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Canonical transformations in three-dimensional phase space

T. Dereli

Physics Department, Koç University, 80910 Sarıyer-Istanbul, TURKEY
tdereli@ku.edu.tr

A. Teğmen¹

Feza Gürsey Institute, 34684 Çengelköy-Istanbul, TURKEY
tegmen@science.ankara.edu.tr

T. Hakioglu

Physics Department, Bilkent University, 06533 Ankara, TURKEY
hakioglu@fen.bilkent.edu.tr

Abstract

Canonical transformation in a three-dimensional phase space endowed with Nambu bracket is discussed in a general framework. Definition of the canonical transformations is constructed as based on canonoid transformations. It is shown that generating functions, transformed Hamilton functions and the transformation itself for given generating functions can be determined by solving Pfaffian differential equations corresponding to that quantities. Types of the generating functions are introduced and all of them is listed. Infinitesimal canonical transformations are also discussed. Finally, we show that decomposition of canonical transformations is also possible in three-dimensional phase space as in the usual two-dimensional one.

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1 Introduction

In 1973 Y. Nambu proposed a generalization of the usual Hamiltonian dynamics, in which odd-dimensional phase spaces are also possible [1]. To his proposal, time evolution of a dynamical variable $f(x_1, \dots, x_n) = f(x)$ over an n -dimensional phase space is given by the so-called Nambu bracket

$$\dot{f} = \{f, H_1, \dots, H_{n-1}\} = \frac{\partial(f, H_1, \dots, H_{n-1})}{\partial(x_1, \dots, x_n)}, \quad (1)$$

where H_1, \dots, H_{n-1} are the functionally independent Hamilton functions and the variables x_1, \dots, x_n stand for the local coordinates of \mathbb{R}^n . The explicit form of the Nambu bracket (1) is given by the

¹On sabbatical from Physics Department, Ankara University 06100 Ankara TURKEY

expression

$$\{f_1, \dots, f_n\} = \frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} = \epsilon_{i_1 \dots i_n} \frac{\partial f_1}{\partial x_{i_1}} \dots \frac{\partial f_n}{\partial x_{i_n}}. \quad (2)$$

(Throughout the text, sum is taken over all repeated indices). The coordinate-free expression of the Nambu bracket is defined by means of the $(n - 1)$ -form $\Gamma = dH_1 \wedge \dots \wedge dH_{n-1}$, namely

$$*(df \wedge \Gamma) = \{f, H_1, \dots, H_{n-1}\}, \quad (3)$$

where d and \wedge denote the usual exterior derivative and exterior product respectively, and $*$ is the Hodge map.

It is well known that canonical transformations (CTs) are a powerful tool in the usual Hamilton mechanics. They serve three main purposes: to describe the evolution of a dynamical system, to show the equivalence of two systems, and mostly to transform a system of interest into a simpler or known one in different variables. In this paper we study CTs in the phase space endowed with canonical Nambu bracket and we will try to gain a deeper insight to the subject in a general framework.

The paper is organized as follows: In Sec.2, a precise definition of CT in three-space is given. Since every CT is a canonoid transformation it is felt that an explicit definition of the canonoid transformations should be given. In doing so, the discussion is kept in its general pattern, i.e., in the time dependent form. Additionally, direct conditions on a CT corresponding to the ones in the usual even-dimensional Hamilton formalism are constructed. Sec.3 is devoted to show how to find the generating functions (GFs) and the new Hamilton functions. This section also contains the way to find the CT for given GFs. It is seen that if one wants to know the GFs, the CT and the new Hamilton functions, one must solve a Pfaffian differential equation related with that quantity. Sec.4 stands for the exemplification of CTs, including the definitions of gauge and point CTs in three-space. Sec.5 deals with the classification of CTs. It gives an extensive number of types. All of the possible eighteen types is listed in six main kinds in Table 1. As an inevitable part of the presentation, we construct the infinitesimal transformations (ICTs) in Sec.6. It is shown that the construction parallels the usual Hamilton formalism such that ICTs can generate finite CTs. In order to complete the discussion, in Sec.7 it is shown that a CT in three-space can be decomposed into a sequence of three minor CTs. This result, in fact, confirms a well known conjecture saying the same thing in the usual classical and quantum mechanics.

2 Definition of Canonical Transformations in Three-Space

In the definition (1), f and Hamilton functions H_1, \dots, H_{n-1} do not contain t explicitly. For the sake of generality we will allow the explicit t dependence. Since, for the local coordinates x_1, x_2, x_3 , the Nambu-Hamilton equations of motion give

$$\dot{x}_i = \epsilon_{ijk} \frac{\partial H_1}{\partial x_j} \frac{\partial H_2}{\partial x_k}, \quad i, j, k = 1, 2, 3, \quad (4)$$

(from now on, all Latin indices will take values 1, 2, 3), total time evolution of a dynamical variable $f(x, t)$ becomes

$$\dot{f} = \{f, H_1, H_2\} + \frac{\partial f}{\partial t}. \quad (5)$$

Hence time evolution of the Hamilton functions amounts to the well known form

$$\dot{H}_\alpha = \frac{dH_\alpha}{dt} = \frac{\partial H_\alpha}{\partial t}, \quad \alpha = 1, 2. \quad (6)$$

Instead of giving directly the definition of a CT in three-space, it may be remarkable to give some interesting situations as a pre-knowledge. First, by using the same terminology developed for the usual Hamilton formalism in the literature [2, 3], we give the definition of a canonoid transformation. The main definition of a CT will be based on this definition.

Definition 2.1. For a dynamical system whose equations of motion are governed by the pair $(H_1(x, t), H_2(x, t))$, the time preserving diffeomorphism $\mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3 \times \mathbb{R}$ such that

$$(x_i, t) \mapsto (X_i(x, t), t) \quad (7)$$

is called a *canonoid* transformation with respect to the pair (H_1, H_2) if there exist a pair $(K_1(X, t), K_2(X, t))$ satisfying

$$\dot{X}_i = \epsilon_{ijk} \frac{\partial K_1}{\partial X_j} \frac{\partial K_2}{\partial X_k}, \quad (8)$$

where $\mathbb{R}^3 \times \mathbb{R}$ is the extended phase space in which t is considered as an additional independent variable.

The invertible transformation (7) (canonoid or not) also changes the basis of vector fields and differential forms:

$$\frac{\partial}{\partial x_i} = \frac{\partial X_j}{\partial x_i} \frac{\partial}{\partial X_j} + \frac{\partial t}{\partial x_i} \frac{\partial}{\partial t} (= 0), \quad \frac{\partial}{\partial X_i} = \frac{\partial x_j}{\partial X_i} \frac{\partial}{\partial x_j} + \frac{\partial t}{\partial X_i} \frac{\partial}{\partial t} (= 0), \quad (9)$$

$$dx_i = \frac{\partial x_i}{\partial X_j} dX_j + \frac{\partial x_i}{\partial t} dt (= 0), \quad dX_i = \frac{\partial X_i}{\partial x_j} dx_j + \frac{\partial X_i}{\partial t} dt. \quad (10)$$

In the time independent case, the extended part drops and the map becomes on \mathbb{R}^3 as expected, i.e.,

$$x_i \mapsto X_i(x). \quad (11)$$

Note that, such a map considers t in any time dependent function $f(x, t)$ as a parameter only.

According to Definition 2.1 it is obvious that K_1 and K_2 serve as Hamilton functions for the new variables and the transformation (7) preserves the Nambu-Hamilton equations.

As an example consider Nambu system

$$\dot{x}_1 = x_2 x_3, \quad \dot{x}_2 = -x_1 x_3, \quad \dot{x}_3 = 0 \quad (12)$$

governed by the Hamilton functions

$$H_1(x) = \frac{1}{2}(x_1^2 + x_2^2), \quad H_2(x) = \frac{1}{2}x_3^2. \quad (13)$$

Let the transformation be

$$X_1 = x_1, X_2 = x_2, X_3 = x_3^2. \quad (14)$$

Now if we choice the new Hamilton functions as

$$K_1(X) = \frac{1}{2}(X_1^2 + X_2^2), K_2(X) = \frac{2}{3}X_3^{3/2}, \quad (15)$$

we see that Nambu-Hamilton equations of motion remain covariant. For a different pair (H_1, H_2) , there may not exist a new pair (K_1, K_2) for the same transformation.

It is well known that the canonicity condition of a transformation must be independent from the forms of the Hamilton functions. We now give a theorem related with this condition. Our theorem is three-dimensional time dependent generalization of the two-dimensional time independent version [4].

Theorem 2.1. The transformation (7) is canonoid with respect to all Hamiltonian pairs iff

$$\{X_1, X_2, X_3\} = \text{constant}. \quad (16)$$

Proof: If we consider the fact that

$$\epsilon_{ijk} \frac{\partial}{\partial X_i} \frac{\partial(K_1, K_2)}{\partial(X_j, X_k)} = 0, \quad (17)$$

it is apparent from (8) that the existence of K_1 and K_2 is equivalent to

$$\frac{\partial \dot{X}_i}{\partial X_i} = 0. \quad (18)$$

Since

$$\dot{X}_i(x, t) = \frac{\partial X_i}{\partial x_j} \dot{x}_j + \frac{\partial X_i}{\partial t}, \quad (19)$$

with the help of (4), (18) reduces to

$$\epsilon_{jkl} \frac{\partial}{\partial X_i} \left(\frac{\partial X_i}{\partial x_j} \frac{\partial H_1}{\partial x_k} \frac{\partial H_2}{\partial x_l} \right) + \frac{\partial}{\partial X_i} \frac{\partial X_i}{\partial t} = 0. \quad (20)$$

Equivalently,

$$\epsilon_{jkl} \left(\frac{\partial}{\partial X_i} \frac{\partial X_i}{\partial x_j} \right) \frac{\partial H_1}{\partial x_k} \frac{\partial H_2}{\partial x_l} + \epsilon_{jkl} \frac{\partial X_i}{\partial x_j} \frac{\partial}{\partial X_i} \left(\frac{\partial H_1}{\partial x_k} \frac{\partial H_2}{\partial x_l} \right) + \frac{\partial}{\partial X_i} \frac{\partial X_i}{\partial t} = 0. \quad (21)$$

If the first transformation rule in (9) is used, the second term of (21) vanishes as

$$\epsilon_{jkl} \frac{\partial}{\partial x_j} \frac{\partial(H_1, H_2)}{\partial(x_k, x_l)} = 0. \quad (22)$$

If we impose the requirement that the transformation is a canonoid transformation independent from the Hamilton functions H_1 and H_2 , the coefficients in the first term of (21) must vanish, namely

$$\frac{\partial}{\partial X_i} \frac{\partial X_i}{\partial x_j} = 0. \quad (23)$$

The last term in (21) is already Hamiltonian independent and it gets directly zero with the condition (23). Therefore the theorem becomes equal to the following statement

$$\frac{\partial}{\partial X_i} \frac{\partial X_i}{\partial x_j} = 0 \quad \Leftrightarrow \quad \{X_1, X_2, X_3\} = \text{constant}. \quad (24)$$

It is straightforward to see, after a bit long but simple calculation, that

$$\partial_{X_m} \{X_1, X_2, X_3\} = 0, \quad (25)$$

if (23) is satisfied. Conversely, the explicit form of (25), for $m = 1$ for instance, is

$$\partial_{X_1} \{X_1, X_2, X_3\} = \epsilon_{jkl} \frac{\partial X_2}{\partial x_k} \frac{\partial X_3}{\partial x_l} \frac{\partial}{\partial X_i} \frac{\partial X_i}{\partial x_j} = 0. \quad (26)$$

Together with the other two values of m , (26) defines a homogeneous system of linear equations for the unknowns

$$\frac{\partial}{\partial X_i} \frac{\partial X_i}{\partial x_j}. \quad (27)$$

The determinant of the matrix of coefficients gives $\{X_1, X_2, X_3\}^2$ and with the condition (16), the unique solution is then the trivial one, i.e., (23). □

Definition 2.2. A *canonical* transformation is a canonoid transformation with

$$\{X_1, X_2, X_3\} = 1. \quad (28)$$

Therefore a CT is a transformation preserving the fundamental Nambu bracket

$$\{x_1, x_2, x_3\} = 1 \quad (29)$$

independently from the forms of the pair (H_1, H_2) . Additionally, if one employs the transformation rule (9) for (29), the canonicity condition gives

$$\{x_1, x_2, x_3\}_X = 1, \quad (30)$$

where the subscript X means that the derivatives in the expansion of the bracket are taken with respect to the new coordinates X_1, X_2, X_3 .

In fact, a brief definition of the CTs in the three-space is given in Ref. [5] as a diffeomorphism of the phase space which preserve Nambu bracket structure. But such a definition bypasses the probability that the transformation is a canonoid transformation.

Remark 2.1. A CT preserves the Nambu bracket of arbitrary functions, i.e.,

$$\{f(x, t), g(x, t), h(x, t)\}_x = \{f(x, t), g(x, t), h(x, t)\}_X. \quad (31)$$

According to the Remark 2.1., one gets

$$\{X_i, H_1, H_2\}_x = \{X_i, H_1, H_2\}_X, \quad (32a)$$

$$\{x_i, H_1, H_2\}_x = \{x_i, H_1, H_2\}_X. \quad (32b)$$

With the help of (9), the first covariance (32a) implies the first group of conditions on a CT

$$\frac{\partial X_i}{\partial x_l} = \frac{\partial(x_m, x_n)}{\partial(X_j, X_k)}, \quad (33)$$

and (32b) implies the second group

$$\frac{\partial x_i}{\partial X_l} = \frac{\partial(X_m, X_n)}{\partial(x_j, x_k)}, \quad (34)$$

where (i, j, k) and (l, m, n) are cycling indices. (33) and (34) are the equations corresponding to the so-called *direct conditions* in Hamilton formalism.

3 Generating Functions

We now discuss how CTs can be generated in the three-space. We will show that to each CT corresponds a particular pair (F_1, F_2) . F_1 and F_2 are the GFs of the transformation defined on $\mathbb{R}^3 \times \mathbb{R}$, and as shown in Sec.5, they can give a complete classification of the CTs.

We start with the three-form

$$\chi = dX_1 \wedge dX_2 \wedge dX_3. \quad (35)$$

When (10) is used for every one-form in (35), we get by (28) that

$$dX_1 \wedge dX_2 \wedge dX_3 = dx_1 \wedge dx_2 \wedge dx_3 + \frac{\partial(X_{[i}, X_j)}{\partial(x_l, x_m)} \frac{\partial X_k]}{\partial t} dx_l \wedge dx_m \wedge dt, \quad (36)$$

where the bracket $[]$ stands for the cyclic sum. The substitution of the term

$$\frac{\partial X_i}{\partial t} = \frac{\partial(K_1, K_2)}{\partial(X_j, X_k)} - \{X_i, H_1, H_2\} \quad (37)$$

obtained by (4), (8) and (19), into (36) gives ultimately that

$$dX_1 \wedge dX_2 \wedge dX_3 = dx_1 \wedge dx_2 \wedge dx_3 - dH_1 \wedge dH_2 \wedge dt + dK_1 \wedge dK_2 \wedge dt. \quad (38)$$

The first property that should be pointed out for (38) is that, for the time independent transformations it reduces simply to

$$dX_1 \wedge dX_2 \wedge dX_3 = dx_1 \wedge dx_2 \wedge dx_3 \quad (39)$$

which is an alternative test for the canonicity. Now let us rewrite (38) as

$$d\Omega = d(x_1 dx_2 \wedge dx_3 - X_1 dX_2 \wedge dX_3 - H_1 dH_2 \wedge dt + K_1 dK_2 \wedge dt) = 0. \quad (40)$$

We assume that the closed two-form Ω can be decomposed as the product of two one-forms dF_1 and dF_2 , then

$$dF_1 \wedge dF_2 = x_1 dx_2 \wedge dx_3 - X_1 dX_2 \wedge dX_3 - H_1 dH_2 \wedge dt + K_1 dK_2 \wedge dt. \quad (41)$$

Equating the coefficients of similar basic two-forms not including dt on both sides of (41) gives

$$\begin{aligned}\frac{\partial(F_1, F_2)}{\partial(x_2, x_3)} &= x_1 - X_1 \frac{\partial(X_2, X_3)}{\partial(x_2, x_3)} := A(x, t), \\ \frac{\partial(F_1, F_2)}{\partial(x_3, x_1)} &= -X_1 \frac{\partial(X_2, X_3)}{\partial(x_3, x_1)} := B(x, t), \\ \frac{\partial(F_1, F_2)}{\partial(x_1, x_2)} &= -X_1 \frac{\partial(X_2, X_3)}{\partial(x_1, x_2)} := C(x, t),\end{aligned}\tag{42}$$

where the relation

$$\frac{\partial A}{\partial x_1} + \frac{\partial B}{\partial x_2} + \frac{\partial C}{\partial x_3} = 0\tag{43}$$

is satisfied independently from the transformation due to the general rule (22) written for the GFs F_1 and F_2 . (42) is a useful set of equations in finding both GFs and CTs: Since we have also

$$\frac{\partial F_\alpha}{\partial x_{[i}} \frac{\partial(F_1, F_2)}{\partial(x_j, x_{k])}} = \epsilon_{ijk} \frac{\partial F_\alpha}{\partial x_i} \frac{\partial F_1}{\partial x_j} \frac{\partial F_2}{\partial x_k} = 0,\tag{44}$$

given CT $X_i(x)$, the GFs appear as the solution to the Pfaffian partial differential equation

$$A(x, t) \frac{\partial F_\alpha}{\partial x_1} + B(x, t) \frac{\partial F_\alpha}{\partial x_2} + C(x, t) \frac{\partial F_\alpha}{\partial x_3} = 0,\tag{45}$$

up to an additive function of t . Conversely, given GFs, (42) provides the differential equation for X_2 and X_3

$$[A(x, t) - x_1] \frac{\partial X_\beta}{\partial x_1} + B(x, t) \frac{\partial X_\beta}{\partial x_2} + C(x, t) \frac{\partial X_\beta}{\partial x_3} = 0 \quad , \quad \beta = 2, 3.\tag{46}$$

Once $X_\beta(x, t)$ has been determined, the complementary part $X_1(x, t)$ of the transformation is immediate by returning to (42).

The general solutions to (45) and (46) are arbitrary functions of some unique arguments. Hence, F_α or X_β do not specify the transformation uniquely. However, by obeying the conventional procedure in the textbooks, through the text we will accept these unique arguments as the solutions so long as they are suitable for our aim.

On the other hand, in (41), the coefficients of the forms including dt gives another useful relation between the GFs, the CT and the new Hamilton functions;

$$\frac{\partial(F_1, F_2)}{\partial(x_i, t)} = -H_1 \frac{\partial H_2}{\partial x_i} + K_1 \frac{\partial K_2}{\partial x_i} - X_1 \frac{\partial(X_2, X_3)}{\partial(x_i, t)}.\tag{47}$$

Given a dynamical system with (H_1, H_2) and a CT, finding the pair (K_1, K_2) is another matter. In order to find the new Hamilton functions, we consider the interior product of ∂_t and the three-form (38) resulting

$$\frac{\partial(K_1, K_2)}{\partial(x_i, x_j)} = \frac{\partial(H_1, H_2)}{\partial(x_i, x_j)} + \frac{\partial(X_{[k}, X_l)}{\partial(x_i, x_j)} \frac{\partial X_m]}{\partial t} =: f_{ij}(x, t).\tag{48}$$

Given f_{ij} , by means of (44) which is also valid for the pair (K_1, K_2) , we obtain the differential equation

$$f_{[ij} \frac{\partial K_\alpha}{\partial x_k]} = 0 \quad (49)$$

whose solutions are the new Hamilton functions.

Alternatively, the Pfaffian partial differential equation

$$\dot{X}_i \frac{\partial K_\alpha}{\partial X_i} = 0, \quad (50)$$

originated from (8) and from the fact

$$\frac{\partial K_\alpha}{\partial X_{[i}} \frac{\partial (K_1, K_2)}{\partial (X_j, X_k)}} = \epsilon_{ijk} \frac{\partial K_\alpha}{\partial X_i} \frac{\partial K_1}{\partial X_j} \frac{\partial K_2}{\partial X_k} = 0, \quad (51)$$

gives the same solution pair but in terms of X . It is apparent that the pairs (F_1, F_2) and (K_1, K_2) must also satisfy (47).

For the time independent CTs, finding the new Hamilton functions is much easier without considering the differential equations given above:

Theorem 3.1. If the CT is time independent, then the new Hamiltonian pair can be found simply as

$$(K_1(X, t), K_2(X, t)) = (H_1(x(X), t), H_2(x(X), t)). \quad (52)$$

Proof:

$$\begin{aligned} \dot{X}_i &= \frac{\partial X_i}{\partial x_j} \dot{x}_j = \epsilon_{jmn} \frac{\partial X_i}{\partial x_j} \frac{\partial H_1}{\partial x_m} \frac{\partial H_2}{\partial x_n} \\ &= \epsilon_{jmn} \frac{\partial X_i}{\partial x_j} \frac{\partial X_k}{\partial x_m} \frac{\partial X_l}{\partial x_n} \frac{\partial H_1}{\partial X_k} \frac{\partial H_2}{\partial X_l} \\ &= \{X_i, X_k, X_l\} \frac{\partial H_1}{\partial X_k} \frac{\partial H_2}{\partial X_l} \\ &= \epsilon_{ikl} \frac{\partial H_1}{\partial X_k} \frac{\partial H_2}{\partial X_l} = \frac{\partial (H_1, H_2)}{\partial (X_k, X_l)} \\ &= \frac{\partial (K_1, K_2)}{\partial (X_k, X_l)}, \end{aligned} \quad (53)$$

where (i, k, l) are cycling indices again and (9) and (2) are used in the first and second lines respectively. □

Note that the new Hamilton functions K_1 and K_2 may contain t explicitly due to $H_1(x, t)$ and $H_2(x, t)$ even if the transformation is time independent.

Before concluding this section, it may be remarkable to point out that in his original paper, as an interesting approach, Nambu considers the CT itself as equations of motion generated by the closed two-form

$$dH(x) \wedge dG(x) = X_1(x) dx_2 \wedge dx_3 + X_2(x) dx_3 \wedge dx_1 + X_3(x) dx_1 \wedge dx_2. \quad (54)$$

Though (54) is a powerful tool to find the CT or the GFs, its closeness property imposes the restriction

$$\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \frac{\partial X_3}{\partial x_3} = 0 \quad (55)$$

on the transformation. Linear CT (64) satisfies the restriction (55) and its analysis via (54) can be found in Ref. [1].

4 Most Known Canonical Transformations and Their Generating Functions

(i) Scaling transformation:

$$X_1 = ax_1, \quad X_2 = bx_2, \quad X_3 = cx_3, \quad abc = 1. \quad (56)$$

Since the transformation is time independent, (41) becomes

$$dF_1 \wedge dF_2 = 0. \quad (57)$$

There exist three possibilities for the GFs: $F_\alpha = \text{constant}$, $F_2 = F_2(F_1)$ and $F_1 = f(x)$, $F_2 = \text{constant}$. We prefer the one compatible with the usual Hamilton formalism, i.e., $F_\alpha = \text{constant}$ which also corresponds to the so-called Methieu transformation [6]. The special case $a = b = c = 1$ is the identity transformation, of course.

As a direct application consider the Euler equations of a rigid body [1]

$$\begin{aligned} \dot{x}_1 &= x_2 \frac{x_3}{I_3} - x_3 \frac{x_2}{I_2}, \\ \dot{x}_2 &= x_3 \frac{x_1}{I_1} - x_1 \frac{x_3}{I_3}, \\ \dot{x}_3 &= x_1 \frac{x_2}{I_2} - x_2 \frac{x_1}{I_1}, \end{aligned} \quad (58)$$

where x_i stands for the components of angular momentum and I_i is the moment of inertia corresponding to the related principal axis. If we take $\gamma_i^2 = -1/I_j + 1/I_k$ with the cycling indices, (58) leads to

$$\dot{x}_1 = \gamma_1^2 x_2 x_3, \quad \dot{x}_2 = \gamma_2^2 x_3 x_1, \quad \dot{x}_3 = \gamma_3^2 x_1 x_2, \quad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 0. \quad (59)$$

If $\gamma_1 \gamma_2 \gamma_3 = 1$ is also satisfied, then the equations of motion are generated by the Hamilton functions

$$H_1 = \frac{1}{2} \left(\frac{x_1^2}{\gamma_1^2} - \frac{x_2^2}{\gamma_2^2} \right), \quad H_2 = \frac{1}{2} \left(\frac{x_1^2}{\gamma_1^2} - \frac{x_3^2}{\gamma_3^2} \right). \quad (60)$$

The scaling transformation

$$X_1 = x_1/\gamma_1, \quad X_2 = x_2/\gamma_2, \quad X_3 = x_3/\gamma_3, \quad (61)$$

converts the Euler system (58) into the Lagrange system [7]

$$\dot{X}_1 = X_2 X_3, \quad \dot{X}_2 = X_3 X_1, \quad \dot{X}_3 = X_1 X_2 \quad (62)$$

which is also called Nahm's system in the theory of static $SU(2)$ -monopoles generated by the transformed Hamilton functions

$$K_1 = \frac{1}{2} (X_1^2 - X_2^2) , K_2 = \frac{1}{2} (X_1^2 - X_3^2) . \quad (63)$$

(ii) Linear transformations:

Three-dimensional version of the linear CT is immediate:

$$\begin{aligned} X_1 &= a_1 x_1 + a_2 x_2 + a_3 x_3, \\ X_2 &= b_1 x_1 + b_2 x_2 + b_3 x_3, \\ X_3 &= c_1 x_1 + c_2 x_2 + c_3 x_3, \end{aligned} \quad (64)$$

satisfying $a_1 \alpha_1 + a_2 \alpha_2 + a_3 \alpha_3 = 1$, where

$$\begin{aligned} \alpha_1 &= b_2 c_3 - b_3 c_2, \\ \alpha_2 &= b_3 c_1 - b_1 c_3, \\ \alpha_3 &= b_1 c_2 - b_2 c_1. \end{aligned} \quad (65)$$

The solutions to (45) appear as the GFs;

$$\begin{aligned} F_1(x) &= \alpha_2 x_3 - \alpha_3 x_2, \\ F_2(x) &= -\frac{1}{2} a_1 x_1^2 + \frac{\alpha_1}{2 \alpha_2} a_2 x_2^2 + \frac{\alpha_1}{2 \alpha_3} a_3 x_3^2 - a_2 x_1 x_2 - a_3 x_1 x_3. \end{aligned} \quad (66)$$

As an application of the linear CTs we consider the Takhtajan's system [5];

$$\dot{x}_1 = x_2 - x_3 , \dot{x}_2 = x_3 - x_1 , \dot{x}_3 = x_1 - x_2. \quad (67)$$

The implicit solution of the system is the trajectory vector $\mathbf{r}(t) = x_1(t) \mathbf{e}_1 + x_2(t) \mathbf{e}_2 + x_3(t) \mathbf{e}_3$ tracing out the curve which is the intersection of the sphere $H_1 = (x_1^2 + x_2^2 + x_3^2)/2$ and the plane $H_2 = x_1 + x_2 + x_3$. $\mathbf{r}(t)$ makes a precession motion with a constant angular velocity around the vector $\mathbf{N} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ normal to the H_2 plane. The linear CT corresponding to the rotation

$$\begin{aligned} X_1 &= \frac{1}{\sqrt{6}} x_1 + \frac{1}{\sqrt{6}} x_2 - \frac{2}{\sqrt{6}} x_3, \\ X_2 &= -\frac{1}{\sqrt{2}} x_1 + \frac{1}{\sqrt{2}} x_2, \\ X_3 &= \frac{1}{\sqrt{3}} x_1 + \frac{1}{\sqrt{3}} x_2 + \frac{1}{\sqrt{3}} x_3 \end{aligned} \quad (68)$$

coincides \mathbf{N} with the \mathbf{e}_3 axis. The new system is then given by the well-known equations of motion of the Harmonic oscillator

$$\dot{X}_1 = \sqrt{3} X_2 , \dot{X}_2 = -\sqrt{3} X_1 , \dot{X}_3 = 0 \quad (69)$$

with the Hamilton functions $K_1 = (X_1^2 + X_2^2 + X_3^2)/2$ and $K_2 = \sqrt{3} X_3$. Therefore inverse of the transformation provides directly an explicit solution to the original system.

(iii) Gauge transformations:

We will define the gauge transformation in our three-dimensional phase space as a model transformation which is similar to the case in the usual Hamilton formalism:

$$X_1 = x_1, X_2 = x_2 + f_1(x_1), X_3 = x_3 + f_2(x_1), \quad (70)$$

where $f_1(x_1)$ and $f_2(x_1)$ are arbitrary functions determined by the GF. Since

$$A(x) = 0, B(x) = x_1 \frac{\partial f_1}{\partial x_1}, C(x) = x_1 \frac{\partial f_2}{\partial x_1}, \quad (71)$$

(45) provides us the GFs as the following form

$$F_1(x) = x_2 \frac{\partial f_2}{\partial x_1} - x_3 \frac{\partial f_1}{\partial x_1}, F_2(x) = -\frac{1}{2} x_1^2. \quad (72)$$

By keeping ourselves in this argument, other possible gauge transformation types can be constructed easily. For instance, a second kind of gauge transformation can be defined by

$$X_1 = x_1 + g_1(x_2), X_2 = x_2, X_3 = x_3 + g_2(x_2) \quad (73)$$

and it is generated by $F_1 = g_1(x_2) x_3$ and $F_2 = x_2$. Another type is

$$X_1 = x_1 + h_1(x_3), X_2 = x_2 + h_2(x_3), X_3 = x_3 \quad (74)$$

and it is generated by $F_1 = h_1(x_3) x_2$ and $F_2 = -x_3$.

(iv) Point transformations:

Our model transformation which is similar to the Hamilton formalism again will be in the form

$$X_1 = f_1(x_1), X_2 = f_2(x_1) x_2, X_3 = f_3(x_1) x_3, \quad (75)$$

where f_1, f_2 and f_3 are arbitrary functions satisfying

$$\frac{\partial f_1}{\partial x_1} f_2 f_3 = 1. \quad (76)$$

(42) says that

$$A(x) = x_1 - f_1 f_2 f_3, B(x) = x_2 f_1 f_3 \frac{\partial f_2}{\partial x_1}, C(x) = x_3 f_1 f_2 \frac{\partial f_3}{\partial x_1}, \quad (77)$$

and to find the GFs we use (45) of course, hence

$$F_1(x) = x_2 \exp\left(-\int \frac{B}{C x_2} dx_1\right), F_2(x) = x_3 \exp\left(-\int \frac{A}{C x_3} dx_1\right), \quad (78)$$

where

$$\exp\left[-\int \frac{1}{C} \left(\frac{A}{x_3} + \frac{B}{x_2}\right) dx_1\right] = C. \quad (79)$$

Other possible types of the point transformation;

$$X_1 = g_1(x_2) x_1, X_2 = g_2(x_2), X_3 = g_3(x_2) x_3, \quad (80)$$

and

$$X_1 = h_1(x_3) x_1, X_2 = h_2(x_3) x_2, X_3 = h_3(x_3) \quad (81)$$

give surprisingly constant GFs.

(v) Rotation in \mathbb{R}^3 :

This last example is chosen as time dependent so that it makes the procedure through a CT more clear. Consider again the system (67) together with the CT

$$X_1 = x_1, X_2 = x_2 \cos t + x_3 \sin t, X_3 = -x_2 \sin t + x_3 \cos t \quad (82)$$

corresponding to the rotation about the x_1 axis. The first attempt to determine the GFs is to consider (45). Since $A(x) = 0$, $B(x) = 0$ and $C(x) = 0$, that equation does not give enough information on the pair (F_1, F_2) . Still things can be put right by considering first (49). For our case it yields

$$(x_2 - x_3) \frac{\partial K_\alpha}{\partial x_1} + (2x_3 - x_1) \frac{\partial K_\alpha}{\partial x_2} + (x_1 - 2x_2) \frac{\partial K_\alpha}{\partial x_3} = 0 \quad (83)$$

with the solution

$$K_1 = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2), K_2 = 2x_1 + x_2 + x_3. \quad (84)$$

Note that one gets, with the aid of the inverse transformation, that

$$K_1 = \frac{1}{2}(X_1^2 + X_2^2 + X_3^2), K_2 = 2X_1 + (\cos t + \sin t)X_2 + (\cos t - \sin t)X_3 \quad (85)$$

and this is also the solution to (50). Now the right hand side of (47) is explicit and the solution

$$F_1 = \frac{x_1}{2} \left(\frac{x_1^2}{3} + x_2^2 + x_3^2 \right), F_2 = t \quad (86)$$

also satisfies (42) or (45).

5 Generating Functions of Type

A CT may admit various independent triplets on $\mathbb{R}^3 \times \mathbb{R}$ apart from (x_1, x_2, x_3) or (X_1, X_2, X_3) . Two main groups are possible; first one is (x_i, x_j, X_k) , and the second one is (X_i, X_j, x_k) , where $i \neq j$ and every group contains obviously nine triplets. In order to show how one can determine the transformation types, two different types of them are treated explicitly. The calculation scheme is the same for all possible types which is listed in Table 1.

First, we consider the triplet (x_1, x_2, X_3) . Then if every term in (41) is written in terms of (x_1, x_2, X_3) , the equivalence of related coefficients of the components on both sides of that equation amounts to

$$\begin{aligned}\frac{\partial(f_1, f_2)}{\partial(x_1, x_2)} &= -x_1 \frac{\partial x_3}{\partial x_1}, \\ \frac{\partial(f_1, f_2)}{\partial(X_3, x_1)} &= X_1 \frac{\partial X_2}{\partial x_1}, \\ \frac{\partial(f_1, f_2)}{\partial(x_2, X_3)} &= x_1 \frac{\partial x_3}{\partial X_3} - X_1 \frac{\partial X_2}{\partial x_2},\end{aligned}\tag{87}$$

and

$$\begin{aligned}\frac{\partial(f_1, f_2)}{\partial(x_1, t)} &= -H_1 \frac{\partial H_2}{\partial x_1} + K_1 \frac{\partial K_2}{\partial x_1}, \\ \frac{\partial(f_1, f_2)}{\partial(x_2, t)} &= -H_1 \frac{\partial H_2}{\partial x_2} + K_1 \frac{\partial K_2}{\partial x_2} + x_1 \frac{\partial x_3}{\partial t}, \\ \frac{\partial(f_1, f_2)}{\partial(X_3, t)} &= -H_1 \frac{\partial H_2}{\partial X_3} + K_1 \frac{\partial K_2}{\partial X_3} + X_1 \frac{\partial X_2}{\partial t},\end{aligned}\tag{88}$$

where $f_\alpha = F_\alpha(x_1, x_2, x_3(x_1, x_2, X_3, t), t)$. Given GFs f_1 and f_2 , these equations do not give always complete information on the transformation. But consider the rearrangement of (87)

$$\begin{aligned}\frac{\partial(f_1, f_2)}{\partial(x_1, x_2)} + \frac{\partial(x_1 x_3, x_2)}{\partial(x_1, x_2)} &= x_3, \\ \frac{\partial(f_1, f_2)}{\partial(X_3, x_1)} + \frac{\partial(x_1 x_3, x_2)}{\partial(X_3, x_1)} &= X_1 \frac{\partial X_2}{\partial x_1}, \\ \frac{\partial(f_1, f_2)}{\partial(x_2, X_3)} + \frac{\partial(x_1 x_3, x_2)}{\partial(x_2, X_3)} &= -X_1 \frac{\partial X_2}{\partial x_2},\end{aligned}\tag{89}$$

which is equivalent to

$$\begin{aligned}df_1 \wedge df_2 + d(x_1 x_3) \wedge dx_2 &= x_3 dx_1 \wedge dx_2 - X_1 dX_2 \wedge dX_3 \\ &\quad - H_1 dH_2 \wedge dt + K_1 dK_2 \wedge dt.\end{aligned}\tag{90}$$

For the functions $F_\alpha(x_1, x_2, X_3, t)$ which are the solutions to the differential equation

$$X_1 \frac{\partial X_2}{\partial x_2} \frac{\partial F_\alpha}{\partial x_1} - X_1 \frac{\partial X_2}{\partial x_1} \frac{\partial F_\alpha}{\partial x_2} - x_3 \frac{\partial F_\alpha}{\partial X_3} = 0\tag{91}$$

obtained from (89); (90) leads to

$$(dF_1 \wedge dF_2)_1 = x_3 dx_1 \wedge dx_2 - X_1 dX_2 \wedge dX_3 - H_1 dH_2 \wedge dt + K_1 dK_2 \wedge dt\tag{92}$$

corresponding to the our first kind transformation. Note, as can be seen from Table 1, that the first kind contains three types. Now x_3 is immediate by

$$\frac{\partial(F_1, F_2)}{\partial(x_1, x_2)} = x_3,\tag{93}$$

and for X_2 one needs to solve

$$\left[\frac{\partial(F_1, F_2)}{\partial(x_2, X_3)} \right] \frac{\partial X_2}{\partial x_1} - \left[\frac{\partial(F_1, F_2)}{\partial(X_3, x_1)} \right] \frac{\partial X_2}{\partial x_2} = 0 \quad (94)$$

which is originated from (89) again. Note that the equivalence of (90) and (92) does not imply in general $F_1 = f_1 + x_1 x_3$ and $F_2 = f_2 + x_2$ unless $df_1 \wedge dx_2 = df_2 \wedge d(x_1 x_3)$. On the other hand, for the transformations $f_2 = x_2$, the equivalence

$$dF_1 \wedge dF_2 = d(f_1 + x_1 x_3) \wedge dx_2 \quad (95)$$

is always possible. To be more explicit about this remark, consider the CT

$$X_1 = x_1 + x_2, \quad X_2 = x_2 + x_3, \quad X_3 = x_3. \quad (96)$$

If the general solutions of (45) are taken as the independent functions $F_1 = x_2 x_3$, $F_2 = x_2$, then the corresponding functions of type become $f_1 = x_2 X_3$, $f_2 = x_2$. Hence by the virtue of (95) the GFs are

$$F_1 = (x_1 + x_2)X_3, \quad F_2 = x_2. \quad (97)$$

Conversely, (97) generates, via (89) and (94), the CT

$$X_1 = x_1 + x_2, \quad X_2 = x_2 + h(x_3), \quad X_3 = x_3. \quad (98)$$

Second, consider the triplet (x_2, x_3, X_1) . This time, for $f_\alpha(x_2, x_3, X_1, t)$, (41) says

$$\begin{aligned} \frac{\partial(f_1, f_2)}{\partial(x_2, x_3)} &= x_1 - X_1 \frac{\partial(X_2, X_3)}{\partial(x_2, x_3)}, \\ \frac{\partial(f_1, f_2)}{\partial(X_1, x_2)} &= -X_1 \frac{\partial(X_2, X_3)}{\partial(X_1, x_2)}, \\ \frac{\partial(f_1, f_2)}{\partial(x_3, X_1)} &= -X_1 \frac{\partial(X_2, X_3)}{\partial(x_3, X_1)} \end{aligned} \quad (99)$$

similar to (42) and

$$\frac{\partial(f_1, f_2)}{\partial(\xi, t)} = -H_1 \frac{\partial H_2}{\partial \xi} + K_1 \frac{\partial K_2}{\partial \xi} - X_1 \frac{\partial(X_2, X_3)}{\partial(\xi, t)}, \quad \xi = x_2, x_3, X_1, \quad (100)$$

similar to (47). This last system of equations says that

$$\begin{aligned} df_1 \wedge df_2 &= dF_1(x_2, x_3, X_1, t) \wedge dF_2(x_2, x_3, X_1, t) \\ &= x_1 dx_2 \wedge dx_3 - X_1 dX_2 \wedge dX_3 - H_1 dH_2 \wedge dt + K_1 dK_2 \wedge dt. \end{aligned} \quad (101)$$

and therefore

$$f_\alpha(x_2, x_3, X_1, t) = F_\alpha(x_2, x_3, X_1, t). \quad (102)$$

Note that $F_\alpha(x_2, x_3, X_1, t)$ serves just like the GF of first type $F_1(q, Q, t)$ of the usual Hamilton formalism. As can be seen in the Table 1, there are six GFs of this type. The example given above obeys also this type of transformation.

As a further consequence, one should note that a CT may be of different types at the same time. For example the scaling transformation given in Sec.4 admits four types simultaneously:

$$\begin{aligned}
F_1 &= \frac{1}{c} x_1 X_3, \quad F_2 = x_2, \\
F_1 &= -\frac{1}{b} x_1 X_2, \quad F_2 = x_3, \\
F_1 &= -c x_3 X_1, \quad F_2 = X_2, \\
F_1 &= b x_2 X_1, \quad F_2 = X_3.
\end{aligned} \tag{103}$$

6 Infinitesimal Canonical Transformations

In the two-dimensional phase space of the usual Hamilton formalism, ICTs are given by the variations in the first order

$$\begin{aligned}
Q &= q + \epsilon \eta_1(q, p) = q + \epsilon \{q, G\} = q + \epsilon \frac{\partial G}{\partial p}, \\
P &= p + \epsilon \eta_2(q, p) = p + \epsilon \{p, G\} = p - \epsilon \frac{\partial G}{\partial q},
\end{aligned} \tag{104}$$

where ϵ is a continuous parameter and $G(q, p)$ is the GF of the ICT. The canonicity condition implies

$$\frac{\partial \eta_1}{\partial q} + \frac{\partial \eta_2}{\partial p} = 0 \tag{105}$$

up to the first order of ϵ . Following the same practice, these results can be extended to the three-space. An ICT in the three-dimensional phase space would then be proposed as

$$X_i = x_i + \epsilon f_i(x) = x_i + \epsilon \{x_i, G_1, G_2\} = x_i + \epsilon \frac{\partial(G_1, G_2)}{\partial(x_j, x_k)}, \tag{106}$$

where $G_1(x)$ and $G_2(x)$ generate directly the ICT via

$$dG_1 \wedge dG_2 = f_1 dx_2 \wedge dx_3 + f_2 dx_3 \wedge dx_1 + f_3 dx_1 \wedge dx_2. \tag{107}$$

One can check easily that, similar to (105), the canonicity condition (39) implies

$$\frac{\partial f_1(x)}{\partial x_1} + \frac{\partial f_2(x)}{\partial x_2} + \frac{\partial f_3(x)}{\partial x_3} = 0 \tag{108}$$

up to the first order of ϵ again.

It is well known that an ICT is a transformation depending on a parameter that moves the system infinitesimally along a trajectory in phase space and therefore a finite CT is the sum of an infinite succession of ICTs giving by the well known expansion

$$\phi = \varphi + \epsilon \{\varphi, G\} + \frac{\epsilon^2}{2!} \{\{\varphi, G\}, G\} + \frac{\epsilon^3}{3!} \{\{\{\varphi, G\}, G\}, G\} + \dots \tag{109}$$

where $\phi = Q, P$ and $\varphi = q, p$ in turn. With the same arguments used for the two-dimensional phase space, the transformation equation of a finite CT generated by the GFs G_1 and G_2 will correspond to

$$\begin{aligned} X_i &= x_i + \epsilon \{x_i, G_1, G_2\} + \frac{\epsilon^2}{2!} \{\{x_i, G_1, G_2\}, G_1, G_2\} \\ &\quad + \frac{\epsilon^3}{3!} \{\{\{x_i, G_1, G_2\}, G_1, G_2\}, G_1, G_2\} + \dots \end{aligned} \quad (110)$$

Equivalently, if we define the vector field

$$\hat{V}_G = f_1(x) \partial_{x_1} + f_2(x) \partial_{x_2} + f_3(x) \partial_{x_3}, \quad (111)$$

it is easy to see that the same transformation is given by

$$e^{\epsilon \hat{V}_G} x_i = X_i. \quad (112)$$

We can give a specific example showing that this construction actually works. For this aim we consider the CT

$$X_1 = x_1, \quad X_2 = x_2 + \epsilon x_3, \quad X_3 = x_3 - \epsilon x_2. \quad (113)$$

The transformation is generated by GFs

$$G_1(x) = \frac{1}{2}(x_2^2 + x_3^2), \quad G_2(x) = x_1 \quad (114)$$

or by vector field

$$\hat{V}_G = x_3 \partial_{x_2} - x_2 \partial_{x_3} \quad (115)$$

which is the generator of rotation about x_1 axis. Therefore it is immediate by means of (110) or (112) that our finite CT is

$$X_1 = x_1, \quad X_2 = x_2 \cos \epsilon + x_3 \sin \epsilon, \quad X_3 = -x_2 \sin \epsilon + x_3 \cos \epsilon, \quad (116)$$

where the parameter ϵ stands clearly for the rotation angle.

7 Decomposition of the Transformations

In classical mechanics a conjecture states surprisingly that any CT in a two dimensional phase space can be decomposed into some sequence of two principal CTs [8]. These are linear and point CTs. Proceeding elaborations of this conjecture in quantum mechanics led to a triplet as a wider class including gauge, point and interchanging transformations [9, 10]. One can check that the same triplet can also be used for the classical CTs. Without giving so many examples here, we give a particular one for the sake of motivation: Consider the CT

$$q \rightarrow p^2 - \frac{q^2}{4p^2}, \quad p \rightarrow -\frac{q}{2p} \quad (117)$$

converting the system with linear potential $H_0 = p^2 + q$ into the free particle $H_1 = p^2$. (In this section, we prefer using the map representation of CTs so that we can perform easily the transformation steps). The decomposition of the transformation can be achieved by the following five steps in turn;

1. interchange $q \rightarrow p, \quad p \rightarrow -q,$
 2. gauge $q \rightarrow q, \quad p \rightarrow p - q^2,$
 3. interchange $q \rightarrow -p, \quad p \rightarrow q,$
 4. point $q \rightarrow q^2, \quad p \rightarrow p/(2q),$
 5. interchange $q \rightarrow -p, \quad p \rightarrow q$
- (118)

corresponding symbolically to the sequence from right to left

$$\mathcal{S} = \mathcal{I}_3 \mathcal{P} \mathcal{I}_2 \mathcal{G} \mathcal{I}_1. \quad (119)$$

As a challenging problem, the statement has not been proven in a generic framework yet. But even though it is not true for every CT, it applies to a huge number of CTs. Parallel to the presentation, we will show that the discussion also applies to the CTs in the three-space.

First we will decompose the linear CT (64). Before doing this note that all the three types (70), (73), (74) of gauge transformation can be generated by the GFs

$$\begin{aligned} \hat{V}_{G_1} &= f_1(x_1)\partial_{x_2} + f_2(x_1)\partial_{x_3}, \\ \hat{V}_{G_2} &= g_1(x_2)\partial_{x_1} + g_2(x_2)\partial_{x_3}, \\ \hat{V}_{G_3} &= h_1(x_3)\partial_{x_1} + h_2(x_3)\partial_{x_2} \end{aligned} \quad (120)$$

respectively when considering (112). Now for the choices

$$\begin{aligned} f_1(x_1) &= \lambda_1 x_1, \quad f_2(x_1) = \lambda_2 x_1, \\ g_1(x_2) &= \mu_1 x_2, \quad g_2(x_2) = \mu_2 x_2, \\ h_1(x_3) &= \nu_1 x_3, \quad h_2(x_3) = \nu_2 x_3, \end{aligned} \quad (121)$$

the sequence

$$\mathcal{S}_L = \mathcal{P} \mathcal{G}_3 \mathcal{G}_2 \mathcal{G}_1 \quad (122)$$

where \mathcal{P} stands for the point transformation generating the scaling transformation (56), generates in turn the transformation chain

1. gauge $x_1 \rightarrow x_1, \quad x_2 \rightarrow x_2 + \lambda_1 x_1, \quad x_3 \rightarrow x_3 + \lambda_2 x_1,$
 2. gauge $x_1 \rightarrow x_1 + \mu_1 x_2, \quad x_2 \rightarrow x_2, \quad x_3 \rightarrow x_3 + \mu_2 x_2,$
 3. gauge $x_1 \rightarrow x_1 + \nu_1 x_3, \quad x_2 \rightarrow x_2 + \nu_2 x_3, \quad x_3 \rightarrow x_3,$
 4. point $x_1 \rightarrow a x_1, \quad x_2 \rightarrow b x_2, \quad x_3 \rightarrow c x_3.$
- (123)

Application of (122) to the coordinates (x_1, x_2, x_3) gives thus the linear CT

$$\begin{aligned} X_1 &= a x_1 + b \mu_1 x_2 + c (\nu_1 + \mu_1 \nu_2) x_3, \\ X_2 &= a \lambda_1 x_1 + b (1 + \lambda_1 \mu_1) x_2 + c [\lambda_1 \nu_1 + (1 + \lambda_1 \mu_1) \nu_2] x_3, \\ X_3 &= a \lambda_2 x_1 + b (\mu_2 + \mu_1 \lambda_2) x_2 + c [1 + \lambda_2 \nu_1 + (\mu_2 + \mu_1 \lambda_2) \nu_2] x_3. \end{aligned} \quad (124)$$

The next example is related with the cylindrical coordinate transformation

$$X_1 = \frac{1}{2}(x_1^2 + x_2^2), \quad X_2 = \tan^{-1} \frac{x_2}{x_1}, \quad X_3 = x_3. \quad (125)$$

The sequence

1. interchange $x_1 \rightarrow -x_2, \quad x_2 \rightarrow x_1, \quad x_3 \rightarrow x_3,$
2. point $x_1 \rightarrow \tan^{-1} x_1, \quad x_2 \rightarrow (1 + x_1^2)x_2, \quad x_3 \rightarrow x_3,$
3. interchange $x_1 \rightarrow x_2, \quad x_2 \rightarrow -x_1, \quad x_3 \rightarrow x_3,$
4. point $x_1 \rightarrow x_1^2/2, \quad x_2 \rightarrow x_2/x_1, \quad x_3 \rightarrow x_3,$

(126)

which can be written in the compact form

$$\mathcal{S}_C = \mathcal{P}_2 \mathcal{I}_2 \mathcal{P}_1 \mathcal{I}_1 \quad (127)$$

is the decomposition of (125).

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Table 1: Types of the canonical transformations in six kinds. ($r = 1, \dots, 6$ and $U = H_1 dH_2 \wedge dt - K_1 dK_2 \wedge dt$).

Independent variables	$(dF_1 \wedge dF_2)_r$
x_1, x_2, X_1	$df_1 \wedge df_2 + d(x_1 x_3) \wedge dx_2 = x_3 dx_1 \wedge dx_2 - X_1 dX_2 \wedge dX_3 - U$
x_1, x_2, X_2	
x_1, x_2, X_3	
x_1, x_3, X_1	$df_1 \wedge df_2 - d(x_1 x_2) \wedge dx_3 = x_2 dx_3 \wedge dx_1 - X_1 dX_2 \wedge dX_3 - U$
x_1, x_3, X_2	
x_1, x_3, X_3	
x_2, x_3, X_1	$df_1 \wedge df_2 = x_1 dx_2 \wedge dx_3 - X_1 dX_2 \wedge dX_3 - U$
x_2, x_3, X_2	
x_2, x_3, X_3	
X_1, X_2, x_1	$df_1 \wedge df_2 - d(X_1 X_3) \wedge dX_2 = x_1 dx_2 \wedge dx_3 - X_3 dX_1 \wedge dX_2 - U$
X_1, X_2, x_2	
X_1, X_2, x_3	
X_1, X_3, x_1	$df_1 \wedge df_2 + d(X_1 X_2) \wedge dX_3 = x_1 dx_2 \wedge dx_3 - X_2 dX_3 \wedge dX_1 - U$
X_1, X_3, x_2	
X_1, X_3, x_3	
X_2, X_3, x_1	$df_1 \wedge df_2 = x_1 dx_2 \wedge dx_3 - X_1 dX_2 \wedge dX_3 - U$
X_2, X_3, x_2	
X_2, X_3, x_3	