Entropy, invertibility and variational calculus of adapted shifts on Wiener space

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Abstract

In this work we study the necessary and sufficient conditions for a positive random variable whose expectation under the Wiener measure is one, to be represented as the Radon–Nikodym derivative of the image of the Wiener measure under an adapted perturbation of identity with the help of the associated innovation process. We prove that the innovation conjecture holds if and only if the original process is almost surely invertible. We also give variational characterizations of the invertibility of the perturbations of identity and the representability of a positive random variable whose total mass is equal to unity. We prove in particular that an adapted perturbation of identity $U = I_W + u$ satisfying the Girsanov theorem, is invertible if and only if the kinetic energy of u is equal to the entropy of the measure induced with the action of U on the Wiener measure μ, in other words U is invertible iff

$$\frac{1}{2} \int_W |u|_H^2 \, d\mu = \int_W \frac{dU\,\mu}{d\mu} \log \frac{dU\,\mu}{d\mu} \, d\mu.$$

The relations with the Monge–Kantorovitch measure transportation are also studied. An application of these results to a variational problem related to large deviations is also given.
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Keywords: Entropy; Invertibility; Monge transportation; Malliavin calculus; Calculus of variations; Large deviations

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Contents

1. Introduction

This paper is devoted to the study of the following question: assume that (W, H, μ) is the classical Wiener space, i.e., $W = C_0([0, 1], \mathbb{R}^d)$, H is the corresponding Cameron–Martin space consisting of the absolutely continuous, $\mathbb{R}^d$-valued functions on $[0, 1]$ with square integrable derivatives. Assume that L is a strictly positive random variable whose expectation with respect to μ is one. We suppose that there exits a map $U : W \to W$ of the form $U = I_W + u$, with $u : W \to H$ such that $\dot{u}$ is adapted to the filtration of the Wiener space and that L is represented by U, i.e.

$$\frac{dU\mu}{d\mu} = L.$$

We suppose also that

$$E[\rho(-\delta u)] = 1,$$

where

$$\rho(-\delta u) = \exp\left[-\int_0^1 (\dot{u}_s, dW_s) - \frac{1}{2} \int_0^1 |\dot{u}_s|^2 \, ds \right].$$

Then $U\mu$ is equivalent to μ and the corresponding Radon–Nikodym derivative L can be represented as an exponential martingale $\rho(-\delta v)$ where $v : W \to H$ satisfies similar properties as those satisfied by u. The question we address is: what are the relations satisfied by the couple (u, v)? For instance, if U and $V = I_W + v$ are inverse to each other then the situation described above happens. However, due to the celebrated example of Tsirelson (cf. [11]), we know that this is not the only case. We concentrate ourselves particularly to this case with the help of associated innovation processes, in terms of which we give necessary and sufficient conditions for the representability (cf. [6]) of a strictly positive density and for the invertibility of the associated

perturbation of identity. The innovation approach leads to a nice result which characterizes the invertibility of an adapted shift in terms of the relative entropy of the measure which it induces. Namely, assume that $U = I_W + u$ as above, then it is invertible if and only if the relative entropy $H(U\mu \mid \mu)$ is equal to the kinetic energy of u, i.e.,

$$H\big(U\mu \mid \mu\big) = \frac{1}{2} E \int_0^1 |\dot{u}_s|^2 \, ds.$$

In Physics the notion of entropy is an indication for the number of accessible states; here it is a remarkable fact that the relative entropy behaves as the physical entropy in the sense that if the system has just enough kinetic energy to fulfill the accessible states, i.e., if this energy is equal to the relative entropy of the probability distribution that it creates then the mapping is invertible. Besides, in general it is always larger or equal to the latter.

We apply this considerations to the innovation problem of the filtering. Namely it is a celebrated question whether the sigma algebra generated by the observation process is equal to that of the innovation process. The case where the signal is independent of the noise has been solved in [1], here we solve the innovation problem in full generality in terms of the entropy of the observed system.

If we represent a density of the form $L = \rho(-\delta v)$ by $U = I_W + u$, then, modulo some integrability hypothesis, the Girsanov theorem implies that $(I_W + v) \circ U = V \circ U$ is a Wiener process. We study then the properties of $U \circ V$ using similar techniques. The relations with the Monge transportation are also exhibited.

In the final part we use the variational methods to characterize the invertibility and representability of densities. As an application we give some new results for a particular case studied in [2]. Namely we give an explicit characterization of the solution of the minimization problem

$$\inf\left(E\left[f \circ U + \frac{1}{2} |u|_H^2 \right] \right),$$

with the help of the entropic characterization of the invertibility explained above, where the inf is taken in the space of adapted, H-valued Wiener functionals with finite energy and f is a 1-convex Wiener functional in the Sobolev space $\mathbb{D}_{2,1}(H)$.

2. Preliminaries and notation

Let W be the classical Wiener space with the Wiener measure μ. The corresponding Cameron–Martin space is denoted by H. Recall that the injection $H \hookrightarrow W$ is compact and its adjoint is the natural injection $W^\star \hookrightarrow H^\star \subset L^2(\mu)$. A subspace F of H is called regular if the corresponding orthogonal projection has a continuous extension to W, denoted again by the same letter. It is well known that there exists an increasing sequence of regular subspaces $(F_n, \ n \geqslant 1)$, called total, such that $\bigcup_n F_n$ is dense in H and in W. Let $\sigma(\pi_{F_n})^1$ be the σ-algebra generated by π_{F_n}, then for any $f \in L^p(\mu)$, the martingale sequence $(E[f \mid \sigma(\pi_{F_n})], \ n \geqslant 1)$ converges to f (strongly if $p < \infty$) in $L^p(\mu)$. Observe that the function $f_n = E[f \mid \sigma(\pi_{F_n})]$ can be identified with a function on the finite dimensional abstract Wiener space (F_n, μ_n, F_n), where $\mu_n = \pi_n \mu$.

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[1] For the notational simplicity, in the sequel we shall denote it by π_n.

Since the translations of μ with the elements of H induce measures equivalent to μ, the Gâteaux derivative in H direction of the random variables is a closable operator on $L^p(\mu)$-spaces and this closure will be denoted by ∇ cf., for example [3,12,13]. The corresponding Sobolev spaces (the equivalence classes) of the real random variables will be denoted as $\mathbb{D}_{p,k}$, where $k \in \mathbb{N}$ is the order of differentiability and $p > 1$ is the order of integrability. If the random variables are with values in some separable Hilbert space, say Φ, then we shall define similarly the corresponding Sobolev spaces and they are denoted as $\mathbb{D}_{p,k}(\Phi)$, $p > 1$, $k \in \mathbb{N}$. Since $\nabla : \mathbb{D}_{p,k} \to \mathbb{D}_{p,k-1}(H)$ is a continuous and linear operator its adjoint is a well-defined operator which we represent by δ. δ coincides with the Itô integral of the Lebesgue density of the adapted elements of $\mathbb{D}_{p,k}(H)$ (cf. [12,13]).

For any $t \geqslant 0$ and measurable $f : W \to \mathbb{R}_+$, we note by

$$P_t f(x) = \int_W f\left(e^{-t}x + \sqrt{1 - e^{-2t}}\, y\right) \mu(dy),$$

it is well known that $(P_t, \ t \in \mathbb{R}_+)$ is a hypercontractive semigroup on $L^p(\mu)$, $p > 1$, which is called the Ornstein–Uhlenbeck semigroup (cf. [3,12,13]). Its infinitesimal generator is denoted by $-\mathcal{L}$ and we call $\mathcal{L}$ the Ornstein–Uhlenbeck operator (sometimes called the number operator by the physicists). The norms defined by

$$\|\phi\|_{p,k} = \left\|(I + \mathcal{L})^{k/2}\phi\right\|_{L^p(\mu)} \tag{2.1}$$

are equivalent to the norms defined by the iterates of the Sobolev derivative ∇. This observation permits us to identify the duals of the space $\mathbb{D}_{p,k}(\Phi)$; $p > 1$, $k \in \mathbb{N}$ by $\mathbb{D}_{q,-k}(\Phi')$, with $q^{-1} = 1 - p^{-1}$, where the latter space is defined by replacing k in (2.1) by $-k$, this gives us the distribution spaces on the Wiener space W (in fact we can take as k any real number). An easy calculation shows that, formally, $\delta \circ \nabla = \mathcal{L}$, and this permits us to extend the divergence and the derivative operators to the distributions as linear, continuous operators. In fact $\delta : \mathbb{D}_{q,k}(H \otimes \Phi) \to \mathbb{D}_{q,k-1}(\Phi)$ and $\nabla : \mathbb{D}_{q,k}(\Phi) \to \mathbb{D}_{q,k-1}(H \otimes \Phi)$ continuously, for any $q > 1$ and $k \in \mathbb{R}$, where $H \otimes \Phi$ denotes the completed Hilbert–Schmidt tensor product (cf., for instance [8,12,13]). Finally, in the case of classical Wiener space, we denote by $\mathbb{D}_{p,k}^a(H)$ the subspace defined by

$$\mathbb{D}_{p,k}^a(H) = \left\{\xi \in \mathbb{D}_{p,k}(H): \dot{\xi} \text{ is adapted}\right\}$$

for $p \geqslant 1$, $k \in \mathbb{R}$.

Let us recall some facts from the convex analysis. Let K be a Hilbert space, a subset S of $K \times K$ is called cyclically monotone if any finite subset $\{(x_1, y_1), \ldots, (x_N, y_N)\}$ of S satisfies the following algebraic condition:

$$\langle y_1, x_2 - x_1 \rangle + \langle y_2, x_3 - x_2 \rangle + \cdots + \langle y_{N-1}, x_N - x_{N-1} \rangle + \langle y_N, x_1 - x_N \rangle \leqslant 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of K. It turns out that S is cyclically monotone if and only if

$$\sum_{i=1}^{N} \langle y_i, x_{\sigma(i)} - x_i \rangle \leqslant 0,$$

for any permutation σ of $\{1, \ldots, N\}$ and for any finite subset $\{(x_i, y_i): i = 1, \ldots, N\}$ of S. Note that S is cyclically monotone if and only if any translate of it is cyclically monotone. By a theorem of Rockafellar, any cyclically monotone set is contained in the graph of the subdifferential of a convex function in the sense of convex analysis [9] and even if the function may not be unique its subdifferential is unique.

Let now (W, μ, H) be an abstract Wiener space; a measurable function $f : W \to \mathbb{R} \cup \{\infty\}$ is called 1-convex if the map

$$h \to f(x + h) + \frac{1}{2}|h|_H^2 = F(x, h)$$

is convex on the Cameron–Martin space H with values in $L^0(\mu)$. Note that this notion is compatible with the μ-equivalence classes of random variables thanks to the Cameron–Martin theorem. It is proven in [4] that this definition is equivalent the following condition: Let $(\pi_n, \ n \geqslant 1)$ be a sequence of regular, finite dimensional, orthogonal projections of H, increasing to the identity map I_H. Denote also by π_n its continuous extension to W and define $\pi_n^\perp = I_W - \pi_n$. For $x \in W$, let $x_n = \pi_n x$ and $x_n^\perp = \pi_n^\perp x$. Then f is 1-convex if and only if

$$x_n \to \frac{1}{2}|x_n|_H^2 + f\left(x_n + x_n^\perp\right)$$

is $\pi_n^\perp \mu$-almost surely convex.

2.1. Preliminaries about the Monge–Kantorovitch measure transportation problem

Definition 1. Let ξ and η be two probabilities on $(W, \mathcal{B}(W))$. We say that a probability γ on $(W \times W, \mathcal{B}(W \times W))$ is a solution of the Monge–Kantorovitch problem associated to the couple (ξ, η) if the first marginal of γ is ξ, the second one is η and if

$$J(\gamma) = \int\limits_{W \times W} |x - y|_H^2 \, d\gamma(x, y) = \inf\left\{ \int\limits_{W \times W} |x - y|_H^2 \, d\beta(x, y) \colon \beta \in \Sigma(\xi, \eta) \right\},$$

where $\Sigma(\xi, \eta)$ denotes the set of all the probability measures on $W \times W$ whose first and second marginals are respectively ξ and η. We shall denote the Wasserstein distance between ξ and η, which is the positive square-root of this infimum, with $d_H(\xi, \eta)$.

Remark. By the weak compactness of probability measures on $W \times W$ and the lower semi-continuity of the strictly convex cost function, the infimum in the definition is attained even if the functional J is identically infinity. In this latter case we say that the solution is degenerate.

The next result, which is the extension of the finite dimensional version of an inequality due to Talagrand [10], gives a sufficient condition for the finiteness of the Wasserstein distance in the case one of the measures is the Wiener measure μ and the second one is absolutely continuous with respect to it. We give a short proof for the sake of completeness:

Theorem 1. *Let $L \in L \log L(\mu)$ be a positive random variable with $E[L] = 1$[2] and let ν be the measure $d\nu = L\, d\mu$. We then have*

$$d_H^2(\nu, \mu) \leqslant 2E[L \log L]. \tag{2.2}$$

Proof. Let us remark first that we can take W as the classical Wiener space $W = C_0([0, 1])$ and, using the stopping techniques of the martingale theory, we may assume that L is upper and lower bounded almost surely. Then a classical result of the Itô calculus implies that L can be represented as an exponential martingale

$$L_t = \exp\left\{ -\int_0^t \dot{u}_\tau \, dW_\tau - \frac{1}{2} \int_0^t |\dot{u}_\tau|^2 \, d\tau \right\},$$

with $L = L_1$, where $(\dot{u}_t, \; t \in [0, 1])$ is a measurable process adapted to the filtration of the canonical Wiener process $(t, x) \to W_t(x) = x(t)$. Let us define $u : W \to H$ as $u(t, x) = \int_0^t \dot{u}_\tau(x) \, d\tau$ and $U : W \to W$ as $U(x) = x + u(x)$. The Girsanov theorem implies that $x \to U(x)$ is a Browian motion under ν, hence the image of the measure ν under the map $U \times I_W : W \to W \times W$ denoted by $\beta = (U \times I)\nu$ belongs to $\Sigma(\mu, \nu)$. Let γ be any optimal measure, then

$$\begin{aligned}
J(\gamma) = d_H^2(\nu, \mu) &\leqslant \int_{W \times W} |x - y|_H^2 \, d\beta(x, y) \\
&= E\big[|u|_H^2 L\big] \\
&= 2E[L \log L],
\end{aligned}$$

where the last equality follows also from the Girsanov theorem and the Itô stochastic calculus. $\quad\square$

The next two theorems, which explain the existence and several properties of the solutions of Monge–Kantorovitch problem and the transport maps have been proven in [5].

Theorem 2 *(General case). Suppose that ρ and ν are two probability measures on W such that*

$$d_H(\rho, \nu) < \infty.$$

Let $(\pi_n, \; n \geqslant 1)$ be a total increasing sequence of regular projections (of H, converging to the identity map of H). Suppose that, for any $n \geqslant 1$, the regular conditional probabilities $\rho(\cdot | \pi_n^\perp = x^\perp)$ vanish $\pi_n^\perp \rho$-almost surely on the subsets of $(\pi_n^\perp)^{-1}(W)$ with Hausdorff dimension $n - 1$. Then there exists a unique solution of the Monge–Kantorovitch problem, denoted by $\gamma \in \Sigma(\rho, \nu)$ and γ is supported by the graph of a Borel map T which is the solution of the Monge problem. $T : W \to W$ is of the form $T = I_W + \xi$, where $\xi \in H$ almost surely. Besides we have

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[2] In the sequel we denote the expectation w.r. to the Wiener measure by E.

$$d_H^2(\rho, v) = \int\limits_{W \times W} |T(x) - x|_H^2 \, d\gamma(x, y)$$

$$= \int\limits_{W} |T(x) - x|_H^2 \, d\rho(x),$$

and for $\pi_n^\perp \rho$-almost almost all $x_n^\perp$, the map $u \to \xi(u + x_n^\perp)$ is cyclically monotone on $(\pi_n^\perp)^{-1}\{x_n^\perp\}$, in the sense that

$$\sum_{i=1}^{N} \left(\xi\left(x_n^\perp + u_i\right), u_{i+1} - u_i\right)_H \leqslant 0$$

$\pi_n^\perp \rho$-almost surely, for any cyclic sequence $\{u_1, \ldots, u_N, u_{N+1} = u_1\}$ from $\pi_n(W)$. Finally, if, for any $n \geqslant 1$, $\pi_n^\perp v$-almost surely, $v(\cdot|\pi_n^\perp = y^\perp)$ also vanishes on the $n - 1$-Hausdorff dimensional subsets of $(\pi_n^\perp)^{-1}(W)$, then T is invertible, i.e, there exists $S : W \to W$ of the form $S = I_W + \eta$ such that $\eta \in H$ satisfies a similar cyclic monotononicity property as ξ and that

$$1 = \gamma\{(x, y) \in W \times W : T \circ S(y) = y\}$$

$$= \gamma\{(x, y) \in W \times W : S \circ T(x) = x\}.$$

In particular we have

$$d_H^2(\rho, v) = \int\limits_{W \times W} |S(y) - y|_H^2 \, d\gamma(x, y)$$

$$= \int\limits_{W} |S(y) - y|_H^2 \, dv(y).$$

Remark 1. In particular, for all the measures ρ which are absolutely continuous with respect to the Wiener measure μ, the second hypothesis is satisfied, i.e., the measure $\rho(\cdot|\pi_n^\perp = x_n^\perp)$ vanishes on the sets of Hausdorff dimension $n - 1$.

The case where one of the measures is the Wiener measure and the other is absolutely continuous with respect to μ is the most important one for the applications. Consequently we give the related results separately in the following theorem where the tools of the Malliavin calculus give more information about the maps ξ and η of Theorem 2:

Theorem 3 (Gaussian case). *Let v be the measure $dv = L \, d\mu$, where L is a positive random variable, with $E[L] = 1$. Assume that $d_H(\mu, v) < \infty$ (for instance $L \in L \log L$). Then there exists a 1-convex function $\phi \in \mathbb{D}_{2,1}$, unique up to a constant, such that the map $T = I_W + \nabla\phi$ is the unique solution of the original problem of Monge. Moreover, its graph supports the unique solution of the Monge–Kantorovitch problem γ. Consequently*

$$(I_W \times T)\mu = \gamma.$$

In particular T maps μ to ν and T is almost surely invertible, i.e., there exists some T^{-1} such that $T^{-1}\nu = \mu$ and that

$$
\begin{aligned}
1 &= \mu\{x\colon T^{-1} \circ T(x) = x\} \\
&= \nu\{y \in W\colon T \circ T^{-1}(y) = y\}.
\end{aligned}
$$

Remark 2. Assume that the operator ∇ is closable with respect to ν, then we have $\eta = \nabla\psi$. In particular, if ν and μ are equivalent, then we have

$$
T^{-1} = I_W + \nabla\psi,
$$

where is ψ is a 1-convex function. ψ is called the dual potential of the MKP(μ, ν) and we have the following relations:

$$
\phi(x) + \psi(y) + \frac{1}{2}|x - y|_H^2 \geqslant 0,
$$

for any $x, y \in W$, and

$$
\phi(x) + \psi(y) + \frac{1}{2}|x - y|_H^2 = 0
$$

γ-almost surely.

Remark 3. Let $(e_n, \, n \in \mathbb{N})$ be a complete, orthonormal in H, denote by V_n the sigma algebra generated by $\{\delta e_1, \ldots, \delta e_n\}$ and let $L_n = E[L|V_n]$. If $\phi_n \in \mathbb{D}_{2,1}$ is the function constructed in Theorem 3, corresponding to L_n, then, using the inequality (2.2) we can prove that the sequence $(\phi_n, \, n \in \mathbb{N})$ converges to ϕ in $\mathbb{D}_{2,1}$.

3. Characterization of the invertible shifts

Let us begin with some results of general interest. Let us first define:

Definition 2. A measurable map $T : W \to W$ is called (μ-) almost surely right invertible if there exists a measurable map $S : W \to W$ such that $S\mu \ll \mu$ and $T \circ S = I_W$ μ-a.s. Similarly, we say that it is left invertible, if $T\mu \ll \mu$ and if there exists a measurable map $S : W \to W$ such that $S \circ T = I_W\mu$-a.s.

The following proposition some parts of which are proven in [15], shows that, whenever an adapted shift has a left inverse almost surely, then it is almost surely invertible and its inverse is also an adapted perturbation of identity and it relates this concept to the existence and uniqueness of strong solutions of stochastic differential equations.

Proposition 1. *Assume $A = I_W + a$, $a \in L^2(\mu, H)$, $\dot{a}$ is adapted, $E[\rho(-\delta a)] = 1$. Suppose that there exists a map $B : W \to W$ such that $B \circ A = I_W$ a.s. Then the following assertions are true*:

(i) *$B\mu$ is equivalent to μ and $A \circ B = I_W$ a.s., i.e., B is also a right inverse.*

(ii) $B = I_W + b$, $b : W \to H$, $\dot{b}$ is also adapted.

(iii) $(t, w) \to B_t(w)$ is the strong solution of

$$dB_t = -\dot{a}_t \circ B \, dt + dW_t,$$
$$B_0 = 0. \tag{3.1}$$

(iv) We have

$$\dot{a}_t + \dot{b}_t \circ A = 0, \tag{3.2}$$
$$\dot{b}_t + \dot{a}_t \circ B = 0, \tag{3.3}$$

$dt \times d\mu$-a.s.

(v) In particular either the property $A\mu \sim \mu$ and the relation (3.2) together or $B\mu \sim \mu$ and the relation (3.3) together imply that $B \circ A = A \circ B = I_W$ a.s.

Proof. For any $f \in C_b(W)$, it follows from the Girsanov theorem

$$E[f \circ B] = E\big[f \circ B \circ A\rho(-\delta a)\big]$$
$$= E\big[f\rho(-\delta a)\big],$$

hence $B\mu$ is equivalent to μ and the corresponding Radon–Nikodym density is $\rho(-\delta a)$. Let

$$D = \big\{w \in W : B \circ A(w) = w\big\}.$$

Since $D \subset A^{-1}(A(D))$ and by the hypothesis $\mu(D) = 1$ we get

$$E[1_{A(D)} \circ A] = 1.$$

Since $A\mu$ is equivalent to μ we have also $\mu(A(D)) = 1$. If $w \in A(D)$, then $w = A(d)$, for some $d \in D$, hence $A \circ B(w) = A \circ B \circ A(d) = A(d) = w$, consequently $A \circ B = I_W$ μ-almost surely and B is the two-sided inverse of A. Evidently, together with the absolute continuity of $B\mu$, this implies that B is of the form $B = I_W + b$, with $b : W \to H$. Moreover, $\dot{a} = -\dot{b} \circ A$, hence the right-hand side is adapted. We can assume that all these processes are uni-dimensional (otherwise we proceed component wise). Let $\dot{b}^n = \max(-n, \min(\dot{b}, n))$. Then $\dot{b}^n \circ A$ is adapted. Let $H \in L^2(dt \times d\mu)$ be an adapted process. Using the Girsanov theorem:

$$E\left[\rho(-\delta a)\int_0^1 \dot{b}_s^n \circ A \, H_s \circ A \, ds\right] = E\left[\int_0^1 \dot{b}_s^n H_s \, ds\right]$$

$$= E\left[\int_0^1 E\big[\dot{b}_s^n \mid \mathcal{F}_s\big] H_s \, ds\right]$$

$$= E\left[\rho(-\delta a)\int_0^1 E\big[\dot{b}_s^n \mid \mathcal{F}_s\big] \circ A \, H_s \circ A \, ds\right].$$

Consequently

$$E[\dot{b}_s^n \mid \mathcal{F}_s] \circ A = \dot{b}_s^n \circ A,$$

almost surely. Since $A\mu$ is equivalent to μ, it follows that

$$E[\dot{b}_s^n \mid \mathcal{F}_s] = \dot{b}_s^n$$

almost surely, hence $\dot{b}^n$ and consequently $\dot{b}$ are adapted. It is now clear that $(B(t), \ t \in [0, 1])$ is a strong solution of (3.1). The uniqueness follows from the fact that, any strong solution of (3.1) would be a right inverse to A, since A is invertible, then this solution is equal to B.

The proof of (v) is quite similar to that of the first part: let $D = \{w \in W: \ A \circ B(w) = w\}$, then $\mu(B^{-1}(B(D))) = 1$, hence $B \circ A = I_W$ μ-a.s. Moreover B can be written as $B = I_W + b$, with $\dot{a} = -\dot{b} \circ A$, proceeding as above, we show that $\dot{b}$ is adapted and the rest of the proof follows. $\quad\square$

The invertibility of A is characterized in terms of the corresponding Wick exponentials as below:

Theorem 4. *Let $A = I_W + a$, $a \in L_a^0(\mu, H)$. Assume that $E[\rho(-\delta a)] = 1$[3] and that*

$$\frac{dA\mu}{d\mu} \circ A\rho(-\delta a) = 1$$

almost surely. Then A is (almost surely) invertible.

Proof. Since $E[\rho(-\delta a)] = 1$, $A\mu$ is equivalent to μ, hence the corresponding Radon–Nikodym derivative can be expressed as an exponential martingale:

$$l = \frac{dA\mu}{d\mu} = \exp\left(-\delta b - \frac{1}{2}|b|_H^2\right),$$

where $b(t, w) = \int_0^t \dot{b}_s(w)\, ds$, with $\dot{b}$ adapted, $\int_0^1 |\dot{b}_s|^2\, ds < \infty$ almost surely and δb is defined in $L^0(\mu)$. The hypothesis implies that

$$\delta(a + b \circ A) + \frac{1}{2}|a + b \circ A|_H^2 = 0 \tag{3.4}$$

almost surely. Define the local martingale (M_t) as

$$M_t = \exp\left(-\int_0^t (\dot{a}_s + \dot{b}_s \circ A)\, dW_s - \frac{1}{2}\int_0^t |\dot{a}_s + \dot{b}_s \circ A|^2\, ds\right).$$

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[3] Here we denote by δa the stochastic integral of the adapted process $\dot{a}$ in $L^0(\mu)$.

The relation (3.4) implies in fact that (M_t) is a uniformly integrable martingale with its final value (at $t = 1$) $M_1 = 1$. Consequently $M_t = 1$ almost surely for any $t \in [0, 1]$ and this implies that

$$\dot{a}_s + \dot{b}_s \circ A = 0$$

$ds \times d\mu$-almost surely. Hence $(I_W + b) \circ A = I_W$ almost surely and the proof is fully completed thanks to Proposition 1. $\square$

Proposition 2. *Assume that $(A_n, n \geqslant 1)$ is a sequence of mappings of the form $A_n = I_W + a_n$, with $a_n : W \to H$, $\dot{a}_n$ is adapted for any n and $(a_n, n \geqslant 1)$ converges to some a in $L^0(\mu, H)$ such that $E[\rho(-\delta a)] = 1$. Suppose that, for any $n \geqslant 1$, $E[\rho(-\delta a_n)] = 1$ and A_n is invertible. If*

$$\lim_{n \to \infty} \frac{d A_n \mu}{d\mu} = l$$

exists in the norm topology of $L^1(\mu)$, then $A = I_W + a$ is also invertible.

Proof. Let us denote by l_n the Radon–Nikodym derivative of $A_n \mu$ with respect to μ. The hypothesis implies that $(l_n, n \geqslant 1)$ is uniformly integrable. Since $(a_n, n \geqslant 1)$ converges in probability, the uniform integrability, combined with the Lusin theorem implies that $(l_n \circ A_n, n \geqslant 1)$ converges in probability to $l \circ A$. Since $(\rho(-\delta a_n), n \geqslant 1)$ converges to $\rho(-\delta a)$ in probability and since, by the invertibility of A_n, we have

$$l_n \circ A_n \rho(-\delta a_n) = 1$$

almost surely for any $n \geqslant 1$, we have also

$$l \circ A \rho(-\delta a) = 1$$

almost surely. The conclusion follows then from Theorem 4. $\square$

The following lemma gives an important information about the Radon–Nikodym density of the measure $A\mu$ with respect to μ:

Lemma 1. *Assume that $A = I_W + a$ with $a \in L^0(\mu, H)$ with $\dot{a}$ adapted. Then*

$$\frac{d A\mu}{d\mu} \circ A E[\rho(-\delta a) \mid A] \leqslant 1$$

almost surely. If we have also $E[\rho(-\delta a)] = 1$, then the above inequality becomes an equality:

$$\frac{d A\mu}{d\mu} \circ A E[\rho(-\delta a) \mid A] = 1$$

almost surely.

Proof. For any positive function $f \in C_b(W)$, using the Girsanov theorem and the Fatou Lemma, we have

$$E[f \circ A] = E\left[f \frac{dA\mu}{d\mu}\right]$$

$$\geqslant E\left[f \circ A \frac{dA\mu}{d\mu} \circ A\rho(-\delta a)\right]$$

$$= E\left[f \circ A \frac{dA\mu}{d\mu} \circ A E\big[\rho(-\delta a) \,\big|\, A\big]\right],$$

which proves the first part of the lemma. For the second part, due to the integrability hypothesis, we can replace the inequality above by the equality and the proof follows. $\quad\square$

4. Properties of non-invertible adapted perturbation of identity

In this section we study the following concept:

Definition 3. A positive random variable whose expectation is equal to one with respect to Wiener measure is said to be representable with a mapping $U : W \to W$ if

$$\frac{dU\mu}{d\mu} = L.$$

We begin with the following

Proposition 3. *Assume that $L = \rho(-\delta v)$, where $v \in L_a^0(\mu, H)$, i.e., $\dot{v}$ is adapted and $\int_0^1 |\dot{v}_s|^2 \, ds < \infty$ a.s. Then there exists $U = I_W + u$, with $u : W \to H$ adapted such that $U\mu = L\mu$ and $E[\rho(-\delta u)] = 1$ if and only if the following condition is satisfied*:

$$1 = L_t \circ U E\big[\rho\big(-\delta u^t\big) \,\big|\, \mathcal{U}_t\big] \tag{4.1}$$

$$= L_t \circ U E\big[\rho(-\delta u) \,\big|\, \mathcal{U}_t\big] \tag{4.2}$$

almost surely for any $t \in [0, 1]$, where u^t is defined as $u^t(\tau) = \int_0^{t \wedge \tau} \dot{u}_s \, ds$ and $\mathcal{U}_t$ is the sigma algebra generated by $(w(\tau) + u(\tau), \ \tau \leqslant t)$.

Proof. Let U_t be defined as $I_W + u^t$, then for any $f \in C_b(W)$ which is $\mathcal{F}_t$-measurable, we have

$$E\big[f \circ U_t L_t \circ U_t \rho\big(-\delta u^t\big)\big] = E[f L_t]$$

$$= E[f \circ U_t].$$

Since, for any $\mathcal{F}_t$-measurable function G, $G \circ U_t$ is $\mathcal{U}_t$ measurable, we get

$$L_t \circ U_t E\big[\rho\big(-\delta u^t\big) \,\big|\, \mathcal{U}_t\big] = 1.$$

Conversely, it follows from the relation (4.1) and from the Girsanov theorem that

$$E[f \circ U] = E\big[f \circ U L \circ U \rho(-\delta u)\big] = E[fL],$$

a similar relation holds when we replace U by U_t. □

Let us calculate $E[\rho(-\delta u^t) \,|\, \mathcal{U}_t] = E[\rho(-\delta u) \,|\, \mathcal{U}_t]$ in terms of the innovation process associated to U. Recall that the term innovation, which originates from the filtering theory is defined as (cf. [7] and [14])

$$Z_t = U_t - \int_0^t E[\dot{u}_s \,|\, \mathcal{U}_s]\, ds$$

and it is a μ-Brownian motion with respect to the filtration $(\mathcal{U}_t, \ t \in [0,1])$. A similar proof as the one in [7] shows that any martingale with respect to the filtration of U can be represented as a stochastic integral with respect to Z. Hence, by the positivity assumption, $E[\rho(-\delta u) \,|\, \mathcal{U}_t]$ can be written as an exponential martingale

$$E\big[\rho(-\delta u) \,|\, \mathcal{U}_t\big] = \exp\!\left(-\int_0^t (\dot{\xi}_s, dZ_s) - \frac{1}{2}\int_0^t |\dot{\xi}_s|^2\, ds\right).$$

Below we give a more detailed result:

Proposition 4. *We have the following explicit result*

$$E\big[\rho(-\delta u) \,|\, \mathcal{U}\big] = \exp\!\left(-\int_0^1 \big(E[\dot{u}_s \,|\, \mathcal{U}_s], dZ_s\big) - \frac{1}{2}\int_0^1 \big|E[\dot{u}_s \,|\, \mathcal{U}_s]\big|^2\, ds\right), \qquad (4.3)$$

hence

$$E\big[\rho(-\delta u) \,|\, \mathcal{U}_t\big] = \exp\!\left(-\int_0^t \big(E[\dot{u}_s \,|\, \mathcal{U}_s], dZ_s\big) - \frac{1}{2}\int_0^t \big|E[\dot{u}_s \,|\, \mathcal{U}_s]\big|^2\, ds\right), \qquad (4.4)$$

almost surely.

Proof. The proof follows from the double utilization of the Girsanov theorem. Let us denote by l_t the Girsanov exponential

$$l_t = \exp\!\left(-\int_0^t \big(E[\dot{u}_s \,|\, \mathcal{U}_s], dZ_s\big) - \frac{1}{2}\int_0^t \big|E[\dot{u}_s \,|\, \mathcal{U}_s]\big|^2\, ds\right).$$

On the first hand, we have, for any $f \in C_b(W)$,

$$E\big[f \circ U \rho(-\delta u)\big] = E[f],$$

and on the other hand, applying the Girsanov theorem to the decomposition

$$U_t = Z_t + \int_0^t E[\dot{u}_s \,|\, \mathcal{U}_s]\, ds,$$

we get

$$E[f \circ U l_1] \leqslant E[f] = E\big[f \circ U \rho(-\delta u)\big]$$

for any positive, measurable f on W. Taking f to be $\mathcal{F}_t$ measurable, we conclude that

$$l_t \leqslant E\big[\rho(-\delta u) \,\big|\, \mathcal{U}_t\big]$$

a.s. for any $t \in [0, 1]$. Consequently $(l_t, \ t \in [0, 1))$ is a uniformly integrable martingale and in particular $E[l_1] = 1$. Hence we have

$$E[f \circ U l_1] = E[f] = E\big[f \circ U \rho(-\delta u)\big],$$

for any $f \in C_b(W)$ which implies that $l_1 = E[\rho(-\delta u)|\mathcal{U}]$ and the proof of (4.3) follows. The relation (4.4) is obvious since $\mathcal{U}_t \subset \mathcal{F}_t$. $\square$

Theorem 5. *A necessary and sufficient condition for the relation* (4.1), *that is to say for the representability of $L = \rho(-\delta v)$ by $U = I_W + u$ is that*

$$E[\dot{u}_t \,|\, \mathcal{U}_t] = -\dot{v}_t \circ U$$

$dt \times d\mu$-almost surely.

Proof. We have

$$L_t \circ U = \exp\left(-\delta v_t \circ U_t - \frac{1}{2}|v_t \circ U_t|_H^2\right).$$

Moreover using the identity

$$\delta v_t \circ U_t = \int_0^t (\dot{v}_s \circ U, dW_s) + \int_0^t (\dot{v}_s \circ U, \dot{u}_s)\, ds,$$

we get

$$L_t \circ U = \exp\left[-\int_0^t \left(\dot{v}_s \circ U, dW_s + \dot{u}_s\, ds + \frac{1}{2}\dot{v}_s \circ U\, ds\right)\right].$$

Substituting all these relations in (4.1) and using the representation (4.3), we obtain

$$1 = L_t \circ U E\big[\rho(-\delta u)\,\big|\,\mathcal{U}_t\big]$$

$$= \exp\left[-\int_0^t \left(\dot{v}_s \circ U, dW_s + \dot{u}_s\,ds + \frac{1}{2}\dot{v}_s \circ U\,ds\right)\right]$$

$$\exp\left(-\int_0^t \big(E[\dot{u}_s\,|\,\mathcal{U}_s], dZ_s\big) - \frac{1}{2}\int_0^t \big|E[\dot{u}_s\,|\,\mathcal{U}_s]\big|^2\,ds\right).$$

But

$$\int_0^t \big(E[\dot{u}_s\,|\,\mathcal{U}_s], dZ_s\big) = \int_0^t \big(E[\dot{u}_s\,|\,\mathcal{U}_s], dW_s + \big(\dot{u}_s - E[\dot{u}_s\,|\,\mathcal{U}_s]\big)\,ds\big).$$

Consequently we get

$$\int_0^t \big(\dot{v}_s \circ U + E[\dot{u}_s\,|\,\mathcal{U}_s], dW_s\big) = 0,$$

almost surely for any $t \in [0, 1]$ and this implies that

$$E[\dot{u}_s\,|\,\mathcal{U}_s] = -\dot{v}_s \circ U$$

$ds \times d\mu$-almost surely. The sufficiency is obvious. $\square$

Corollary 1. *A necessary and sufficient condition for the relation* (4.1) *is that*

$$V \circ U = Z,$$

in other words

$$U_t = Z_t - \int_0^t \dot{v}_s \circ U\,ds$$

almost surely, where Z is the innovation process associated to U.

Proof. The condition in Theorem 5 reads as

$$\dot{v}_t \circ U + E[\dot{u}_t\,|\,\mathcal{U}_t] = 0 \tag{4.5}$$

almost surely. Hence

$$(V \circ U)(t) = U(t) + (v \circ U)(t)$$

$$= Z(t) + \int_0^t E[\dot{u}_s \,|\, \mathcal{U}_s] \, ds + \int_0^t \dot{v}_s \circ U \, ds$$

$$= Z_t,$$

by the relation (4.5). □

Corollary 2. *Suppose that the innovation process Z is an $(\mathcal{F}_t, \ t \in [0, 1])$-local martingale, then U is almost surely invertible and its inverse is V.*

Proof. We have

$$U_t = W_t + \int_0^t \dot{u}_s \, ds = Z_t + \int_0^t E[\dot{u}_s \,|\, \mathcal{U}_s] \, ds,$$

hence $(W_t - Z_t, \ t \in [0, 1])$ is a continuous local martingale of finite variation. This implies that Z and W are equal hence

$$\dot{u}_t = E[\dot{u}_t \,|\, \mathcal{U}_t],$$

$dt \times d\mu$-almost surely. From Theorem 5, it follows that $u + v \circ U = 0$ almost surely, i.e., $V \circ U = I_W$ almost surely. It follows from Proposition 1 that

$$U \circ V = I_W$$

also μ-almost surely. □

We can give a complete characterization of the representable random variables as follows:

Theorem 6. *Assume that $L = \rho(-\delta v)$, $V = I_W + v$, $v \in L_a^0(\mu, H)$. Assume that $U = I_W + u$ is also an adapted perturbation of identity with $E[\rho(-\delta u)] = 1$. Assume that $V \circ U = B$ is a Brownian motion with respect to its own filtration. We have $U\mu = L \cdot \mu$ if and only if B is a local martingale with respect to the filtration generated by U and in this case B is equal to the innovation associated to U.*

Proof. The necessity has already been proven, for the sufficiency, note that, we have $U = B - v \circ U$. On the other hand we can always represent U by its innovation process as

$$U_t = Z_t + \int_0^t E[\dot{u}_s \,|\, \mathcal{U}_s] \, ds = B_t - \int_0^t \dot{v}_s \circ U \, ds$$

where Z is the innovation process associated to U, which is a Brownian motion with respect to $(\mathcal{U}_t, \ t \in [0, 1])$. Consequently

$$-\dot{v}_s \circ U = E[\dot{u}_s \,|\, \mathcal{U}_s],$$

$ds \times d\mu$-almost surely and the proof follows from Theorem 5. □

5. Relations with entropy

Assume that $u \in \mathbb{D}_{2,0}^a(H)$ with $E[\rho(-\delta u)] = 1$ and let $L \in \mathbb{L}\log\mathbb{L}(\mu)$ be the Radon–Nikodym density of $U\mu = (I_W + u)\mu$ with respect to μ. Let us represent L as $\rho(-\delta v)$. Denote $E[\rho(-\delta u)|\mathcal{U}]$ by $\hat{\rho}$. Then, due to the Girsanov theorem, we have

$$E[\hat{\rho} \log \hat{\rho}] = \frac{1}{2} E[\hat{\rho}|v \circ U|_H^2]$$

$$= \frac{1}{2} E[\rho(-\delta u)|v \circ U|_H^2]$$

$$= \frac{1}{2} E[|v|_H^2].$$

In particular, the Jensen inequality implies that

$$E[|v|_H^2] \leqslant 2E[\rho(-\delta u) \log \rho(-\delta u)]$$

$$= E[\rho(-\delta u)|u|_H^2].$$

Proposition 5. *Let P_ε denote the Ornstein–Uhlenbeck semigroup and denote by v_ε the regularization $P_\varepsilon v$ and denote by u_ε the H-valued mapping which is defined as $I_W + u_\varepsilon = (I_W + v_\varepsilon)^{-1}$ whose existence follows from [15]. The set $(u_\varepsilon, \varepsilon > 0)$ has a unique weak accumulation point $\tilde{u} \in \mathbb{D}_{2,0}(H)$. If the relation (4.1) holds then $\tilde{u}$ satisfies the following relation:*

$$\frac{d}{ds}\tilde{u}(s) \circ Z = -E[\dot{v}_s \circ U \mid \mathcal{Z}_s] = E[\dot{u}_s \mid \mathcal{Z}_s]$$

$ds \times d\mu$-almost surely, where $\mathcal{Z}$ denotes the sigma algebra generated by the innovation Z associated to U.

Proof. From [15], $V_\varepsilon = I_W + v_\varepsilon$ is almost surely invertible and its inverse can be written as $U_\varepsilon = I_W + u_\varepsilon$. Moreover $u_\varepsilon = -v_\varepsilon \circ U_\varepsilon$. Hence $(u_\varepsilon, \varepsilon > 0)$ is bounded in $L^2(\mu, H)$. Consequently, there exists a subnet which converges weakly to some $\tilde{u}$. Let ξ be an H-valued, bounded continuous function on W. Denoting by $\langle \cdot, \cdot \rangle$ the duality bracket of $L^2(\mu, H)$, we get

$$\langle u_\varepsilon, \xi \rangle = \langle u_\varepsilon \circ V_\varepsilon, \xi \circ V_\varepsilon \, \rho(-\delta v_\varepsilon) \rangle$$

$$= -\langle v_\varepsilon, \xi \circ V_\varepsilon \rho(-\delta v_\varepsilon) \rangle$$

$$\to -\langle v, \xi \circ V \rho(-\delta v) \rangle.$$

Hence

$$\langle \tilde{u}, \xi \rangle = -\langle v, \xi \circ V \rho(-\delta v) \rangle.$$

Consequently $\tilde{u}$ is unique, i.e., the net $(u_\varepsilon, \varepsilon > 0)$ has only one accumulation point in the weak topology of $\mathbb{D}_{2,0}(H) = L^2(\mu, H)$. From the last hypothesis

$$\frac{dU\mu}{d\mu} = \rho(-\delta v).$$

Hence

$$\langle \tilde{u}, \xi \rangle = -\big(v, \xi \circ V\rho(-\delta v)\big)$$

$$= -\langle v \circ U, \xi \circ V \circ U \rangle$$

$$= -\langle v \circ U, \xi \circ Z \rangle$$

$$= -E \int_0^1 E[\dot{v}_s \circ U \mid \mathcal{Z}_s]\dot{\xi}_s \circ Z \, ds.$$

Since Z is a Brownian motion, we also have

$$\langle \tilde{u}, \xi \rangle = \langle \tilde{u} \circ Z, \xi \circ Z \rangle,$$

hence the proof is completed. $\square$

Remark 4. We draw the attention of the reader to the fact that in general the weak convergence does not imply the strong convergence. The situation illustrated above is a typical example for this; in fact if there were also a strong convergence, then $I + v$ would have been invertible and we would have $I_W + \tilde{u} = I_W + u = (I_W + v)^{-1}$ (cf. [15]).

Remark 5. Similarly, suppose that v is bounded and that

$$E\big[|\tilde{u}|_H^2\big] = 2E[L \log L]. \tag{5.1}$$

Then $V = I_W + v$ is invertible and its inverse is $U = I_W + u$ with $u = \tilde{u}$. In fact this follows from the hypothesis (5.1), which implies that

$$\lim_{\varepsilon \to 0} E\big[|u_\varepsilon|_H^2\big] = \lim_{\varepsilon \to 0} E\big[|v_\varepsilon|_H^2 L_\varepsilon\big]$$

$$= E\big[|v|_H^2 L\big]$$

$$= 2E[L \log L]$$

$$= E\big[|\tilde{u}|_H^2\big].$$

Since $\mathbb{D}_{2,0}(H)$ is a Hilbert space, the convergence of the norms implies that $\lim_{\varepsilon \to 0} u_\varepsilon = \tilde{u}$ in the norm topology of $\mathbb{D}_{2,0}(H)$. Therefore V is invertible as proven in [15]. Consequently, in the case where the mapping V is not invertible, this equality cannot take place.

The remark above suggests the following claim:

Theorem 7. *Assume that* $u \in \mathbb{D}_{2,0}^a(H)$, $E[\rho(-\delta u)] = 1$ *and*

$$\frac{dU\mu}{d\mu} = \rho(-\delta v) = L,$$

such that $v \in L_a^0(\mu, H)$. $U = I_W + u$ is then almost surely invertible with its inverse $V = I_W + v$ if and only if

$$2E[L \log L] = E\big[|u|_H^2\big].$$

In other words, U is invertible if and only if

$$H(U\mu \mid \mu) = \frac{1}{2} \|u\|_{\mathbb{D}_{2,0}(H)}^2,$$

where $H(U\mu \mid \mu)$ denotes the entropy of $U\mu$ with respect to μ.

Proof. Since U represents $Ld\mu$, we have $E[\dot{u}_s | \mathcal{U}_s] + \dot{v}_s \circ U = 0$ $ds \times d\mu$-almost surely. Hence, from the Jensen inequality $E[|v \circ U|_H^2] \leqslant E[|u|_H^2]$. Moreover the Girsanov theorem gives

$$2E[L \log L] = E\big[|v|_H^2 L\big] = E\big[|v \circ U|_H^2\big] = E\left[\int_0^1 |E[\dot{u}_s \mid \mathcal{U}_s]|^2 \, ds\right].$$

Hence the hypothesis implies that

$$E\big[|u|_H^2\big] = E\left[\int_0^1 |E[\dot{u}_s \mid \mathcal{U}_s]|^2 \, ds\right].$$

From which we deduce that $\dot{u}_s = E[\dot{u}_s \mid \mathcal{U}_s]$ $ds \times d\mu$-almost surely. Finally we get $\dot{u}_s + \dot{v}_s \circ U = 0$ $ds \times d\mu$, which is a necessary and sufficient condition for the claim. The necessity is obvious. $\quad\square$

Remark 6. This theorem says that U is invertible if and only if the "kinetic energy" of U is equal to the entropy of the measure that it induces. Moreover U is non-invertible if and only if we have

$$H(U\mu \mid \mu) < \frac{1}{2} \|u\|_{\mathbb{D}_{2,0}(H)}^2.$$

The above relation between the entropy and the (kinetic) energy can be generalized to the maps $I_W + u$, where $u \in L^0(\mu, H)$ which do not fulfill necessarily the integrability condition $E[\rho(-\delta u)] = 1$ as follows:

Theorem 8. *Assume that $u \in L_a^2(\mu, H)$, let $U = I_W + u$ and define L as to be*

$$L = \frac{dU\mu}{d\mu}.$$

We then have

$$H(U\mu \mid \mu) = E[L \log L] \leqslant \frac{1}{2} E\big[|u|_H^2\big].$$

Proof. If $|u|_H \in L^\infty(\mu)$, the claim is obvious from above. For the general case, let $(T_n, \ n \geqslant 1)$ be a sequence of stopping times increasing to infinity such that $|u^n|_H$ is bounded, where $u^n(t) = \int_0^t 1_{[0,T_n]}(s)\dot{u}_s \, ds$. Denote by L_n the Radon–Nikodym derivative of $(I_W + u^n)\mu$ w.r.t. μ. From Remark 6, it follows that the sequence $(L_n, \ n \geqslant 1)$ is uniformly integrable, hence it converges to L in the weak topology of $L^1(\mu)$. From the lower semi-continuity of the entropy w.r.t. this topology, we get

$$E[L \log L] \leqslant \liminf_n E[L_n \log L_n] \leqslant \lim \frac{1}{2} E\big[|u^n|_H^2\big] = \frac{1}{2} E\big[|u|_H^2\big]. \qquad \square$$

6. Relations with the innovation conjecture of the filtering

Let us briefly explain the question (cf. [16,1,7] for further details): Assume that we are given a process of the form

$$y_t(w, \beta) = W_t(w) + \int_0^t h_s(w, \beta) \, ds,$$

called the observation, where β is independent of the Wiener path w, $s \to h_s(w, \beta) \in L^2([0,1], ds)$ almost surely and adapted to some filtration in which the filtration of (W_t) can be injected. The question is whether the filtration of $y = (y_t, \ t \in [0,1])$ is equal to the filtration of the innovation process defined as before:

$$v_t = y_t - \int_0^t E[h_s \mid \mathcal{Y}_s] \, ds \tag{6.1}$$

where $(\mathcal{Y}_s, s \in [0,1])$ is the filtration of y, called the observation process. The following result gives a complete answer to the innovation conjecture in the general case to which the above problem can be translated:

Theorem 9. *Assume that $U = I_W + u$ is an adapted perturbation of identity such that $u \in \mathbb{D}_{2,0}(H)$ and that $E[\rho(-\delta u)] = 1$. Define L as the Radon–Nikodym density*

$$L = \frac{dU \mu}{d\mu}$$

and define $v \in L_a^0(\mu, H)$ as $L = \rho(-\delta v)$. Let $\mathcal{U} = (\mathcal{U}_t, \ t \in [0,1])$ be its filtration eventually completed with μ-null sets. Let Z be the innovation process associated to U as defined above, denote by $\mathcal{Z} = (\mathcal{Z}_t, \ t \in [0,1])$ its filtration. Then $\mathcal{U} = \mathcal{Z}$ if and only if there exists some $\hat{u} \in L_a^0(\mu, H)$ such that $\hat{U} = I_W + \hat{u}$ is almost surely invertible with inverse $V = I_W + v$ and $U = \hat{U} \circ Z$ almost surely.

Proof. Sufficiency: We have $\mathcal{Z} \subset \mathcal{U}$ by the construction of Z, on the other hand the relation $U = \hat{U} \circ Z$ implies that $\mathcal{U} \subset \mathcal{Z}$, hence the sufficiency is proved.

Necessity: Suppose now that $\mathcal{Z} = \mathcal{U}$, let L be the Radon–Nikodym derivative

$$L = \frac{dU\mu}{d\mu}.$$

Since $L > 0$ almost surely, there exists some $v : W \to H$ such that $\dot{v}$ is adapted and that L can be represented as $L = \rho(-\delta v)$. Hence the random variable L is represented by U, this implies that $V \circ U = Z$ almost surely, where $V = I_W + v$. Since $\mathcal{U} = \mathcal{Z}$, we can write U as a function of Z, i.e., $U = \hat{U}(Z)$. Then

$$
\begin{aligned}
1 = \mu\{V \circ U = Z\} &= \mu\{V \circ \hat{U}(Z) = Z\} \\
&= \mu\{V \circ \hat{U}(w) = w\},
\end{aligned}
$$

since $Z\mu = \mu$. Consequently, $\hat{U}$ is a right inverse of V. Moreover $\hat{U}\mu = \hat{U} \circ Z\mu = U\mu \sim \mu$ hence it follows from Proposition 1 that $V \circ \hat{U} = \hat{U} \circ V = I_W$ μ-almost surely. $\square$

Corollary 3. *Assume that we are in the situation described by the relation* (6.1). *Let us denote by* $\hat{H} : W \to H$ *defined by*

$$\hat{H}(t, y) = \int_0^t E[h_s \mid \mathcal{Y}_s]\, ds.$$

Denote by V the mapping defined by $V = I_W - \hat{H}$. Then the filtration generated by the innovation v is equal to the filtration of the observation y if and only if

$$E\left[\frac{dV}{d\mu} \log \frac{dV\mu}{d\mu}\right] = \frac{1}{2} E\left[|\hat{H}|_H^2\right].$$

Proof. It follows from Theorem 9, that the invertibility of V is a necessary and sufficient condition, then we apply Theorem 7. $\square$

Remark 7. In [1], the authors treat the case where the noise is independent of the signal, this amounts to say that u is independent of w, here on the contrary we are in a situation where the things are correlated.

7. The properties of $U \circ V$

Assume that L, $U = I_W + u$ and $V = I_W + v$ be as in Section 4. We know then that the mapping $V \circ U$ preserves the Wiener measure μ. On the other hand we have, from the Girsanov theorem

$$
\begin{aligned}
E[f \circ U \circ V L] &= E\left[f \circ U \circ V \rho(-\delta v)\right] \\
&= E[f \circ U] \\
&= E[fL],
\end{aligned}
$$

for any $f \in C_b(W)$. In other words $U \circ V$ preserves the measure ν which is defined by $d\nu = L\,d\mu$. Let us denote $U \circ V$ with M. This mapping is of the form $M = I_W + m$, where $m = v + u \circ V$ is an adapted, H-valued mapping.

Proposition 6. *Assume that m satisfies the following hypothesis*:

$$E\big[\rho(-\delta m)\big] = 1,$$

where δm denotes the Itô integral of $(\dot{m}_s, s \in [0,1])$ in $L^0(\mu)$-sense.[4] Then the mapping $M = U \circ V$ satisfies the following probabilistic Monge–Ampère equation:

$$L \circ M E\big[\rho(-\delta m)\,\big|\,\mathcal{M}\big] = E[L\,|\,\mathcal{M}], \tag{7.1}$$

almost surely, where $\mathcal{M}$ denotes the sigma-algebra generated by M.

Proof. Let us note that the hypothesis $E[\rho(-\delta m)] = 1$ is satisfied as soon as $E[\rho(-\delta u)] = 1$ and $E[\rho(-\delta v)] = 1$. Now, from the Girsanov theorem, for any $f \in C_b(W)$, we get

$$E[fL] = E\big[f \circ M L \circ M \rho(-\delta m)\big].$$

On the other hand M preserves the measure $d\nu = L\,d\mu$, hence

$$E[f \circ M L] = E[fL].$$

Therefore

$$E\big[f \circ M L \circ M \rho(-\delta m)\big] = E[f \circ M L],$$

for any $f \in C_b(W)$ and this proves the claim. $\quad\square$

Let us denote by $(\mathcal{M}_t,\ t \in [0,1])$ the filtration generated by M and let us suppose that $m = v + u \circ V$ is in $L^1(\mu, H)$. This last hypothesis is amply sufficient to ensure the existence of the dual predictable projection $\hat{m}$ of m with respect to the filtration $(\mathcal{M}_t,\ t \in [0,1])$. It can be calculated as in Proposition 4

$$\hat{m}(t) = \int_0^t E[\dot{m}_s\,|\,\mathcal{M}_s]\,ds, \quad t \in [0,1].$$

Besides, the innovation process $(R_t,\ t \in [0,1])$ associated to M, defined by

$$R_t = M_t - \int_0^t E[\dot{m}_s\,|\,\mathcal{M}_s]\,ds$$

---

[4] This is an abuse of notation since the divergence coincides with the Itô integral only for the adapted elements of $L^p(\mu, H)$ with $p > 1$.

is an $(\mathcal{M}_t,\ t \in [0,1])$-Brownian motion and again from [7], any martingale of this filtration can be represented as a stochastic integral with respect to this innovation process. Consequently, the martingale $E[\rho(-\delta m) \mid \mathcal{M}_t]$ can be represented as in Proposition 4:

$$E\big[\rho(-\delta m) \mid \mathcal{M}_t\big] = \exp\left(-\int_0^t \big(E[\dot{m}_s \mid \mathcal{M}_s], dR_s\big) - \frac{1}{2}\int_0^t \big|E[\dot{m}_s \mid \mathcal{M}_s]\big|^2 ds\right).$$

From the Itô representation theorem, there exists an $(\mathcal{M}_t,\ t \in [0,1])$-adapted process $(\dot{\gamma}_t,\ t \in [0,1])$ such that $\int_0^1 |\dot{\gamma}_t|^2\, dt < \infty$ almost surely and that

$$E[L \mid \mathcal{M}_t] = \exp\left(-\int_0^t (\dot{\gamma}_s, dR_s) - \frac{1}{2}\int_0^t |\dot{\gamma}_s|^2 ds\right).$$

Let us calculate the terms at the right of the relation (7.1):

$$L \circ M = \exp\left(-\delta v \circ M - \frac{1}{2}|v \circ M|_H^2\right).$$

Using the identity

$$\delta v \circ M = \delta(v \circ M) + (v \circ M, m)_H$$

and taking into account the exponents of the relation (7.1), we get

$$\delta(v \circ M) + (v \circ M, m)_H + \frac{1}{2}|v \circ M|_H^2 + \int_0^1 \big(E[\dot{m}_s \mid \mathcal{M}_s], dR_s\big) + \frac{1}{2}\int_0^1 \big|E[\dot{m}_s \mid \mathcal{M}_s]\big|^2 ds$$

$$= \int_0^1 (\dot{\gamma}_s, dR_s) + \frac{1}{2}|\gamma|_H^2,$$

where the letters without "dot" denote the primitives of those with "dot". If we restrict all these calculations to the time interval $[0,t]$, for any $t \in [0,1]$, similar relation holds, consequently we have proven

Theorem 10. *If $U\mu = v = L \cdot \mu$ and if $L = \rho(-\delta v)$, where u and v are adapted and if $E[\rho(-\delta m)] = 1$ and if $m = v + u \circ V \in L^1(\mu, H)$, then we have the following relation between v and m:*

$$\dot{v}_t \circ M = E_v[\dot{v}_t - \dot{m}_t \mid \mathcal{M}_t] \tag{7.2}$$

$dt \times d\mu$-almost surely, where the conditional expectation is taken w.r.t. the measure v.

Proof. Let us calculate the conditional expectation $E[L \mid \mathcal{M}]$: by a stopping argument, it suffices to suppose that L is bounded. Let ξ be a bounded, $(\mathcal{M}_t, \ t \in [0, 1])$-adapted process. Then

$$E\left[L \int_0^1 \xi_s \, dR_s \right]$$

$$= E\left[\left(-\int_0^1 \dot{v}_s L_s \, dW_s \right) \left(\int_0^1 \xi_s \, dR_s \right) \right]$$

$$= E\left[\left(\int_0^1 E\left[-\dot{v}_s L_s + L_s \big(\dot{m}_s - E[\dot{m}_s \mid \mathcal{M}_s] \big) \mid \mathcal{M}_s \right] dR_s \right) \left(\int_0^1 \xi_s \, dR_s \right) \right]$$

$$= E\left[\left(\int_0^1 \big\{ E_v\left[-\dot{v}_s + \dot{m}_s - E[\dot{m}_s \mid \mathcal{M}_s] \mid \mathcal{M}_s \right] \big\} E[L_s \mid \mathcal{M}_s] dR_s \right) \left(\int_0^1 \xi_s \, dR_s \right) \right].$$

Consequently

$$E[L \mid \mathcal{M}] = \exp\left(-\int_0^1 E_v\left[\dot{v}_s - \big(\dot{m}_s - E[\dot{m}_s \mid \mathcal{M}_s] \big) \mid \mathcal{M}_s \right] dR_s \right.$$

$$\left. -\frac{1}{2} \int_0^1 \left| E_v\left[\dot{v}_s - \big(\dot{m}_s - E[\dot{m}_s \mid \mathcal{M}_s] \big) \mid \mathcal{M}_s \right] \right|^2 ds \right)$$

and the proof follows from the relation 7.1 of Proposition 6.   □

8. Relations with the Monge's transport map

Assume that the density L is in the class $L \log L(\mu)$. It follows from [5] that there exists an $H - 1$-convex element φ of $\mathbb{D}_{2,1}$ such that the perturbation of identity T defined as

$$T(w) = w + \nabla\varphi(w)$$

maps the Wiener measure μ to $\nu = L \cdot \mu$ and also there is another map $S = I_W + \nabla\psi$, $\psi \in \mathbb{D}_{2,1}$ also $H - 1$-convex such that

$$\mu\big(\{w: \ S \circ T(w) = w\} \big) = 1$$

and

$$\nu\big(\{w: \ T \circ S(w) = w\} \big) = 1.$$

In particular, whenever μ and ν are equivalent, then T and S are inverse to each other μ-almost surely. Let us remark that neither T nor S are adapted to the filtration $(\mathcal{F}_t)$. We shall assume in

the sequel that L is μ-almost surely strictly positive and represented as before as an exponential density $L = \rho(-\delta v)$. Let us denote by $(\mathcal{T}_t, \ t \in [0,1])$ the filtration generated by $(T_t, \ t \in [0,1])$, where T_t is defined as $T_t(w) = w(t) + \nabla \varphi(t)$ with $\nabla \varphi(t) = \int_0^t D_s \varphi \, ds$. We have

Theorem 11. *Assume further that* $L \in L^{1+\varepsilon}(\mu)$ *for some* $\varepsilon > 0$, *then* T *is a* μ-*semimartingale with respect to* $(\mathcal{T}_t)$ *and it has the following decomposition*:

$$T_t = B_t + \int\limits_0^t \left(\frac{E[D_s L \mid \mathcal{F}_s]}{E[L \mid \mathcal{F}_s]} \right) \circ T \, ds, \tag{8.1}$$

where $B = (B_t)$ *is a* $(\mathcal{T}_t)$-*Brownian motion. Moreover* (8.1) *can be also expressed as*

$$T_t = B_t - \int\limits_0^t \dot{v}_s \circ T \, ds, \tag{8.2}$$

where $\dot{v}$ *is defined as* $L = \rho(-\delta v)$.

Proof. Since $(W_t, \ t \in [0,1])$ is the canonical Brownian motion, the equality $\mathcal{T}_t = T^{-1}(\mathcal{F}_t)$ is immediate. Consequently, for any positive, measurable function f, we have the following identity:

$$E[f \circ T \mid \mathcal{T}_t] = E_\nu[f \mid \mathcal{F}_t] \circ T.$$

This relation implies that $(T_t, \ t \in [0,1])$ is a $(\mu, (\mathcal{T}_t))$- quasimartingale if and only if $(W_t, \ t \in [0,1])$ is a $(\nu, (\mathcal{F}_t))$-quasimartingale. This latter property is immediate since $V = W + v$ is a $(\nu, (\mathcal{F}_t))$-Brownian motion and $E_\nu[|v|_H^2] = 2E[L \log L] < \infty$. Let us calculate the drift of $(T_t, \ t \in [0,1])$: if θ is a bounded, $\mathcal{F}_t$-measurable cylindrical function, we have, using the integration by parts formula

$$
\begin{aligned}
\frac{1}{h} E\big[(T_{t+h} - T_t)\theta \circ T\big] &= \frac{1}{h} E\big[(W_{t+h} - W_t)\theta L\big] \\
&= \frac{1}{h} E\left[\theta \int\limits_t^{t+h} D_s L \, ds\right] \\
&\to E[\theta D_t L] \\
&= E\big[\theta E[D_t L \mid \mathcal{F}_t]\big] \\
&= E\left[\theta E[D_t L \mid \mathcal{F}_t] \frac{L}{L_t}\right] \\
&= E\left[\theta \circ T \frac{E[D_t L \mid \mathcal{F}_t]}{L_t} \circ T\right],
\end{aligned}
$$

as $h \to 0$, where $L_s = E[L \mid \mathcal{F}_s]$. Moreover, the local martingale part is a continuous process with $\langle B^i, B^j \rangle_t = \delta_{i,j} t$, hence it is a Brownian motion and (T_t) has the decomposition given

by the formula (8.1) which is equivalent to the decomposition given by (8.2). In fact L can be represented as

$$L = 1 + \int_0^1 E[D_s L \mid \mathcal{F}_s] \, dW_s.$$

On the other hand from the Itô's formula, we have

$$L = 1 - \int_0^1 \dot{v}_s L_s \, dW_s$$

hence $L_s \dot{v}_s = -E[D_s L \mid \mathcal{F}_s] \, ds \times d\mu$-almost surely. $\square$

Remark 8. We could have guessed this theorem by observing simply that the mapping $B = V \circ T$ preserves the Wiener measure due to the Girsanov theorem. Therefore the process $(t, w) \to B(w)(t)$ is a Brownian motion with respect to its own filtration. However the theorem says that it is also a Brownian motion with respect to the larger filtration $(\mathcal{T}_t, \ t \in [0, 1])$.

Theorem 12. *Assume that $L = \rho(-\delta v)$ satisfies the hypothesis of Theorem 11, let $V = I_W + v$. The map V is not invertible, i.e., the equation*

$$U_t = W_t - \int_0^t \dot{v}_s \circ U \, ds \tag{8.3}$$

has no strong solution if and only if the equation

$$T_t = B_t - \int_0^t \dot{v}_s \circ T \, ds \tag{8.4}$$

has no strong solution.

Proof. Assume that T is a strong solution, then by definition T should be adapted to the filtration of the Brownian motion $B = (B_t)$, hence it is of the form $T = \hat{T} \circ B$. Then

$$1 = \mu\{B = \hat{T} \circ B + v \circ \hat{T} \circ B\}$$
$$= \mu\{w = \hat{T}(w) + v \circ \hat{T}(w)\}$$
$$= \mu\{w : V \circ \hat{T}(w) = w\} = \mu(D),$$

hence $\hat{T}$ is a right inverse to V. Moreover, for any $f \in C_b(W)$,

$$E[f \circ \hat{T}] = E[f \circ \hat{T} \circ B] = E[f \circ T] = E[fL].$$

Therefore $\hat{T}\mu$ is equivalent to μ. Since

$$1_{\hat{T}(D)} \circ \hat{T} \geqslant 1_D,$$

we obtain $\mu(\hat{T}(D)) = 1$ which means that $\hat{T}$ is almost surely surjective, consequently it is also a left inverse and it follows from Proposition 1 that $\hat{T}$ is a strong solution to Eq. (8.3), which is a contradiction. To show the sufficiency suppose that Eq. (8.3) has a strong solution U, then U and V are inverse to each other almost surely. moreover $B = V \circ T$ is also invertible hence $U = T \circ B^{-1}$ is $(\mathcal{F}_t)$-adapted and this implies that T is $(B^{-1}(\mathcal{F}_t))$-adapted, consequently Eq. (8.4) has a strong solution which is a contradiction. $\square$

9. Variational techniques for representability and invertibility

In this section we shall derive a necessary and sufficient condition for a large class of adapted perturbation of identity. We begin with some technical results:

Lemma 2. *Assume that $f \in \mathbb{D}_{2,1}$ and $\eta \in \mathbb{D}_{2,0}^a(H)$ such that $|\eta|_H \in L^\infty(\mu)$. Then we have*

$$f(w + \eta(w)) = f(w) + \int_0^1 \nabla_\eta f(w + t\eta(w))\, dt$$

μ-almost surely.

Proof. If f is Fréchet differentiable or if it is $H - C^1$, then the identity is obvious. Assume that $(f_n,\ n \geqslant 1)$ is a sequence of such functions converging to f in $\mathbb{D}_{2,1}$ and denote $I_W + \eta$ by T_η. Then we have on the one hand

$$E\big[|f_n \circ T_\eta - f_m \circ T_\eta|\big] = E\left[|f_n - f_m| \frac{dT_\eta\mu}{d\mu}\right]$$

$$\leqslant E\big[|f_n - f_m|^2\big]^{1/2} E\left[\left(\frac{dT_\eta\mu}{d\mu}\right)^2\right]^{1/2}.$$

From Lemma 1, we have

$$E\left[\left(\frac{dT_\eta\mu}{d\mu}\right)^2\right] = E\left[\left(\frac{dT_\eta\mu}{d\mu}\right) \circ T_\eta\right]$$

$$= E\left[\frac{1}{E[\rho(-\delta\eta) \mid T_\eta]}\right]$$

$$\leqslant E\left[\frac{1}{\rho(-\delta\eta)}\right]$$

$$= E\left[\exp\left(\delta\eta + \frac{1}{2}|\eta|_H^2\right)\right] < \infty$$

since $|\eta|_H \in L^\infty(\mu)$. Hence we get that

$$\lim_{n,m\to\infty} E\big[|f_n \circ T_\eta - f_m \circ T_\eta|\big] = 0.$$

Similarly

$$
\begin{aligned}
E \int_0^1 |\nabla_\eta f_n - \nabla_\eta f_m|_H \circ T_{t\eta} \, dt &= E\left[|\nabla_\eta f_n - \nabla_\eta f_m|_H \int_0^1 \frac{dT_{t\eta}\mu}{d\mu} \, dt \right] \\
&\leqslant \|f_n - f_m\|_{2,1} \left(E \int_0^1 \left(\frac{dT_{t\eta}\mu}{d\mu} \right)^2 \right)^{1/2} \\
&\leqslant \|f_n - f_m\|_{2,1} \left(E \int_0^1 \exp\left(t\delta\eta + \frac{t^2}{2}|\eta|_H^2 \right) dt \right)^{1/2} \\
&\to 0
\end{aligned}
$$

as $n, m \to \infty$. $\square$

Corollary 4. *Assume that $f \in \mathbb{D}_{2,1}$ is $\mathcal{F}_{t_0}$-measurable for some fixed $t_0 < 1$. Then the conclusion of Lemma 2 holds for any $u \in \mathbb{D}_{2,0}^a(H)$.*

Proof. Let (τ_n) be a sequence of stopping times increasing to infinity such that $|u^{\tau_n}|$ is essentially bounded where u^{τ_n} is defined as

$$u^{\tau_n}(t) = \int_0^t 1_{[0,\tau_n]}(s)\dot{u}_s \, ds.$$

From Lemma 2, it follows trivially that

$$f\big(w + u^{\tau_n}(w)\big) = f(w) + \int_0^1 \big(\nabla f\big(w + tu^{\tau_n}(w)\big), u^{\tau_n}(w)\big)_H \, dt,$$

moreover, on the set $\{\tau_n > t_0\}$, we have $f(w + u^{\tau_n}(w)) = f(w + u(w))$ and

$$\big(\nabla f\big(w + tu^{\tau_n}(w)\big), u^{\tau_n}(w)\big)_H = \big(\nabla f\big(w + tu(w)\big), u(w)\big)_H$$

almost surely. $\square$

Theorem 13. *Assume that $v \in \mathbb{D}_{2,2}^a(H)$ such that $|v|_H \in L^\infty(\mu)$ and that*

$$E\big[\exp \varepsilon \|\nabla v\|_{\mathrm{op}}^2\big] < \infty$$

for some $\varepsilon > 0$, where $\|\nabla v\|_{\mathrm{op}}$ denotes the operator norm of ∇v. If the following infimum

$$\inf\left(\frac{1}{2}E\big[|\xi + v \circ (I_W + \xi)|^2\big]:\ \xi \in \mathbb{D}_{2,0}^a(H)\right),$$

is attained for some u, then its value is zero and $U = I_W + u$ is inverse of the shift $I_W + v$.

Proof. The main point is to show the validity of the variational formula:

$$v\big(w + u(w) + \eta(w)\big) = v\big(w + u(w)\big) + \int_0^1 \nabla_\eta v\big(w + u(w) + t\eta(w)\big)\, dt \qquad (9.1)$$

almost surely where $\eta \in \mathbb{D}_{2,0}^a(H)$ with $|\eta|_H \in L^\infty(\mu)$ and that these terms are properly integrable in such a way that the Gâteaux derivative at u of $F(u)$ is well defined. Let us denote by v_n the regularization of v defined as $P_{1/n}v$, where $P_{1/n}$ is the Ornstein–Uhlenbeck semigroup. Since v_n is H-differentiable, we get trivially the identity:

$$v_n\big(w + u(w) + \eta(w)\big) = v_n\big(w + u(w)\big) + \int_0^1 \nabla_\eta v_n\big(w + u(w) + t\eta(w)\big)\, dt. \qquad (9.2)$$

By the Jensen inequality we have

$$\sup_n E\big[\exp \varepsilon \|\nabla v_n\|_{\mathrm{op}}\big] < \infty. \qquad (9.3)$$

Let us denote by T_t the shift $I_W + u + t\eta$. Then

$$E\int_0^1 |\nabla_\eta v_n \circ T_t|_H\, dt \leqslant \|\eta\|_{L^\infty(\mu)} E \int_0^1 \|\nabla v_n\|_{\mathrm{op}} l_t\, dt$$

where l_t is the Radon–Nikodym derivative of $T_t\mu$ with respect to μ. Using the Young inequality for the dual convex functions exp and $x\log x$ we obtain, for any $\kappa > 0$,

$$\|\nabla v_n\|_{\mathrm{op}} l_t \leqslant \exp \kappa \|\nabla v_n\|_{\mathrm{op}} + \frac{1}{\kappa} l_t \log l_t. \qquad (9.4)$$

It is clear that, from the hypothesis and the Jensen lemma, the sequence $(\exp \kappa \|\nabla v_n\|,\ n \geqslant 1)$ is uniformly integrable for small $\kappa > 0$. From Lemma 1

$$l_t \circ T_t E\big[\rho\big(-\delta(u + t\eta)\big)\,\big|\, T_t\big] \leqslant 1,$$

hence

$$E[l_t \log l_t] = E[\log l_t \circ T_t]$$
$$\leqslant E\big[-\log E\big[\rho\big(-\delta(u + t\eta)\big) \mid T_t\big]\big]$$
$$\leqslant E\big[-\log \rho\big(-\delta(u + t\eta)\big)\big]$$
$$= \frac{1}{2} E\big[|u + t\eta|_H^2\big]$$
$$\leqslant E\big[|u|_H^2\big] + E\big[|\eta|_H^2\big].$$

Hence $(l_t, \ t \in [0, 1])$ is uniformly integrable, but we also need to prove the uniform integrability of $(l_t \log l_t, \ t \in [0, 1])$. For this, let A be any measurable subset of W, we have, again from Lemma 1,

$$E[1_A l_t \log l_t] = E[1_A \circ T_t \log l_t \circ T_t]$$
$$= E\big[1_A \circ T_t\big(-\log E\big[\rho\big(-\delta(u + t\eta)\big) \mid T_t\big]\big)\big]$$
$$\leqslant E\Big[1_A \circ T_t\Big(\delta(u + t\eta) + \frac{1}{2}|u + t\eta|_H^2\Big)\Big]$$
$$\leqslant E\big[1_A \circ T_t \delta(u + t\eta)\big] + E\Big[1_A \circ T_t \frac{1}{2}|u + t\eta|_H^2\Big].$$

The last two terms are equivalent, hence it suffices to show that the second terms can be chosen arbitrarily small by choosing $\mu(A)$ small enough. However this is obvious from the integrability of $|u|_H^2$ and from the uniform integrability of $(l_t, \ t \in [0, 1])$. From this and from the inequality (9.3), we see that the left-hand side of (9.4) is uniformly integrable. Consequently we can pass to the limit in the relation (9.2) in $L^1(\mu)$ and obtain the relation (9.1). We can now calculate the Gâteaux derivative of F at u in any direction $\eta \in \mathbb{D}_{2,0}^a(H)$ with $|\eta|_H \in L^\infty(\mu)$ (instead of $\eta \circ U$) as follows:

$$F(u + \lambda\eta) - F(u)$$
$$= E \int_0^\lambda \big(u + t\eta + v \circ (I_W + u + t\eta), (I_H + \nabla v) \circ (I_W + u + t\eta)[\eta]\big)_H \, dt. \qquad (9.5)$$

Let us remark that

$$E\big[|u|_H \big\|\nabla v \circ (I_W + u + t\eta)\big\|_{\mathrm{op}}\big]$$
$$\leqslant E\big[|u|_H^2\big]^{1/2} E\big[\big\|\nabla v \circ (I_W + u + t\eta)\big\|_{\mathrm{op}}^2\big]^{1/2}$$
$$\leqslant E\big[|u|_H^2\big]^{1/2} E\Big[\exp \varepsilon \|\nabla v\|_{\mathrm{op}}^2 + \frac{1}{\varepsilon} l_{t\eta,u} \log l_{t\eta,u}\Big]^{1/2}, \qquad (9.6)$$

where

$$l_{t\eta,u} = \frac{d(I_W + u + t\eta)\mu}{d\mu}$$

and from Lemma 1, we know that

$$E[l_{t\eta,u} \log l_{t\eta,u}] \leqslant \frac{1}{2} E\big[|u + t\eta|_H^2\big].$$

Hence we can commute the expectation with the Lebesgue integral in the formula (9.5). Let us denote the expectation of the integrand of (9.5) by $F'(u + t\eta)[\eta]$. Since $v \in \mathbb{D}_{2,2}^a(H)$, using the formula (9.1) for ∇v instead of v and the inequality (9.6), we see that the map $t \to F'(u + t\eta)[\eta]$ is continuous on $[0, 1]$. Since u is minimal, we should have $F'(u)[\eta] \geqslant 0$ for any η as above. Writing the things explicitly:

$$
\begin{aligned}
F'(u)[\eta] &= E\big[\big(u + v \circ U, (I_H + \nabla v \circ U)\eta\big)_H\big] \\
&= E\big[\big((I_H + \nabla v \circ U)^\star (u + v \circ U), \eta\big)_H\big] \\
&\geqslant 0.
\end{aligned}
$$

By the invertibility of $I_H + \nabla v$, we get

$$u + v \circ U = 0$$

almost surely and this is equivalent to the fact that $U = I_W + u$ and $V = I_W + v$ are inverse to each other. In particular $F(u) = 0$. $\quad\square$

As an application of these kind of variational calculations in relation with the representability, consider the problem of calculation of

$$\inf\left(E\left[\frac{1}{2}|\alpha|_H^2 + f \circ (I_W + \alpha)\right] : \alpha \in \mathbb{D}_{2,0}^a(H)\right),$$

where $f : W \to \mathbb{R}$ is a fixed Wiener functional. In fact, as it is shown in [2], this infimum is equal to $-\log E[\exp - f]$ which is also equal to

$$\inf\left(\int_W f \, d\gamma + \int_W \frac{d\gamma}{d\mu} \log \frac{d\gamma}{d\mu} \, d\mu \right) \tag{9.7}$$

where the infimum is taken w.r.to all the probability measures on $(W, \mathcal{B}(W))$ and the latter is uniquely attained at

$$d\gamma_0 = \frac{1}{\int e^{-f} d\mu} e^{-f} \, d\mu.$$

In the next theorem we shall give sufficient conditions under which it is attained:

Theorem 14. *Assume that $f \in \mathbb{D}_{2,1}$ is a 1-convex, bounded Wiener functional such that*

$$E\big[\exp \varepsilon |\nabla f|_H\big] < \infty,$$

for some $\varepsilon > 0$. Then the infimum

$$\inf\left(E\left[\frac{1}{2}|\alpha|_H^2 + f \circ (I_W + \alpha) \right] \colon \alpha \in \mathbb{D}_{2,0}^a(H) \right)$$

is attained at some $u \in \mathbb{D}_{2,0}^a(H)$ and this adapted vector field satisfies the following relation:

$$\dot{u}_t + E[D_t f \circ U \mid \mathcal{F}_t] = 0$$

$dt \times d\mu$-almost surely, where $U = I_W + u$. Besides we have

(1)
$$\frac{dU\mu}{d\mu} = \exp\left(-\int_0^1 E_{U\mu}[D_t f \mid \mathcal{F}_t] \, dW_t - \frac{1}{2}\int_0^1 \left| E_{U\mu}[D_t f \mid \mathcal{F}_t] \right|^2 dt \right),$$

where $E_{U\mu}$ denotes the expectation with respect to the measure $U\mu$, i.e., the image of μ under U.

(2) *Let $\dot{v}_t = E_{U\mu}[D_t f \mid \mathcal{F}_t]$, denote by Z the innovation process associated to U, i.e., $Z_t = U_t - \int_0^t E[\dot{u}_s \mid \mathcal{U}_s] \, ds$, and define l as*

$$l = \exp\left(-\int_0^1 E[\dot{u}_t \mid \mathcal{U}_t] \, dZ_t - \frac{1}{2}\int_0^1 \left| E[\dot{u}_t \mid \mathcal{U}_t] \right|^2 dt \right),$$

where $\mathcal{U}_t$ is the sigma algebra $U^{-1}(\mathcal{F}_t) = \sigma(W_s + u(s), \ s \leqslant t)$. Then $E[l] = 1$ and we have

$$l\frac{dU\mu}{d\mu} \circ U = l\rho(-\delta v) \circ U = 1$$

almost surely.

Proof. Let $J(\alpha)$ the expectation above without inf. For $\lambda > 0$, let $D_\lambda = \{\alpha \in \mathbb{D}_{2,0}^a(H) \colon J(\alpha) \leqslant \lambda\}$. Then, for sufficiently large λ, D_λ is a non-empty, convex set. Moreover, if $(\alpha_n, n \geqslant 1) \subset D_\lambda$ converges to some α in $\mathbb{D}_{2,0}^a(H)$, then, writing $A_n = I_W + \alpha_n$, we have

$$E\left[\frac{dA_n\mu}{d\mu} \log \frac{dA_n\mu}{d\mu} \right] \leqslant \frac{1}{2} E\left[|\alpha_n|_H^2 \right].$$

Hence the sequence of Radon–Nikodym densities $(\frac{dA_n\mu}{d\mu}, n \geqslant 1)$ is uniformly integrable. This property, combined with Lusin theorem implies that $(f \circ A_n, n \geqslant 1)$ converges to $f \circ A$ in $L^p(\mu)$ for any $p \geqslant 0$, where $A = I_W + \alpha$. Therefore D_λ is closed, since it is convex, it is also weakly closed in $\mathbb{D}_{2,0}^a(H)$. This implies that $\alpha \to J(\alpha)$ is weakly lower semi continuous (l.s.c.). Since D_λ is weakly compact, J attains its infimum on D_λ and the convexity of J implies that this infimum is a global one. The scalar version of Proposition 13 implies that

$$\begin{aligned} 0 &= E\left[(u, \alpha)_H + (\nabla f \circ U, \alpha)_H \right] \\ &= E\left[(u, \alpha)_H + \left(\pi(\nabla f \circ U), \alpha \right)_H \right], \end{aligned}$$

for any bounded $\alpha \in \mathbb{D}_{2,0}^a(H)$, where π denotes the dual predictable projection. Hence we get

$$\dot{u}_t + E[D_t f \circ U \mid \mathcal{F}_t] = 0$$

$dt \times d\mu$-almost surely. Taking the conditional expectation of this relation with respect to $\mathcal{U}_t$, we obtain immediately

$$E[\dot{u}_t \mid \mathcal{U}_t] + E_{U\mu}[D_t f \mid \mathcal{F}_t] \circ U = 0 \tag{9.8}$$

$dt \times d\mu$-almost surely and the expression for $dU\mu/d\mu$ follows from Theorem 5. It is a simple calculation to see that Eq. (9.8) implies

$$l\rho(-\delta v) \circ U = 1$$

almost surely. From the Girsanov theorem, we get

$$1 = E\big[l\rho(-\delta v) \circ U\big] \leqslant E\big[\rho(-\delta v)\big],$$

therefore $E[\rho(-\delta v)] = 1$. Similarly, for any positive, measurable g on W, we have

$$E[g \circ U] = E\big[g \circ U l \rho(-\delta v) \circ U\big] \leqslant E\big[g\rho(-\delta v)\big],$$

therefore

$$\frac{dU\mu}{d\mu} \leqslant \rho(-\delta v),$$

since both are probability densities, they are equal μ-almost surely. To prove $E[l] = 1$ it suffices to write $l = 1/\rho(-\delta v) \circ U$, then

$$
\begin{aligned}
E[l] &= E\left[\frac{1}{\rho(-\delta v)} \circ U\right] \\
&= E\left[\rho(-\delta v).\frac{1}{\rho(-\delta v)}\right] \\
&= 1
\end{aligned}
$$

and this completes the proof. $\quad\square$

Remark 9. Suppose that $\|\nabla^2 f\|_{\mathrm{op}} \leqslant c < 1$ almost surely, where $c > 0$ is a fixed constant and the norm is the operator norm on H. Then the map $\Phi : \mathbb{D}_{2,0}^a(H) \to \mathbb{D}_{2,0}^a(H)$ defined by

$$\Phi(\xi) = -\pi\big(\nabla f \circ (I_W + \xi)\big),$$

where π denotes the dual predictable projection, is a strict contraction, hence there exists a unique $u \in \mathbb{D}_{2,0}^a(H)$ which satisfies the equation

$$\dot{u}_t + E[D_t f \circ U \mid \mathcal{F}_t] = 0$$

$dt \times d\mu$-almost surely.

Corollary 5. *Let $u \in \mathbb{D}_{2,0}^a(H)$ be a minimizer whose existence is assured by Theorem* 14*. Define* $U = I_W + u$. *Then*

$$\frac{dU\mu}{d\mu} = \frac{e^- f}{E[e^- f]} = L$$

if and only if U *is a.s. invertible.*

Proof. Since

$$J(u) = E[fL] + E[L \log L] = E[f \circ U] + \frac{1}{2} E\big[|u|_H^2\big]$$

and since by the hypothesis we have $E[fL] = E[f \circ U]$, we obtain

$$E[L \log L] = \frac{1}{2} E\big[|u|_H^2\big].$$

On the other hand, from Theorem 14,

$$
\begin{aligned}
E[L \log L] &= E[\log L \circ U] \\
&= E[-\log l] \\
&= \frac{1}{2} E\Bigg[\int_0^1 \big|E[\dot{u}_s \mid \mathcal{U}_s]\big|^2 \, ds \Bigg].
\end{aligned}
$$

Consequently, $\dot{u}_s = E[\dot{u}_s \mid \mathcal{U}_s] \, ds \times d\mu$-almost surely. This implies that $E[\rho(-\delta u)] = 1$, hence the hypothesis of Theorem 7 is satisfied and the invertibility of U follows. Conversely, suppose that U is invertible, let M be the Radon–Nikodym density of $U\mu$ w.r.t. μ. Then we have

$$J(u) = \int_W f M \, d\mu + \int_W M \log M \, d\mu,$$

hence $M d\mu = L d\mu$ by the uniqueness of the solution of the minimization problem (9.7). $\quad\square$

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