On the basis of the Burnside ring of a fusion system ☆

Matthew Gelvin [a], Sune Precht Reeh [b], Ergün Yalçın [c],*

[a] *Mathematics and Computer Science Department, Wesleyan University, Middletown, CT 06459-0128, USA*
[b] *Department of Mathematical Sciences, University of Copenhagen, Copenhagen, Denmark*
[c] *Department of Mathematics, Bilkent University, 06800 Bilkent, Ankara, Turkey*

A R T I C L E   I N F O

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We consider the Burnside ring $A(\mathcal{F})$ of $\mathcal{F}$-stable S-sets for a saturated fusion system $\mathcal{F}$ defined on a p-group S. It is shown by S.P. Reeh that the monoid of $\mathcal{F}$-stable sets is a free commutative monoid with canonical basis $\{\alpha_P\}$. We give an explicit formula that describes α_P as an S-set. In the formula we use a combinatorial concept called broken chains which we introduce to understand inverses of modified Möbius functions.

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1. Introduction

For a finite group G, the Burnside ring $A(G)$ is defined as the Grothendieck ring of the isomorphism classes of G-sets with addition given by disjoint union and multiplication

by Cartesian product. The Burnside ring $A(G)$ is free as an abelian group, with basis given by isomorphism classes of transitive G-sets $[G/H]$. In particular the basis elements are in one-to-one correspondence with G-conjugacy classes of subgroups of G.

One often studies the Burnside ring of a finite group G using the mark homomorphism $\Phi : A(G) \to \mathbb{Z}^{Cl(G)}$, where $Cl(G)$ is the set of G-conjugacy classes of subgroups of G. For $K \le G$, the K'th coordinate of Φ is defined by $\Phi_K(X) = |X^K|$ when X is a G-set, extended linearly for the rest of $A(G)$. The ring $A^*(G) := \mathbb{Z}^{Cl(G)}$ is the ring of super class functions $f : Cl(G) \to \mathbb{Z}$ with multiplication given by coordinate-wise multiplication. It is called the ghost ring of G and it plays an important role for explaining G-sets using their fixed point data. In particular, it is shown that the mark homomorphism is an injective map with a finite cokernel. This means that using rational coefficients, one can express the idempotent basis of $A^*(G)$ in terms of basis elements $[G/H]$ (see D. Gluck [6]).

Given a saturated fusion system $\mathcal{F}$ on a p-group S, one can define the Burnside ring $A(\mathcal{F})$ of the fusion system $\mathcal{F}$ as a subring of $A(S)$ formed by elements $X \in A(S)$ such that $\Phi_P(X) = \Phi_{\varphi(P)}(X)$ for every morphism $\varphi : P \to S$ in $\mathcal{F}$. This subring is also the Grothendieck ring of $\mathcal{F}$-stable S-sets (see (2.3) for a definition). It is proved by S.P. Reeh [10] that the monoid of $\mathcal{F}$-stable S-sets is a free commutative monoid with a canonical basis satisfying certain properties. Our primary interest is to identify the elements of this basis, so we describe it in more detail here.

For every $X \in A(S)$, let $c_Q(X)$ denote the number of $[S/Q]$-orbits in X so that $X = \sum c_Q(X)[S/Q]$, where the sum is taken over the set of S-conjugacy classes of subgroups of S. For each $\mathcal{F}$-conjugacy class of subgroups P of S, there is a unique (up to S-isomorphism) $\mathcal{F}$-stable set α_P satisfying

 (i) $c_Q(\alpha_P) = 1$ if Q is fully normalized and $\mathcal{F}$-conjugate to P,
(ii) $c_Q(\alpha_P) = 0$ if Q is fully normalized and not $\mathcal{F}$-conjugate to P.

The set $\{\alpha_P\}$ over all $\mathcal{F}$-conjugacy classes of subgroups form an (additive) basis for $A(\mathcal{F})$ (see Proposition 2.2).

The main purpose of this paper is to give explicit formulas for the number of fixed points $|(\alpha_P)^Q|$ and for the coefficients $c_Q(\alpha_P)$ of $[S/Q]$-orbits, for the basis element α_P. Our first observation is that the matrix of fixed points $\mathcal{F}Mark_{Q,P} = |(\alpha_P)^Q|$ can be described using a simple algorithm in linear algebra. We now explain this algorithm.

Let $M\ddot{o}b = Mark^{-1}$ denote the inverse matrix of the usual table of marks for S. For each $\mathcal{F}$-conjugacy class of subgroups of S, take the sum of the corresponding columns of $M\ddot{o}b$, obtaining a non-square matrix. Then, from the set of rows corresponding to an $\mathcal{F}$-conjugacy class, select one representing a fully $\mathcal{F}$-normalized subgroup; delete the others. The resulting matrix $\mathcal{F}M\ddot{o}b$ is a square matrix with dimension equal to the number of $\mathcal{F}$-conjugacy classes of subgroups. Then we observe that the inverse matrix

FMark := *FMöb*$^{-1}$ is the matrix of marks for $A(\mathcal{F})$. In other words, we prove the following:

Theorem 1.1. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Let the square matrix FMöb be constructed as above, with rows and columns corresponding to the $\mathcal{F}$-conjugacy classes of subgroup in S. Then FMöb is invertible, and the inverse FMark := FMöb^{-1} is the matrix of marks for $A(\mathcal{F})$, i.e.*

$$\mathit{FMark}_{Q^*,P^*} = \left| (\alpha_{P^*})^{Q^*} \right|.$$

Here Q^* and P^* denote the chosen $\mathcal{F}$-conjugacy class representatives. This theorem is proved as Theorem 3.1 in the paper. We also give a detailed calculation to illustrate this method (see Example 3.2). This is all done in Section 3.

In Section 4 we look closely at the above matrix method and analyze it using Möbius inversion. We observe that the entries of *FMark*, the table of marks for $\mathcal{F}$, can be explained by a combinatorial formula using a concept called (tethered) broken chains (see Definition 4.7). This formula is proved in Theorem 4.9.

In Section 5, we prove the main theorem of the paper, which gives a formula for the coefficients $c_Q(\alpha_P)$ in the linear combination $\alpha_P = \sum c_Q(\alpha_P)[S/Q]$. As in the case of fixed point orders, here also the formula is given in terms of an alternating sum of the number of broken chains linking Q to P (see Definition 5.1). The main theorem of the paper is the following:

Theorem 1.2. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Let $\mathfrak{BC}_{\mathcal{F}}(Q, P)$ denote the set of $\mathcal{F}$-broken chains linking Q to P. Then the number of $[S/Q]$-orbits in each irreducible $\mathcal{F}$-stable set α_P, denoted $c_Q(\alpha_P)$, can be calculated as*

$$c_Q(\alpha_P) = \frac{|W_S P^*|}{|W_S Q|} \cdot \sum_{\sigma \in \mathfrak{BC}_{\mathcal{F}}(Q,P)} (-1)^{\ell(\sigma)}$$

for Q, P subgroups of S, where $P^* \sim_{\mathcal{F}} P$ is fully normalized.

In the above formula, $\ell(\sigma)$ denotes the length of a broken chain $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_k)$ linking Q to P defined as the integer $\ell(\sigma) := k + |\sigma_0| + \cdots + |\sigma_k|$ (see Definition 5.1). This theorem is proved in Section 5 as Theorem 5.2. In Example 5.5, we illustrate how this combinatorial formula can be used to calculate the coefficients $c_Q(\alpha_P)$ for some subgroups Q, P for the fusion system $\mathcal{F} = \mathcal{F}_{D_8}(A_6)$.

In Section 6, we prove some simplifications for the formula in Theorem 1.2. These simplifications come from observations about broken chains and from properties of Möbius functions. Then in Section 7 we give an application of our main theorem to characteristic bisets. Since understanding the characteristic bisets was one of the motivations for this work, we now say a few more words about this application.

Let S be a p-group and $\mathcal{F}$ be a fusion system on S as before. A characteristic biset for the fusion system $\mathcal{F}$ is an (S, S)-biset Ω satisfying certain properties (see Definition 7.1). These bisets were first introduced by Linckelmann and Webb, and they play an important role in fusion theory. One of the properties of a characteristic biset is stability under $\mathcal{F}$-conjugation, namely for every $\varphi : Q \to S$, the (Q, S)-bisets $_\varphi\Omega$ and $_Q\Omega$ are isomorphic. Since each (S, S)-biset is a left $(S \times S)$-set, we can convert this stability condition to a stability condition for the fusion system $\mathcal{F} \times \mathcal{F}$ on the p-group $S \times S$ and consider characteristic bisets as elements in $A(\mathcal{F} \times \mathcal{F})$.

It is shown by M. Gelvin and S.P. Reeh [5] that every characteristic biset includes a unique minimal characteristic biset, denoted by $\Omega_{\min}$. The minimal biset can be described as the basis element $\alpha_{\Delta(S, id)}$ of the fusion system $A(\mathcal{F} \times \mathcal{F})$, where for a morphism $\varphi : Q \to S$ in $\mathcal{F}$, the subgroup $\Delta(P, \varphi)$ denotes the diagonal subgroup $\{(\varphi(s), s) \mid s \in P\}$ in $S \times S$. Now Theorem 1.2 can be used to give formulas for the coefficients $c_{\Delta(P, \varphi)}(\Omega_{\min})$. Such formulas are important for various other applications of these bisets (see for example [12]). Using the new interpretation of these coefficients we were able to give a proof for the statement that all the stabilizers $\Delta(P, \varphi)$ appearing in $\Omega_{\min}$ must satisfy $P \geq O_p(\mathcal{F})$ where $O_p(\mathcal{F})$ denotes the largest normal p-subgroup of $\mathcal{F}$. This was originally proved in [5, Proposition 9.11], the proof we give in Proposition 7.3 uses broken chains and is much simpler.

2. Burnside rings for groups and fusion systems

In this section we recall the Burnside ring of a finite group S and how to describe its structure in terms of the homomorphism of marks, which embeds the Burnside ring into a suitable ghost ring. We also recall the Burnside ring of a saturated fusion system $\mathcal{F}$ on a p-group S, in the sense of [10].

Let S be a finite group. We use the letter S instead of G for a finite group since in all the applications of these results the group S will be a p-group. The isomorphism classes of finite S-sets form a semiring with disjoint union as addition and Cartesian product as multiplication. The Burnside ring of S, denoted $A(S)$, is then defined as the additive Grothendieck group of this semiring, and $A(S)$ inherits the multiplication as well. Given a finite S-set X, we let $[X]$ denote the isomorphism class of X as an element of $A(S)$. The isomorphism classes $[S/P]$ of transitive S-sets form an additive basis for $A(S)$, and two transitive sets S/P and S/Q are isomorphic if and only if the subgroups P and Q are conjugate in S.

For each element $X \in A(S)$ we define $c_P(X)$, with $P \leq S$, to be the coefficients when we write X as a linear combination of the basis elements $[S/P]$ in $A(S)$, i.e.

$$X = \sum_{[P] \in Cl(S)} c_P(X) \cdot [S/P],$$

where $Cl(S)$ denotes the set of S-conjugacy classes of subgroup in S. The resulting maps $c_P : A(S) \to \mathbb{Z}$ are group homomorphisms, but they are *not* ring homomorphisms.

To describe the multiplication of $A(S)$, it is enough to know the products of basis elements $[S/P]$ and $[S/Q]$. By taking the Cartesian product $(S/P) \times (S/Q)$ and considering how it breaks into orbits, one reaches the following double coset formula for the multiplication in $A(S)$:

$$[S/P] \cdot [S/Q] = \sum_{\bar{s} \in P \backslash S / Q} \left[S/(P \cap {}^s Q) \right], \tag{2.1}$$

where $P \backslash S / Q$ is the set of double cosets PsQ with $s \in S$.

Instead of counting orbits, an alternative way of characterizing a finite S-set is counting the fixed points for each subgroup $P \leq S$. For every $P \leq S$ and S-set X, we denote the number of P-fixed points by $\Phi_P(X) := |X^P|$. This number only depends on P up to S-conjugation. Since we have

$$\left| (X \sqcup Y)^P \right| = \left| X^P \right| + \left| Y^P \right| \quad \text{and} \quad \left| (X \times Y)^P \right| = \left| X^P \right| \cdot \left| Y^P \right|$$

for all S-sets X and Y, the *fixed point map* Φ_P for S-sets extends to a ring homomorphism $\Phi_P : A(S) \to \mathbb{Z}$. On the basis elements $[S/P]$, the number of fixed points is given by

$$\Phi_Q \big([S/P] \big) = \left| (S/P)^Q \right| = \frac{|N_S(Q,P)|}{|P|}, \tag{2.2}$$

where $N_S(Q,P) = \{ s \in S \mid {}^s Q \leq P \}$ is the transporter in S from Q to P. In particular, $\Phi_Q([S/P]) \neq 0$ if and only if $Q \lesssim_S P$ (Q is subconjugate to P).

We have one fixed point homomorphism Φ_P per conjugacy class of subgroups in S, and we combine them into the *homomorphism of marks*

$$\Phi = \Phi^S : A(S) \xrightarrow{\ \prod_{[P]} \Phi_P\ } \prod_{[P] \in Cl(S)} \mathbb{Z}.$$

This ring homomorphism maps $A(S)$ into the product ring $A^*(S) := \prod_{[P] \in Cl(S)} \mathbb{Z}$, the so-called *ghost ring* for the Burnside ring $A(S)$.

We think of the elements in the ghost ring $A^*(S)$ as superclass functions $Cl(S) \to \mathbb{Z}$ defined on the subgroups of S and constant on every S-conjugacy class. For an element $\xi \in A^*(S)$ we write $\xi(Q)$, with $Q \leq S$, to denote the value of the class function ξ on the S-conjugacy class of Q. We think of $\xi(Q)$ as the number of Q-fixed points for ξ, even though ξ might not be the fixed point vector for an actual element of $A(S)$. The ghost ring $A^*(S)$ has a natural basis consisting of e_P for each $[P] \in Cl(S)$, where e_P is the class function with value 1 on the class $[P]$, and 0 on all the other classes. The elements $\{ e_P \mid [P] \in Cl(S) \}$ are the primitive idempotents of $A^*(S)$.

Results by tom Dieck and others show that the mark homomorphism is injective, but not every $\xi \in A^*(S)$ is the fixed point vector for an element of $A(S)$. The cokernel of Φ

contains the obstruction to ξ being the fixed point vector of a (virtual) S-set, hence we speak of this cokernel as the *obstruction group* $Obs(S) := \prod_{[P] \in Cl(S)} (\mathbb{Z}/|W_S P|\mathbb{Z})$, where $W_S P := N_S P/P$. These statements are combined in the following proposition, the proof of which can be found in [2, Chapter 1], [3], and [13].

Proposition 2.1. *Let $\Psi = \Psi^S : A^*(S) \to Obs(S)$ be given by the $[P]$-coordinate functions*

$$\Psi_P(\xi) := \sum_{\bar{s} \in W_S P} \xi\big(\langle s \rangle P\big) \quad \big(\mathrm{mod}\ |W_S P|\big).$$

Then, the following sequence of abelian groups is exact:

$$0 \to A(S) \xrightarrow{\ \Phi\ } A^*(S) \xrightarrow{\ \Psi\ } Obs(S) \to 0.$$

Note that in the exact sequence above Φ is a ring homomorphism, but Ψ is just a group homomorphism.

2.1. The Burnside ring of a saturated fusion system

Let S be a finite p-group, and suppose that $\mathcal{F}$ is a saturated fusion system on S (see [1] for necessary definitions on fusion systems). We say that a finite S-set is $\mathcal{F}$-stable if the action is unchanged up to isomorphism whenever we act through morphisms of $\mathcal{F}$. More precisely, if $P \leq S$ is a subgroup and $\varphi : P \to S$ is a homomorphism in $\mathcal{F}$, we can consider X as a P-set by using φ to define the action $g.x := \varphi(g)x$ for $g \in P$. We denote the resulting P-set by ${}_{P,\varphi}X$. In particular when $incl : P \to S$ is the inclusion map, ${}_{P,incl}X$ has the usual restriction of the S-action to P.

Restricting the action of S-sets along φ extends to a ring homomorphism $r_\varphi : A(S) \to A(P)$, and we let ${}_{P,\varphi}X$ denote the image $r_\varphi(X)$ for all elements $X \in A(S)$. We say that an element $X \in A(S)$ is $\mathcal{F}$-*stable* if it satisfies

$$\begin{aligned} {}_{P,\varphi}X = {}_{P,incl}X \quad \text{inside } A(P), \quad \text{for all } P \leq S \quad \text{and} \\ \text{homomorphisms } \varphi : P \to S \quad \text{in } \mathcal{F}. \end{aligned} \tag{2.3}$$

The $\mathcal{F}$-stability condition originally came from considering action maps $S \to \Sigma_X$ into the symmetric group on X that are maps of fusion systems $\mathcal{F} \to \mathcal{F}_{\Sigma_X}$.

Alternatively, one can characterize $\mathcal{F}$-stability in terms of fixed points and the mark homomorphism, and the following three properties are equivalent for all $X \in A(S)$:

(i) X is $\mathcal{F}$-stable.
(ii) $\Phi_P(X) = \Phi_{\varphi P}(X)$ for all $\varphi \in \mathcal{F}(P, S)$ and $P \leq S$.
(iii) $\Phi_P(X) = \Phi_Q(X)$ for all pairs $P, Q \leq S$ with $P \sim_\mathcal{F} Q$.

A proof of this claim can be found in [4, Proposition 3.2.3] or [10]. We shall primarily use (ii) and (iii) to characterize $\mathcal{F}$-stability.

It follows from property (iii) that the $\mathcal{F}$-stable elements form a subring of $A(S)$. We define the *Burnside ring of* $\mathcal{F}$ to be the subring $A(\mathcal{F}) \subseteq A(S)$ consisting of all the $\mathcal{F}$-stable elements. Equivalently, we can consider the actual S-sets that are $\mathcal{F}$-stable: The $\mathcal{F}$-stable sets form a semiring, and we define $A(\mathcal{F})$ to be the Grothendieck group hereof. These two constructions give rise to the same ring $A(\mathcal{F})$ – see [10].

According to [10], every $\mathcal{F}$-stable S-set decomposes uniquely (up to S-isomorphism) as a disjoint union of irreducible $\mathcal{F}$-stable sets, where the irreducible $\mathcal{F}$-stable sets are those that cannot be written as disjoint unions of smaller $\mathcal{F}$-stable sets. Each irreducible $\mathcal{F}$-stable set corresponds to an $\mathcal{F}$-conjugacy class $[P]_{\mathcal{F}} = \{Q \leq S \mid Q \text{ is isomorphic to } P \text{ in } \mathcal{F}\}$ of subgroups, and they satisfy the following characterization:

Proposition 2.2. *(See [10, Proposition 4.8 and Theorem A].) Let $\mathcal{F}$ be a saturated fusion system over S. For each conjugacy class in $\mathcal{F}$ of subgroups $[P]_{\mathcal{F}}$ there is a unique (up to S-isomorphism) $\mathcal{F}$-stable set α_P satisfying*

(i) *$c_Q(\alpha_P) = 1$ if Q is fully normalized and $\mathcal{F}$-conjugate to P,*
(ii) *$c_Q(\alpha_P) = 0$ if Q is fully normalized and not $\mathcal{F}$-conjugate to P.*

The sets α_P form an additive basis for the monoid of all $\mathcal{F}$-stable S-sets. In addition, by construction in [10] the stabilizer of any point in α_P is $\mathcal{F}$-conjugate to a subgroup of P.

3. Fixed point orders of the irreducible $\mathcal{F}$-stable sets

Let *Mark* be the matrix of marks for the Burnside ring of S, i.e. the matrix for the mark homomorphism $\Phi: A(S) \to A^*(S)$ with entries

$$Mark_{Q,P} = \left| (S/P)^Q \right| = \frac{|N_S(Q,P)|}{|P|}.$$

The rows and columns of *Mark* correspond to the S-conjugacy classes $[P]_S \in Cl(S)$ of subgroups in S. We order the subgroup classes by increasing order of the subgroups, in particular the trivial group 1 corresponds to the first row and column, and S itself corresponds to the last row and column. This way *Mark* becomes upper triangular.

Over the rational numbers the mark homomorphism $\Phi: A(S) \otimes \mathbb{Q} \xrightarrow{\cong} A^*(S) \otimes \mathbb{Q}$ is an isomorphism, and we let $M\ddot{o}b = Mark^{-1}$ be the inverse rational matrix.

From $M\ddot{o}b$ we construct a further matrix $\mathcal{F}M\ddot{o}b$ as follows: For each $\mathcal{F}$-conjugacy class of subgroups in S we take the sum of the corresponding columns of $M\ddot{o}b$ to be the columns of $\mathcal{F}M\ddot{o}b$. For each $\mathcal{F}$-conjugacy class of subgroups we choose a fully normalized representative of the class, and then we delete all rows that do not correspond to one of the chosen representatives. The resulting matrix $\mathcal{F}M\ddot{o}b$ is guaranteed to be a square

matrix with dimension equal to the number of $\mathcal{F}$-conjugacy classes of subgroups; the rows and columns correspond to the chosen representatives of the $\mathcal{F}$-conjugacy classes (see Example 3.2).

For each class $[P]_{\mathcal{F}}$ let P^* be the chosen representative. The precise description of the entries $\mathcal{F}M\ddot{o}b_{Q^*,P^*}$ in terms of $M\ddot{o}b_{Q,P}$ is then

$$\mathcal{F}M\ddot{o}b_{Q^*,P^*} := \sum_{[P]_S \subseteq [P^*]_{\mathcal{F}}} M\ddot{o}b_{Q^*,P} \, .$$

Theorem 3.1. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Let the square matrix $\mathcal{F}M\ddot{o}b$ be constructed as above, with rows and columns corresponding to the $\mathcal{F}$-conjugacy classes of subgroup in S. Then $\mathcal{F}M\ddot{o}b$ is invertible, and the inverse $\mathcal{F}Mark := \mathcal{F}M\ddot{o}b^{-1}$ is the matrix of marks for $A(\mathcal{F})$, i.e.*

$$\mathcal{F}Mark_{Q^*,P^*} = \left| (\alpha_{P^*})^{Q^*} \right| .$$

Proof. In the rational ghost ring $A^*(S) \otimes \mathbb{Q} = \prod_{[P]_S} \mathbb{Q}$ the unit vector e_P is the superclass function with value 1 for the class $[P]$ and value 0 for the other subgroup classes. We have one unit vector e_P corresponding to each conjugacy class $[P]_S$, and the matrix of marks *Mark* expresses the usual basis for $A(S) \otimes \mathbb{Q}$, consisting of the transitive sets $[S/P]$, in terms of the idempotents e_P. Conversely, the inverse $M\ddot{o}b = Mark^{-1}$ then expresses the idempotents e_P as (rational) linear combinations of the orbits $[S/P]$.

An element $X \in A(S) \otimes \mathbb{Q}$ is $\mathcal{F}$-stable if the number of fixed points $|X^Q|$ and $|X^P|$ are the same for $\mathcal{F}$-conjugate subgroups $Q \sim_{\mathcal{F}} P$, i.e. if the coefficients of X with respect to the idempotents e_Q and e_P are the same for $\mathcal{F}$-conjugate subgroups. The $\mathcal{F}$-stable elements of $A^*(\mathcal{F}) \otimes \mathbb{Q} \le A^*(S) \otimes \mathbb{Q}$ thus have an idempotent basis consisting of

$$e_P^{\mathcal{F}} := \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} e_{P'},$$

with a primitive $\mathcal{F}$-stable idempotent $e_P^{\mathcal{F}}$ corresponding to each $\mathcal{F}$-conjugacy class of subgroups. To express the idempotent $e_P^{\mathcal{F}}$ as a linear combination of orbits $[S/P]$, we just have to take the sum of the columns in $M\ddot{o}b$ associated to the conjugacy class $[P]_{\mathcal{F}}$. Hence counting the number of $[S/Q]$-orbits in $e_P^{\mathcal{F}}$, we get

$$c_Q\left(e_P^{\mathcal{F}}\right) = \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} M\ddot{o}b_{Q,P'} \, .$$

Let $\mathcal{P}^*$ denote the set of chosen fully normalized representatives for each $\mathcal{F}$-conjugacy class of subgroups in S. For each $P^* \in \mathcal{P}^*$, we have an irreducible $\mathcal{F}$-stable set α_{P^*}, and by Proposition 2.2 any linear combination X of the $\{\alpha_{P^*}\}$ can be determined solely by counting the number of $[S/P^*]$-orbits for each $P^* \in \mathcal{P}^*$.

Because $e_P^{\mathcal{F}}$ is $\mathcal{F}$-stable, it is a (rational) linear combination of the $\{\alpha_{Q^*}\}$. The coefficients of this linear combination, coincide with the number of $[S/Q^*]$-orbits, so to

express the idempotents $e_P^{\mathcal{F}}$ in terms of the $\{\alpha_{Q^*}\}$ we only care about the rows of *Möb* corresponding to $Q^* \in \mathcal{P}^*$, and ignore all the other rows. Consequently, $\mathcal{F}M\ddot{o}b$ is the matrix that expresses the idempotents $e_P^{\mathcal{F}}$ in terms of the $\{\alpha_{Q^*}\}$.

The inverse $\mathcal{F}Mark = \mathcal{F}M\ddot{o}b^{-1}$ therefore expresses the irreducible $\mathcal{F}$-stable sets α_P in terms of the $\mathcal{F}$-stable idempotents $e_Q^{\mathcal{F}}$, which exactly reduces to counting the Q-fixed points of α_P. $\square$

Example 3.2. Let $S = D_8$ be the dihedral group of order 8 and $\mathcal{F} = \mathcal{F}_{D_8}(A_6)$ denote the fusion system induced by the finite group A_6. Let $\mathcal{P}$ be the entire subgroup poset of D_8 and $\mathcal{P}/\mathcal{F}$ the poset of $\mathcal{F}$-conjugacy classes of subgroups in D_8:

where V_4^* is an elementary abelian 2-group of order 4, C_n^* is a cyclic group of order n, $Z \cong C_2$ is the centre of D_8, and square brackets denote the $\mathcal{F}$-conjugacy class. The horizontal squiggly lines indicate subgroups' being in the same D_8-conjugacy class and dashed lines means that they are in the same $\mathcal{F}$-conjugacy class.

The table of marks *Mark* and its inverse $M\ddot{o}b = Mark^{-1}$ are given below

Mark	1	C_2^1	Z	C_2^2	V_4^1	C_4	V_4^2	D_8
1	8	4	4	4	2	2	2	1
C_2^1		2	0	0	2	0	0	1
Z			4	0	2	2	2	1
C_2^2				2	0	0	2	1
V_4^1					2	0	0	1
C_4						2	0	1
V_4^2							2	1
D_8								1

Möb	1	C_2^1	Z	C_2^2	V_4^1	C_4	V_4^2	D_8
1	1/8	−1/4	−1/8	−1/4	1/4	0	1/4	0
C_2^1		1/2	0	0	−1/2	0	0	0
Z			1/4	0	−1/4	−1/4	−1/4	1/2
C_2^2				1/2	0	0	−1/2	0
V_4^1					1/2	0	0	−1/2
C_4						1/2	0	−1/2
V_4^2							1/2	−1/2
D_8								1

Below we give the matrix for $\mathcal{F}M\ddot{o}b$ and its inverse $\mathcal{F}Mark = \mathcal{F}M\ddot{o}b^{-1}$. Recall that the matrix for $\mathcal{F}M\ddot{o}b$ is obtained by adding the columns of *Möb* for the subgroups which are $\mathcal{F}$-conjugate, and then by choosing a fully normalized subgroup in every $\mathcal{F}$-conjugacy class on the rows. Here Z is the unique fully normalized subgroup in its $\mathcal{F}$-conjugacy class.

$\mathcal{F}M\ddot{o}b$	1	Z	V_4^1	C_4	V_4^2	D_8
1	1/8	−5/8	1/4	0	1/4	0
Z		1/4	−1/4	−1/4	−1/4	1/2
V_4^1			1/2	0	0	−1/2
C_4				1/2	0	−1/2
V_4^2					1/2	−1/2
D_8						1

$\mathcal{F}Mark$	1	Z	V_4^1	C_4	V_4^2	D_8
1	8	20	6	10	6	1
Z		4	2	2	2	1
V_4^1			2	0	0	1
C_4				2	0	1
V_4^2					2	1
D_8						1

From this we obtain the $\Phi(\alpha_P)$ by reading off the columns of $\mathcal{F}Mark$ (since $e_Z^{\mathcal{F}} = e_{C_2^1} + e_Z + e_{C_2^2}$):

$$\Phi(\alpha_1) = 8e_1^{\mathcal{F}} \qquad\qquad = 8e_1$$

$$\Phi(\alpha_Z) = 20e_1^{\mathcal{F}} + 4e_Z^{\mathcal{F}} \qquad = 20e_1 + 4e_{C_2^1} + 4e_Z + 4e_{C_2^2}$$

$$\Phi(\alpha_{V_4^1}) = 6e_1^{\mathcal{F}} + 2e_Z^{\mathcal{F}} + 2e_{V_4^1}^{\mathcal{F}} = 6e_1 + 2e_{C_2^1} + 2e_Z + 2e_{C_2^2} + 2e_{V_4^1}$$

$$\Phi(\alpha_{C_4}) = 10e_1^{\mathcal{F}} + 2e_Z^{\mathcal{F}} + 2e_{C_4}^{\mathcal{F}} = 10e_1 + 2e_{C_2^1} + 2e_Z + 2e_{C_2^2} + e_{C_4}$$

$$\Phi(\alpha_{V_4^2}) = 6e_1^{\mathcal{F}} + 2e_Z^{\mathcal{F}} + 2e_{V_4^2}^{\mathcal{F}} = 6e_1 + 2e_{C_2^1} + 2e_Z + 2e_{C_2^2} + 2e_{V_4^2}$$

$$\Phi(\alpha_{D_8}) = e_1^{\mathcal{F}} + e_Z^{\mathcal{F}} + e_{V_4^1}^{\mathcal{F}} = e_1 + e_{C_2^1} + e_Z + e_{C_2^2}$$
$$+ e_{C_4}^{\mathcal{F}} + e_{V_4^2}^{\mathcal{F}} + e_{D_8}^{\mathcal{F}} \qquad + e_{V_4^1} + e_{C_4} + e_{V_4^2} + e_{D_8}$$

Finally, applying the matrix *Möb* to each of these fixed point vectors yields the *S*-orbit description of the α_P:

$$\alpha_1 = [S/1]$$
$$\alpha_Z = [S/Z] + 2[S/C_2^1] + 2[S/C_2^2]$$
$$\alpha_{V_4^1} = [S/V_4^1] + [S/C_2^2]$$
$$\alpha_{C_4} = [S/C_4] + [S/C_2^1] + [S/C_2^2]$$
$$\alpha_{V_4^2} = [S/V_4^2] + [S/C_2^1]$$
$$\alpha_{D_8} = [S/D_8]$$

There is an explicit formula for expressing the idempotent basis $\{e_Q\}$ in terms of the transitive *S*-set basis $\{[S/P]\}$ using the combinatorics of the subgroup poset, which is often referred as the Gluck's idempotent formula [6]. In the following two sections we find similar explicit formulas for the coefficients of α_{P^*} with respect to the idempotent basis $\{e_Q\}$ and then with respect to the *S*-set basis $\{[S/P]\}$. For this we need to look at the Möbius inversion in Gluck's idempotent formula more closely.

4. Fixed point orders and Möbius inversion

In this section we discuss how a more explicit formula can be obtained for fixed point orders of basis elements using Möbius inversion. We first introduce basic definitions about Möbius inversion. For more details, we refer the reader to [11].

Let $\mathcal{P}$ be a finite poset. The *incidence function of* $\mathcal{P}$ is defined as the function

$$\zeta_\mathcal{P} : \mathcal{P} \times \mathcal{P} \to \mathbb{Z} : (a, b) \mapsto \begin{cases} 1 & a \leq b, \\ 0 & \text{else.} \end{cases}$$

The *incidence matrix* of $\mathcal{P}$ is the $|\mathcal{P}| \times |\mathcal{P}|$-matrix $(\zeta_\mathcal{P})$ with entries $(\zeta_\mathcal{P})_{a,b} = \zeta_\mathcal{P}(a, b)$. When labelling the rows/columns we respect the partial order of $\mathcal{P}$, such that $a \leq b$ in $\mathcal{P}$ implies that the a-row/-column precedes the b-row/-column. This way the incidence matrix is always upper unitriangular (an upper triangular matrix with all diagonal entries equal to 1).

Definition 4.1. The *Möbius function* for a poset $\mathcal{P}$ is $\mu_\mathcal{P} : \mathcal{P} \times \mathcal{P} \to \mathbb{Q}$ defined by

$$\sum_{a \in \mathcal{P}} \zeta_\mathcal{P}(x, a) \mu_\mathcal{P}(a, y) = \delta_{x,y} = \sum_{a \in \mathcal{P}} \mu_\mathcal{P}(x, a) \zeta_\mathcal{P}(a, y)$$

for all $x, y \in \mathcal{P}$. If the corresponding $|\mathcal{P}| \times |\mathcal{P}|$ *Möbius matrix* is $(\mu_\mathcal{P})$, we have $(\mu_\mathcal{P}) = (\zeta_\mathcal{P})^{-1}$.

Lemma 4.2. $\mu_\mathcal{P}(a, b) \in \mathbb{Z}$ *for all* $a, b \in \mathcal{P}$.

Proof. By our labelling convention, $(\zeta_{\mathcal{P}})$ is upper unitriangular. Therefore we can write $(\zeta_{\mathcal{P}}) = I + (\eta_{\mathcal{P}})$, where $(\eta_{\mathcal{P}})_{i,j} = 1$ when $a_i < a_j$ and vanishes elsewhere. Then $\eta_{\mathcal{P}}$ is strictly upper triangular, and $(\eta_{\mathcal{P}})^{|\mathcal{P}|} = 0$, so

$$(\mu_{\mathcal{P}}) = (\zeta_{\mathcal{P}})^{-1} = \left(I + (\eta_{\mathcal{P}})\right)^{-1}$$
$$= I - (\eta_{\mathcal{P}}) + (\eta_{\mathcal{P}})^2 - (\eta_{\mathcal{P}})^3 + \ldots + (-1)^{|\mathcal{P}|-1}(\eta_{\mathcal{P}})^{|\mathcal{P}|-1}$$

has all integral entries. $\quad\square$

Each of the matrices $(\eta_{\mathcal{P}}^k) := (\eta_{\mathcal{P}})^k$ has an interpretation in terms of chains in the poset $\mathcal{P}$.

Definition 4.3. A *chain of length k in $\mathcal{P}$* is a totally ordered subset of $k + 1$ elements $\sigma = \{a_0 < a_1 < \ldots < a_k\}$. Such a chain *links a_0 to a_k*.

Let $\mathfrak{C}_{\mathcal{P}}^k(a, b)$ be the set of chains of length k linking a to b, and $\mathfrak{C}_{\mathcal{P}}^k$ the set of all chains of length k in $\mathcal{P}$. $\mathfrak{C}_{\mathcal{P}}^0$ is the set of elements of $\mathcal{P}$. Similarly, let $\mathfrak{C}_{\mathcal{P}}(a, b)$ be the set of all chains linking a to b, $\mathfrak{C}_{\mathcal{P}}$ the set of all chains in $\mathcal{P}$, and for any chain $\sigma \in \mathfrak{C}_{\mathcal{P}}$ let $|\sigma|$ denote the length of σ.

Lemma 4.4. $(\eta_{\mathcal{P}}^k)_{a,b} = |\mathfrak{C}_{\mathcal{P}}^k(a, b)|$.

Proof. With $a_0 := a$ and $a_k := b$, the definition of matrix multiplication gives us

$$\left(\eta_{\mathcal{P}}^k\right)_{a_0, a_k} = \sum_{a_1, \ldots, a_{k-1} \in \mathcal{P}} \eta_{\mathcal{P}}(a_0, a_1) \cdot \eta_{\mathcal{P}}(a_1, a_2) \cdots \eta_{\mathcal{P}}(a_{k-1}, a_k).$$

By definition of the incidence function $\eta_{\mathcal{P}}$, each factor $\eta_{\mathcal{P}}(a_i, a_{i+1})$ is 1 if $a_i < a_{i+1}$ and zero otherwise. The product $\eta_{\mathcal{P}}(a_0, a_1) \cdots \eta_{\mathcal{P}}(a_{k-1}, a_k)$ is therefore nonzero and equal to 1 precisely when $a_0 < a_1 < \cdots < a_k$ is a k-chain in $\mathcal{P}$ linking a_0 to a_k. $\quad\square$

Proposition 4.5. *For all $a, b \in \mathcal{P}$,*

$$\mu_{\mathcal{P}}(a, b) = \sum_{k=0}^{\infty} (-1)^k |\mathfrak{C}_{\mathcal{P}}^k(a, b)| = \sum_{\sigma \in \mathfrak{C}_{\mathcal{P}}(a, b)} (-1)^{|\sigma|}.$$

Proof. Immediate from Lemmas 4.2 and 4.4 and their proofs. $\quad\square$

Remark 4.6. From the formula above it is clear that the Möbius function can be expressed as the reduced Euler characteristic of a subposet in $\mathcal{P}$. More specifically, for $a < b$, let $(a, b)_{\mathcal{P}}$ denote the poset of all $c \in \mathcal{P}$ with $a < c < b$. Then $\mu_{\mathcal{P}}(a, b)$ is equal to the reduced Euler characteristic $\widetilde{\chi}((a, b)_{\mathcal{P}})$ of the subposet $(a, b)_{\mathcal{P}}$ for every $a, b \in \mathcal{P}$ such that $a < b$.

4.1. Möbius functions and fixed points

On the next few pages we go through the construction of the matrix $\mathcal{F}Mark$ in Theorem 3.1 again, but this time we follow the calculations in detail using the framework of incidence and Möbius functions. For a finite p-group S, we let $\mathcal{P}$ be the poset of subgroups ordered by inclusion. This poset has incidence and Möbius functions $\zeta_{\mathcal{P}}$ and $\mu_{\mathcal{P}}$ as described in the previous section, denoted thereafter by ζ, respectively μ.

The matrix of marks $Mark$ for the Burnside ring of S has entries $Mark_{[Q],[P]} = |N_S(Q, P)|/|P|$ defined for pairs $([Q], [P])$ of S-conjugacy classes of subgroups in S. Each column is divisible by the diagonal entry, which is the order of the Weyl group $W_S P = N_S P/P$. If we divide the $[P]_S$-column by $|W_S P|$, we get

$$Mark_{[Q],[P]} \cdot \frac{1}{|W_S P|} = \frac{|N_S(Q, P)|}{|N_S P|} = \frac{|\{s \in S \mid {}^s Q \leq P\}|}{|N_S P|} = \frac{|\{s \in S \mid Q \leq P^s\}|}{|N_S P|}$$

$$= \left|\{P' \leq S \mid P' \sim_S P \text{ and } Q \leq P'\}\right| = \sum_{P' \sim_S P} \zeta(Q, P').$$

We denote this value by $\widetilde{\zeta}_S([Q], [P])$, and we call $\widetilde{\zeta}_S$ the *modified incidence function* for the S-conjugacy classes of subgroups. We have $(\widetilde{\zeta}_S)_{[Q],[P]} = Mark_{[Q],[P]}/|W_S P|$, so the modified incidence matrix $(\widetilde{\zeta}_S)$ is upper unitriangular (see Example 4.10 for the computations in the case $S = D_8$).

Inverting the matrix $(\widetilde{\zeta}_S)$, we define $(\widetilde{\mu}_S) := (\widetilde{\zeta}_S)^{-1}$ which gives rise to a *modified Möbius function* $\widetilde{\mu}_S$ for S-conjugacy classes of subgroups. Since $M\ddot{o}b = Mark^{-1}$ is the inverse of the matrix of marks, we have $(\widetilde{\mu}_S)_{[Q],[P]} = |W_S Q| \cdot M\ddot{o}b_{Q,P}$. As $(\widetilde{\zeta}_S)$ is triangular with diagonal entries 1, we also have $(\widetilde{\mu}_S) = (\widetilde{\zeta}_S)^{-1} = \sum_{k=0}^{\infty} (-1)^k \cdot ((\widetilde{\zeta}_S) - I)^k$ as in the proof of Lemma 4.2, which we use to calculate the entries of $(\widetilde{\mu}_S)$:

$$\widetilde{\mu}_S([Q], [P]) = \sum_{k=0}^{\infty} (-1)^k \cdot ((\widetilde{\zeta}_S) - I)^k$$

$$= \sum_{([R_0],[R_1],\ldots,[R_k]) \in \mathcal{T}_S} (-1)^k \widetilde{\zeta}_S([R_0], [R_1]) \cdots \widetilde{\zeta}_S([R_{k-1}], [R_k])$$

where $\mathcal{T}_S$ consists of all tuples $([R_0], [R_1], \ldots, [R_k])$, for $k \geq 0$, of S-conjugacy classes of subgroups $[R_i] \in Cl(S)$ such that $[R_0] = [Q]$, $[R_k] = [P]$, and $|R_0| < |R_1| < \cdots < |R_k|$. Since we have

$$\widetilde{\zeta}_S([R_i], [R_j]) = \sum_{R_j' \sim_S R_j} \zeta(R_i, R_j')$$

for all i, j, we obtain that $\widetilde{\mu}_S([Q], [P])$ is equal to the sum

$$\sum_{\substack{([R_0],[R_1],\ldots,[R_k])\in\mathcal{T}_S \\ R_0=Q}} \sum_{R_1'\sim_S R_1} (-1)^k \zeta\big(R_0,R_1'\big)\widetilde{\zeta}_S\big([R_1'],[R_2]\big)\cdots\widetilde{\zeta}_S\big([R_{k-1}],[R_k]\big)$$

$$\vdots$$

$$= \sum_{\substack{([R_0],[R_1],\ldots,[R_k])\in\mathcal{T}_S \\ R_0=Q}} \sum_{R_1'\sim_S R_1} \sum_{R_2'\sim_S R_2} \cdots$$

$$\times \sum_{R_k'\sim_S R_k} (-1)^k \zeta\big(R_0,R_1'\big)\zeta\big(R_1',R_2'\big)\cdots\zeta\big(R_{k-1}',R_k'\big)$$

$$= \sum_{\substack{R_0 < R_1' < \cdots < R_k' \\ \text{s.t. } R_0=Q,\, R_k'\sim_S P}} (-1)^k = \sum_{P'\sim_S P}\sum_{k=0}^{\infty}(-1)^k\big|\mathfrak{C}_{\mathcal{P}}^k(Q,P')\big| = \sum_{P'\sim_S P}\mu(Q,P').$$

Therefore, the matrix *Möb*, the inverse of the matrix of marks, has entries

$$M\ddot{o}b_{[Q],[P]} = \frac{1}{|W_S Q|}\widetilde{\mu}_S\big([Q],[P]\big) = \frac{1}{|W_S Q|}\sum_{P'\sim_S P}\mu(Q,P').$$

This concludes the part of our investigation concerning only the subgroup structure of S, and for the calculations below we include the extra data of a saturated fusion system $\mathcal{F}$ on S.

In order to determine the number of fixed points $|(\alpha_P)^Q|$ as in Theorem 3.1, we wish to calculate the $\mathcal{F}$-analogs of *Mark* and *Möb* above. To do this, we first choose a fully normalized representative P^* for each $\mathcal{F}$-conjugacy class $[P]_{\mathcal{F}}$ of subgroups, and as before let $\mathcal{P}^*$ be the collection of these representatives. Recall that the matrix $\mathcal{F}M\ddot{o}b$ is constructed from *Möb* by picking out the rows corresponding to $Q^* \in \mathcal{P}^*$, and the column in $\mathcal{F}M\ddot{o}b$ corresponding to $P^* \in \mathcal{P}^*$ is the sum of the columns in *Möb* corresponding to $[P]_S$ with $P \sim_{\mathcal{F}} P^*$. More explicitly, we have

$$\mathcal{F}M\ddot{o}b_{Q^*,P^*} := \sum_{[P]_S\subseteq[P^*]_{\mathcal{F}}} M\ddot{o}b_{[Q^*],[P]} = \frac{1}{|W_S Q^*|}\sum_{P\sim_{\mathcal{F}} P^*}\mu(Q^*,P).$$

We define the *modified Möbius function* $\widetilde{\mu}_{\mathcal{F}}:\mathcal{P}^*\times\mathcal{P}^*\to\mathbb{Z}$ for the (representatives of) $\mathcal{F}$-conjugacy classes of subgroups, to be

$$\widetilde{\mu}_{\mathcal{F}}\big(Q^*,P^*\big) := \big|W_S Q^*\big|\cdot\mathcal{F}M\ddot{o}b_{Q^*,P^*} = \sum_{P\sim_{\mathcal{F}} P^*}\mu\big(Q^*,P\big),$$

summing the usual Möbius function. The associated matrix $(\widetilde{\mu}_{\mathcal{F}})$ is then upper unitriangular.

The modified incidence matrix for $\mathcal{F}$ is defined as the inverse $(\widetilde{\zeta}_{\mathcal{F}}) := (\widetilde{\mu}_{\mathcal{F}})^{-1}$, with the associated function $\widetilde{\zeta}_{\mathcal{F}}:\mathcal{P}^*\times\mathcal{P}^*\to\mathbb{Z}$. By Theorem 3.1 we then have

$$\big|(\alpha_P)^Q\big| = \mathcal{F}Mark_{Q^*,P^*} = \big|W_S P^*\big|\cdot\widetilde{\zeta}_{\mathcal{F}}\big(Q^*,P^*\big)$$

where $\mathcal{F}Mark := \mathcal{F}M\ddot{o}b^{-1}$. Recall that for each subgroup $R \leq S$, we denote by R^* the chosen fully normalized representative for the $\mathcal{F}$-conjugacy class of R. As previously, the fact that $(\widetilde{\mu}_{\mathcal{F}})$ is unitriangular implies

$$\widetilde{\zeta}_{\mathcal{F}}(Q^*, P^*) = \sum_{k=0}^{\infty} (-1)^k \cdot \left((\widetilde{\mu}_{\mathcal{F}}) - I \right)^k$$

$$= \sum_{(R_0^*, R_1^*, \ldots, R_k^*) \in \mathcal{T}_{\mathcal{F}}} (-1)^k \widetilde{\mu}_{\mathcal{F}}(R_0^*, R_1^*) \cdots \widetilde{\mu}_{\mathcal{F}}(R_{k-1}^*, R_k^*)$$

where $\mathcal{T}_{\mathcal{F}}$ consists of all tuples $(R_0^*, R_1^*, \ldots, R_k^*)$, for all $k \geq 0$, of $\mathcal{F}$-conjugacy class representatives $R_i^* \in \mathcal{P}^*$ such that $R_0^* = Q^*$, $R_k^* = P^*$, and $|R_0^*| < |R_1^*| < \cdots < |R_k^*|$. Since we have

$$\widetilde{\mu}_{\mathcal{F}}(R_i^*, R_j^*) = \sum_{R_j \sim_{\mathcal{F}} R_j^*} \mu(R_i^*, R_j),$$

for all $R_i^*, R_j^* \in \mathcal{P}^*$, we obtain that

$$\widetilde{\zeta}_{\mathcal{F}}(Q^*, P^*) = \sum_{(R_0^*, R_1^*, \ldots, R_k^*) \in \mathcal{T}_{\mathcal{F}}} \sum_{R_1 \sim_{\mathcal{F}} R_1^*} (-1)^k \mu(R_0^*, R_1) \widetilde{\mu}_{\mathcal{F}}(R_1^*, R_2) \cdots \widetilde{\mu}_{\mathcal{F}}(R_{k-1}^*, R_k)$$

$$\vdots$$

$$= \sum_{\substack{(R_0^*, R_1^*, \ldots, R_k^*) \in \mathcal{T}_{\mathcal{F}}, \\ R_1, \ldots, R_k \in \mathcal{P} \text{ s.t. } R_i \sim_{\mathcal{F}} R_i^*}} (-1)^k \mu(R_0^*, R_1) \mu(R_1^*, R_2) \cdots \mu(R_{k-1}^*, R_k)$$

$$= \sum_{\substack{(R_0^*, R_1^*, \ldots, R_k^*) \in \mathcal{T}_{\mathcal{F}}, \\ R_1, \ldots, R_k \in \mathcal{P} \text{ s.t. } R_i \sim_{\mathcal{F}} R_i^*}} \sum_{\substack{\sigma_i \in \mathfrak{C}_{\mathcal{P}}(R_{i-1}^*, R_i) \\ \text{for } 1 \leq i \leq k}} (-1)^{k + |\sigma_1| + \cdots + |\sigma_k|}. \qquad (4.1)$$

To calculate $\widetilde{\zeta}_{\mathcal{F}}(Q^*, P^*)$ we hence have to count sequences of chains $(\sigma_1, \ldots, \sigma_k)$ such that the end R_i of σ_i is $\mathcal{F}$-conjugate to the start R_i^* of σ_{i+1}, and the first chain σ_1 has to start at Q^* while the final chain σ_k only has to end at P^* up to $\mathcal{F}$-conjugation. We give these sequences a name:

Definition 4.7. A *tethered $\mathcal{F}$-broken chain in $\mathcal{P}$ linking $Q^* \in \mathcal{P}^*$ to $P \in \mathcal{P}$* is a sequence of chains $(\sigma_1, \ldots, \sigma_k)$ in $\mathcal{P}$ subject to the following requirements. With each chain written as $\sigma_i = (a_0^i, \ldots, a_{n_i}^i)$ they must satisfy

- $a_{n_i}^i \sim_{\mathcal{F}} a_0^{i+1}$ for all $1 \leq i \leq k-1$, so the endpoints of the chains fit together up to conjugation in $\mathcal{F}$.
- $a_0^i \in \mathcal{P}^*$ for all $1 \leq i \leq k$. Every chain starts at one of the chosen representatives.
- $|\sigma_i| = n_i > 0$, for all $1 \leq i \leq k$.
- $a_0^1 = Q^*$ and $a_{n_k}^k \sim_{\mathcal{F}} P$.

If $Q^* \sim_{\mathcal{F}} P$, we allow the trivial broken chain with $k = 0$. Let $\mathfrak{t}\mathfrak{BC}_{\mathcal{F}}(Q^*, P)$ be the set of tethered $\mathcal{F}$-broken chains linking Q^* to P. The *total length* of a tethered broken chain $\sigma = (\sigma_1, \ldots, \sigma_k)$ is defined to be

$$\ell(\sigma) := k + |\sigma_1| + \cdots + |\sigma_k|.$$

We visualize a tethered broken chain as a zigzag diagram in the following way:

$$
\begin{array}{ccccc}
a_0^1 & < & \cdots & < & a_{n_1}^1 \\
& & & \wr & \\
& a_0^2 & < & \cdots & < & a_{n_2}^2 \\
& & & & \wr & \\
& & & & & \ddots \\
& & & & \wr & \\
& & & a_0^k & < & \cdots & < & a_{n_k}^k
\end{array}
$$
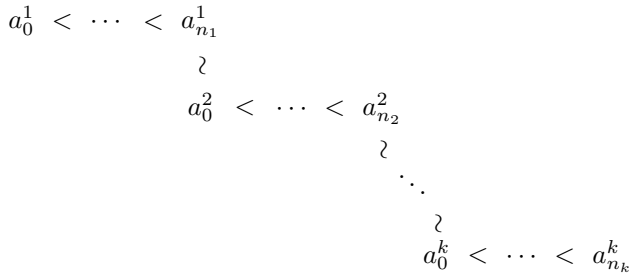
The total length of the tethered broken chain is then the total number of $<$ and $\sim$ signs plus 1. The added 1 can be viewed as an additional hidden $Q \sim_{\mathcal{F}} Q^*$ in front of the broken chain, and this interpretation matches the description, in Remark 5.3 below, of tethered broken chains as a special case of the *broken chains* defined in Section 5.

With the terminology of tethered broken chains, the calculations above translate to the following statements:

Proposition 4.8. *The modified incidence function $\widetilde{\zeta}_{\mathcal{F}}$ for a saturated fusion system $\mathcal{F}$, can be calculated as*

$$\widetilde{\zeta}_{\mathcal{F}}(Q^*, P^*) = \sum_{\sigma \in \mathfrak{t}\mathfrak{BC}_{\mathcal{F}}(Q^*, P^*)} (-1)^{\ell(\sigma)} = \sum_{(\sigma_1, \ldots, \sigma_k) \in \mathfrak{t}\mathfrak{BC}_{\mathcal{F}}(Q^*, P^*)} (-1)^{k + |\sigma_1| + \cdots + |\sigma_k|}$$

for all fully normalized representatives $Q^*, P^* \in \mathcal{P}^*$.

We now state the main result of this section.

Theorem 4.9. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $\mathcal{P}^*$ be a set of fully normalized representatives for the $\mathcal{F}$-conjugacy classes of subgroups in S. Let $\mathfrak{t}\mathfrak{BC}_{\mathcal{F}}(Q^*, P^*)$ denote the set of all tethered $\mathcal{F}$-broken chains linking Q^* to P^*. Then the numbers of fixed points for the irreducible $\mathcal{F}$-stable sets α_{P^*}, $P^* \in \mathcal{P}^*$, can be calculated as*

$$\left| (\alpha_{P^*})^{Q^*} \right| = \left| W_S P^* \right| \cdot \sum_{\sigma \in \mathfrak{t}\mathfrak{BC}_{\mathcal{F}}(Q^*, P^*)} (-1)^{\ell(\sigma)}$$

for $Q^*, P^* \in \mathcal{P}^*$.

Proof. Immediate from the proposition since $|(\alpha_{P^*})^{Q^*}| = |W_S P^*| \cdot \widetilde{\zeta}_{\mathcal{F}}(Q^*, P^*)$. $\quad\square$

Example 4.10. Let $S = D_8$ and $\mathcal{F} = \mathcal{F}_S(A_6)$ as before. The incidence matrix $\zeta_{\mathcal{P}}$ and the Möbius matrix $\mu_{\mathcal{P}}$ are given as follows.

$\zeta_{\mathcal{P}}$	1	C_2^1	$C_2^{1\prime}$	Z	$C_2^{2\prime}$	C_2^2	V_4^1	C_4	V_4^2	D_8
1	1	1	1	1	1	1	1	1	1	1
C_2^1		1					1			1
$C_2^{1\prime}$			1				1			1
Z				1			1	1	1	1
$C_2^{2\prime}$					1			1	1	1
C_2^2						1		1	1	1
V_4^1							1			1
C_4								1		1
V_4^2									1	1
D_8										1

$\mu_{\mathcal{P}}$	1	C_2^1	$C_2^{1\prime}$	Z	$C_2^{2\prime}$	C_2^2	V_4^1	C_4	V_4^2	D_8
$\underline{1}$	1	−1	−1	−1	−1	−1	2	0	2	0
$\underline{C_2^1}$		1					−1			0
$C_2^{1\prime}$			1				−1			0
$\underline{Z}$				1			−1	−1	−1	2
$C_2^{2\prime}$					1			−1		0
$\underline{C_2^2}$						1		−1		0
$\underline{V_4^1}$							1			−1
$\underline{C_4}$								1		−1
$\underline{V_4^2}$									1	−1
$\underline{D_8}$										1

Below we see the matrices for $\widetilde{\mu}_S$ and $\widetilde{\zeta}_S$ obtained by summing over the columns of subgroups belonging to the same S-conjugacy class and choosing an S-conjugacy class representative on the rows.

$\widetilde{\mu}_S$	1	C_2^1	Z	C_2^2	V_4^1	C_4	V_4^2	D_8
1	1	−2	−1	−2	2	0	2	0
C_2^1		1	0	0	−1	0	0	0
Z			1	0	−1	−1	−1	2
C_2^2				1	0	0	−1	0
V_4^1					1	0	0	−1
C_4						1	0	−1
V_4^2							1	−1
D_8								1

$\widetilde{\zeta}_S$	1	C_2^1	Z	C_2^2	V_4^1	C_4	V_4^2	D_8
1	1	2	1	2	1	1	1	1
C_2^1		1	0	0	1	0	0	1
Z			1	0	1	1	1	1
C_2^2				1	0	0	1	1
V_4^1					1	0	0	1
C_4						1	0	1
V_4^2							1	1
D_8								1

$\mathbf{W}_{D_8}$	1	C_2^1	Z	C_2^2	V_4^1	C_4	V_4^2	D_8
1	8							
C_2^1		2						
Z			4					
C_2^2				2				
V_4^1					2			
C_4						2		
V_4^2							2	
D_8								1

$\mathbf{m}_{D_8}$	1	C_2^1	Z	C_2^2	V_4^1	C_4	V_4^2	D_8
1	8	4	4	4	2	2	2	1
C_2^1		2	0	0	2	0	0	1
Z			4	0	2	2	2	1
C_2^2				2	0	0	2	1
V_4^1					2	0	0	1
C_4						2	0	1
V_4^2							2	1
D_8								1

The last two matrices above are the diagonal matrix $\mathbf{W}_{D_8}$ with entries $(\mathbf{W}_{D_8})_{[P],[P]} = |W_S(P)|$, and the matrix $\mathbf{m}_{D_8} = \widetilde{\zeta}_S \cdot \mathbf{W}_{D_8}$ which is the same as matrix of the mark homomorphism $Mark$. So we also have $M\ddot{o}b = \mathbf{W}_{D_8}^{-1} \cdot \widetilde{\mu}_S$.

Now let $\widetilde{\mu}_{\mathcal{F}}$ be the matrix obtained by summing columns of $\widetilde{\mu}_S$ over the $\mathcal{F}$ conjugacy classes and picking fully normalized representatives for the rows. Let $\widetilde{\zeta}_{\mathcal{F}} = (\widetilde{\mu}_{\mathcal{F}})^{-1}$.

$\widetilde{\mu}_{\mathcal{F}}$	1	Z	V_4^1	C_4	V_4^2	D_8
1	1	−5	2	0	2	0
Z		1	−1	−1	−1	2
V_4^1			1	0	0	−1
C_4				1	0	−1
V_4^2					1	−1
D_8						1

$\widetilde{\zeta}_{\mathcal{F}}$	1	Z	V_4^1	C_4	V_4^2	D_8
1	1	5	3	5	3	1
Z		1	1	1	1	1
V_4^1			1	0	0	1
C_4				1	0	1
V_4^2					1	1
D_8						1

From the definition of $\mathcal{F}M\ddot{o}b$, it is easy to see that $\mathcal{F}M\ddot{o}b = \mathbf{W}_{\mathcal{F}}^{-1} \cdot \widetilde{\mu}_{\mathcal{F}}$ and $\mathcal{F}Mark = \mathcal{F}M\ddot{o}b^{-1} = \widetilde{\zeta}_{\mathcal{F}} \cdot \mathbf{W}_{\mathcal{F}}$ where $\mathbf{W}_{\mathcal{F}}$ is the diagonal matrix with entries $(\mathbf{W}_{\mathcal{F}})_{P^*,P^*} = |W_S(P^*)|$ for all $P^* \in \mathcal{P}^*$. Theorem 4.9 says that we can calculate the entries of the matrix $\widetilde{\zeta}_{\mathcal{F}}$ by counting the number of tethered broken chains. For example, $\widetilde{\zeta}_{\mathcal{F}}(1, Z) = 5$ because there are 5 tethered broken chains linking 1 to Z. We give more complicated examples of tethered broken chain calculations in Example 5.5.

Remark 4.11. Note that the modified incidence matrix with respect to S-conjugations and the modified Möbius function on S-conjugate subgroups (coming from the poset of subgroups) are constructed in the same way: Add the columns of S-conjugate subgroups, pick out any row from each class. It is interesting that performing the same operation on the originals of the incidence function and the Möbius inverse ends up giving you inverse matrices; in particular, this is not what happens for modifications with respect to $\mathcal{F}$-conjugation which is what is done in the rest of the paper. We think that this shows that the S-conjugation action on the subgroup poset is more special that the $\mathcal{F}$-conjugation action.

5. Broken chains and the main theorem

Now that we have formulas for the number of fixed points of α_P, we will determine how each α_P decomposes into S-orbits. For every element $X \in A(S)$ of the Burnside ring, we let $c_Q(X)$ denote the number of (virtual) $[S/Q]$-orbits, i.e. the coefficients of the linear combination $X = \sum_{[Q]_S} c_Q(X) \cdot [S/Q]$. The matrix of marks *Mark* encodes the number of fixed points in terms of the number of orbits, so the numbers $|X^Q|$ form a fixed point vector $\varphi := Mark \cdot (c_Q(X))$. Recall that *Möb* is the inverse of *Mark*. Given any fixed point vector φ, we can therefore recover the orbit decomposition as $(c_Q(X)) = Möb \cdot \varphi$.

For α_P we already have a formula for the number of fixed points $|(\alpha_P)^Q|$, which we write in the form of

$$\left|(\alpha_P)^Q\right| = \mathcal{F}Mark_{Q^*,P^*} = \left|W_S P^*\right| \cdot \widetilde{\zeta}_{\mathcal{F}}\left(Q^*, P^*\right)$$

where $\widetilde{\zeta}_{\mathcal{F}}(Q^*, P^*)$ has a complicated Möbius formula given in (4.1). We also know how *Möb* is given in terms of Möbius functions. The number of $[S/Q]$-orbits in α_P must therefore be

$$c_Q(\alpha_P) = \sum_{[R] \in Cl(S)} Möb_{Q,R} \cdot \left|(\alpha_P)^R\right| = \frac{1}{|W_S Q|} \sum_{[R] \in Cl(S)} \widetilde{\mu}_S\left([Q],[R]\right) \cdot \left|(\alpha_{P^*})^{R^*}\right|$$

$$= \frac{1}{|W_S Q|} \sum_{R \in \mathcal{P}} \mu(Q, R) \cdot \left(\left|W_S P^*\right| \cdot \widetilde{\zeta}_{\mathcal{F}}\left(R^*, P^*\right)\right)$$

$$= \frac{|W_S P^*|}{|W_S Q|} \sum_{R \in \mathcal{P}} \mu(Q, R)$$

$$\cdot \sum_{\substack{(R_0^*, R_1^*, \ldots, R_k^*) \in \mathcal{T}_{\mathcal{F}}, \\ R_1, \ldots, R_k \in \mathcal{P} \text{ s.t. } R_i \sim_{\mathcal{F}} R_i^*}} (-1)^k \mu\left(R_0^*, R_1\right) \mu\left(R_1^*, R_2\right) \cdots \mu\left(R_{k-1}^*, R_k\right)$$

where the sum is over $\mathcal{T}_{\mathcal{F}}$ of all k-tuples, for all $k \geq 0$, of (prefixed) $\mathcal{F}$-conjugacy class representatives $R_i^* \in \mathcal{P}^*$ such that $R_0^* = R^*$, $R_k^* = P^*$, and $|R_0^*| < |R_1^*| < \cdots < |R_k^*|$. From this we obtain that

$$c_Q(\alpha_P)$$

$$= \frac{|W_S P^*|}{|W_S Q|} \sum_{\substack{R_0, R_1, \ldots, R_k \in \mathcal{P} \\ \text{s.t. } R_k \sim_{\mathcal{F}} P^*, \\ |Q| \leq |R_0| < |R_1| < \cdots < |R_k|}} (-1)^k \mu(Q, R_0) \mu\left(R_0^*, R_1\right) \mu\left(R_1^*, R_2\right) \cdots \mu\left(R_{k-1}^*, R_k\right)$$

$$= \frac{|W_S P^*|}{|W_S Q|} \sum_{\substack{R_0, R_1, \ldots, R_k \in \mathcal{P} \\ \text{s.t. } R_k \sim_{\mathcal{F}} P^*, \\ |Q| \leq |R_0| < |R_1| < \cdots < |R_k|}} \sum_{\sigma_0 \in \mathfrak{C}_{\mathcal{P}}(Q, R_0)} \sum_{\substack{\sigma_i \in \mathfrak{C}_{\mathcal{P}}(R_{i-1}^*, R_i) \\ \text{for } 1 \leq i \leq k}} (-1)^{k + |\sigma_0| + |\sigma_1| + \cdots + |\sigma_k|}.$$

The resulting formula is very similar to the calculations for fixed points in the previous section, except that we have an additional (possibly trivial) chain σ_0 in front. We combine this additional chain with the definition of tethered broken chains and arrive at the following definition:
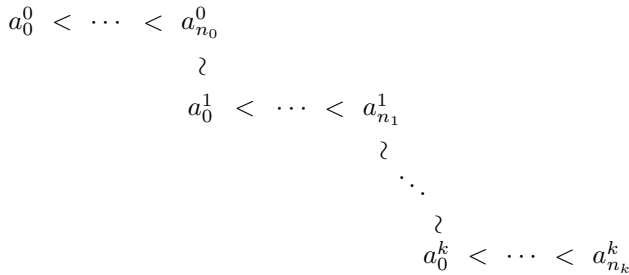
Definition 5.1. An $\mathcal{F}$-*broken chain in* $\mathcal{P}$ *linking* $Q \in \mathcal{P}$ *to* $P \in \mathcal{P}$ is a sequence of chains $(\sigma_0, \sigma_1, \dots, \sigma_k)$ in $\mathcal{P}$ subject to the following requirements. With each chain written as $\sigma_i = (a_0^i, \dots, a_{n_i}^i)$ they must satisfy

- $a_{n_i}^i \sim_{\mathcal{F}} a_0^{i+1}$ for all $0 \leq i \leq k - 1$, so the endpoints of the chains fit together up to conjugation in $\mathcal{F}$.
- $a_0^i \in \mathcal{P}^*$ for all $1 \leq i \leq k$. Every chain except for σ_0 starts at one of the chosen representatives.
- $|\sigma_i| = n_i > 0$, for all $1 \leq i \leq k$. Note that σ_0 is allowed to be trivial.
- $a_0^0 = Q$ and $a_{n_k}^k \sim_{\mathcal{F}} P$.

As before, if $Q \sim_{\mathcal{F}} P$, we allow the trivial broken chain with $k = 0$ and σ_0 trivial. Let $\mathfrak{BC}_{\mathcal{F}}(Q, P)$ be the set of $\mathcal{F}$-broken chains linking Q to P. We define the *total length* of a broken chain $\sigma = (\sigma_0, \dots, \sigma_k)$ to be

$$\ell(\sigma) := k + |\sigma_0| + \cdots + |\sigma_k|.$$

To visualize a broken chain, we represent it by the diagram

$$
\begin{array}{ccccc}
a_0^0 & < & \cdots & < & a_{n_0}^0 \\
& & & \wr & \\
& a_0^1 & < & \cdots & < & a_{n_1}^1 \\
& & & & \wr \\
& & & \ddots & \\
& & & & \wr \\
& & & a_0^k & < & \cdots & < & a_{n_k}^k
\end{array}
$$

The total length of the represented broken chain is then equal to the number of $<$ and $\sim$ signs put together.

Now we state our main theorem:

Theorem 5.2. *Let* $\mathcal{F}$ *be a saturated fusion system over a finite p-group* S. *Let* $\mathfrak{BC}_{\mathcal{F}}(Q, P)$ *denote the set of* $\mathcal{F}$-*broken chains linking* Q *to* P. *Then the number of* $[S/Q]$-*orbits in each irreducible* $\mathcal{F}$-*stable set* α_P, *denoted* $c_Q(\alpha_P)$, *can be calculated as*
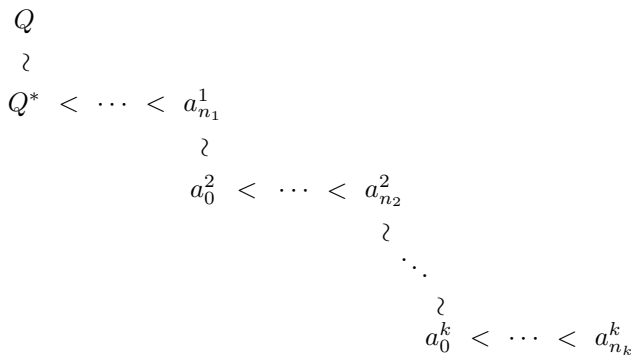
$$c_Q(\alpha_P) = \frac{|W_S P^*|}{|W_S Q|} \cdot \sum_{\sigma \in \mathfrak{BC}_{\mathcal{F}}(Q,P)} (-1)^{\ell(\sigma)}$$

for $Q, P \in \mathcal{P}$, where $P^* \sim_{\mathcal{F}} P$ is fully normalized.

Proof. Immediate from the argument at the beginning of the section. $\square$

Remark 5.3. If a broken chain $(\sigma_0, \sigma_1, \ldots, \sigma_k) \in \mathfrak{BC}_{\mathcal{F}}(Q,P)$ happens to have σ_0 equal to the trivial chain, i.e. $|\sigma_0| = 0$, then Q is the endpoint of σ_0 so σ_1 has to start at Q^*. The converse is also true, if σ_1 starts at Q^*, then σ_0 has to be trivial. In this case $(\sigma_1, \ldots, \sigma_k)$ is exactly the data of a *tethered* broken chain linking Q^* to P.

Hence the tethered broken chains $(\sigma_1, \ldots, \sigma_k) \in \mathfrak{tBC}_{\mathcal{F}}(Q^*, P)$ correspond precisely to the broken chains $(\sigma_0, \sigma_1, \ldots, \sigma_k) \in \mathfrak{BC}_{\mathcal{F}}(Q,P)$ where σ_0 is the trivial chain. This way, in diagram form, a tethered broken chain linking Q (or rather Q^*) to P looks like

$$
\begin{array}{ccccc}
Q & & & & \\
\wr & & & & \\
Q^* & < & \cdots & < & a^1_{n_1} \\
& & & & \wr \\
& & a^2_0 & < & \cdots & < & a^2_{n_2} \\
& & & & & & \wr \\
& & & & & & \ddots \\
& & & & & & \wr \\
& & & & & & a^k_0 & < & \cdots & < & a^k_{n_k}
\end{array}
$$

with $a^k_{n_k} \sim_{\mathcal{F}} P$. Drawn in this form, the total length of the tethered broken chain is the total number of $<$ and $\sim$ symbols, where the initial $Q \sim Q^*$ adds the necessary $+1$ in comparison with Definition 4.7.

Theorem 4.9 can thus be reformulated as

Corollary 5.4 (*Theorem 4.9 revisited*). *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. The numbers of fixed points for each irreducible $\mathcal{F}$-stable set α_P can be calculated as*

$$\left|(\alpha_P)^Q\right| = \left|W_S P^*\right| \cdot \sum_{\substack{\sigma=(\sigma_0,\ldots,\sigma_k)\in\mathfrak{BC}_{\mathcal{F}}(Q,P) \\ |\sigma_0|=0}} (-1)^{\ell(\sigma)}$$

for $Q, P \in \mathcal{P}$, where $P^* \sim_{\mathcal{F}} P$ is fully normalized.

Example 5.5. Let $S = D_8$ and $\mathcal{F} = \mathcal{F}_S(A_6)$ as before. We showed earlier that $c_Q(\alpha_P) = 1$ when $Q = C_2^1$ and $P = V_4^2$. Note that in this case $|W_S P^*| = |W_S Q| = 2$ and there is only one broken chain from C_2^1 to V_4^2 which is

$$
\begin{array}{c}
C_2^1 \\
\wr \\
Z \;\; < \;\; V_4^2
\end{array}
$$

Note that this is also a tethered broken chain. So we have $|(\alpha_P)^Q| = |W_S P^*| \cdot 1 = 2$ for $Q = C_2^1$ and $P = V_4^2$.

If we repeat the same calculation for $Q = C_2^1$ and $P = D_8$, then we observe that there are 10 broken chains from C_2^1 to D_8 which are

$$
\begin{array}{ccc}
C_2^1 \;\; < \;\; D_8 & C_2^1 & C_2^1 \;\; < \;\; V_4^1 \\
& \wr & \wr \\
C_2^1 \;\; < \;\; V_4^1 \;\; < \;\; D_8 & Z \;\; < \;\; D_8 & V_4^1 \;\; < \;\; D_8
\end{array}
$$

$$
\begin{array}{ccc}
C_2^1 & C_2^1 & C_2^1 \\
\wr & \wr & \wr \\
Z \;\; < \;\; V_4^1 \;\; < \;\; D_8 & Z \;\; < \;\; C_4 \;\; < \;\; D_8 & Z \;\; < \;\; V_4^2 \;\; < \;\; D_8
\end{array}
$$

$$
\begin{array}{ccc}
C_2^1 & C_2^1 & C_2^1 \\
\wr & \wr & \wr \\
Z \;\; < \;\; V_4^1 & Z \;\; < \;\; C_4 & Z \;\; < \;\; V_4^2 \\
\wr & \wr & \wr \\
V_4^1 \;\; < \;\; D_8 & C_4 \;\; < \;\; D_8 & V_4^2 \;\; < \;\; D_8
\end{array}
$$

If we sum the signs $(-1)^{\ell(\sigma)}$ over all the broken chains above, and multiply it with $|W_S P^*|/|W_S Q|$, we get

$$
c_Q(\alpha_P) = \frac{1}{2}(1 - 2 + 4 - 3) = 0.
$$

Note that if we only consider the tethered broken chains, then we obtain

$$
\left|(\alpha_P)^Q\right| = \left|W_S P^*\right|(1 - 3 + 3) = 1.
$$

Note that in the above example some of the broken chains naturally can be paired with each other to cancel their contributions. For example, all the broken chains on the

second row cancels with the broken chains on the third row. In the next section we prove that the broken chain calculations for calculating $c_Q(\alpha_P)$ and $|(\alpha_P)^Q|$ can be simplified.

6. Computational simplifications

In this section, we show that certain types of broken chains can be naturally paired with certain other types of broken chains in such a way that their contributions in the summation in Theorem 5.2 cancel each other. This gives a modified version of the formula in Theorem 5.2 where we only consider broken chains which are not in either type. We start with a definition of these types.

Definition 6.1. Let $\sigma = (\sigma_0, \ldots, \sigma_k)$ be a broken chain in $\mathcal{F}$ with $\sigma_i = (a_0^i, \ldots, a_{n_i}^i)$. Suppose that a subgroup a_j^i in the broken chain is S-conjugate to the chosen representative $(a_j^i)^* \in \mathcal{P}^*$. We say that such an a_j^i is *a $*$-group of type* 1 if $0 < j < n_i$, or if $i = j = 0$ and $n_0 > 0$. We say that a_j^i is *a $*$-group of type* 2 if $j = n_i$ and $0 \leq i < k$. In the remaining cases we either have $j = 0$ and $i > 0$, in which case $a_j^i \in \mathcal{P}^*$ is always required, or we have $i = k$ and $j = n_k$ with a_j^i as the very last group. In either of these last cases, a_j^i is *not* a $*$-group.

In diagram form the two types of $*$-groups are as follows:

$$
\begin{array}{ccccc}
\ddots & & & & \\
& \cdots \; < \; (a_j^i)^* \; < \; \cdots & & \text{or} \qquad (a_0^0)^* \; < \; \cdots & \text{(Type 1)} \\
& & \ddots & & \ddots
\end{array}
$$

$$
\begin{array}{ccccc}
\ddots & & & & \\
& \cdots \; < \; (a_j^i)^* & & \text{or} \qquad (a_0^0)^* & \text{(Type 2)} \\
& \wr & & \wr & \\
& a_0^{i+1} \; < \; \cdots & & a_0^1 \; < \; \cdots & \\
& & \ddots & & \ddots
\end{array}
$$

If a broken chain σ contains at least one $*$-group, we say that σ is *sparkling of type* 1 *or* 2 where the type of σ is determined by the type of the smallest $*$-group in σ. A broken chain is *drab* if it has no $*$-groups at all.

Example 6.2. Consider the last calculation in Example 5.5, where $Q = C_2^1$ and $P = D_8$. The broken chains on the second row are all sparkling of type 1. More specifically in all these, the second chains include $*$-groups of type 1 which are V_4^1, C_4, and V_4^2. Note also that the second broken chain on the first row is a sparkling broken chain of the type 1. The fourth chain on the first row and all the chains on the third row are sparkling broken chains of type 2. We will see below that these type 1 and type 2 chains can be paired

in an obvious way. The only drab broken chains in this example are the first and third broken chains on the first row.

Proposition 6.3. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. In calculating the coefficients $c_Q(\alpha_P)$ by [Theorem 5.2](#), it is sufficient to consider only the drab broken chains:*

$$c_Q(\alpha_P) = \frac{|W_S P^*|}{|W_S Q|} \cdot \sum_{\substack{\sigma \in \mathfrak{BC}_\mathcal{F}(Q,P) \\ \sigma \text{ is drab}}} (-1)^{\ell(\sigma)}$$

for $Q, P \in \mathcal{P}$, where $P^* \sim_\mathcal{F} P$ is fully normalized.

Proof. By [Theorem 5.2](#) we have

$$c_Q(\alpha_P) = \frac{|W_S P^*|}{|W_S Q|} \cdot \sum_{\sigma \in \mathfrak{BC}_\mathcal{F}(Q,P)} (-1)^{\ell(\sigma)}$$

for $Q, P \in \mathcal{P}$, where $P^* \sim_\mathcal{F} P$ is fully normalized. For each subgroup $R \leq S$ we will consider all the sparkling broken chains that have R as their smallest $*$-group and links Q to P. For each R we will show that these broken chains cancel each other in the sum above, leaving only the drab broken chains at the end. In order for R to be a $*$-group at all, R must be S-conjugate to the chosen representative $R^* \in \mathcal{P}^*$. We can therefore choose an $s \in S$ such that $^s R = R^*$, and we let s be fixed for the remainder of the proof.

Let $\sigma \in \mathfrak{BC}_\mathcal{F}(Q,P)$ be a broken chain with R as its smallest $*$-group. Suppose $\sigma = (\ldots, \sigma_*, \ldots)$ where σ_* is the chain containing R as a $*$-group. If R is at the end of σ_*, then σ is type 2, otherwise σ is type 1.

If σ is type 1, then we write $\sigma_* = (A_0, \ldots, A_{m-1}, R, B_1, \ldots, B_n)$ where $n \geq 1$. We can then conjugate the entire second part of the chain with s to get subgroups $C_i := {}^s B_i$. These form a chain $(R^*, C_1, \ldots, C_n)$ which starts at $R^* \in \mathcal{P}^*$ and has length at least 1 (see the illustration [(6.1)](#) below). We also have $C_n \sim_S B_n \sim_\mathcal{F} B_n^*$, so we can "break" σ_* at R into two chains and get a legal broken chain $\sigma' := (\ldots, (A_0, \ldots, A_{m-1}, R), (R^*, C_1, \ldots, C_n), \ldots)$ where we don't change any other part of σ. The new broken chain σ' is type 2 with R as its smallest $*$-group. Since σ' has one extra break compared to σ, $\ell(\sigma') = \ell(\sigma) + 1$.

If alternatively σ has type 2, we write $\sigma_* = (A_0, \ldots, A_{m-1}, R)$ and let $(R^*, C_1, \ldots, C_n)$ be the chain of σ that follows σ_* (such a chain exists since R is not the very last group of σ). We conjugate every C_i with s from the right $B_i := C_i^s$, and they form a chain $(R, B_1, \ldots, B_n)$ starting at R and satisfying $B_n \sim_S C_n \sim_\mathcal{F} C_n^*$. We can then combine σ_* with the B_i-chain to get a single chain, and a new broken chain $\sigma' := (\ldots, (A_0, \ldots, A_{m-1}, R, B_1, \ldots, B_n), \ldots)$ of type 1 with R as its smallest $*$-group. We also have $\ell(\sigma') = \ell(\sigma) - 1$.

The two operations are inverses to each other and are illustrated below:

$$
\begin{array}{c}
\ddots \\
\wr \\
A_0 \; < \; \cdots \; < \; A_{m-1} \; < \; R \; < \; B_1 \; < \; \cdots \; < \; B_n \\
\end{array}
\qquad \text{Type 1} \tag{6.1}
$$

$$
\wr s \qquad \wr s \qquad \qquad \wr s
$$

$$
R^* \; < \; C_1 \; < \; \cdots \; < \; C_n
$$

$$
\text{Type 2} \qquad\qquad\qquad \wr
$$

$$
\ddots
$$

Because any two corresponding broken chains have lengths that differ by 1, they cancel in the sum of Theorem 5.2. $\quad\square$

Another way to reduce the number of terms in the sum of Theorem 5.2, is to limit the sizes of the individual chains in a broken chain. This stems from the fact that the usual Möbius function for subgroups of p-groups has $\mu(A, B) = 0$ unless $B \leq N_S A$ with B/A elementary abelian (see [7, Corollary 3.5], [8, Proposition 2.4]).

Proposition 6.4. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. In calculating the coefficients $c_Q(\alpha_P)$ by Theorem 5.2, it is sufficient to consider only broken chains $(\sigma_0, \ldots, \sigma_k)$ where every $\sigma_i = (a_0^i, \ldots, a_{n_i}^i)$ has $a_{n_i}^i \leq N_S(a_0^i)$ with $a_{n_i}^i/a_0^i$ elementary abelian. Therefore, we have*

$$
c_Q(\alpha_P) = \frac{|W_S P^*|}{|W_S Q|} \cdot \sum_{\substack{\sigma = ((a_j^i)_{j=0}^{n_i})_{i=0}^k \in \mathfrak{B}\mathfrak{C}_{\mathcal{F}}(Q, P), \\ \text{s.t. each } a_{n_i}^i/a_0^i \text{ is elm.ab.}}} (-1)^{\ell(\sigma)}
$$

for $Q, P \in \mathcal{P}$, where $P^* \sim_{\mathcal{F}} P$ is fully normalized.

Proof. In the proof of Theorem 5.2 we consider the sum

$$
c_Q(\alpha_P) = \frac{|W_S P^*|}{|W_S Q|} \sum_{\substack{R_0, R_1, \ldots, R_k \in \mathcal{P} \\ \text{s.t. } R_k \sim_{\mathcal{F}} P^*, \\ |Q| \leq |R_0| < |R_1| < \cdots < |R_k|}} (-1)^k \mu(Q, R_0) \mu(R_0^*, R_1) \mu(R_1^*, R_2) \cdots \mu(R_{k-1}^*, R_k)
$$

A term of this sum is only nonzero if $Q \lhd R_0$ and $R_{i-1} \lhd R_i$ with elementary abelian quotients for all i. Hence the sum reduces to

$$
\begin{aligned}
&c_Q(\alpha_P) \\
&= \frac{|W_S P^*|}{|W_S Q|} \sum_{\substack{R_0, R_1, \ldots, R_k \in \mathcal{P} \\ \text{s.t. } R_k \sim_{\mathcal{F}} P^*, \\ |Q| \leq |R_0| < |R_1| < \cdots < |R_k|, \\ R_0/Q \text{ and } R_i/R_{i-1} \text{ are elm.ab.}}} (-1)^k \mu(Q, R_0) \mu(R_0^*, R_1) \mu(R_1^*, R_2) \cdots \mu(R_{k-1}^*, R_k)
\end{aligned}
$$

As in the proof of Theorem 5.2 we then replace each product of Möbius functions by broken chains and arrive at the formula in the proposition. □

Remark 6.5. Sadly the two reductions of Propositions 6.3 and 6.4 cannot be combined, as that would require canceling the same broken chain with two different other broken chains. To see this, let $\mathcal{F} = \mathcal{F}_{D_8}(A_6)$ be as in Example 3.2, where we showed that $\alpha_{D_8} = [S/D_8]$. Let us show that if we exclude both the sparkling broken chains and those that violate the hypothesis of Proposition 6.4, then we would not be able to compute the coefficient of the orbit $[S/C_2^1]$ in α_{D_8} correctly.

As it is listed in Example 5.5, there are a total of 10 broken chains linking C_2^1 to D_8. Of these, only $(C_2^1 < D_8)$ and $(C_2^1, Z < D_8)$ are drab, and of those, only the second would be counted in Proposition 6.4. Thus there is no chance for cancelation, and the intersections of Propositions 6.3 and 6.4 would yield $c_{C_2^1}(\alpha_{D_8}) = 1/2$, which is obviously false. The issue is that there can be cancelation between sparkling subgroups and subgroups that violate the hypothesis of Proposition 6.4, so that by combining both conditions we may undercount the cancelations needed in the proof of Theorem 5.2.

7. An application to characteristic bisets

In this section we demonstrate how we can use Theorem 5.2 to give structural results for the minimal characteristic biset associated to a saturated fusion system.

Definition 7.1. We consider (S, S)-bisets, i.e. finite sets equipped with both a left S-action and a right S-action, and such that the actions commute. The structure of such a biset X is equivalent to an action of $S \times S$ on X with $(s_1, s_2).x = s_1.x.(s_2)^{-1}$, and for each point $x \in X$ we speak of the stabilizer $Stab_{S \times S}(x)$ as a subgroup of $S \times S$.

An $\mathcal{F}$-*characteristic biset* for a fusion system $\mathcal{F}$ on S is a biset Ω satisfying three properties originally suggested by Linckelmann and Webb:

(i) For every point $\omega \in \Omega$ the stabilizer $Stab_{S \times S}(\omega)$ has the form of a *graph/twisted diagonal* $\Delta(P, \varphi)$ for some $\varphi \in \mathcal{F}(P, S)$ and $P \leq S$, where the twisted diagonal $\Delta(P, \varphi) \leq S \times S$ is defined as

$$\Delta(P, \varphi) = \big\{ \big(\varphi(s), s\big) \mid s \in P \big\}.$$

(ii) Ω is $\mathcal{F}$-stable with respect to both S-actions. For bisets that satisfy property (i) this boils down to checking that the number of fixed points satisfy

$$\big| \Omega^{\Delta(P, id)} \big| = \big| \Omega^{\Delta(P, \varphi)} \big| = \big| \Omega^{\Delta(\varphi P, id)} \big|$$

for all $\varphi \in \mathcal{F}(P, S)$ and $P \leq S$.

(iii) The prime p does not divide $|\Omega|/|S|$ (which is an integer because of (i)). This ensures that Ω is not degenerate.

In [9] it is shown that there exists a characteristic biset for $\mathcal{F}$ if and only if $\mathcal{F}$ is saturated, and it is shown how to reconstruct $\mathcal{F}$ given any $\mathcal{F}$-characteristic biset. In [5] two of the authors of this paper give a parametrization of all the characteristic bisets for a given saturated fusion system $\mathcal{F}$. In particular it is shown that there is a unique minimal $\mathcal{F}$-characteristic biset $\Lambda_{\mathcal{F}}$, and every other $\mathcal{F}$-characteristic biset contains at least one copy of $\Lambda_{\mathcal{F}}$.

Theorem 7.2. *(See [5, Theorem 5.3 and Corollary 5.4].) Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S, and consider the product fusion system $\mathcal{F} \times \mathcal{F}$ on $S \times S$. According to Proposition 2.2 there is an irreducible $(\mathcal{F} \times \mathcal{F})$-stable $(S \times S)$-set $\alpha_{\Delta(S,id)}$ corresponding to the diagonal $\Delta(S, id) \leq S \times S$. Denote this $(S \times S)$-set or (S,S)-biset by $\Lambda_{\mathcal{F}} := \alpha_{\Delta(S,id)}$.*

The biset $\Lambda_{\mathcal{F}}$ is then $\mathcal{F}$-characteristic, and every $\mathcal{F}$-characteristic biset contains a copy of $\Lambda_{\mathcal{F}}$ (up to isomorphism). Hence $\Lambda_{\mathcal{F}}$ is the unique minimal characteristic biset for $\mathcal{F}$.

In order to apply Theorem 5.2 to study $\Lambda_{\mathcal{F}}$ we need to figure out what broken chains look like in the context of bisets and the fusion system $\mathcal{F} \times \mathcal{F}$.

In a product fusion system the conjugation is defined coordinate-wise. Hence two twisted diagonals $\Delta(P, \varphi)$ and $\Delta(P', \varphi')$ are conjugate in $\mathcal{F} \times \mathcal{F}$ if and only if there are additional isomorphisms $\psi, \rho \in \mathcal{F}$ such that $\varphi' = \psi \circ \varphi \circ \rho^{-1}$. Consequently, every $\Delta(P, \varphi)$ with $\varphi \in \mathcal{F}(P, S)$ is conjugate to $\Delta(P, id)$ which is conjugate to $\Delta(P', id)$ for all $P' \sim_{\mathcal{F}} P$. In addition the subgroups of $S \times S$ that are subconjugate to $\Delta(S, id)$ in $\mathcal{F} \times \mathcal{F}$ are precisely all the twisted diagonals $\Delta(P, \varphi)$ with $\varphi \in \mathcal{F}(P, S)$ and $P \leq S$. To study $\Lambda_{\mathcal{F}} = \alpha_{\Delta(S,id)}$ we therefore have to consider broken chains where all the groups are twisted diagonals coming from maps in $\mathcal{F}$.

Two twisted diagonals satisfy $\Delta(Q, \psi) \leq \Delta(P, \varphi)$ exactly when φ extends ψ, i.e. $Q \leq P$ and $\psi = \varphi|_Q$. Every $(\mathcal{F} \times \mathcal{F})$-conjugacy class of twisted diagonals contains a fully normalized representative on the form $\Delta(P^*, id)$ where P^* is fully $\mathcal{F}$-normalized, suppose for Theorem 5.2 that we have chosen such at fully normalized representative $\Delta(P^*, id)$ for each conjugacy class. The broken chains that we consider are chains of inclusions connected by $(\mathcal{F} \times \mathcal{F})$-conjugations.

- Every chain of inclusions $\Delta(P_1, \varphi_1) \leq \cdots \leq \Delta(P_k, \varphi_k)$ is a sequence of extensions with $\varphi_i = \varphi_k|_{P_i}$.
- Every chain (except for the 0'th chain) starts with a diagonal of the form $\Delta(P^*, id)$ where P^* a fully normalized representative for the $\mathcal{F}$-conjugacy class.

With this insight we can now apply Theorem 5.2 and relate $\Lambda_{\mathcal{F}}$ to the largest normal subgroup in $\mathcal{F}$. Here normality is in the sense of [1, Definition 4.3] where $P \leq S$ is normal in $\mathcal{F}$ if every homomorphism $\varphi \in \mathcal{F}(Q, R)$ extends to some $\widetilde{\varphi} \in \mathcal{F}(QP, RP)$ with $\widetilde{\varphi}(P) = P$. For each fusion system $\mathcal{F}$ there is a largest normal subgroup, denoted $O_p(\mathcal{F})$.

Proposition 7.3. *(See [5, Proposition 9.11].) Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S, and let $\Lambda_\mathcal{F}$ be the minimal characteristic biset for $\mathcal{F}$. Denote by $O_p(\mathcal{F})$ the largest normal subgroup of $\mathcal{F}$. Then for each point $\omega \in \Lambda_\mathcal{F}$ the stabilizer $\mathrm{Stab}_{S \times S}(\omega) = \Delta(P, \varphi)$ satisfies $P \geq O_p(\mathcal{F})$.*

The original proof in [5] is quite involved. In contrast the proof below, using broken chains, is actually quite elementary once you have the idea of pairing broken chains of opposite sign together.

Proof. Let $\Delta(R, \rho)$ with $\rho \in \mathcal{F}(R, S)$ be such that R does not contain $O_p(\mathcal{F})$. We then wish to show that $c_{\Delta(R,\rho)}(\Lambda_\mathcal{F}) = 0$. Because $\Lambda_\mathcal{F} = \alpha_{\Delta(S,id)}$, we can apply Theorem 5.2 and consider all $(\mathcal{F} \times \mathcal{F})$-broken chains linking $\Delta(R, \rho)$ with $\Delta(S, id)$:

$$c_{\Delta(R,\rho)}(\Lambda_\mathcal{F}) = \sum_{\sigma \in \mathfrak{BC}_{\mathcal{F} \times \mathcal{F}}(\Delta(R,\rho),\Delta(S,id))} (-1)^{\ell(\sigma)}. \tag{7.1}$$

We will then show that all these broken chains cancel in pairs of two broken chains with opposite signs.

Consider a broken chain $\sigma \in \mathfrak{BC}_{\mathcal{F} \times \mathcal{F}}(\Delta(R, \rho), \Delta(S, id))$. Since S contains $O_p(\mathcal{F})$ and R does not, there is a first twisted diagonal $\Delta(P, \varphi)$ in σ with $P \geq O_p(\mathcal{F})$, and necessarily $P > R$. Note that $\Delta(P, \varphi)$ cannot be in the beginning of any chain in σ, since normality of $O_p(\mathcal{F})$ implies that the end of the previous chain would also contain $O_p(\mathcal{F})$.

Let $\Delta(Q, \psi)$ be the twisted diagonal coming just before $\Delta(P, \varphi)$ in σ. Because $O_p(\mathcal{F})$ is normal in $\mathcal{F}$, hence also in S and P, the product $Q \cdot O_p(\mathcal{F})$ is a well-defined subgroup of P. If we restrict φ to $QO_p(\mathcal{F})$ we then have inclusions

$$\Delta(Q, \psi) < \Delta\big(QO_p(\mathcal{F}), \varphi\big) \leq \Delta(P, \varphi).$$

If $P \neq QO_p(\mathcal{F})$, then the broken chain σ looks like

$$\ddots \quad \Delta(Q,\psi) \;<\; \Delta(P,\varphi) \qquad , \\ \ddots$$

and we can add $\Delta(QO_p(\mathcal{F}), \varphi)$ in the middle to make the broken chain one step longer. Conversely, if $P = QO_p(\mathcal{F})$, and if $\Delta(P, \varphi) = \Delta(QO_p(\mathcal{F}), \varphi)$ is not at the end of a chain in σ, then σ looks like

$$\ddots \quad \Delta(Q,\psi) \;<\; \Delta(QO_p(\mathcal{F}),\varphi) \;<\; \Delta(T,\eta) \qquad , \\ \ddots$$

and we can remove $\Delta(P, \varphi) = \Delta(QO_p(\mathcal{F}), \varphi)$ to make the broken chain one step shorter. These two constructions are inverse to each other, hence the broken chains with $P \neq QO_p(\mathcal{F})$ are paired with the broken chains where $P = QO_p(\mathcal{F})$ and $\Delta(P, \varphi)$ is not at the end of a chain, and the pairing is such that the total length changes by 1. Hence these broken chains cancel each other in (7.1), and we are left with the broken chains σ where $\Delta(P, \varphi) = \Delta(QO_p(\mathcal{F}), \varphi)$ is at the end of a chain in σ.

All the remaining broken chains look like

$$
\begin{array}{ccc}
\iddots & & \\
\Delta(Q, \psi) & < & \Delta(QO_p(\mathcal{F}), \varphi) \\
& \wr & \\
& \Delta(T, \eta) & < \quad \cdots \\
& & \ddots
\end{array}
$$

Let σ_i be the chain in σ that contains the segment $\Delta(Q, \psi) < \Delta(QO_p(\mathcal{F}), \varphi)$. We divide the remaining broken chains into two types: Those broken chains where σ_i consists only of $\Delta(Q, \psi) < \Delta(QO_p(\mathcal{F}), \varphi)$ and has $i \geq 1$; we call these Type A. The remaining broken chains form Type B, i.e. the broken chains where the chain σ_i contains twisted diagonals before $\Delta(Q, \psi)$, or where $i = 0$. We will finish the proof by canceling broken chains of Type A with those of Type B and vice versa.

For each possible choice of Q, there is a chosen representative $Q^* \sim_{\mathcal{F}} Q$ such that $\Delta(Q^*, id)$ is fully normalized in the $(\mathcal{F} \times \mathcal{F})$-conjugacy class of $\Delta(Q, id)$ and $\Delta(Q, \psi)$ for all $\psi \in \mathcal{F}(Q, S)$. For each $Q \sim_{\mathcal{F}} Q^*$, we choose a particular $\mathcal{F}$-isomorphism $\chi_Q : Q \to Q^*$. For each $Q \sim_{\mathcal{F}} Q^*$ and each homomorphism $\psi \in \mathcal{F}(Q, S)$, we also make the choice of an extension $\widetilde{\psi} : QO_p(\mathcal{F}) \to \psi(Q)O_p(\mathcal{F})$ in $\mathcal{F}$ such that $\widetilde{\psi}|_Q = \psi$ and $\widetilde{\psi}(O_p(\mathcal{F})) = O_p(\mathcal{F})$. In particular, we have isomorphisms $\widetilde{\chi}_Q : QO_p(\mathcal{F}) \to Q^*O_p(\mathcal{F})$.

If σ is Type A, then by the definition of broken chains we must have $\Delta(Q, \psi) = \Delta(Q^*, id)$ because it is the start of the chain σ_i and $i \geq 1$. Hence σ looks like

$$
\begin{array}{ccc}
\iddots & & \\
\Delta(Q', \psi') & & \\
\wr & & \\
\Delta(Q^*, id) & < & \Delta(Q^*O_p(\mathcal{F}), \varphi) \qquad \qquad \text{(Type A)} \\
& \wr & \\
& \Delta(T, \eta) & < \quad \cdots \\
& & \ddots
\end{array}
$$

We pair this with the following chain of Type B and a total length that has decreased by 1:

$$\begin{array}{c}
\ddots \\
\Delta(Q', \psi') \;<\; \Delta(Q'O_p(\mathcal{F}), \widetilde{\psi'} \circ (\widetilde{\chi}_{Q'})^{-1} \circ \varphi \circ \widetilde{\chi}_{Q'}) \qquad\qquad \text{(Type B)} \\
\wr \\
\Delta(T, \eta) \qquad\qquad < \qquad\qquad \cdots \\
\ddots
\end{array}$$

Conversely, if σ is Type B, then it has the shape

$$\begin{array}{c}
\ddots \\
\Delta(Q, \psi) \;<\; \Delta(QO_p(\mathcal{F}), \varphi) \qquad\qquad\qquad \text{(Type B)} \\
\wr \\
\Delta(T, \eta) \quad < \quad \cdots \\
\ddots
\end{array}$$

and we can split σ_i into two chains, thereby increasing the total length by 1:

$$\begin{array}{c}
\ddots \\
\Delta(Q, \psi) \\
\wr \\
\Delta(Q^*, id) \;<\; \Delta(Q^*O_p(\mathcal{F}), \widetilde{\chi}_Q \circ (\widetilde{\psi})^{-1} \circ \varphi \circ (\widetilde{\chi}_Q)^{-1}) \qquad\qquad \text{(Type A)} \\
\wr \\
\Delta(T, \eta) \qquad\qquad < \qquad\qquad \cdots \\
\ddots
\end{array}$$

When we split σ_i this way, $\Delta(Q, \psi)$ is still in a chain of length at least 1 if $i \geq 1$, and if $i = 0$, then $\Delta(Q, \psi)$ is allowed to form a trivial chain by itself.

This completes the proof as all the remaining broken chains of Type A cancel in (7.1) with all those of Type B. □

Acknowledgments

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