THE EULER CLASS OF A SUBSET COMPLEX

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Abstract

The subset complex $\Delta(G)$ of a finite group G is defined as the simplicial complex whose simplices are non-empty subsets of G. The oriented chain complex of $\Delta(G)$ gives a $\mathbb{Z}G$-module extension of $\mathbb{Z}$ by $\tilde{\mathbb{Z}}$, where $\tilde{\mathbb{Z}}$ is a copy of integers on which G acts via the sign representation of the regular representation. The extension class $\zeta_G \in \mathrm{Ext}_{\mathbb{Z}G}^{|G|-1}(\mathbb{Z}, \tilde{\mathbb{Z}})$ of this extension is called the Ext class or the Euler class of the subset complex $\Delta(G)$. This class was first introduced by Reiner and Webb [The combinatorics of the bar resolution in group cohomology, *J. Pure Appl. Algebra* **190** (2004), 291–327] who also raised the following question: What are the finite groups for which ζ_G is non-zero?

In this paper, we answer this question completely. We show that ζ_G is non-zero if and only if G is an elementary abelian p-group or G is isomorphic to $\mathbb{Z}/9$, $\mathbb{Z}/4 \times \mathbb{Z}/4$ or $(\mathbb{Z}/2)^n \times \mathbb{Z}/4$ for some integer $n \geq 0$. We obtain this result by first showing that ζ_G is zero when G is a non-abelian group, then by calculating ζ_G for specific abelian groups. The key ingredient in the proof is an observation by Mandell which says that the Ext class of the subset complex $\Delta(G)$ is equal to the (twisted) Euler class of the augmentation module of the regular representation of G.

We also give some applications of our results to group cohomology, to filtrations of modules and to the existence of Borsuk–Ulam type theorems.

1. Introduction

Let G be a finite group and let X be a finite G-set, say of order $n + 1$. The *subset complex* of X is defined as the simplicial complex $\Delta(X)$ whose simplices are non-empty subsets of X. One can choose an orientation on $\Delta(X)$ by choosing an ordering $x_0 < \cdots < x_n$ for elements of X. The oriented chain complex of $\Delta(X)$ augmented by a copy of the trivial module gives an exact sequence of $\mathbb{Z}G$-modules

$$\varepsilon_X: \quad 0 \longrightarrow \tilde{\mathbb{Z}} \longrightarrow C_{n-1}(\Delta(X)) \longrightarrow \cdots \longrightarrow C_0(\Delta(X)) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where $\tilde{\mathbb{Z}}$ is a copy of the trivial module on which G acts via the sign representation of X. Associated to this exact sequence, there is an extension class $\zeta_X \in \mathrm{Ext}_{\mathbb{Z}G}^n(\mathbb{Z}, \tilde{\mathbb{Z}})$. This class was first introduced by Reiner and Webb [**14**] and is called *the Ext class of the subset complex* of X.

In [**14**], it is shown that the extension class ζ_X is an essential class when $X = G/1$ is the transitive G-set with a point stabilizer. In this case, $\Delta(X)$ is the subset complex of the group G, and we denote the associated Ext class by ζ_G. Note that if ζ_G is a non-trivial class, then this means that the group G has

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non-trivial essential cohomology. On the other hand, if ζ_G is zero, then the hypercohomology spectral sequence associated to the subset complex collapses at the E_2-page. Because of these consequences, Reiner and Webb asked for which finite groups $\zeta_G \neq 0$.

In this paper, we give a complete answer to this question. Our starting point is the observation that the extension class ζ_G is, in fact, the (twisted) Euler class of the augmentation module $I_G = \ker\{\mathbb{R}G \to \mathbb{R}\}$. In their paper, Reiner and Webb [14] attribute this observation to Mandell, but they do not provide a proof. Since our arguments are based on this observation, we give a proof of this fact in Section 2. Because of this observation, we sometimes call the Ext class ζ_G, *the Euler class of the subset complex* to emphasize that it is an Euler class.

It has been shown in [14] that ζ_G is zero when G is not a p-group. Therefore throughout the paper we consider only p-groups. The main result of the paper is the following:

THEOREM 1.1 *If G is a finite non-abelian group, then $\zeta_G = 0$.*

The proof follows from a reduction argument. We show that if ζ_G is non-zero then $\zeta_{H/K}$ is non-zero for every subquotient H/K of G. Thus the minimal counterexample to Theorem 1.1 should have all proper subquotients abelian. It is easy to classify non-abelian p-groups whose proper subquotients are all abelian. Such p-groups are either of order p^3 or isomorphic to the modular group M_{p^k} with $k \geq 4$. By direct calculation, we show that the Euler classes of the augmentation modules of these groups are all zero. This proves Theorem 1.1.

In the rest of the paper, we consider abelian p-groups. First we show that ζ_G is zero when G is isomorphic to $\mathbb{Z}/8$ or $(\mathbb{Z}/4)^2 \times \mathbb{Z}/2$, and hence conclude that any group which has a subquotient isomorphic to one of these groups has zero Euler class. This shows that if G is a 2-group with $\zeta_G \neq 0$, then G is either elementary abelian or is isomorphic to $\mathbb{Z}/4 \times \mathbb{Z}/4$ or $(\mathbb{Z}/2)^n \times \mathbb{Z}/4$ for some $n \geq 0$. Then, we show that the Euler class ζ_G for these groups is non-zero. For $p > 2$, we use similar arguments and obtain that ζ_G is non-zero if and only if G is elementary abelian or isomorphic to $\mathbb{Z}/9$. Hence, we conclude the following:

THEOREM 1.2 *Let G be a finite abelian group. Then, ζ_G is non-zero if and only if G is either an elementary abelian p-group or is isomorphic to $\mathbb{Z}/9, \mathbb{Z}/4 \times \mathbb{Z}/4$ or $(\mathbb{Z}/2)^n \times \mathbb{Z}/4$ for some integer $n \geq 0$.*

There are many consequences of Theorems 1.1 and 1.2. In Section 6, we discuss some immediate consequences such as the classification of p-groups for which the mod p-reduction of ζ_G is zero. We also give a complete list of p-groups which has $(\zeta_G)^2 \neq 0$. In Section 7, we give some applications of our results to filtrations of modules. Another application of our results is to the existence of Borsuk–Ulam type theorems which are in turn related to the Tverberg problem in combinatorics. We discuss these in Section 8.

In Section 9, we consider the hypercohomology spectral sequences associated to the boundary of the subset complex. When G is not one of the groups listed in Theorem 1.2, the cohomology spectral sequence collapses at the E_2-page. In particular, the edge homomorphism $H_G^*(\text{pt}, \mathbb{Z}) \to H_G^*(\partial \Delta(G), \mathbb{Z})$ induced by the constant map $\partial \Delta(G) \to \text{pt}$ is injective. Since the G-complex $\partial \Delta(G)$ has no fixed points, this gives, for example, that the essential ideal is nilpotent with nilpotency degree $\leq |G|$. The collapsing of the cohomology spectral sequence also allows one to look at the isotropy spectral sequence to obtain information about the integral cohomology of G.

In Section 10, we consider the barycentric subdivision of the subset complex. The reason for introducing the barycentric subdivision is that as a G-complex, the subset complex is not admissible.

This causes problems when we want to calculate the product structure of the cohomology ring using the hypercohomology spectral sequence associated to the subset complex. Our main result in this section is a formula for the permutation modules appearing on the chain complex of the barycentric subdivision. We use a power series approach which was introduced by Webb in [20] and later used in [14].

2. The twisted Euler class of a real representation

The main purpose of this section is to show that the Ext class ζ_X is equal to the twisted Euler class of the augmentation module $I_X = \ker\{\mathbb{R}X \to \mathbb{R}\}$ after suitable identifications. By twisted Euler class, we mean the Euler class of a not necessarily orientable real vector bundle. We first discuss the notion of the twisted Euler class of a real representation.

Given a real representation V of a finite group G, we can form a real vector bundle $EG \times_G V \to BG$ using the Borel construction. The Euler class of V is defined as the Euler class of this vector bundle. In general this bundle is not orientable, but there is a notion of orientation for non-orientable bundles and depending on the choice of orientation, one defines a twisted Euler class in a similar way the usual Euler class is defined. Although the definition and the properties of a twisted Euler class are standard, it is hard to find in the literature. A short note on the twisted Euler class of a real representation is given in [6, Appendix]. We include some of this material here for the convenience of the reader and also to introduce the notation.

Let B be a connected (pointed) CW-complex and $\xi : V \to E \xrightarrow{\pi} B$ be an n-dimensional real vector bundle over B. The Stiefel–Whitney class $w_1(\xi) \in H^1(B, \mathbb{Z}/2)$ can be considered as a homomorphism $\pi_1(B) \to \{\pm 1\}$. Let $\mathbb{Z}(\xi)$ denote the one-dimensional integral representation of $\pi_1(B)$, where the action is given by $w_1(\xi)$. When $w_1(\xi)$ is trivial the bundle is called orientable. If $w_1(\xi)$ is not trivial, then we say the bundle is non-orientable. In this case, $H^n(\pi^{-1}(b_0), \pi^{-1}(b_0) \setminus \{0\})$ is isomorphic to $\mathbb{Z}(\xi)$ as a $\pi_1(B)$-module. An orientation of ξ is defined as a $\pi_1(B)$-module isomorphism

$$\omega : H^n(\pi^{-1}(b_0), \pi^{-1}(b_0)\setminus\{0\}) \to \mathbb{Z}(\xi).$$

For each vector bundle, there are two orientations, ω and $-\omega$, which are opposite to each other. A bundle ξ on which the orientation ω is fixed is denoted by ξ^ω. We sometimes call such a bundle oriented, although it may not be an orientable bundle in the above sense. For an oriented bundle ξ^ω, the Thom isomorphism

$$\phi_L : H^r(B, L) \longrightarrow H^{r+n}(E, E_0; L \otimes \mathbb{Z}(\xi))$$

is given by $\phi_L(x) = \pi^*(x) \cdot U$, where U is the Thom class and L is a coefficient bundle over B. Here the map $\pi^* : H^*(B, L) \to H^*(E, L)$ is induced from the projection $\pi : E \to B$. The *Euler class* of ξ^ω is defined by

$$e(\xi^\omega) = \phi_{\mathbb{Z}(\xi)}^{-1}(U^2) \in H^n(B, \mathbb{Z}(\xi)).$$

Sometimes this Euler class is referred to as the twisted Euler class to emphasize that the coefficients are twisted, but we will not make this distinction here. The following are some basic properties of the Euler class.

LEMMA 2.1 *Let ξ^ω and η^ν be two oriented real vector bundles of dimension n and m, respectively. Then,*

(1) $e(\xi^\omega) = -e(\xi^{-\omega})$.
(2) *The mod 2 reduction of $e(\xi^\omega)$ is equal to the top Stiefel–Whitney class of ξ.*
(3) $e(\xi^\omega)e(\eta^\nu) = e((\xi \oplus \eta)^{\omega \oplus \nu})$.

Proof. The proofs of these statements are standard and can be found, for example, in [**11**]. Note that in the third statement to get the equality the cup product on the left is defined as the composition

$$H^n(B, \mathbb{Z}(\omega)) \otimes H^m(B, \mathbb{Z}(\nu)) \to H^{n+m}(B, \mathbb{Z}(\omega) \otimes \mathbb{Z}(\nu)) \to H^{n+m}(B, \mathbb{Z}(\omega \oplus \nu)),$$

where the second map is induced by the canonical isomorphism $\mathbb{Z}(\omega) \otimes \mathbb{Z}(\nu) \cong \mathbb{Z}(\omega \oplus \nu)$, and $\omega \oplus \nu$ is the orientation of $\xi \oplus \eta$ given by $w_1(\xi) + w_1(\eta)$.

For a real representation V of G, the first Stiefel–Whitney class of V is defined as the first Stiefel–Whitney class of the bundle $\xi_V : EG \times_G V \to BG$ and is given by the composition

$$\mathrm{sgn}(V) : G \to GL(V) \xrightarrow{\det} \mathbb{R}^\times.$$

This one-dimensional representation is usually called the *sign representation* of V. The bundle ξ_V associated to V is orientable if and only if the sign representation of V is trivial. Note that choosing an orientation $\omega : H^n(V, V \setminus \{0\}) \to \mathbb{Z}(\xi_V)$ for a (not necessarily orientable) bundle ξ_V is the same as choosing an orientation for V. If V is a real representation with a fixed orientation, then we denote the module $\mathbb{Z}(\xi_V)$ by $\tilde{\mathbb{Z}}$, and the associated Euler class by $e(V)$.

The Euler class can also be defined as the first obstruction to the existence of a non-zero section. Let us choose an arbitrary G-invariant inner product on V and let $S(V)$ be the set of all unit vectors in V with respect to this inner product. Associated to the G-space $S(V)$, there is a sphere bundle $EG \times_G S(V) \to BG$ with fibers $S(V)$.

LEMMA 2.2 *Let V be an n-dimensional real representation of G with a fixed orientation. The Euler class $e(V) \in H^n(BG, \tilde{\mathbb{Z}})$ is the first obstruction for an existence of a section for the sphere bundle $EG \times_G S(V) \to BG$ associated to $S(V)$. Equivalently, the Euler class $e(V)$ is the first obstruction $o_n \in H_G^n(EG, \tilde{\mathbb{Z}})$ for finding a G-map $f : EG \to S(V)$.*

Proof. See Milnor and Stasheff [**11**].

Given a finite G-CW-complex Y which has the homology of a sphere, say of dimension $n - 1$, there is a concept of polarization which is commonly used to fix a k-invariant for the complex. A polarization of Y is a pair of isomorphisms $\varphi : H_0(Y) \to \mathbb{Z}$ and $\psi : H_{n-1}(Y) \to \tilde{\mathbb{Z}}$, and associated to each polarization there is a unique k-invariant $\zeta \in H^n(BG, \tilde{\mathbb{Z}})$ defined as follows: Note that given a polarized G-CW-complex Y with polarizations φ and ψ, we get an extension of $\mathbb{Z}G$-modules of the form

$$\varepsilon_V : 0 \to \tilde{\mathbb{Z}} \to C_{n-1}(Y) \to \cdots \to C_0(Y) \to \mathbb{Z} \to 0,$$

using the polarizations at the ends of the extension to get $\mathbb{Z}$ and $\tilde{\mathbb{Z}}$. This defines a unique extension class $\zeta(Y, \varphi, \psi) \in \mathrm{Ext}_{\mathbb{Z}G}^n(\mathbb{Z}, \tilde{\mathbb{Z}})$. The k-invariant is defined as the corresponding class in $H^n(BG, \tilde{\mathbb{Z}})$.

It is easy to see that if one fixes the polarization $\varphi : H_0(Y) \to \mathbb{Z}$ to be the one given by augmentation map $C_0(Y) \to \mathbb{Z}$, then the choice of the second polarization corresponds to the choice of orientation for the chain complex. In the case of the unit sphere $S(V)$ of a real representation V, the polarization

$\psi : H_{n-1}(S(V)) \to \mathbb{Z}$ corresponds to an orientation of V. If V comes with a fixed orientation, then there is a unique extension class $\zeta_V \in \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, \tilde{\mathbb{Z}})$ associated to $S(V)$. The following is a standard result known by experts in the field.

PROPOSITION 2.3 *Let V be an n-dimensional real representation of G with a fixed orientation. Let $\tilde{\mathbb{Z}}$ denote the one-dimensional integral representation induced from the sign representation of V (see the definition given above). Consider the exact sequence*

$$\varepsilon_V : \ 0 \longrightarrow \tilde{\mathbb{Z}} \longrightarrow C_{n-1}(S(V)) \longrightarrow \cdots \longrightarrow C_0(S(V)) \longrightarrow \mathbb{Z} \longrightarrow 0$$

obtained by applying the polarizations coming from the fixed orientation of V. Let $\zeta_V \in \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, \tilde{\mathbb{Z}})$ be the extension class of this extension. Then, the image of ζ_V is equal to the Euler class $e(V)$ under the canonical isomorphism $\mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, \tilde{\mathbb{Z}}) \cong H^n(G, \tilde{\mathbb{Z}})$.

Proof. Since $\pi_i(S(V)) = 0$ for $i \leq n-1$, we can construct a G-map $f_{n-1} : EG^{(n-1)} \to S(V)$ in such a way that the induced map on the 0th homology is identity. By Lemma 2.2, the obstruction to extending f_{n-1} to a G-map $f_n : EG^{(n)} \to S(V)$ is the Euler class $e(V)$. By obstruction theory, this obstruction class is represented by a cocycle in $\mathrm{Hom}_G(C_n(EG), H_{n-1}(S(V)))$ which is defined by the composition

$$o_n : C_n(EG) = \oplus_{\sigma_n} H_n(\sigma_n, \partial\sigma_n) \xrightarrow{\partial} \oplus_{\sigma_n} H_{n-1}(\partial\sigma_n) \xrightarrow{H_{n-1}(f)} H_{n-1}(S(V)).$$

Now, consider the following commutative diagram:

It is clear from this diagram that o_n is the lifting of the identity, so $e(V)$ corresponds to the extension class of the bottom extension under the isomorphism $H^n(G, H_{n-1}(S(V))) \cong \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, H_{n-1}(S(V)))$. Note that since V has a fixed orientation, there is a canonical isomorphism $H_{n-1}(S(V), \mathbb{Z}) \cong \tilde{\mathbb{Z}}$ which we can use to replace $H_{n-1}(S(V), \mathbb{Z})$ with $\tilde{\mathbb{Z}}$ in the above argument. So, $e(V)$ corresponds to ζ_V under the isomorphism $\mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, \tilde{\mathbb{Z}}) \cong H^n(G, \tilde{\mathbb{Z}})$.

Now, we are ready to show that the extension class ζ_X is the same as the Euler class of the augmentation module $I_X = \ker\{\mathbb{R}X \to \mathbb{R}\}$. Recall that in Section 1 we defined ζ_X as the extension class of the extension

$$\varepsilon_X : 0 \longrightarrow \tilde{\mathbb{Z}} \longrightarrow C_{n-1}(\Delta(X)) \longrightarrow \cdots\cdots \longrightarrow C_0(\Delta(X)) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where $\Delta(X)$ is the subset complex of X. It is easy to see that this extension is equivalent to the extension

$$0 \longrightarrow H_{n-1}(\partial\Delta(X)) \longrightarrow C_{n-1}(\partial\Delta(X)) \longrightarrow \cdots \longrightarrow C_0(\partial\Delta(X)) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where $\partial\Delta(X)$ denotes the boundary of the subset complex $\Delta(X)$. For the boundary of the subset complex, we have the following observation.

LEMMA 2.4 ([**2**, Lemma 2.2]) *Let G be a finite group and X be a finite G-set. Suppose that $S(I_X)$ denotes the unit sphere of the augmentation ideal $I_X = \ker\{\mathbb{R}X \to \mathbb{R}\}$ and $|\partial\Delta(X)|$ denotes the realization of the boundary of the subset complex $\Delta(X)$. Then, there is a G-homeomorphism between the topological spaces $S(I_X)$ and $|\partial\Delta(X)|$.*

Proof. Let $\{x_0, \ldots, x_n\}$ be the set of elements of X. We can regard I_X as the normal space of the vector $(1, \ldots, 1)$ and x_i as the ith unit vector. Let v_i be the unit vector of the projection of x_i into I_X. Then the set $\{v_0, \ldots, v_n\}$ is an affinely independent set of vectors in $S(I_X)$. Let $\Delta'(X)$ be the n-simplex with vertex set $\{v_0, \ldots, v_n\}$. Let us define a map $\phi : \Delta(X) \to \Delta'(X)$ by $\phi(x_i) = v_i$. It is easy to see that ϕ is a G-homeomorphism and it sends $\partial\Delta(X)$ to the boundary of $\Delta'(X)$, hence it induces a G-homeomorphism between the associated topological spaces $S(I_X)$ and $|\partial\Delta(X)|$. ∎

Now, we are ready to prove our main theorem in this section.

THEOREM 2.5 *Let G be a finite group and X be a finite G-set. Then the Ext class ζ_X is equal to the Euler class $e(I_X)$ of the augmentation module I_X under the canonical isomorphism $\mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, \tilde{\mathbb{Z}}) \cong H^n(G, \tilde{\mathbb{Z}})$.*

Proof. Fix an ordering of elements in X so that we have a fixed orientation throughout. By Lemma 2.4, the chain complexes of $\partial\Delta(X)$ and $S(I_X)$ are chain homotopic. This means that the extension class ζ_X also represents the following exact sequence

$$0 \longrightarrow \tilde{\mathbb{Z}} \longrightarrow C_{n-1}(S(I_X)) \longrightarrow \cdots \longrightarrow C_0(S(I_X)) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

However by Proposition 2.3, this extension is represented by the Euler class $e(I_X)$. Therefore, the image of ζ_X is equal to $e(I_X)$ under the isomorphism $\mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, \tilde{\mathbb{Z}}) \cong H^n(G, \tilde{\mathbb{Z}})$. ∎

The above theorem reduces Reiner and Webb's question to a question about the Euler class of the augmentation module. Since the augmentation module decomposes to irreducible real representations, this observation makes it much easier to calculate the Ext class of the subset complex. In fact, from now on we will call the Ext class, *the Euler class of the subset complex* to emphasize on the fact that it is actually an Euler class.

Finally, we would like to note that, although in our calculations we use the group cohomology with twisted coefficients, the Euler class itself often lies in the integral cohomology of G (with no twisting). In fact, Reiner and Webb showed that ζ_G lies in the cohomology with twisted coefficients if and only if G has a non-trivial cyclic Sylow 2-subgroup (see [**14**, Lemma 5.4]). In particular, if G is a p-group, then the Euler class is twisted only when G is a cyclic 2-group in which case the coefficients are given by the unique non-trivial map $G \to \mathbb{Z}/2$.

3. Proof of Theorem 1.1

The key ingredient in the proof of Theorem 1.1 is the following reduction argument.

PROPOSITION 3.1 *If the Euler class ζ_G is non-zero, then the Euler class $\zeta_{H/K}$ is non-zero for every subquotient H/K of G.*

Proof. In [**14**], it is shown that for any subgroups $K \leq H \leq G$, the extension class $\zeta_{G/K}$ is equivalent to the cup product of $\zeta_{G/H}$ and $\mathcal{N}_H^G(\zeta_{H/K})$, where $\mathcal{N}$ denotes the Evens' norm map (see, [**14**, Proposition 7.13]). Therefore, for subgroups $1 \leq K \leq H \leq G$ we have

$$\zeta_G = \zeta_{G/H} \cdot \mathcal{N}_H^G(\zeta_{H/K} \cdot \mathcal{N}_K^H(\zeta_K)).$$

From this it follows immediately that if $\zeta_{H/K} = 0$ for some subquotient H/K of G, then $\zeta_G = 0$.

Proposition 3.1 implies that a minimal counterexample to Theorem 1.1 must be a non-abelian p-group whose proper subquotients are all abelian. First we classify such groups and then we show that $\zeta_G = 0$ for all groups in the list. This means that there cannot be any counterexamples to Theorem 1.1, hence it completes the proof. The classification of all non-abelian p-groups whose proper subquotients are all abelian is given as follows:

PROPOSITION 3.2 *Let G be a non-abelian p-group whose proper subquotients are all abelian. Then either G has order p^3 or G is isomorphic to the modular p-group M_{p^k} for some $k \geq 4$.*

Proof. Since every group of order p^2 is abelian, non-abelian groups of order p^3 obviously satisfy the assumption of the theorem. So, let us assume that G is a non-abelian p-group of order $|G| > p^3$ whose proper subquotients are all abelian. Let c be a central element of order p in G. Since $G/\langle c \rangle$ is abelian and G is non-abelian, $\langle c \rangle$ is the commutator group of G. Similarly, any central subgroup of order p is the commutator group and hence G has only one central subgroup of order p. This means that the center $Z(G)$ of G is cyclic. Note that G has an element of order p which is not central because otherwise G has a unique subgroup of order p which implies that G is either cyclic or a generalized quaternion group (see, [**3**, Theorem 4.3]). But these groups do not satisfy our starting assumption. So, G has an element of order p which is not central, say a. Let s be an element of G which does not commute with a. Since the subgroup generated by s and a is non-abelian, we must have $G = \langle a, s \rangle$. Note that $asa^{-1}s^{-1} = c^t$ for some $t \not\equiv 0 \mod p$. This gives $as^p a^{-1} = c^{tp} s^p = s^p$, so s^p is central in G. This forces the Frattini subgroup of G to be the subgroup generated by c and s^p. Thus, the Frattini subgroup is central, and hence cyclic. Since $|G| > p^3$, the element s^p cannot be trivial. So, we have $c^r = s^{p^{k-2}}$ for some $r \not\equiv 0 \mod p$, where p^{k-1} is the order of s. Note that we

can also assume $r = t = 1$ by replacing a and c with appropriate powers of themselves. Therefore, G has a presentation

$$G = \langle a, s | a^p = s^{p^{k-1}} = 1, asa^{-1} = s^{p^{k-2}+1} \rangle.$$

Hence, it is isomorphic to the modular group of order p^k with $k \geq 4$.

In order to prove Theorem 1.1, we need to show that the Euler class ζ_G is zero for all the groups listed in Proposition 3.2. We will use different arguments for p is odd and $p = 2$. Let us first deal with the case where $p = 2$.

LEMMA 3.3 *The Euler class ζ_G is zero when $G \cong Q_8$ or $G \cong M_{2^k}$ with $k \geq 3$.*

Proof. In both cases the Frattini subgroup $\Phi(G)$ of G is cyclic and central, and the quotient $G/\Phi(G)$ is isomorphic to the elementary abelian group of order 4. Therefore G has a central extension of the form

$$0 \longrightarrow \Phi(G) \longrightarrow G \longrightarrow \mathbb{Z}/2 \times \mathbb{Z}/2 \longrightarrow 0.$$

Let us consider the Lyndon–Hochschild–Serre spectral sequence corresponding to this extension. By Proposition 7.2 in [**21**], the generator μ of the group $H^3(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{Z}) \cong \mathbb{Z}/2$ is in the image of the differential d_3. This means that $\inf_{G/\Phi(G)}^G(\mu)$ is zero in $H^3(G, \mathbb{Z})$.

If $\{x_1, x_2\}$ is a set of generators for the mod 2 cohomology of $G/\Phi(G)$, then the mod 2 reduction of μ is $x_1 x_2 (x_1 + x_2)$ which is the top Stiefel–Whitney class of $I_{G/\Phi(G)}$. Since the mod 2 reduction is injective for elementary abelian groups, this gives $\mu = e(I_{G/\Phi(G)})$. Therefore $\zeta_{G/\Phi(G)} = \text{Inf}_{G/\Phi(G)}^G e(I_{G/\Phi(G)}) = \text{Inf}_{G/\Phi(G)}^G(\mu) = 0$, and hence $\zeta_G = 0$ by Proposition 3.1.

Now, we consider the case $p > 2$. The modular p-group M_{p^k} with $k \geq 4$ has an abelian subgroup isomorphic to $\mathbb{Z}/p^2 \times \mathbb{Z}/p$. Thus, the fact that $\zeta_G = 0$ for $G = M_{p^k}$ with $k \geq 4$ is a consequence of the following lemma.

LEMMA 3.4 *The Euler class ζ_G is zero when $G \cong \mathbb{Z}/p^2 \times \mathbb{Z}/p$ and p is odd.*

Proof. Let $G = \langle a, b | a^{p^2} = b^p = 1, ab = ba \rangle$. Then $H^*(G, \mathbb{Z}) = \mathbb{Z}[\alpha, \beta] \otimes \wedge(\chi)$, where $\deg\alpha = \deg\beta = 2$, $\deg\chi = 3$ and $p^2\alpha = p\beta = p\chi = 0$ (see [**8**]). Since the Chern class c_1 defines an isomorphism $\text{Hom}(G, \mathbb{C}^\times) \cong H^2(G, \mathbb{Z})$, we can consider the generators α and β of $H^2(G, \mathbb{Z})$ as the Chern classes of the representations $V_1 : a \to \omega, b \to 1$ and $V_2 : a \to 1, b \to \omega^p$, where ω is the primitive p^2th root of unity. With this notation, the Chern class $c_1(V_3)$ of the one-dimensional complex representation $V_3 : a \to \omega^p, b \to 1$ is equal to $p\alpha$.

Let W_2 and W_3 be the underlying two-dimensional real representations of V_2 and V_3, respectively. Then we have

$$e(W_2 \oplus W_3) = e(W_2)e(W_3) = c_1(V_2)c_1(V_3) = p\alpha\beta = 0.$$

Since $W_2 \oplus W_3$ is a direct summand of the augmentation module I_G, this gives $\zeta_G = 0$.

Now it remains to consider the non-abelian groups of order p^3 for $p > 2$. The following lemma solves the problem for this case and hence completes the proof of Theorem 1.1.

LEMMA 3.5 *Let G be a non-abelian p-group of order p^3 with $p > 2$. Then, the Euler class ζ_G is zero.*

Proof. If G is a p-group of order p^3 with $p > 2$, then G is either isomorphic to the extra-special group E_{p^3} of exponent p or to the modular group M_{p^3} of exponent p^2. In both cases, the exponent

of $H^i(G, \mathbb{Z})$ is p when i is not divisible by $2p$ (see [8]). Therefore, the mod p reduction map $H^{p^3-1}(G, \mathbb{Z}) \to H^{p^3-1}(G, \mathbb{F}_p)$ is injective. So, it is sufficient to show that the mod p reduction of ζ_G is zero.

In the mod p cohomology of the groups E_{p^3} and M_{p^3}, there are relations of the form $xy = 0$ and $\beta(x) + xy = 0$, respectively, where x, y denote the generators of one-dimensional cohomology for each group. Consider the operator $\beta P \beta$, where P denotes the Steenrod reduced pth power operator and β denotes the Bockstein operator. If we apply $\beta P \beta$ to these relations, then we obtain

$$\beta(x) \prod_{j=0}^{p-1} \beta(jx + y) = 0.$$

This product is a factor of the mod p reduction of ζ_G. Hence, $\zeta_G = 0$.

4. Calculations for some abelian 2-groups

In this section we calculate the Euler class ζ_G for some small abelian 2-groups. This allows us to narrow the range for the search of abelian 2-groups with non-zero Euler class.

PROPOSITION 4.1 *The Euler class ζ_G is zero when $G \cong \mathbb{Z}/8$.*

Proof. Let H be the maximal subgroup of G. We have $\zeta_G = \zeta_{G/H} \cdot e(W)$ where W is the direct sum of all irreducible two-dimensional representations of G. The Euler class $\zeta_{G/H}$ is represented by the extension

$$0 \longrightarrow \tilde{\mathbb{Z}} \longrightarrow \mathbb{Z}[G/H] \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Consider the long exact sequence associated with this short exact sequence:

$$\cdots \longrightarrow H^6(G, \tilde{\mathbb{Z}}) \longrightarrow H^6(H, \mathbb{Z}) \xrightarrow{\text{tr}} H^6(G, \mathbb{Z}) \xrightarrow{\cdot \zeta_{G/H}} H^7(G, \tilde{\mathbb{Z}}) \longrightarrow \cdots$$
$$\Big\| \qquad\qquad\qquad \Big\|$$
$$\mathbb{Z}/|H|\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/|G|\mathbb{Z}$$

It is clear from the above diagram that to show that $\zeta_G = 0$, it suffices to show that $e(W) \in H^6(G, \mathbb{Z})$ is divisible by 2. Note that $W = W_1 + W_2 + W_3$, where W_i is the two-dimensional real representation such that the action is given by $2\pi i/8$ degree rotation. Also note that if $\alpha = e(W_1)$, then $e(W_j) = j\alpha$ for all $j = 1, 2, 3$. This gives $e(W) = 6\alpha^3$ which is divisible by 2 as desired.

The calculation above shows that the Euler class ζ_G is zero for all cyclic 2-groups with order greater or equal to 8. An easy calculation shows that the Euler class for $\mathbb{Z}/4$ is not zero. We will show later that ζ_G is not zero also when $G = \mathbb{Z}/4 \times \mathbb{Z}/4$. The next group we consider is $G = (\mathbb{Z}/4)^2 \times \mathbb{Z}/2$. We show that the Euler class for this group is zero. For this calculation, we need the structure of the cohomology of the group $\mathbb{Z}/4 \times \mathbb{Z}/4$ with integer coefficients. In [18], Townsley completely describes the integral cohomology of all abelian groups. We quote the result from [18], but since what we need is a very special case of Townsley's calculations, we provide a proof for the convenience of the reader.

PROPOSITION 4.2 (Townsley [18]) *Let* $G = \mathbb{Z}/4 \times \mathbb{Z}/4$. *Then,*

$$H^*(G, \mathbb{Z}) \cong \mathbb{Z}[\mu_1, \mu_2, \mu_{12}]/\langle 4\mu_1 = 4\mu_2 = 4\mu_{12} = 0, \ \mu_{12}^2 = 2\mu_1\mu_2(\mu_1 + \mu_2)\rangle.$$

Proof. Let us consider the Lyndon–Hochschild–Serre spectral sequence with E_2-page

$$E_2^{p,q} \cong H^p(G/K, H^q(K, \mathbb{Z})) \Rightarrow H^{p+q}(G, \mathbb{Z}),$$

where K is a cyclic subgroup of G of order 4. Let t_1 and t_2 be the generators of $H^*(K, \mathbb{Z})$ and $H^*(G/K, \mathbb{Z})$. Since $H^2(G, \mathbb{Z}) \cong H^2(G/K, \mathbb{Z}) \oplus H^2(K, \mathbb{Z})$, we have $d_2(t_1) = 0$, which implies that $d_2 = 0$. By dimension reasons $d_i = 0$ for any $i \geq 2$, hence the spectral sequence collapses at E_2-page. Thus, the Poincaré series of $H^*(G, \mathbb{Z})$ is given by

$$P_{H^*(G,\mathbb{Z})}(t) = \frac{1 + t^3}{(1 - t^2)^2}.$$

Since $E_2^{1,2} = H^1(G/K, H^2(K, \mathbb{Z})) \cong \mathbb{Z}/4$, the cohomology ring $H^*(G, \mathbb{Z})$ has at least three generators. Let μ_1 and μ_2 be the generators of degree 2 and let μ_{12} be the generator of degree 3. Without loss of generality, we can assume $\mathrm{res}_K^G \mu_1 = t_1$ and $\mathrm{Inf}_{G/K}^G t_2 = \mu_2$. We claim that μ_1 and μ_2 are algebraically independent. Indeed, if

$$f(\mu_1, \mu_2) = \sum_{i=0}^{k} a_i \mu_1^i \mu_2^{k-i}$$

is a relation with the smallest degree then the restriction of $f(\mu_1, \mu_2)$ to the subgroup K gives $a_k t_1^k = 0$ and hence $a_k = 0$. Therefore, $f(\mu_1, \mu_2) = \mu_2 g(\mu_1, \mu_2)$ for some polynomial $g(\mu_1, \mu_2)$ with smaller degree. Since μ_2 is a non-zero divisor, this gives $g(\mu_1, \mu_2) = 0$ which contradicts the minimality of $f(\mu_1, \mu_2)$. Therefore, μ_1 and μ_2 are algebraically independent as claimed.

Let us assume for the moment that $\mu_{12}^2 = 2\mu_1\mu_2(\mu_1 + \mu_2)$ is the only relation for the generators μ_1, μ_2 and μ_{12} aside from the modular relations. It implies that

$$S = \mathbb{Z}[\mu_1, \mu_2, \mu_{12}]/\langle 4\mu_1, \ 4\mu_2, \ 4\mu_{12}, \ \mu_{12}^2 - 2\mu_1\mu_2(\mu_1 + \mu_2)\rangle$$

is a subring of $H^*(G, \mathbb{Z})$. On the other hand, S and $H^*(G, \mathbb{Z})$ have the same Poincaré series. So, we obtain $H^*(G, \mathbb{Z}) = S$ as desired. Now we prove that $\mu_{12}^2 = 2\mu_1\mu_2(\mu_1 + \mu_2)$ is the only relation. Since μ_{12} has an odd degree, we have $2\mu_{12}^2 = 0$. This implies that $(\mu_{12})^2 = 2f(\mu_1, \mu_2)$ for some polynomial $f(\mu_1, \mu_2) \in H^6(G, \mathbb{Z})$. It is easy to show that μ_{12}^2 is not zero by considering the spectral sequence associated to the extension

$$0 \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to G \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to 0.$$

So, we can assume $f(\mu_1, \mu_2) = a_1\mu_1^3 + a_2\mu_2^3 + a_3\mu_1^2\mu_2 + a_4\mu_1\mu_2^2$, where at least one of the a_i is non-zero. Since the restriction of μ_{12} to any cyclic subgroup H of G is zero, we get $a_1 = a_2 = 0$ and $a_3 = a_4 = 1$. So, $\mu_{12}^2 = 2\mu_1\mu_2(\mu_1 + \mu_2)$. Suppose now that there is another relation. Then, it must be of the form

$$\mu_{12} \cdot g(\mu_1, \mu_2) + h(\mu_1, \mu_2) = 0,$$

but this is impossible since the degree of μ_1 and μ_2 are 2 and the degree of μ_{12} is 3.

Now we are ready to do the following calculation.

PROPOSITION 4.3 *The Euler class ζ_G is zero when $G \cong (\mathbb{Z}/4)^2 \times \mathbb{Z}/2$.*

Proof. The Euler class ζ_G includes $\inf^G_{G/\Phi(G)} \zeta_{G/\Phi(G)}$ as a factor. So, it is enough to prove that this factor is zero. The cohomology ring $H^*(G/\Phi(G), \mathbb{Z})$ is generated by the elements u_I, where the indices I run through the subsets of $\{1, 2, 3\}$. The mod 2 reduction of u_I is given by the formula

$$m_2(u_I) = \Big(\prod_{i \in I} x_i\Big)\Big(\sum_{i \in I} x_i\Big),$$

where x_1, x_2 and x_3 are the generators of $H^*(G/\Phi(G), \mathbb{Z}/2)$. By direct calculation, one can show that the mod 2 reduction of $u_1^2 u_{23} + u_2^2 u_{13} + u_3^2 u_{12}$ is equal to the product of all non-trivial one-dimensional classes, hence it is equal to the top Stiefel–Whitney class of $I_{G/\Phi(G)}$. Since the reduction modulo 2 is an injective map for elementary abelian groups, we conclude that

$$\zeta_{G/\Phi(G)} = u_1^2 u_{23} + u_2^2 u_{13} + u_3^2 u_{12}.$$

Now we show that the inflation of this element is zero. Let a, b, c be the generators of G with $a^4 = b^4 = c^2 = 1$. Suppose μ_1, μ_2 and μ_3 are the generators of $H^2(G, \mathbb{Z}) = \text{Hom}(G, \mathbb{C}^\times)$ which are dual to a, b, c, respectively. We can choose the generators u_1, u_2, u_3 for $H^2(G/\Phi(G), \mathbb{Z})$ in a compatible way and assume that $\text{Inf}^G_{G/\Phi(G)} u_i = 2\mu_i$ for $i = 1, 2$ and $\text{Inf}^G_{G/\Phi(G)} u_3 = \mu_3$. Since the exponent of $H^*(G, \mathbb{Z})$ is 4, we get $\text{Inf}^G_{G/\Phi(G)} u_i^2 = 0$ for $i = 1, 2$. Hence

$$\text{Inf}^G_{G/\Phi(G)}\big(u_1^2 u_{23} + u_2^2 u_{13} + u_3^2 u_{12}\big) = \mu_3^2 \cdot \text{Inf}^G_{G/\Phi(G)} u_{12}.$$

On the other hand, we have

$$\text{Inf}^G_{G/\Phi(G)} u_{12} = \text{Inf}^G_{\overline{G}} \text{Inf}^{\overline{G}}_{\overline{G}/\Phi(\overline{G})} u_{12} = \text{Inf}^G_{\overline{G}} 2\mu_{12} = 2\text{Inf}^G_{\overline{G}} \mu_{12},$$

where $\overline{G} = G/\langle c \rangle \cong \mathbb{Z}/4 \times \mathbb{Z}/4$ and μ_{12} is the generator of $H^3(\overline{G}, \mathbb{Z}) = \mathbb{Z}/4$. Since $2\mu_3 = 0$, we get $\text{Inf}^G_{G/\Phi(G)}\big(u_1^2 u_{23} + u_2^2 u_{13} + u_3^2 u_{12}\big) = 0$. This completes the proof. $\blacksquare$

5. Proof of Theorem 1.2

We first consider abelian 2-groups. In the previous section we showed that if G is isomorphic to $\mathbb{Z}/8$ or $(\mathbb{Z}/4)^2 \times \mathbb{Z}/2$, then the Euler class of the augmentation module I_G is zero. This implies that ζ_G is zero if G has a subquotient isomorphic to $\mathbb{Z}/8$ or $(\mathbb{Z}/4)^2 \times \mathbb{Z}/2$. But, the only abelian 2-groups that do not have any such subquotients are either elementary abelian or isomorphic to $\mathbb{Z}/4 \times \mathbb{Z}/4$ or $(\mathbb{Z}/2)^n \times \mathbb{Z}/4$ for some n. This proves one direction of Theorem 1.2 for 2-groups. For the other direction, we need to show that ζ_G is non-zero for these groups. We start with the calculation of ζ_G for $G = \mathbb{Z}/4 \times \mathbb{Z}/4$.

PROPOSITION 5.1 *Let $G = \mathbb{Z}/4 \times \mathbb{Z}/4$. Then, the Euler class ζ_G is non-zero.*

Proof. Let $G = \langle a, b \rangle$, and let μ_1, μ_2 be the generators of $H^2(G, \mathbb{Z})$ dual to a and b, respectively, and let μ_{12} be a generator of $H^3(G, \mathbb{Z})$. We have $I_G \cong \mathrm{Inf}^G_{G/\Phi(G)} I_{G/\Phi(G)} \oplus W$, where W is the direct sum of all irreducible two-dimensional real representations of G. Since $G/\Phi(G)$ is an elementary abelian group of order 4, it follows from an argument similar to that used in the proof of Proposition 4.3 that $\zeta_{G/\Phi(G)} = u_{12}$. Hence we get

$$e(\mathrm{Inf}^G_{G/\Phi(G)} I_{G/\Phi(G)}) = \mathrm{Inf}^G_{G/\Phi(G)} \zeta_{G/\Phi(G)} = \mathrm{Inf}^G_{G/\Phi(G)} u_{12} = 2\mu_{12}.$$

Now, we calculate the Euler class of W. Every two-dimensional real representation of G is the underlying real representation of a one-dimensional complex representation. If θ is the real representation associated to the complex representation $\rho : G \to \mathbb{C}^\times$, then $e(\theta) = c_1(\rho)$. Note that for each two-dimensional real representation, the kernel is a cyclic group of order 4. In fact, there is a one-to-one correspondence between two-dimensional real representations of G and its cyclic subgroups of order 4. The cyclic subgroups of G are $\langle a \rangle$, $\langle ab^2 \rangle$, $\langle ab \rangle$, $\langle ab^3 \rangle$, $\langle a^2 b \rangle$ and $\langle b \rangle$. Therefore

$$e(W) = \mu_2(2\mu_1 + \mu_2)(\mu_1 + \mu_2)(\mu_1 + 3\mu_2)\mu_1(\mu_1 + 2\mu_2) = \mu_1^2 \mu_2^2 (\mu_1 + \mu_2)^2$$

and hence $\zeta_G = 2\mu_{12}\mu_1^2\mu_2^2(\mu_1 + \mu_2)^2$. It is clear from Proposition 4.2 that this class is not zero in $H^*(G, \mathbb{Z})$.

It remains to show that the Euler class is non-zero when G is either elementary abelian or isomorphic to $(\mathbb{Z}/2)^n \times \mathbb{Z}/4$ for some $n \geq 0$. For these groups we show that the Euler class is non-zero by showing that its mod 2 reduction is non-zero. Recall that the mod 2 reduction of the Euler class of a real representation V is equal to the top Stiefel–Whitney class $w_{\mathrm{top}}(V)$ of V. To conclude that $w_{\mathrm{top}}(I_G)$ is non-zero, we give an explicit formula for it in terms of the generators of the cohomology ring of G. It is often more convenient to express the formula for $w_{\mathrm{top}}(I_G)$ in terms of the polynomial f where

$$f(a_1, a_2, \ldots, a_m) = \prod_{(\alpha_1, \ldots, \alpha_m) \in (\mathbb{F}_2)^m \setminus \{0\}} (\alpha_1 a_1 + \cdots + \alpha_m a_m)$$

for tuples $(a_1, \ldots, a_m)$. The formula for the top Stiefel–Whitney class of an elementary abelian 2-group appears in many places (see, for example, Turygin [17]). In this case, we have

$$w_{\mathrm{top}}(I_G) = f(x_1, \ldots, x_n)$$

where $\{x_1, \ldots, x_n\}$ is a set of generators of the cohomology ring $H^*(BG, \mathbb{F}_2)$. Note that this is the top Dickson invariant of the polynomial algebra $\mathbb{F}_2[x_1, \ldots, x_n]$. In particular, the Euler class ζ_G is non-zero when G is an elementary abelian 2-group.

Now, we perform a similar calculation for the group $G = (\mathbb{Z}/2)^n \times \mathbb{Z}/4$. We should note that it is possible to conclude that the top Stiefel–Whitney class is non-zero for these groups without obtaining an explicit formula, but we believe that the formula itself might also be useful. Before stating the result, we need some further notation. Let V_i be the one-dimensional non-trivial representation inflated from the ith term in the product $G/\Phi(G) \cong (\mathbb{Z}/2)^{n+1}$ and let W_{n+1} be the irreducible two-dimensional representation inflated from the $\mathbb{Z}/4$ term of the product $G = (\mathbb{Z}/2)^n \times \mathbb{Z}/4$. Notice that

the cohomology of the group G with $\mathbb{F}_2$ coefficients is

$$H^*(G, \mathbb{F}_2) \cong \mathbb{F}_2[x_1, \ldots, x_n, s] \otimes \wedge[t],$$

where $w_{\text{top}}(V_i) = x_i$ for $1 \leq i \leq n$, $w_{\text{top}}(V_{n+1}) = t$ and $w_{\text{top}}(W_{n+1}) = s$. With this notation the formula for $w_{\text{top}}(I_G)$ can be expressed as follows:

PROPOSITION 5.2 *Let* $G \cong (\mathbb{Z}/2)^n \times \mathbb{Z}/4$ *with* $n \geq 0$. *Then,*

$$w_{\text{top}}(I_G) = f(x_1, \ldots, x_n, t) \frac{f(x_1^2, \ldots, x_n^2, s)}{f(x_1^2, \ldots, x_n^2)}.$$

In particular, $w_{\text{top}}(V)$ *is non-zero in* $H^*(G, \mathbb{F}_2)$.

Proof. Let V be the direct sum of all the non-trivial one-dimensional real representations of G. Since each non-trivial one-dimensional real representation is a tensor product of elements contained in some non-empty subset of $\{V_1, \ldots, V_{n+1}\}$, we have $w_{\text{top}}(V) = f(x_1, \ldots, x_n, t)$. On the other hand, $I_G = V \oplus W$ where

$$W = \bigoplus_{(\gamma_1, \ldots, \gamma_n) \in S_n} V_1^{\gamma_1} \otimes \cdots \otimes V_n^{\gamma_n} \otimes W_{n+1}.$$

Here S_n is the set of n-tuples $(\gamma_1, \ldots, \gamma_n)$ such that $\gamma_i \in \{1, 2\}$ for all i. By the tensor product formula for one-dimensional real vector bundles, we have

$$w(V_1^{\gamma_1} \otimes \cdots \otimes V_n^{\gamma_n}) = 1 + \alpha_1 x_1 + \cdots + \alpha_n x_n,$$

where α_i denotes the mod 2 reduction of γ_i for each i. Using the splitting principle, we can regard the rth Stiefel–Whitney class of the vector bundle W_{n+1} as the rth elementary symmetric function of indeterminates a_1 and a_2 so that $w_1(W_{n+1}) = a_1 + a_2 = 0$ and $w_2(W_{n+1}) = a_1 a_2 = s$. Then, we have

$$w(V_1^{\gamma_1} \otimes \cdots \otimes V_n^{\gamma_n} \otimes W_{n+1}) = \prod_{i=1}^{2} (1 + \alpha_1 x_1 + \cdots + \alpha_n x_n + a_i)$$

$$= 1 + \alpha_1 x_1^2 + \cdots + \alpha_n x_n^2 + s.$$

Thus, the top Stiefel–Whitney class of W is given by

$$w_{\text{top}}(W) = \prod_{(\gamma_1, \ldots, \gamma_n) \in S_n} w_{\text{top}}(V_1^{\gamma_1} \otimes \cdots \otimes V_n^{\gamma_n} \otimes W_{n+1})$$

$$= \prod_{(\alpha_1, \ldots, \alpha_n) \in (\mathbb{F}_2)^n} (\alpha_1 x_1^2 + \cdots + \alpha_n x_n^2 + s) = \frac{f(x_1^2, \ldots, x_n^2, s)}{f(x_1^2, \ldots, x_n^2)}.$$

The formula for $w_{\text{top}}(I_G)$ follows from the identity $w_{\text{top}}(I_G) = w_{\text{top}}(V) w_{\text{top}}(W)$. Note that since $t^2 = 0$, we can rewrite the top Stiefel–Whitney class as

$$w_{\text{top}}(I_G) = t \big(f(x_1, \ldots, x_n) \big)^2 \frac{f(x_1^2, \ldots, x_n^2, s)}{f(x_1^2, \ldots, x_n^2)}.$$

From this it is clear that $w_{\text{top}}(I_G)$ is non-zero in $H^*(G, \mathbb{F}_2)$.

This completes the proof of the Theorem 1.2 for 2-groups. Now, we consider the case $p > 2$. We begin with the calculations for cyclic groups. Let $G = \langle g \rangle$ be a cyclic group of order p^n with $p > 2$. All non-trivial representations of G are two-dimensional which are the underlying real representations of one-dimensional complex representations. A complete list of corresponding complex representations can be given as $V_j : g \to \omega^j$, where ω is the p^nth root of unity and $1 \leq j \leq (p^n - 1)/2$. We can take $\alpha = c_1(V_1)$ as the generator of $H^2(G, \mathbb{Z}) \cong \mathbb{Z}/p^n$, then we have $c_1(V_j) = j\alpha$ for all j. This gives

$$e(I_G) = \prod_{j=1}^{(p^n-1)/2} c_1(V_j) = \left(\frac{p^n - 1}{2} \right)! \, \alpha^{(p^n-1)/2}.$$

From this we conclude the following:

LEMMA 5.3 *Let p be an odd prime and G be a cyclic p-group. Then the Euler class ζ_G is non-zero if and only if G has order p or is isomorphic to $\mathbb{Z}/9$.*

Proof. Suppose that G has order p^n. Then, $H^{2k}(G, \mathbb{Z}) \cong \mathbb{Z}/p^n$ for all $k \geq 1$. It follows that $e(I_G) = 0$ if and only if

$$\left(\frac{p^n - 1}{2} \right)! \equiv 0 \pmod{p^n}.$$

This is a consequence of the formula for $e(I_G)$ given above. It is easy to see that this equation holds for all p and n except when $n = 1$ or when $p = 3$ and $n = 2$.

The above lemma implies that the Euler class ζ_G of an abelian p-group with $p > 3$ vanishes if G is not elementary abelian. For $p = 3$, we need to be more careful. Since ζ_G is not zero for $\mathbb{Z}/9$, we need to consider the next possibility, which is $\mathbb{Z}/9 \times \mathbb{Z}/3$. But, this is the special case of Lemma 3.4, so $\zeta_G = 0$ in this case as well. This proves one direction of Theorem 1.2 for $p > 2$. For the other direction, we need to show that ζ_G is non-zero when G is an elementary abelian p-group with $p > 2$. This follows easily from the structure of cohomology of elementary abelian p-groups since the two-dimensional classes in $H^*((\mathbb{Z}/p)^n, \mathbb{Z})$ generate a polynomial subalgebra. So, the proof of Theorem 1.2 is complete.

6. Some consequences of Theorems 1.1 and 1.2

In this section, we state and prove some corollaries of Theorems 1.1 and 1.2. We first consider the mod p reduction of the Euler class. Since most of the group cohomology calculations are done in mod p coefficients, it makes sense to consider the mod p reduction of ζ_G.

COROLLARY 6.1 *Let G be a finite group and let $\overline{\zeta}_G$ denote the mod p reduction of the Euler class ζ_G. Then, $\overline{\zeta}_G$ is non-zero if and only if G is an elementary abelian p-group or is isomorphic to $(\mathbb{Z}/2)^n \times \mathbb{Z}/4$ for some $n \geq 0$.*

Proof. We only need to consider the groups where ζ_G is non-zero. We have already seen that when $G = \mathbb{Z}/4 \times \mathbb{Z}/4$ or $G = \mathbb{Z}/9$, the Euler class is divisible by p. So, the mod p reduction of ζ_G is zero in these cases. For groups isomorphic to $(\mathbb{Z}/2)^n \times \mathbb{Z}/4$, we have already shown that ζ_G is non-zero by

showing that its mod 2 reduction is non-zero. For elementary abelian p-groups, the mod p-reduction map is injective. So, the mod p-reduction of ζ_G is also non-zero for these groups.

For $p > 2$, the corollary above also follows from a theorem of Serre [16]. To see this, first observe that to prove the corollary, it is enough to consider p-groups. When G is a p-group, the quotient $G/\Phi(G)$ is an elementary abelian p-group and

$$\inf_{G/\Phi(G)}^{G} \overline{\zeta}_{G/\Phi(G)} = \lambda \left(\prod_{x \in S} \beta(x) \right)^{(p-1)/2},$$

where λ is a non-zero scalar, S is a set of representatives of non-zero elements on each line in $H^1(G, \mathbb{F}_p)$, and $\beta(x)$ is the image of x under the Bockstein operator. In [16], Serre proves that the product $\prod_{x \in S} \beta(x)$ is zero when G is not elementary abelian. This implies that when G is a p-group which is not elementary abelian, $\inf_{G/\Phi(G)}^{G} \overline{\zeta}_{G/\Phi(G)} = 0$, hence $\overline{\zeta}_G = 0$. It is clear that when G is an elementary abelian p-group, $\overline{\zeta}_G$ is non-zero.

The $p = 2$ case is slightly different, since in this case $\inf_{G/\Phi(G)}^{G} \overline{\zeta}_{G/\Phi(G)}$ is equal to the product of all one-dimensional classes. Serre's theorem in [16] only gives that the product of Booksteins of one-dimensional classes is zero, so the same argument does not work in this case. The $p = 2$ version of Serre's theorem has been considered in [22] and it has been proved that the product of one-dimensional classes is zero exactly when G is one of the 2-groups given in the above corollary. So, the $p = 2$ case of the above corollary follows from the results in [22].

In the other direction, one can obtain Serre's theorem as a consequence of Corollary 6.1. For this one needs to reduce Serre's theorem to extra-special p-groups and apply Corollary 6.1 together with some other facts from group cohomology. For example, one needs to use the fact that if a cohomology class is detected by a central subgroup of order p, then it is a non-zero divisor. Since there are many different proofs for Serre's theorem, this does not really provide a new way of looking at this theorem. In fact, one can see that the proof of Theorem 1.1 has many similarities to the proof of integer coefficient version of Serre's theorem given by Evens (see, [7, Theorem 6.4.1]).

Now, we consider the square of the Euler class ζ_G. Note that $(\zeta_G)^2$ is the Euler class of the representation $I_G \oplus I_G$ which can be considered as the underlying real representation of complexification of I_G. Note that the complexification of I_G is the kernel of the augmentation map $\mathbb{C}G \to \mathbb{C}$. This shows that $(\zeta_G)^2$ is nothing but the top Chern class of the augmentation module of the regular complex representation of G. So, it is interesting to find exactly when this class is zero.

COROLLARY 6.2 *Let G be a finite group. Then, $(\zeta_G)^2 \neq 0$ if and only if G is an elementary abelian p-group or is isomorphic to $(\mathbb{Z}/2)^n \times \mathbb{Z}/4$ for some $n \geq 0$. On the other hand, the mod p reduction of $(\zeta_G)^2$ is non-zero if and only if G is an elementary abelian p-group.*

Proof. The proof is similar to the proof of Corollary 6.1. The cases $G = \mathbb{Z}/4 \times \mathbb{Z}/4$ and $G = \mathbb{Z}/9$ do not appear here since in these cases ζ_G is divisible by p and both groups have cohomology with exponent p^2. The only group we have to be careful about is $G = (\mathbb{Z}/2)^n \times \mathbb{Z}/4$. Note that the calculation given in Proposition 5.2 shows that the mod 2 reduction of $(\zeta_G)^2$ is non-zero. But, the same calculation repeated for the top Chern class will give us that $(\zeta_G)^2 \neq 0$. The elementary abelian case is much easier to analyze. The square of ζ_G is the top Dickson invariant of the polynomial subalgebra generated by two-dimensional classes in $H^*(G, \mathbb{Z})$. So, both $(\zeta_G)^2$ and its mod p reduction are non-zero.

From this we can conclude the following:

COROLLARY 6.3 *Let G be a finite group. Then, ζ_G is nilpotent if and only if G is not an elementary abelian p-group.*

Recall that the Euler class ζ_G of a group G is a special case of the Euler class of a G-set X, denoted by ζ_X. We defined the Euler class ζ_X as the Euler class of the subset complex of the G-set X. One can state a similar problem: Find all G-sets X where ζ_X is non-zero. It is already given in [14] that ζ_X is zero if X is not transitive. So, we can assume $X = G/H$ for some subgroup $H \leq G$. Note that if H is normal in G, then $\zeta_{G/H}$ is the inflation of the Euler class of the quotient group G/H. In general, we can use similar ideas to reduce the calculation of ζ_X to smaller subquotients and smaller G-sets. The answer depends on X and the ring structure of the integral cohomology of G. At this point, we do not know any general statements that hold for all X.

7. Filtrations of modules

In this section, we discuss some applications of our results to filtrations of modules. Suppose k denotes a field of characteristic p and kG denotes the group algebra over k. Throughout the section, we assume all kG-modules are finitely generated. For a kG-module M, we denote the jth Heller shift of M by $\Omega^j(M)$. Let L_ζ denote the kG-module defined as the kernel of the homomorphism $\Omega^n(k) \to k$ representing the class ζ. In [1], it has been shown that if $\zeta \in H^n(G, k)$ is represented by the kG-module extension

$$E : 0 \to k \to M_{n-1} \to \cdots \to M_0 \to k \to 0,$$

then there is a projective kG-module P such that $L_\zeta \oplus P$ has a filtration

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n = L_\zeta \oplus P$$

with $L_i/L_{i-1} \cong \Omega^{n-i+1}(M_{i-1})$ for $i = 1, \ldots, n$. When $\zeta = 0$, the module L_ζ is defined as the direct sum $\Omega(k) \oplus \Omega^n(k)$. So, if E is an extension whose extension class is zero, then there is a filtration of the form

$$0 = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = k \oplus \Omega^{n-1}(k) \oplus P$$

with $K_i/K_{i-1} \cong \Omega^{n-i}(M_{i-1})$ for $i = 1, \ldots, n$ where P is a projective kG-module. Tensoring this sequence with a kG-module M, we get a filtration

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n = M \oplus \Omega^{n-1}(M) \oplus Q$$

with $N_i/N_{i-1} \cong \Omega^{n-i}(M_{i-1} \otimes M)$ for $i = 1, \ldots, n$, where Q is a projective kG-module.

If the extension E is an extension with the property that all the M_i are permutation modules with no trivial summands, then we say E is an *extension of proper permutation modules*. Split extensions of proper permutation modules give rise to induction theorems. Using this method Carlson [4] proved that any kG-module M is a direct summand of a module that is filtered by modules induced from elementary abelian p-subgroups. He proved this result using specific split extensions of proper permutation modules, where the permutation modules are induced from maximal subgroups. Such extensions exist when G is not elementary abelian as a consequence of Serre's theorem.

Now, we will illustrate how the chain complex for the subset complex can be used to obtain similar filtrations. Note that the chain complex of the subset complex of G with coefficients in k gives an extension

$$0 \to k \to C_{n-1}(\Delta(G)) \otimes_{\mathbb{Z}} k \to \cdots \to C_0(\Delta(G)) \otimes_{\mathbb{Z}} k \to k \to 0$$

whose extension class is the image of ζ_G under the map $H^n(G, \mathbb{Z}) \to H^n(G, k)$ induced by tensor product with k. So, if G is a p-group which is not one of the groups listed in Theorem 1.2, then for each kG-module M, there is a projective kG-module P and a filtration

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n = M \oplus \Omega^{n-1}(M) \oplus P$$

such that $L_i/L_{i-1} \cong \Omega^{n-i}(C_{i-1}(\Delta(G)) \otimes_{\mathbb{Z}} M)$ for $i = 1, \ldots, n$. Instead of the chain complex, we could have used the cochain complex of the subset complex. In that case, we get the following:

COROLLARY 7.1 *Let G be a finite group and let k be a field of characteristic p. If G is not one of the groups listed in Theorem* 1.2, *then for every kG-module M, there is a projective kG-module P and a filtration*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M \oplus \Omega^{n-1}(M) \oplus P$$

such that $M_i/M_{i-1} \cong \Omega^{n-i}(C^{n-i}(\Delta(G)) \otimes_{\mathbb{Z}} M)$ for $i = 1, \ldots, n$.

In [14], an explicit formula for permutation modules appearing in $C^j(\Delta(X))$ is given. So, using this formula we can give upper bounds for the dimension of a kG-module M in terms of the Heller shifts of the projective free parts of the restrictions of M to its proper subgroups. This approach was used to obtain upper bounds for essential endotrivial modules of extraspecial p-groups in [5]. For small groups our bounds are not as good as the bounds obtained in [5], but we expect that for groups with large cohomology lengths, this method will give better bounds than the ones given in [5]. Unfortunately, we were not able to verify this.

8. Borsuk–Ulam type theorems

In this section, we give some applications of our results to the existence of Borsuk–Ulam type theorems. Recall that the Borsuk–Ulam theorem says that every continuous map $f : S^n \to \mathbb{R}^m$ maps some pair of antipodal points on S^n to the same point in $\mathbb{R}^m$ provided that $m \leq n$. To generalize this theorem, one defines the notion of coincidence set. Suppose that M is a topological manifold with a free G-action, and let $f : M \to \mathbb{R}^m$ be a continuous map. The coincidence set of f is defined as the set

$$A(f) = \{x \in M | f(x) = f(gx) \quad \text{for all } g \in G\}.$$

We say that M has a *Borsuk–Ulam type theorem* if there is an appropriate condition on m such that the coincidence set is non-empty. The Borsuk–Ulam theorem corresponds to the case where $M = S^n$ and $G = \mathbb{Z}/2$ acting with antipodal action on M. Another example is given recently by Turygin [17] where M is a product of (mod p) homology spheres and G is an elementary abelian p-group acting on M by a product action. Some other generalizations of these types of theorems are discussed in [17]. It is interesting to note that one of these generalizations is related to the well-known theorem of Milnor [10] which states that if a finite group G acts freely on a sphere, then every element of order 2 in G must be central.

Now we will discuss a topological condition for the existence of a Borsuk–Ulam type theorem for a manifold M with a free G action. Consider the real vector bundle

$$\xi : M \times_G I_G^{\oplus m} \to M/G$$

with classifying map $q : M/G \to BG$, where I_G denotes the augmentation module. It is easy to see that there exists a continuous map $f : M \to \mathbb{R}^m$ with empty coincidence set if and only if ξ has a non-zero section (see, [17, Lemma 1.1]). So, to prove a Borsuk–Ulam type theorem for M, it is enough to show that, under suitable conditions on m, the bundle ξ has a non-trivial Euler class. The Euler class of this bundle is the image of the Euler class of the universal bundle $\hat{\xi} : EG \times_G I_G^{\oplus m} \to BG$ under the map $q^* : H^*(BG, \mathbb{Z}) \to H^*(M/G, \mathbb{Z})$. Note that the Euler class of $\hat{\xi}$ is the same as the mth power of ζ_G. We can summarize these in the following way:

PROPOSITION 8.1 (Sarkaria [15], Turygin [17]) *Let M be a topological manifold with a free action of a finite group G. Suppose that $\left(q^*(\zeta_G)\right)^m \neq 0$, where $q^* : H^*(BG, \mathbb{Z}) \to H^*(M/G, \mathbb{Z})$ is the map induced by the classifying map $q : M/G \to BG$. Then, every continuous map $f : M \to \mathbb{R}^m$ has a non-empty coincidence set.*

This shows that we can find some Borsuk–Ulam type theorems for the groups listed in Theorem 1.2 if we choose the manifold M in an appropriate way. This is how Turygin [17] obtains a Borsuk–Ulam type theorem for products of spheres with free elementary abelian group action. However, it is clear that when $\zeta_G = 0$, we cannot find any G-free space M satisfying the condition in Proposition 8.1. So, we conclude the following.

COROLLARY 8.2 *If G is not one of the groups listed in Theorem 1.2, then one cannot obtain Borsuk–Ulam type theorems for any G-free topological manifold using the cohomological Euler class.*

This does not say that there are no Borsuk–Ulam theorems since it is still possible to use some other cohomology theory and conclude that the bundle $\xi : M \times_G I_G^{\oplus m} \to M/G$ has no non-zero section.

Now, we explain another application of our results. For this we need to define the Borsuk–Ulam property. Note that another way to state the Borsuk–Ulam theorem is the following: For every antipodal preserving map $f : S^n \to \mathbb{R}^n$, there exists an $x \in S^n$ such that $f(x) = 0$. Note that if we take $G = \mathbb{Z}/2$, then $\mathbb{R}^n$ can be thought of as the $\mathbb{R}G$-module $I_G^{\oplus n}$ and S^n can be thought of as the $(n+1)$-fold join $G^{*(n+1)}$ with usual G-action. One generalizes this situation as follows:

DEFINITION 8.3 Let G be a finite group and let V be an n-dimensional real representation of G. The representation V is said to have the Borsuk–Ulam property if any continuous G-map $f : G^{*(n+1)} \to V$ has a zero.

The Borsuk–Ulam theorem is equivalent to the statement that $I_G^{\oplus n}$ has the Borsuk–Ulam property when $G = \mathbb{Z}/2$. One can ask if the similar statement holds for other groups. In [15], Sarkaria shows that an n-dimensional real representation V has the Borsuk–Ulam property if and only if the Euler class $e(V) \in H^n(G, \tilde{\mathbb{Z}})$ is non-zero (see, [15, Theorem 1]). As a consequence of Theorems 1.1 and 1.2, we obtain the following:

COROLLARY 8.4 *Let G be a finite group. The augmentation module I_G has the Borsuk–Ulam property if and only if G is one of the groups listed in Theorem* 1.2.

The Borsuk–Ulam property is closely related to the continuous version of the Tverberg Theorem in combinatorics. The Tverberg Theorem can be stated as follows:

THEOREM 8.5 (Tverberg) *For any affine map f from the standard $(q-1)(d+1)$-simplex to an affine d-space, there are q disjoint faces $\sigma_1, \ldots, \sigma_q$ such that the intersection $f(\sigma_1) \cap \cdots \cap f(\sigma_q)$ is non-empty*.

The continuous Tverberg theorem is a generalization of this theorem to continuous maps. It is known that the continuous Tverberg theorem holds for all prime powers $q = p^k$. There is a very nice paper by Sarkaria [15], where the connection between the continuous Tverberg problem and the Borsuk–Ulam property has been described. In this article, Sarkaria shows that the continuous Tverberg theorem holds for some q and d, if there exists a group G of order q for which $I_G^{\oplus(d+1)}$ has the Borsuk–Ulam property. There is a theorem by Özaydın [12], proved independently by Volovikov [19], which states the following:

PROPOSITION 8.6 ([15, Theorem 4]) *Every representation V of an elementary abelian group which does not contain the trivial representation has the Borsuk–Ulam property.*

The Özaydın–Volovikov theorem solves the topological Tverberg problem when q is a prime power. Note that Corollary 6.3 not only implies the Özaydın–Volovikov theorem, but also shows that elementary abelian p-groups are the only groups where the Özaydın–Volovikov theorem holds. It is still an open problem to show if the Tverberg theorem holds when q is a composite number. For more detailed discussion about this problem, we refer the reader to [9, 15].

9. Hypercohomology spectral sequences

In this section, we consider the hypercohomology spectral sequences associated to the subset complex. Given a cochain complex C^* of $\mathbb{Z}G$-modules, we can form a double complex $\mathrm{Hom}_{\mathbb{Z}G}(P_*, C^*)$, where P_* is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. The cohomology of this cochain complex is called the hypercohomology of G with coefficients in C^* and usually denoted by $\mathcal{H}^*(G, C^*)$. Since this is the cohomology of a double complex, there are two spectral sequences which converge to $\mathcal{H}^*(G, C^*)$. The first spectral sequence, which is usually called *the cohomology spectral sequence*, has E_2-page

$$E_2^{p,q} = H^p(G, H^q(C^*, \mathbb{Z})) \Rightarrow \mathcal{H}^{p+q}(G, C^*).$$

We have a second spectral sequence with E_1-page

$$E_1^{p,q} = H^q(G, C^p) \Rightarrow \mathcal{H}^{p+q}(G, C^*)$$

which is called *the isotropy spectral sequence*. We refer the reader to the book by Brown [3] for more information on hypercohomology spectral sequences.

In [**14**], Reiner and Webb consider the spectral sequences which are isomorphic to the hypercohomology spectral sequences for $C^*(\Delta(G))$. Since $\Delta(G)$ is contractible, the cohomology spectral sequence gives

$$\mathcal{H}^*(G, C^*(\Delta(G))) \cong H^*(G, \mathbb{Z}).$$

Reiner and Webb study the isotropy spectral sequence for this hypercohomology group to obtain information about integral cohomology groups of G. For this, they obtain a formula which gives the (signed) permutation modules appearing in $C_n(\Delta(G))$ for each n. The formula is in the form of a power series with coefficients in the Burnside ring $B(G)$ of G. Once this power series is calculated, it becomes possible to calculate the E_1 page of the isotropy spectral sequence. Reiner and Webb illustrate how this method works by giving estimates for the orders of some integral cohomology groups of the Dihedral group of order 8. Theorems 1.1 and 1.2 have some immediate applications to Reiner and Webb's hypercohomology calculations. For example, by Corollary 7.7 in [**14**], we obtain the following:

COROLLARY 9.1 *Let G be a finite group which is not one of the groups listed in Theorem* 1.2*. Then,* $E_\infty^{r,|G|-1} = 0$ *for all r in the isotropy spectral sequence for the subset complex* $\Delta(G)$.

Another hypercohomology calculation that is interesting to consider is the hypercohomology of G with coefficients in the cochain complex of the boundary $\partial\Delta(G)$ of the subset complex. Since $\partial\Delta(G)$ is homeomorphic to a sphere, the E_2-page of the cohomology spectral sequence is a two line spectral sequence. If $|G| = n + 1$, then $E_2^{p,q}$ is non-zero only when $q = 0$ or $q = n - 1$. Moreover, the differential d_n is given by the multiplication with $\zeta_G \in H^n(G, \tilde{\mathbb{Z}})$. So, if G is not one of the groups listed in Theorem 1.2, then the cohomology spectral sequence for $\partial\Delta(G)$ collapses at E_2-page, and we have

$$\mathcal{H}^i(G, C^*(\partial\Delta(G))) \cong \begin{cases} H^i(G, \mathbb{Z}) & \text{if } i < n - 1 \\ H^i(G, \mathbb{Z}) \oplus H^{i-n+1}(G, \mathbb{Z}) & \text{if } i \geq n - 1. \end{cases}$$

The collapsing of this spectral sequence is important for theoretical reasons as well as computational reasons. Note that when the spectral sequence collapses at the E_2-page, the edge homomorphism

$$\pi^* : H^*(G, \mathbb{Z}) \longrightarrow \mathcal{H}^*(G, C_*(\partial\Delta(G)))$$

is injective. The hypercohomology of the chain complex $C_*(\partial\Delta(G))$ is isomorphic to the equivariant cohomology of the G-complex $\partial\Delta(G)$. It is known that the equivariant cohomology of a finite G-CW-complex X has an injective edge homomorphism if X has a fixed point. When G is an elementary abelian p-group, the converse is also known to hold in mod p coefficients as a consequence of the well-known Localization theorem. The boundary of the subset complex has no fixed points but its edge homomorphism is zero when G is not one of the groups listed in Theorem 1.2. Therefore, $\partial\Delta(G)$ can be considered as a counterexample to the Localization theorem for arbitrary groups. Other examples of such spaces are also known, for example, the projectivization of an irreducible

complex representation has zero edge homomorphism but has no fixed points when the representation is of dimension greater than 2.

In [**13**], Pakianathan and Yalçın show that if there exists a topological G-space Y which has no G-fixed point and for which the edge homomorphism $\pi^* : H^*(G) \to H^*_G(Y)$ is injective, then the essential ideal $\mathcal{E}ss^*(G)$ is nilpotent with nilpotency degree less than or equal to the dimension of Y. Recall that the essential ideal of the group G is defined as

$$\mathcal{E}ss^*(G) = \mathrm{Ker}\left\{\prod \mathrm{res}^G_H : H^*(G) \longrightarrow \prod_{H < G} H^*(H)\right\}.$$

Since $\partial\Delta(G)$ has no fixed points, our main theorem also implies the nilpotency of essential ideal for non-abelian groups. The nilpotency of essential ideal has many consequences in group cohomology such as Quillen's F-injectivity theorem and Serre's theorem.

Although the hypercohomology spectral sequence for the subset complex is a useful tool to obtain information about the cohomology of the group, we believe that it would be very difficult to calculate group cohomology using this method. One of the difficulties is that we do not know at which page the isotropy spectral sequence converges. It is an interesting problem to find group theoretical conditions which make the isotropy spectral sequence converge at a certain page.

Another problem with the isotropy spectral sequence of $\partial\Delta(G)$ is that the subset complex is not an admissible simplicial complex. Recall that a simplicial complex with a G-action is called admissible if it satisfies the condition that if a simplex is fixed by an element in G then its vertices are also fixed. When the G-action is not admissible, the product structure on the spectral sequence is not compatible with the product structure of the integral cohomology ring. That means that we can only calculate cohomology groups of G, but not the cohomology ring of G. The following is an example of a subset complex, where the product structure of the isotropy spectral sequence is not compatible with the product structure of the group cohomology.

EXAMPLE 9.2 Let $G = \langle a, b \rangle$ be an elementary abelian group of order 4. Then the corresponding complex $\partial\Delta(G)$ can be pictured as a tetrahedron and the Euler class of $\Delta(G)$ is $\zeta_G = \mu_{12}$, where μ_{12} is a generator of $H^3(G, \mathbb{Z})$. Recall that in this case the integral cohomology ring of G has the structure

$$H^*(G, \mathbb{Z}) \cong \mathbb{Z}[\mu_1, \mu_2, \mu_{12}]\big/\langle 2\mu_1 = 2\mu_2 = 2\mu_{12} = 0, \ \mu_{12}^2 = \mu_1\mu_2(\mu_1 + \mu_2)\rangle.$$

The differential d_3 is given by multiplication with μ_{12} which is a non-zero divisor. This gives that $\mathcal{H}^n(G, C^*) \cong H^n(G, \mathbb{Z})\big/\langle\mu_{12}\rangle$ for $n \neq 2$ and $\mathcal{H}^2(G, C^*) \cong H^2(G, \mathbb{Z}) \oplus \mathbb{Z}$. On the other hand, the chain complex of $\partial\Delta(G)$ is

$$0 \longrightarrow \mathbb{Z}[G] \longrightarrow \oplus^3_{i=1}\tilde{\mathbb{Z}}[G/H_i] \longrightarrow \mathbb{Z}[G] \longrightarrow 0,$$

where H_1, H_2, H_3 are the cyclic subgroups of order 2 in G. So, the E_1-page of the isotropy spectral sequence is as follows:

The $E_1 -$ page
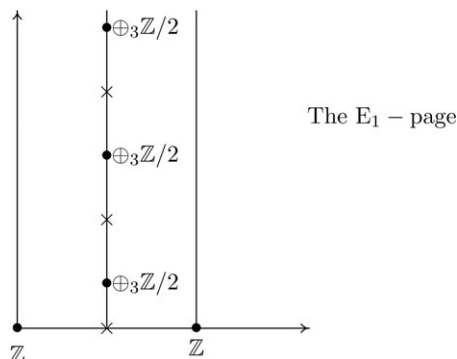
It is easy to see that all differentials d_i are zero for $i \geq 1$ and hence $E_\infty = E_1$. This gives

$$H^*(G, \mathbb{Z}) \cong \bigoplus_{i=1}^{3} H^*(H_i, \tilde{\mathbb{Z}})$$

except at the dimensions 0 and 2. Note that the product $E_1^{1,1} \otimes E_1^{1,1} \to E_1^{2,2}$ is zero, so all the two-dimensional classes are nilpotent with respect to the product in the spectral sequence, but this is not true in the integral cohomology of the group. This shows that the product structure of spectral sequence is not compatible with the product structure in the integral cohomology of the group.

It is known that when a simplicial complex has an admissible G-action, the cohomology structure is compatible with the product structure of the associated isotropy spectral sequence (see [**3**]). The induced action on barycentric subdivision of a simplicial complex is an admissible action. Therefore to avoid problems arising with the product structure, one should consider the barycentric subdivision of the complex $\partial \Delta(G)$.

10. Barycentric subdivision of the subset complex

In [**14**], Reiner and Webb give a formula which provides a complete description of the (signed) permutation modules involved in the chain complex of $\Delta(G)$. Using this formula, one calculates the E_1-page of the isotropy spectral sequence for $\partial \Delta(G)$. To be able to do similar calculations with the barycentric subdivision of $\partial \Delta(G)$, we need to find a similar formula for the barycentric subdivision of $\partial \Delta(G)$. We first describe the formula given by Reiner and Webb and show how a similar formula for the barycentric subdivision of $\partial \Delta(G)$ can be obtained.

A graded G-set is a G-set Ω partitioned as $\Omega = \Omega(0) \cup \Omega(1) \cup \Omega(2) \cup \cdots$, where each component $\Omega(i)$ is a G-set. Given a graded G-set, one can form a Poincaré series

$$P_\Omega(t) = \sum_{i \geq 0} [\Omega(i)] t^i,$$

where $[\Omega(i)]$ denotes the isomorphism class of the set $\Omega(i)$. We can consider this series as a power series with coefficients in the Burnside ring $B(G)$. By using the properties of the Burnside ring and

the Gluck idempotent formula, Reiner and Webb [**14**] give a formula for this Poincaré series as

$$P_\Omega(t) = \sum_{K \leq H \leq G} \frac{[G/K]\mu(K, H)f_H(t)}{|G : K|}, \tag{1}$$

where $f_H(t) = \sum_{i \geq 0} |\Omega(i)^H| t^i$ and μ denotes the Möbius function on the poset of subgroups of G. Note that in the above formula $[G/K]$ denotes the isomorphism class of the transitive G-set with isotropy subgroup K.

For a G-simplicial complex Γ, the Poincaré series $P_\Gamma(t)$ associated to Γ is defined as the Poincare series of the graded G-set

$$\Gamma = \Gamma(0) \cup \Gamma(1) \cup \Gamma(2) \cup \cdots,$$

where $\Gamma(0) = [G/G]$ and $\Gamma(i)$ is the isomorphism class of the set of $(i - 1)$-simplices of Γ for $i \geq 1$. If we apply this definition to the subset complex $\Delta(X)$, we get a Poincaré series of the form

$$P_{\Delta(X)}(t) = \Lambda^0(X) + \Lambda^1(X)t + \cdots + \Lambda^{n+1}(X)t^{n+1},$$

where $|X| = n + 1$ and $\Lambda^i(X)$ is the ith exterior power of X. In [**20**], Webb simplifies the formula in Equation (1) for this case and obtains the following:

PROPOSITION 10.1 ([**20**, Proposition 1.4]) *Let X be a finite G-set. Then*

$$P_{\Delta(X)}(t) = \sum_{K \leq H \leq G} \frac{[G/K]\mu(K, H)f_H(t)}{|G : K|},$$

where $f_H(t) = \prod_{i=1}^m (1 + t^{|X_i|})$ when $X = X_1 \cup \cdots \cup X_m$ is the decomposition of X into H orbits.

For a G-set X, let $\text{Bary}(X)$ denote the barycentric subdivision of $\partial \Delta(X)$. We are interested in finding a similar formula for the Poincaré series of $\text{Bary}(X)$. Let us call $\text{Bary}(X)(i)$ the ith T-power of the set X and denote it with $T^i(X)$. With this notation we have

$$P_{\text{Bary}(X)}(t) = T^0(X) + T^1(X)t + \cdots + T^n(X)t^n,$$

where $|X| = n + 1$. We have the following alternative description for T-powers of a G-set.

LEMMA 10.2 *For a G-set X, the ith T-power $T^i(X)$ is isomorphic to the set of surjective maps $f : X \to \{1, 2, \ldots, i + 1\}$.*

Proof. By definition, the ith T-power of X is the set of proper chains

$$\emptyset \subset A_1 \subset A_2 \subset \cdots \subset A_i \subset X.$$

It is easy to see that this set is isomorphic to the set of sequences $B_1, B_2, \ldots, B_{i+1}$ of non-empty disjoint sets whose union is X. But, this is isomorphic to the set of surjections $f : X \to \{1, 2, \ldots, i + 1\}$.

Let us denote the size of the set $T^i(\{1, 2, \ldots, m\})$ by $\begin{bmatrix} m \\ i \end{bmatrix}$. This is analogous to the similar notation $|\Lambda^i(\{1, 2, \ldots, m\})| = \binom{m}{i}$ for exterior powers. Note that

$$\begin{bmatrix} m \\ i \end{bmatrix} = (i+1)! \, S(m, i+1),$$

where $S(m, i+1)$ denotes the Stirling number of second type, which is the number of ways to partition the set $\{1, \cdots, m\}$ into $i+1$ disjoint subsets. The Poincaré series for $\mathrm{Bary}(X)$ is given by the following formula.

PROPOSITION 10.3 *Let X be a G-set and let $\mathrm{Bary}(X)$ denote the barycentric subdivision of the boundary of the subset complex $\Delta(X)$. Then,*

$$P_{\mathrm{Bary}(X)}(t) = \sum_{K \leq H \leq G} \frac{[G/K]\mu(K, H)F_{|X/H|}(t)}{|G : K|}$$

where $F_m(t) = \begin{bmatrix} m \\ 0 \end{bmatrix} + \begin{bmatrix} m \\ 1 \end{bmatrix} t + \cdots + \begin{bmatrix} m \\ m-1 \end{bmatrix} t^{m-1}$ for $m \geq 1$.

Proof. By Equation (1), it is enough to show that $|T^i(X)^H| = \begin{bmatrix} |X/H| \\ i \end{bmatrix}$. Note that by Lemma 10.2, the fixed point set $T^i(X)^H$ is in one-to-one correspondence with surjective maps $f : X/H \to \{1, 2, \ldots, i+1\}$. So, the result follows. ∎

The polynomials $F_m(t)$ can be easily calculated using the induction formula

$$\begin{bmatrix} m \\ i \end{bmatrix} = \left(\begin{bmatrix} m-1 \\ i \end{bmatrix} + \begin{bmatrix} m-1 \\ i-1 \end{bmatrix} \right)(i+1).$$

The first six terms of the series $F_m(t)$ are as follows:

$$F_1(t) = 1$$
$$F_2(t) = 1 + 2t$$
$$F_3(t) = 1 + 6t + 6t^2$$
$$F_4(t) = 1 + 14t + 36t^2 + 24t^3$$
$$F_5(t) = 1 + 30t + 150t^2 + 240t^3 + 120t^4$$
$$F_6(t) = 1 + 62t + 540t^2 + 1560t^3 + 1800t^4 + 720t^5.$$

We conclude this section with an example that illustrates how one can obtain information about the product structure using the complex $\mathrm{Bary}(X)$.

EXAMPLE 10.4 Let G be an elementary abelian group of order 4, and let H_1, H_2, H_3 denote its subgroups of order 2. Consider $\mathrm{Bary}(G)$, the barycentric subdivision of $\partial\Delta(G)$. The cohomology

spectral sequence for this complex is the same as the one given in Example 9.2. Let us look at the isotropy spectral sequence closely. By Proposition 10.3, the coefficient of $[G/H_i]$ in $P_{\mathrm{Bary}(X)}(t)$ is

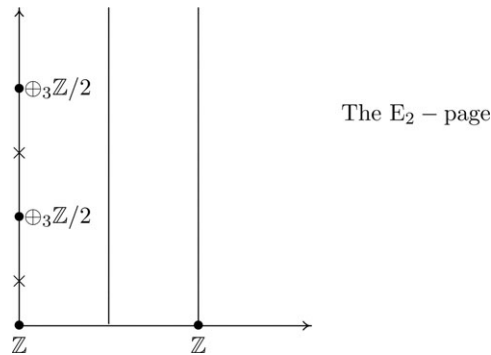$$\frac{1}{2}[F_2(t) - F_1(t)] = \frac{1}{2}[(1 + 2t) - 1] = t$$

for each $i = 1, 2, 3$ and the coefficient of $[G/1]$ is

$$\frac{1}{4}[F_4(t) - 3F_2(t) + 2F_1(t)] = 2t + 9t^2 + 6t^3.$$

This means that the chain complex for Bary(X) is given by

$$0 \longrightarrow \oplus_6 \mathbb{Z}[G/1] \longrightarrow \oplus_9 \mathbb{Z}[G/1] \longrightarrow \oplus_i \mathbb{Z}[G/H_i] \oplus (\oplus_2 \mathbb{Z}[G/1]) \longrightarrow 0.$$

The differential d_1 on the E_1-page is non-zero only on the bottom line and can easily be calculated. We obtain that the E_2-page looks like the following:

The E_2 – page

Since there are no further differentials, the spectral sequence collapses at the E_2-page. Note that if u_1 and u_2 are generators of $H^2(G, \mathbb{Z})$ corresponding to projections on each factor, then they map to $(1, 1, 0)$ and $(0, 1, 1)$ in $E_3^{0,2}$. So, we obtain the relation $u_1 u_2(u_1 + u_2) = 0$ as a consequence of the product structure on E_3. This shows that the product structure of the spectral sequence is compatible with the product structure of the group cohomology.

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