 2\varphi^2(q)\,, \tag{84}$$

$$V_1(q) = V_0(q) + 2\hbar\partial_q\varphi(q)\,, \tag{85}$$

where $\varphi(q)$ is the solution to the Riccati equation (85) which can be linearized by the Darboux transformation $\varphi(q) = -\hbar\partial_q\phi(q)/\phi(q)$ to give the Schrödinger equation with zero eigenvalue

$$-\hbar^2\partial_q^2\phi(q) + V_0(q)\phi(q) = 0\,. \tag{86}$$

Now let us return to the ansatz (83). By (29), this is the gauge transformation

$$L(q,p) = e^{-\int\varphi(q)dq/\hbar}\star p\star e^{\int\varphi(q)dq/\hbar}\,. \tag{87}$$

If we replace p in (87) by the decomposition

$$p = e^{i(q^2+p^2)/\hbar}\star e^{\ln q}\star[e^{i(q^2+p^2)/\hbar}]^{-1}\,, \tag{88}$$

we conclude that $L(q,p)$ is a sequence of the fundamental transformations and that intertwining is a CT. With the help of (18), the explicit definition of the transformation $L(q,p)$ is then given by the equations

$$L(q,p) \star q \star L^{-1}(q,p) = q - i\hbar(p - i\varphi)^{-1}\,, \tag{89}$$

$$L(q,p) \star p \star L^{-1}(q,p) = p + \hbar\partial_q\varphi \star (p - i\varphi)^{-1}\,. \tag{90}$$

As a final remark note that (82) can also be used, besides the intertwining, to determine the GF for the given any potential pair V_0 and V_1. For example for the pair $V_0 = q$ and $V_1 = 0$ that is a nonintertwining transformation from the linear potential to the free particle, the solution is

$$L(q,p) = e^{-2i(q\,p+4p^3/3)/\hbar}\,. \tag{91}$$

This fact is the most remarkable property of (82), that is it may relate any two systems without considering their potentials are intertwined or not.

On the other hand, without regarding (82), the same transformation; i.e. $H_0 = p^2 + q \to H_1 = p^2$ can be obtained by the five-step sequence

$$L(q,p) = e^{i(q^2+p^2)/\hbar} \star e^{\lambda[f(q)p+g(q)]} \star \left[e^{i(q^2+p^2)/\hbar}\right]^{-1} \star e^{iq^3/3\hbar} \star e^{i(q^2+p^2)/\hbar} \tag{92}$$

giving

$$L(q,p) \star (p^2 + q) \star L^{-1}(q,p) = p^2\,, \tag{93}$$

where

$$f(q) = \frac{i}{\hbar\lambda}\left[2q + 1 + (1+8q)^{1/2}\right] \tag{94}$$

and

$$g(q) = \frac{1}{2\lambda}\ln\left[\frac{1 + (1+8q)^{1/2}}{1+8q}\right] \tag{95}$$

correspond to the transformation $Q = q^2$, $P = p/2q$. It seems at first sight that the transformation (92) can be converted easily into the $\star$-exponential form with the help of (23) and (69), but note that the fourth step remains unclear since the determination of $f(q)$ for $A(q) = q^2$ in (5) is not so easy.

6. Summary and Conclusions

In quantum mechanics a conjecture states that any quantum CT can be generated as a sequence of three fundamental CT's. It is seen that when the isomorphism (15) is used to write the fundamental quantum CT's in $\star$-product formalism, gauge and point transformations are immediate but the interchange is not. But the system of differential equations (24) allows us to get the generator of the interchange transformation surprisingly in an ordinary exponential form. Parallel to this result, it is shown that the others can also be obtained in the ordinary form. The convertibility of (16) into a system of differential equations allows us a powerful tool in determining both the GF and the CT. Moreover, if point transformation is defined by (32), the gauge transformation is unnecessary and this reduces the number of independent transformations to two.

It is also shown that the approach developed above offers results compatible with the well-known behaviors of functions under linear and nonlinear CT's. On the other hand, the intertwining method can also be investigated within this framework. As an extra advantage, (82) offers a relation between any two systems even though their potentials are not intertwined.

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