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On Darboux-integrable semi-discrete chains

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Abstract
A differential-difference equation $\frac{\mathrm{d}}{\mathrm{d}x}t(n+1,x) = f(x, t(n,x), t(n+1,x), \frac{\mathrm{d}}{\mathrm{d}x}t(n,x))$ with unknown $t(n, x)$ depending on the continuous and discrete variables x and n is studied. We call an equation of such kind Darboux integrable if there exist two functions (called integrals) F and I of a finite number of dynamical variables such that $D_x F = 0$ and $DI = I$, where D_x is the operator of total differentiation with respect to x and D is the shift operator: $Dp(n) = p(n+1)$. It is proved that the integrals can be brought to some canonical form. A method of construction of an explicit formula for a general solution to Darboux-integrable chains is discussed and such solutions are found for a class of chains.

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1. Introduction

In this paper we study the Darboux-integrable semi-discrete chains of the form

$$\frac{\mathrm{d}}{\mathrm{d}x}t(n+1,x) = f\left(x, t(n,x), t(n+1,x), \frac{\mathrm{d}}{\mathrm{d}x}t(n,x)\right). \tag{1}$$

Here the unknown function $t = t(n, x)$ depends on the discrete and continuous variables n and x respectively, the function $f = f(x, t, t_1, t_x)$ is assumed to be locally analytic and $\frac{\partial f}{\partial t_x}$ is not identically zero. In the last two decades the discrete phenomena have become very popular due to various important applications (for more details see [1–3] and references therein).

Below we use a subindex to indicate the shift of the discrete argument: $t_k = t(n+k, x)$, $k \in \mathbb{Z}$, and the derivatives with respect to x: $t_{[1]} = t_x = \frac{\mathrm{d}}{\mathrm{d}x}t(n,x)$, $t_{[2]} = t_{xx} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}t(n,x)$ and $t_{[m]} = \frac{\mathrm{d}^m}{\mathrm{d}x^m}t(n,x)$, $m \in \mathbb{N}$. We introduce *the set of dynamical variables* containing $\{t_k\}_{k=-\infty}^{\infty}$; $\{t_{[m]}\}_{m=1}^{\infty}$. We denote through D and D_x the shift operator and the operator of the total derivative with respect to x correspondingly. For instance, $Dh(n,x) = h(n+1,x)$ and $D_x h(n,x) = \frac{\mathrm{d}}{\mathrm{d}x} h(n,x)$.

The functions I and F, both depending on x, n and a finite number of dynamical variables, are respectively called *n- and x-integrals* of (1), if $DI = I$ and $D_x F = 0$ (see also [4]). Clearly, any function depending on n only is an x-integral, and any function depending on x only is an n-integral. Such integrals are called *trivial* integrals. One can see that any n-integral I does not depend on variables t_m, $m \in \mathbb{Z}\backslash\{0\}$, and any x-integral F does not depend on variables $t_{[m]}$, $m \in \mathbb{N}$.

Chain (1) is called *Darboux integrable* if it admits a nontrivial n-integral and a nontrivial x-integral.

The basic ideas on the integration of partial differential equations (PDEs) of the hyperbolic type go back to classical works by Laplace, Darboux, Goursat, Vessiot, Monge, Ampere, Legendre, Egorov, etc. Note that the understanding of integration as finding an explicit formula for a general solution was later replaced by other, in a sense less obligatory, definitions. For instance, the Darboux method for the integration of hyperbolic-type equations consists of searching for integrals in both directions followed by the reduction of the equation to two ordinary differential equations (ODEs). In order to find integrals, provided that they exist, Darboux used the Laplace cascade method. An alternative, more algebraic approach based on the characteristic vector fields was used by Goursat and Vessiot. Namely, this method allowed Goursat to get a list of integrable equations [5]. An important contribution to the development of the algebraic method investigating Darboux integrable equations was made by A B Shabat who introduced the notion of the characteristic algebra of the hyperbolic equation

$$u_{xy} = f(x, y, u, u_x, u_y). \tag{2}$$

It turned out that the operator D_y of total differentiation, with respect to the variable y, defines a derivative in the characteristic algebra in the direction of x. Moreover, the operator ad_{D_y} defined according to the rule $ad_{D_y} X = [D_y, X]$ acts on the generators of the algebra in a very simple way. This makes it possible to obtain effective integrability conditions for equation (2) (see [6]).

A V Zhiber and F Kh Mukminov investigated the structure of the characteristic algebras for the so-called quadratic systems containing the Liouville equation and the sine-Gordon equation (see [7]). In [7] and [8], the very nontrivial connection between the characteristic algebras and Lax pairs of the hyperbolic S-integrable equations and systems of equations is studied, and the perspectives on the application of the characteristic algebras to classify such kinds of equations are discussed.

Recently, the concept of the characteristic algebras has been defined for discrete models. In our articles [9–11] an effective algorithm was worked out to classify Darboux-integrable models. By using this algorithm some new classification results were obtained. In [12], a method of classification of S-integrable discrete models is suggested based on the concept of characteristic algebra.

This paper is organized as follows. In section 2 characteristic algebras are defined for chain (1). In section 3 we describe the structure of n-integrals and x-integrals of the Darboux-integrable chains of the general form (1) (see theorems 3.1 and 3.2). Then we show that one can choose the minimal-order n-integral and the minimal-order x-integral of a special canonical form, important for the purpose of classification (see theorems 3.3 and 3.4).

The complete classification of a particular case $t_{1x} = t_x + d(t_1, t)$ in [11] was done due to the finiteness of the characteristic algebras in both directions. However, algebras themselves were not described. In subsections 4.1 and 4.2 we fill up this gap and represent the tables of multiplications for all of these algebras.

The problem of finding explicit solutions for Darboux-integrable models is rather complicated. Even in the mostly studied case of the PDE $u_{xy} = f(x, y, u, u_x, u_y)$, this problem is not completely solved. In subsection 4.3 we give the explicit formulas for the general solutions of the integrable chains in the particular case $t_{1x} = t_x + d(t, t_1)$ (see theorem 4.2).

It is remarkable that the classification of Darboux-integrable chains is closely connected with the classical problem of the description of ODE admitting a fundamental system of solutions (following Vessiot–Guldberg–Lie); for the details, one can see [13] and the references therein. Indeed any n-integral defines an ODE $F(n, x, t, t_x, \ldots, t_{[k]}) = p(x)$ for which the corresponding x-integral gives a formula $I(n, x, t, t_1, \ldots, t_m) = c_n$ allowing one to find a new solution t_m for the given set of solutions $t, t_1, \ldots, t_{m-1}$. Iterating this way one finds a solution $t_N = H(t, t_1, \ldots, t_{m-1}, x, c_1, c_2, \ldots, c_k)$ depending on k arbitrary constants. In the case when I does not depend on x explicitly, this formula gives a general solution in a desired form. Examples are given in the remark in section 4.

2. Characteristic algebras of discrete models

Let us introduce the characteristic algebras for chain (1). Due to the requirement of $\frac{\partial f}{\partial t_x} \neq 0$, we can rewrite (at least locally) chain (1) in the inverse form

$$t_x(n-1, x) = g(x, t(n, x), t(n-1, x), t_x(n, x)).$$

Since the x-integral F does not depend on the variables $t_{[k]}$, $k \in \mathbb{N}$, the equation $D_x F = 0$ becomes $KF = 0$, where

$$K = \frac{\partial}{\partial x} + t_x \frac{\partial}{\partial t} + f \frac{\partial}{\partial t_1} + g \frac{\partial}{\partial t_{-1}} + f_1 \frac{\partial}{\partial t_2} + g_{-1} \frac{\partial}{\partial t_{-2}} + \cdots. \tag{3}$$

Also, $XF = 0$, with $X = \frac{\partial}{\partial t_x}$. Consider the linear space over the field of locally analytic functions depending on a finite number of dynamical variables spanned by all multiple commutators of K and X. This set is closed with respect to three operations: addition, multiplication by a function and taking the commutator of two elements. It is called the characteristic algebra L_x of chain (1) in the x-direction. Therefore, any vector field from algebra L_x annulates F. The relation between the Darboux integrability of chain (1) and its characteristic algebra L_x is given by the following important criterion.

Theorem 2.1 (see [11]). *Chain (1) admits a nontrivial x-integral if and only if its characteristic algebra L_x is of finite dimension.*

The equation $DI = I$, defining an n-integral I, in an enlarged form becomes

$$I(x, n+1, t_1, f, f_x, \ldots) = I(x, n, t, t_x, t_{xx}, \ldots). \tag{4}$$

The left-hand side contains the variable t_1, while the right-hand side does not. Hence we have $D^{-1} \frac{\mathrm{d}}{\mathrm{d}t_1} DI = 0$, i.e. the n-integral is in the kernel of the operator

$$Y_1 = D^{-1} Y_0 D,$$

where

$$Y_1 = \frac{\partial}{\partial t} + D^{-1}(Y_0 f) \frac{\partial}{\partial t_x} + D^{-1} Y_0(f_x) \frac{\partial}{\partial t_{xx}} + D^{-1} Y_0(f_{xx}) \frac{\partial}{\partial t_{xxx}} + \cdots, \tag{5}$$

and

$$Y_0 = \frac{\mathrm{d}}{\mathrm{d}t_1}. \tag{6}$$

One can show that $D^{-j}Y_0D^jI = 0$ for any natural j. Direct calculations show that

$$D^{-j}Y_0D^j = X_{j-1} + Y_j, \qquad j \geqslant 2,$$

where

$$Y_{j+1} = D^{-1}(Y_j f)\frac{\partial}{\partial t_x} + D^{-1}Y_j(f_x)\frac{\partial}{\partial t_{xx}} + D^{-1}Y_j(f_{xx})\frac{\partial}{\partial t_{xxx}} + \cdots, \qquad j \geqslant 1, \tag{7}$$

$$X_j = \frac{\partial}{\partial_{t_{-j}}}, \qquad j \geqslant 1. \tag{8}$$

Define by N^* the dimension of the linear space spanned by the operators $\{Y_j\}_1^\infty$. Introduce the linear space over the field of locally analytic functions depending on a finite number of dynamical variables spanned by all multiple commutators of the vector fields from $\{Y_j\}_1^{N^*} \cup \{X_j\}_1^{N^*}$. This linear space is closed with respect to three operations: addition, multiplication by a function and taking the commutator of two elements. It is called the characteristic algebra L_n of chain (1) in the n-direction.

Theorem 2.2 (see [9]). *Equation (1) admits a nontrivial n-integral if and only if its characteristic algebra L_n is of finite dimension.*

3. On the structure of nontrivial *x*- and *n*-integrals

We define the *order of a nontrivial n-integral* $I = I(x, n, t, t_x, \ldots, t_{[k]})$ with $\frac{\partial I}{\partial t_{[k]}} \neq 0$, as the number k.

Theorem 3.1. *Assume equation (1) admits a nontrivial n-integral. Then for any nontrivial n-integral $I^*(x, n, t, t_x, \ldots, t_{[k]})$ of the smallest order and any n-integral I, we have*

$$I = \phi(x, I^*, D_xI^*, D_x^2I^*, \ldots), \tag{9}$$

where ϕ is some function.

Proof. Denote by $I^* = I^*(x, n, t, \ldots, t_{[k]})$ an n-integral of the smallest order. Let I be any other n-integral, $I = I(x, n, t, \ldots, t_{[r]})$. Clearly $r \geqslant k$. Let us introduce new variables $x, n, t, t_x, \ldots, t_{[k-1]}, I^*, D_xI^*, \ldots, D_x^{r-k}I^*$ instead of the variables $x, n, t, t_x, \ldots, t_{[k-1]}, t_{[k]}, t_{[k+1]}, \ldots, t_{[r]}$. Now, $I = I(x, n, t, t_x, \ldots, t_{[k-1]}, I^*, D_xI^*, \ldots, D_x^{r-k}I^*)$. We write the power series for the function I in the neighborhood of the point $\left((I^*)_0, (D_xI^*)_0, \ldots, \left(D_x^{r-k}I^*\right)_0\right)$:

$$I = \sum_{i_0,i_1,\ldots,i_{r-k}} E_{i_0,i_1,\ldots,i_{r-k}}(I^* - (I^*)_0)^{i_0}(D_xI^* - (D_xI^*)_0)^{i_1}\cdots\left(D_x^{r-k}I^* - \left(D_x^{r-k}I^*\right)_0\right)^{i_{r-k}}. \tag{10}$$

Then

$$DI = \sum_{i_0,i_1,\ldots,i_{r-k}} DE_{i_0,i_1,\ldots,i_{r-k}}(DI^* - (I^*)_0)^{i_0}(DD_xI^* - (D_xI^*)_0)^{i_1} \cdots\left(DD_x^{r-k}I^* - \left(D_x^{r-k}I^*\right)_0\right)^{i_{r-k}}.$$

Since $DI = I$, $DD_x^jI^* = D_x^jDI^* = D_x^jI^*$ and the power series representation for the function I is unique, $DE_{i_0,i_1,\ldots,i_{r-k}} = E_{i_0,i_1,\ldots,i_{r-k}}$, i.e. $E_{i_0,i_1,\ldots,i_{r-k}}(x, n, t, \ldots, t_{[k-1]})$ are all n-integrals. Due to the fact that the minimal n-integral depends on $x, n, t, \ldots, t_{[k]}$, we conclude that all $E_{i_0,i_1,\ldots,i_{r-k}}(x, n, t, \ldots, t_{[k-1]})$ are the trivial n-integrals, i.e. functions depending only on x. Now equation (9) follows immediately from (10). □

We define the *order of a nontrivial x-integral* $F = F(x, n, t_k, t_{k+1}, \ldots t_m)$ with $\frac{\partial F}{\partial t_{[m]}} \neq 0$, as the number $m-k$.

Theorem 3.2. *Assume equation (1) admits a nontrivial x-integral. Then for any nontrivial x-integral $F^*(x, n, t, t_1, \ldots, t_m)$ of the smallest order and any x-integral F, we have*

$$F = \xi(n, F^*, DF^*, D^2F^*, \ldots), \tag{11}$$

where ξ is some function.

Proof. Denote by $F^* = F^*(x, n, t, t_1, \ldots, t_m)$ an x-integral of the smallest order. Let F be any other x-integral, $F = F(x, n, t, t_1, \ldots, t_l)$. Clearly, $l \geqslant m$. Let us introduce new variables $x, n, t, t_1, \ldots, t_{m-1}, F^*, DF^*, \ldots, D^{l-m}F^*$ instead of variables $x, n, t, t_1, \ldots, t_{m-1}, t_m, \ldots, t_l$. Now, $F = F(x, n, t, t_1, \ldots, t_{m-1}, F^*, DF^*, \ldots, D^{l-m}F^*)$. We write the power series representation of the function F in the neighborhood of the point $((F^*)_0, (DF^*)_0, \ldots, (D^{l-m}F^*)_0)$:

$$F = \sum_{i_0, i_1, \ldots, i_{l-m}} K_{i_0, i_1, \ldots, i_{l-m}} (F^* - (F^*)_0)^{i_0} (DF^* - (DF^*)_0)^{i_1} \cdots (D^{l-m}F^* - (D^{l-m}F^*)_0)^{i_{l-m}}. \tag{12}$$

Then

$$\begin{aligned} D_x F = &\sum_{i_0, i_1, \ldots, i_{l-m}} D_x\{K_{i_0, i_1, \ldots, i_{l-m}}\}(F^* - (F^*)_0)^{i_0}(DF^* - (DF^*)_0)^{i_1} \\ &\cdots (D^{l-m}F^* - (D^{l-m}F^*)_0)^{i_{l-m}} \\ &+ \sum_{i_0, i_1, \ldots, i_{l-m}} K_{i_0, i_1, \ldots, i_{l-m}} D_x\{(F^* - (F^*)_0)^{i_0}(DF^* - (DF^*)_0)^{i_1} \\ &\cdots (D^{l-m}F^* - (D^{l-m}F^*)_0)^{i_{l-m}}\}. \end{aligned}$$

Since $D_x D^j F^* = D^j D_x F^* = 0$,

$$D_x\{(F^* - (F^*)_0)^{i_0}(DF^* - (DF^*)_0)^{i_1} \cdots (D^{l-m}F^* - (D^{l-m}F^*)_0)^{i_{l-m}}\} = 0.$$

Therefore,

$$\begin{aligned} 0 = D_x F = &\sum_{i_0, i_1, \ldots, i_{l-m}} D_x\{K_{i_0, i_1, \ldots, i_{l-m}}\}(F^* - (F^*)_0)^{i_0}(DF^* - (DF^*)_0)^{i_1} \\ &\cdots (D^{l-m}F^* - (D^{l-m}F^*)_0)^{i_{l-m}}. \end{aligned}$$

Due to the unique representation of the zero power series, we have that $D_x\{K_{i_0, i_1, \ldots, i_{l-m}}\} = 0$, i.e. all $K_{i_0, i_1, \ldots, i_{l-m}}(x, n, t, \ldots, t_{m-1})$ are x-integrals. Since the minimal nontrivial x-integral is of order m, all $K_{i_0, i_1, \ldots, i_{l-m}}$ are trivial x-integrals, i.e. functions depending on n only. Now equation (11) follows from (12). □

The next two theorems are the discrete versions of lemma 1.2 from [14].

Theorem 3.3. *Among all the nontrivial n-integrals $I^*(x, n, t, t_x, \ldots, t_{[k]})$ of the smallest order, with $k \geqslant 2$, there is an n-integral $I^0(x, n, t, t_x, \ldots, t_{[k]})$ such that*

$$I^0(x, n, t, t_x, \ldots, t_{[k]}) = a(x, n, t, t_x, \ldots, t_{[k-1]})t_{[k]} + b(x, n, t, t_x, \ldots, t_{[k-1]}). \tag{13}$$

Proof. Consider a nontrivial minimal n-integral $I^*(x, n, t, t_x, \ldots, t_{[k]})$ with $k \geqslant 2$. The equality $DI^* = I^*$ can be rewritten as

$$I^*(x, n+1, t_1, f, f_x, \ldots, f_{[k-1]}) = I^*(x, n, t, t_x, \ldots, t_{[k]}).$$

We differentiate both sides of the last equality with respect to $t_{[k]}$:

$$\frac{\partial I^*(x, n+1, t_1, f, \ldots, f_{[k-1]})}{\partial f_{[k-1]}} \cdot \frac{\partial f_{[k-1]}}{\partial t_{[k]}} = \frac{\partial I^*(x, n, t, \ldots, t_{[k]})}{\partial t_{[k]}}. \tag{14}$$

In virtue of $\frac{\partial f_{[j]}}{\partial t_{[j+1]}} = f_{t_x}$, equation (14) can be rewritten as

$$\frac{\partial I^*(x, n+1, t_1, f, \ldots, f_{[k-1]})}{\partial f_{[k-1]}} f_{t_x} = \frac{\partial I^*(x, n, t, \ldots, t_{[k]})}{\partial t_{[k]}}. \tag{15}$$

Let us differentiate once more with respect to $t_{[k]}$ both sides of the last equation; we have

$$\frac{\partial^2 I^*(x, n+1, t_1, f, \ldots, f_{[k-1]})}{\partial^2 f_{[k-1]}} f_{t_x}^2 = \frac{\partial^2 I^*(x, n, t, \ldots, t_{[k]})}{\partial t_{[k]}^2},$$

or similarly

$$D\left\{\frac{\partial^2 I^*}{\partial t_{[k]}^2}\right\} f_{t_x}^2 = \frac{\partial^2 I^*}{\partial t_{[k]}^2},$$

where $I^* = I^*(x, n, t, \ldots, t_{[k]})$. It follows from (15) that

$$D\left\{\frac{\partial^2 I^*}{\partial t_{[k]}^2}\right\}\left\{\frac{\partial I^*}{\partial t_{[k]}}\right\}^2 = \frac{\partial^2 I^*}{\partial t_{[k]}^2} D\left\{\left(\frac{\partial I^*}{\partial t_{[k]}}\right)^2\right\},$$

or similarly the function

$$J := \frac{\frac{\partial^2 I^*}{\partial t_{[k]}^2}}{\left(\frac{\partial I^*}{\partial t_{[k]}}\right)^2}$$

is an n-integral; and by theorem 3.1, we have that $J = \phi(x, I^*)$. Therefore,

$$\frac{\partial^2 I^*}{\partial t_{[k]}^2} = \frac{\partial H(x, I^*)}{\partial I^*}\left(\frac{\partial I^*}{\partial t_{[k]}}\right)^2, \qquad \text{where} \qquad \frac{\partial H}{\partial I^*} = J,$$

or

$$\frac{\partial}{\partial t_{[k]}}\left\{\ln \frac{\partial I^*}{\partial t_{[k]}} - H(x, I^*)\right\} = 0.$$

Hence, $e^{-H(x,I^*)} \frac{\partial I^*}{\partial t_{[k]}} = e^g$ for some function $g(x, n, t, t_x, \ldots, t_{[k-1]})$. Introduce W in such a way that $\frac{\partial W}{\partial I^*} = e^{-H(x,I^*)}$. Then $\frac{\partial W}{\partial t_{[k]}} = e^g$ and $W = e^{g(x,n,t,\ldots,t_{[k-1]})} t_{[k]} + l(x, n, t, \ldots, t_{[k-1]})$ is an n-integral, where $l(x, n, t, \ldots, t_{[k-1]})$ is some function. □

Theorem 3.4. *Among all the nontrivial x-integrals* $F^*(x, n, t_{-1}, t, t_1, \ldots, t_m)$ *of the smallest order, with* $m \geqslant 1$, *there is an x-integral* $F^0(x, n, t_{-1}, t, t_1, \ldots, t_m)$ *such that*

$$F^0(x, n, t_{-1}, t, t_1, \ldots, t_m) = A(x, n, t_{-1}, t, \ldots, t_{m-1}) + B(x, n, t, t_1, \ldots, t_m). \tag{16}$$

Proof. Consider a nontrivial x-integral $F^*(x, n, t_{-1}, t, t_1, \ldots, t_m)$ of minimal order. Since $D_x F^* = 0$,

$$\frac{\partial F^*}{\partial x} + g\frac{\partial F^*}{\partial t_{-1}} + t_x\frac{\partial F^*}{\partial t} + f\frac{\partial F^*}{\partial t_1} + Df\frac{\partial F^*}{\partial t_2} + \cdots + D^{m-1} f\frac{\partial F^*}{\partial t_m} = 0. \tag{17}$$

We differentiate both sides of (17) with respect to t_m and with respect to t_{-1} separately and have the following two equations:

$$\left\{D_x + \frac{\partial}{\partial t_m}(D^{m-1}f)\right\}\frac{\partial F^*}{\partial t_m} = 0, \tag{18}$$

$$\left\{D_x + \frac{\partial g}{\partial t_{-1}}\right\}\frac{\partial F^*}{\partial t_{-1}} = 0. \tag{19}$$

Let us differentiate (18) with respect to t_{-1}; we have

$$D_x\frac{\partial^2 F^*}{\partial t_m \partial t_{-1}} + \frac{\partial g}{\partial t_{-1}}\frac{\partial^2 F^*}{\partial t_m \partial t_{-1}} + \frac{\partial}{\partial t_m}(D^{m-1}f)\frac{\partial^2 F^*}{\partial t_m \partial t_{-1}} = 0. \tag{20}$$

It follows from (18) and (19) that $\frac{\partial}{\partial t_m}(D^{m-1}f) = -\frac{D_x F^*_{t_m}}{F^*_{t_m}}$, $\frac{\partial g}{\partial t_{-1}} = -\frac{D_x F^*_{t_{-1}}}{F^*_{t_{-1}}}$. Equation (20) becomes

$$D_x\left\{\ln\frac{F^*_{t_m t_{-1}}}{F^*_{t_m}F^*_{t_{-1}}}\right\} = 0.$$

By theorem 3.2, we have $\frac{F^*_{t_m t_{-1}}}{F^*_{t_m}F^*_{t_{-1}}} = \xi(n, F^*)$, or

$$\frac{F^*_{t_m t_{-1}}}{F^*_{t_m}} = F^*_{t_{-1}}\xi(n, F^*) = H'(F^*)F^*_{t_{-1}} = \frac{\partial}{\partial t_{-1}}H(F^*), \qquad \text{where} \qquad \xi(n, F^*) = H'(n, F^*).$$

Thus, $\frac{\partial}{\partial t_{-1}}\{\ln F^*_{t_m} - H(n, F^*)\} = 0$, or $\mathrm{e}^{-H(n,F^*)}F^*_{t_m} = C(x, n, t, t_1, \ldots, t_m)$ for some function $C(x, n, t, t_1, \ldots, t_m)$. Denote such a function by $\tilde{H}^*(n, F)$ so that $\tilde{H}'(n, F^*) = \mathrm{e}^{-H(n,F^*)}$. Then $\frac{\partial \tilde{H}(n,F^*)}{\partial t_m} = C(x, n, t, t_1, \ldots, t_m)$. Hence, $\tilde{H}(n, F^*) = B(x, n, t, t_1, \ldots, t_m) + A(x, n, t_{-1}, t, \ldots, t_{m-1})$. Since $D_x\tilde{H}(F^*) = \tilde{H}'(n, F^*)D_x(F^*) = 0$, $\tilde{H}(n, F^*)$ is an x-integral in the desired form (16). □

4. Particular case: $t_{1x} = t_x + d(t, t_1)$

The finiteness of the characteristic algebras L_x and L_n was used in [10] and [11] to classify the Darboux-integrable semi-discrete chains of the special form

$$t_{1x} = t_x + d(t, t_1). \tag{21}$$

The statement of this classification result is given by the next theorem from [11].

Theorem 4.1. *Chain (21) admits the nontrivial x- and n-integrals if and only if it is one of the following kinds:*

(a)

$$t_{1x} = t_x + A(t_1 - t), \tag{22}$$

where $A(t_1 - t)$ is given in the implicit form $A(t_1 - t) = \frac{\mathrm{d}}{\mathrm{d}\theta}P(\theta)$, $t_1 - t = P(\theta)$, with $P(\theta)$ being an arbitrary quasi polynomial, i.e. a function satisfying an ODE $P^{(N+1)} = \mu_N P^{(N)} + \cdots + \mu_1 P' + \mu_0 P$ with the constant coefficients μ_k, $0 \leqslant k \leqslant N$,

(b)

$$t_{1x} = t_x + C_1\left(t_1^2 - t^2\right) + C_2(t_1 - t), \tag{23}$$

(c)

$$t_{1x} = t_x + \sqrt{C_3\,\mathrm{e}^{2\alpha t_1} + C_4\,\mathrm{e}^{\alpha(t_1+t)} + C_3\,\mathrm{e}^{2\alpha t}}, \tag{24}$$

(d)

$$t_{1x} = t_x + C_5(\mathrm{e}^{\alpha t_1} - \mathrm{e}^{\alpha t}) + C_6(\mathrm{e}^{-\alpha t_1} - \mathrm{e}^{-\alpha t}), \tag{25}$$

where $\alpha \neq 0$, C_i, $1 \leqslant i \leqslant 6$, are arbitrary constants. Moreover, some nontrivial x-integrals F and n-integrals I in each of the cases are *(i)* $F = x - \int^{t_1 - t} \frac{ds}{A(s)}$, $I = L(D_x)t_x$, *where* $L(D_x)$ *is a differential operator which annihilates* $\frac{d}{d\theta}P(\theta)$ *where* $D_x\theta = 1$.

(ii) $F = \frac{(t_3-t_1)(t_2-t)}{(t_3-t_2)(t_1-t)}$, $I = t_x - C_1t^2 - C_2t$,

(iii) $F = \operatorname{arcsinh}(a\,e^{\alpha(t_1-t_2)} + b) + \operatorname{arcsinh}(a\,e^{\alpha(t_1-t)} + b)$, $a = 2C_3\Big/\sqrt{4C_3^2 - C_4^2}$, $b = C_4/\sqrt{4C_3^2 - C_4^2}$, $I = 2t_{xx} - \alpha t_x^2 - \alpha C_3\,e^{2\alpha t}$,

(iv) $F = \frac{(e^{\alpha t} - e^{\alpha t_2})(e^{\alpha t_1} - e^{\alpha t_3})}{(e^{\alpha t} - e^{\alpha t_3})(e^{\alpha t_1} - e^{\alpha t_2})}$, $I = t_x - C_5e^{\alpha t} - C_6\,e^{-\alpha t}$.

Note that all the integrals in theorem 4.1 are given in their canonical forms (see theorems 3.3 and 3.4).

Remark. In case (c) equation (24) is closely connected with the well-known Steen–Ermakov equation (see [15] and the references therein)

$$y'' + q(x)y = c_3y^{-3}. \tag{26}$$

Indeed for $\alpha = 2$ its n-integral $2t_{xx} - \alpha t_x^2 - \alpha C_3\,e^{2\alpha t} = p(x)$ is reduced to the form (26) by substituting $y = e^{-t}$. Now it follows from the x-integral F that for the three arbitrary solutions $y(x), z(x), w(x)$ of the Steen–Ermakov equation, the following function

$$R(y, z, w) = \operatorname{arcsinh}(aw^2y^{-2} + b) + \operatorname{arcsinh}(az^2y^{-2} + b)$$

does not depend on x. Recall that the Riccati equation connected with the cases (b) and (d) has a similar property: the cross-ratio of its four solutions is a constant.

4.1. *Characteristic algebras L_x for Darboux-integrable equations $t_{1x} = t_x + d(t, t_1)$*

It was proved (see [10]) that if the equation $t_{1x} = t_x + d(t, t_1)$ admits a nontrivial x-integral, then it admits a nontrivial x-integral not depending on x. Introduce new vector fields

$$\tilde{X} = [X, K] = \sum_{k=-\infty,}^{\infty} \frac{\partial}{\partial t_k} \qquad J := [\tilde{X}, K].$$

4.1.1. *Case 1: $t_{1x} = t_x + A(t_1 - t)$.*

Direct calculations show that the multiplication table for the characteristic algebra L_x is as follows:

L_x	X	K	$\tilde{X}$
X	0	$\tilde{X}$	0
K	$-\tilde{X}$	0	0
$\tilde{X}$	0	0	0

4.1.2. *Case 2: $t_{1x} = t_x + C_1\left(t_1^2 - t^2\right) + C_2(t_1 - t)$.*

Direct calculations show that

$$J = 2C_1 \sum_{k=-\infty, k\neq 0}^{\infty} (t_k - t)\frac{\partial}{\partial t_k}$$

and

$$[J, K] = 2C_1^2 \sum_{k=-\infty, k\neq 0}^{\infty} (t_k - t)^2\frac{\partial}{\partial t_k} = 2C_1(K - t_x\tilde{X}) - (2C_1t + C_2)J,$$

and the multiplication table for the characteristic algebra L_x is as follows:

L_x	X	K	$\tilde{X}$	J
X	0	$\tilde{X}$	0	0
K	$-\tilde{X}$	0	$-J$	$-2C_1(K - t_x\tilde{X}) + (2C_1t + C_2)J$
$\tilde{X}$	0	J	0	0
J	0	$2C_1(K - t_x\tilde{X}) - (2C_1t + C_2)J$	0	0

4.1.3. Case 3: $t_{1x} = t_x + \sqrt{C_3\,e^{2\alpha t_1} + C_4\,e^{\alpha(t_1+t)} + C_3\,e^{2\alpha t}}$. Direct calculations show that $[\tilde{X}, K] = \alpha K - \alpha t_x\tilde{X}$, and the multiplication table for characteristic algebra L_x is as follows:

L_x	X	K	$\tilde{X}$
X	0	$\tilde{X}$	0
K	$-\tilde{X}$	0	$-\alpha K + \alpha t_x\tilde{X}$
$\tilde{X}$	0	$\alpha K - \alpha t_x\tilde{X}$	0

4.1.4. Case 4: $t_{1x} = t_x + C_5(e^{\alpha t_1} - e^{\alpha t}) + C_6(e^{-\alpha t_1} - e^{-\alpha t})$. Direct calculations show that

$$J = \alpha \sum_{k=-\infty, k\neq 0}^{\infty} \{C_5(e^{\alpha t_k} - e^{\alpha t}) - C_6(e^{-\alpha t_k} - e^{-\alpha t})\}\frac{\partial}{\partial t_k}$$

and

$$\begin{aligned}[J, K] &= 2C_5C_6\alpha^2 \sum_{k=-\infty, k\neq 0}^{\infty} \{e^{\alpha(t-t_k)} + e^{\alpha(t_k-t)} - 2\}\frac{\partial}{\partial t_k} \\ &= \alpha^2(C_5\,e^{\alpha t} + C_6\,e^{-\alpha t})(K - t_x\tilde{X}) + \alpha(C_6\,e^{-\alpha t} - C_5\,e^{\alpha t})J.\end{aligned}$$

Denote by

$$\beta_1 = \alpha^2(C_5\,e^{\alpha t} + C_6\,e^{-\alpha t}), \qquad \beta_2 = \alpha(C_6\,e^{-\alpha t} - C_5\,e^{\alpha t}).$$

The multiplication table for the characteristic algebra L_x is

L_x	X	K	$\tilde{X}$	J
X	0	$\tilde{X}$	0	0
K	$-\tilde{X}$	0	$-J$	$-\beta_1(K - t_x\tilde{X}) - \beta_2 J$
$\tilde{X}$	0	J	0	$\alpha^2 K - \alpha^2\tilde{X}$
J	0	$\beta_1(K - t_x\tilde{X}) + \beta_2 J$	$\alpha^2\tilde{X} - \alpha^2 K$	0

4.2. Characteristic algebras L_n for the Darboux-integrable equation $t_{1x} = t_x + d(t, t_1)$

4.2.1. Case 1: $t_{1x} = t_x + A(t_1 - t)$. The characteristic algebra L_n is generated only by the two vector fields X_1 and Y_1, and can be of any finite dimension. If $A(t_1 - t) = t_1 - t + c$, where c is some constant, then the characteristic algebra L_n is trivial, consisting of X_1 and Y_1 only, with commutativity relation $[X_1, Y_1] = 0$. If $A(t_1 - t) \neq t_1 - t + c$, one can choose a basis in L_n consisting of $W = \frac{\partial}{\partial\theta}$, $Z = \sum_{k=0}^{k=\infty} D_x^k p(\theta)\partial/\partial t_{[k]}$, with $\theta = x + \alpha_n$, $C_1 = [W, Z]$, $C_{k+1} = [W, C_k]$, $1 \leqslant k \leqslant N-1$. Its multiplication table for L_n is as follows:

L_n	W	Z	C_1	C_2	$\dots$	C_k	$\dots$	C_{N-1}	C_N
W	0	C_1	C_2	C_3	$\dots$	C_{k+1}	$\dots$	C_N	K
Z	$-C_1$	0	0	0	$\dots$	0	$\dots$	0	0
C_1	$-C_2$	0	0	0	$\dots$	0	$\dots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
C_N	$-K$	0	0	0	$\dots$	0	$\dots$	0	0

where $K = \mu_0 Z + \mu_1 C_1 + \dots + \mu_N C_N$.

4.2.2. Cases 2 and 4: $t_{1x} = t_x + C_1\left(t_1^2 - t^2\right) + C_2(t_1 - t)$ *and* $t_{1x} = t_x + C_5(\mathrm{e}^{\alpha t_1} - \mathrm{e}^{\alpha t}) + C_6(\mathrm{e}^{-\alpha t_1} - \mathrm{e}^{-\alpha t})$. In both cases the characteristic algebra L_n is trivial, consisting of X_1 and Y_1 only, with commutativity relation $[X_1, Y_1] = 0$.

4.2.3. Case 3: $t_{1x} = t_x + \sqrt{C_3\,\mathrm{e}^{2\alpha t_1} + C_4\,\mathrm{e}^{\alpha(t_1+t)} + C_3\,\mathrm{e}^{2\alpha t}}$. Denote by $\tilde{X}_1 = A(\tau_{-1})\,\mathrm{e}^{-\alpha\tau_{-1}}\frac{\partial}{\partial\tau_{-1}}$ and $\tilde{Y}_1 = A(\tau_{-1})Y_1$, $C_2 = [\tilde{X}_1, \tilde{Y}_1]$. Direct calculations show that the multiplication table for the algebra L_n is as follows:

L_n	$\tilde{X}_1$	$\tilde{Y}_1$	C_2
$\tilde{X}_1$	0	C_2	$\alpha^2 C_3\tilde{Y}_1 + C_4/(2C_3)\tilde{X}_1$
$\tilde{Y}_1$	$-C_2$	0	K
C_2	$-\alpha^2 C_3\tilde{Y}_1 - C_4/(2C_3)\tilde{X}_1$	$-K$	0

where $K = -(\alpha^2 C_4/2)\tilde{Y}_1 + (2\alpha^2 C_4\,\mathrm{e}^{\alpha\tau_{-1}} - \alpha^2 C_3)\tilde{X}_1$.

4.3. Explicit solutions for Darbour-integrable chains from theorem 4.1

Below we find the explicit solutions for Darboux-integrable chains of special form (21).

Theorem 4.2.

(a) *The explicit solution of equation (22) is*

$$t(n,x) = t(0,x) + \sum_{j=0}^{n-1} R(x + P_j), \tag{27}$$

where $t(0,x)$ *and* P_j *are the arbitrary functions of* x *and* j*, respectively, and* $A(\tau) = R'(\theta)$, $t_1 - t = R(\theta)$.

(b) *The explicit solution of equation (23) is*

$$t(n,x) = \frac{1}{C_1}\left(\frac{\psi_{xx}}{2\psi_x} - \frac{\psi_x}{P_n + \psi}\right) - \frac{C_2}{2C_1}, \tag{28}$$

where $\psi = \psi(x)$ *is an arbitrary function depending on* x *and* P_n *is an arbitrary function depending on* n *only.*

(c) *The explicit solution t(n,x) of equation (24) satisfies*

$$\mathrm{e}^{\alpha t(n,x)} = \frac{\mu'(x)(R_1(P_n - P_{n+1}))}{0.25\alpha^2(\mu(x) + (P_n + P_{n+1}) + R_3(P_n - P_{n+1}))^2 - C_3(R_1(P_n - P_{n+1}))^2}, \tag{29}$$

where $R_1 = 2\alpha/\sqrt{2C_3 + C_4}$, $R_3 = \sqrt{2C_3 - C_4}/\sqrt{2C_3 + C_4}$, *and* μ *and* P_n *are the arbitrary functions depending respectively on the variables* x *and* n.

(d) Equation (25) does not admit any explicit formula for a general solution of the form

$$t = H(x, \psi(x), \psi'(x), \ldots, \psi^{(k)}(x), P_n, P_{n+1}, \ldots, P_{n+m}). \tag{30}$$

However, equation (25) admits a general solution in a more complicated form

$$\mathrm{e}^{\alpha t(n,x)} = \frac{1}{\alpha C_5}\left(\frac{\psi_{xx}}{2\psi_x} - \frac{\psi_x}{P_n + \psi}\right) + \frac{1}{\alpha C_5} w, \tag{31}$$

where the nonlocal variable $w = w(x)$ is a solution of the first-order ODE

$$w'_x + w^2 - \alpha^2 C_5 C_6 = -\left(\frac{\psi_{xx}}{2\psi_x}\right)_x + \left(\frac{\psi_{xx}}{2\psi_x}\right)^2.$$

Proof. In a trivial case (a), the explicit solution was described in [11].

In case (b), equation (23) has an n-integral $I = t_x - C_1 t^2 - C_2 t$. Since $DI = I$, we have the following Riccati equation $t_x - C_1 t^2 - C_2 t = C(x)$ to solve and obtain the explicit solution (28).

In case (c), to find the explicit solution of equation (24), we look for the Cole–Hopf-type substitution $t = H(v, v_1, v_x)$ that reduces the equation to the semi-discrete D'Alembert equation $v_{1x} = v_x$ for which the solution is $v = \mu(x) + P_n$. Let us find the function $H(v, v_1, v_x)$. Since $v_{1x} = v_x$, $t_1 = H(v_1, v_2, v_x) =: \bar{H}$ and $t_x = v_{xx} H_{v_x} + v_x(H_v + H_{v_1})$, $t_{1x} = v_{xx}\bar{H}_{v_x} + v_x(\bar{H}_{v_1} + \bar{H}_{v_2})$. In new variables, equation (24) becomes

$$v_{xx}\{\bar{H}_{v_x} - H_{v_x}\} = v_x(H_v + H_{v_1} - \bar{H}_{v_1} - \bar{H}_{v_2}) + \sqrt{C_3\,\mathrm{e}^{2\alpha\bar{H}} + C_4\,\mathrm{e}^{\alpha(H+\bar{H})} + C_3\,\mathrm{e}^{2\alpha H}}\,. \tag{32}$$

The right-hand side of (32) does not depend on v_{xx}, but the left-hand side does unless $\bar{H}_{v_x} = H_{v_x}$. It implies that $H(v, v_1, v_x) = \psi(v_x) + A(v, v_1)$, where ψ is a function of one variable v_x and A is a function depending on v and v_1. Now, $t = H(v, v_1, v_x) = \psi(v_x) + A(v, v_1)$, $t_1 = \psi(v_x) + A(v_1, v_2) =: \psi(v_x) + \bar{A}$, and equality (32) becomes

$$(\bar{A}_{v_1} + \bar{A}_{v_2} - A_v - A_{v_1})v_x = \mathrm{e}^{\alpha\psi(v_x)}\sqrt{C_3\,\mathrm{e}^{2\alpha\bar{A}} + C_4\,\mathrm{e}^{\alpha(A+\bar{A})} + C_3\,\mathrm{e}^{2\alpha A}} \tag{33}$$

that shows $\mathrm{e}^{\alpha\psi(v_x)} = R v_x$, where R is some constant. We have

$$\mathrm{e}^{\alpha H(v, v_1, v_x)} = R v_x\,\mathrm{e}^{\alpha A(v, v_1)}. \tag{34}$$

Let us find the function $A(v, v_1)$. In the variables v_k, $v_{[k]}$ an n-integral $I = 2t_{xx} - \alpha t_x^2 - \alpha C_3\,\mathrm{e}^{2\alpha t} = q(x)$ of equation (24) becomes

$$I = \left(\frac{2v_{xxx}}{\alpha v_x} - \frac{3v_{xx}^2}{\alpha v_x^2}\right) + v_x^2(2A_{vv} + 4A_{vv_1} + 2A_{v_1v_1} - \alpha(A_v + A_{v_1})^2 - \alpha C_3 R^2\,\mathrm{e}^{2\alpha A})$$
$$=: s(v_x, v_{xx}, v_{xxx}) + v_x^2 p(v, v_1) = q(x).$$

One can see

$$p(v, v_1) = 2A_{vv} + 4A_{vv_1} + 2A_{v_1v_1} - \alpha(A_v + A_{v_1})^2 - \alpha C_3 R^2\,\mathrm{e}^{2\alpha A} = 0,$$

that can be rewritten in variables $\xi = (v + v_1)/2$ and $\eta = (v - v_1)/2$ in the following form:

$$2A_{\xi\xi} - \alpha(A_\xi)^2 - \alpha C_3 R^2\,\mathrm{e}^{2\alpha A} = 0$$

that implies

$$\mathrm{e}^{-\alpha A(\xi,\eta)} = \frac{\alpha^2}{4} C_1(\eta)(\xi + C_2(\eta))^2 - \frac{C_3 R^2}{C_1(\eta)}, \tag{35}$$

where $C_1(\eta)$ and $C_2(\eta)$ are some functions depending on η only. Equation (33) implies not only $e^{\alpha\psi(v_x)} = Rv_x$, but also

$$\bar{A}_{v_1} + \bar{A}_{v_2} - A_v - A_{v_1} = R\sqrt{C_3\,e^{2\alpha\bar{A}} + C_4\,e^{\alpha(A+\bar{A})} + C_3\,e^{2\alpha A}}$$

that in the variables ξ and η becomes

$$\bar{A}_{\xi_1} - A_\xi = R\sqrt{C_3\,e^{2\alpha\bar{A}} + C_4\,e^{\alpha(A+\bar{A})} + C_3\,e^{2\alpha A}}. \tag{36}$$

We find the function A from (35) and substitute it into (36), remembering that $\xi_1 = \xi - \eta - \eta_1$. We have

$$\frac{4}{\alpha^2 C_3 R^2}\left\{-\frac{\xi-\eta-\eta_1+C_2(\eta_1)}{(\xi-\eta-\eta_1+C_2(\eta_1))^2-4\alpha^{-2}C_3R^2}+\frac{\xi+C_2(\eta)}{(\xi+C_2(\eta))^2-4\alpha^{-2}C_3R^2}\right\}^2$$
$$= \left(\frac{4\alpha^{-2}C_1^{-1}(\eta_1)}{(\xi-\eta-\eta_1+C_2(\eta_1))^2-4\alpha^{-2}C_3R^2C_1^{-2}(\eta_1)}\right)^2$$
$$+\left(\frac{4\alpha^{-2}C_1^{-1}(\eta)}{(\xi+C_2(\eta))^2-4\alpha^{-2}C_3R^2C_1^{-2}(\eta)}\right)^2$$
$$+\frac{16\alpha^{-4}C_4C_3^{-1}C_1^{-1}(\eta)C_1^{-1}(\eta_1)}{\big((\xi+C_2(\eta))^2-4\alpha^{-2}C_3R^2C_1^{-2}(\eta)\big)\big((\xi-\eta-\eta_1+C_2(\eta_1))^2-4\alpha^{-2}C_3R^2C_1^{-2}(\eta_1)\big)},$$

or, equivalently,

$$(C_2(\eta)+\eta-(C_2(\eta_1)-\eta_1))^2-4\alpha^{-2}C_3R^2\big(C_1^{-2}(\eta)+C_1^{-2}(\eta_1)\big)$$
$$-4\alpha^{-2}C_4R^2C_1^{-1}(\eta)C_1^{-1}(\eta_1)=0. \tag{37}$$

To find $C_1(\eta)$ and $C_2(\eta)$, let us differentiate both sides of (37) with respect to η and then with respect to η_1. We have

$$\frac{-2C_4R^2C_1'(\eta)}{C_1^2(\eta)(C_2'(\eta)+1)} = \frac{(C_2'(\eta_1)-1)C_1^2(\eta_1)}{C_1'(\eta_1)}.$$

We use the fact that the left-hand side of the last equation depends on η only, but the right-hand side depends only on η_1 and obtain

$$C_1(\eta) = \frac{1}{R_1\eta+R_2}, \qquad C_2(\eta) = R_3\eta + R_4, \tag{38}$$

where $R_1 = 2(D+2\alpha^{-2}C_4R^2D^{-1})^{-1}$, $R_3 = (-D+2\alpha^{-2}C_4R^2D^{-1})(D+2\alpha^{-2}C_4R^2D^{-1})^{-1}$, and D, R_2, R_4 are some constants. Combining formulas (34), (35) and (38), we see that

$$e^{\alpha t} = \frac{\mu'(x)R(R_1(P_n-P_{n+1})+R_2)}{0.25\alpha^2(\mu(x)+(P_n+P_{n+1})+R_3(P_n-P_{n+1})+R_4)^2-C_3(R_1(P_n-P_{n+1})+R_2)^2}. \tag{39}$$

Without loss of generality, we may assume $R = 1$ and $R_4 = 0$. The substitution of (39) into (24) shows that $R_2 = 0$, $R_1 = 2\alpha/\sqrt{2C_3+C_4}$ and $R_3 = \sqrt{2C_3-C_4}/\sqrt{2C_3+C_4}$.

Let us study case (d). For the sake of convenience, we set $C_5 = C_6 = \alpha = 1$. Suppose that there exists a function H such that for any choice of functions $\psi(x)$ and P_n, function (30) solves (25). Consider the variables $\psi, \psi', \ldots, \psi^{(j)}, \ldots, P_n, P_{n\pm1}, \ldots, P_{n\pm k}, \ldots$ as new dynamical variables. The substitution of $t = H$, $t_1 = H(x, \psi(x), \psi'(x), \ldots, \psi^{(k)}, P_{n+1}, P_{n+2}, \ldots, P_{n+m+1}) =: \bar{H}$, $t_x = H_x + \psi' H_\psi + \psi'' H_{\psi'} + \cdots + \psi^{(k+1)} H_{\psi^{(k)}}$, $t_{1x} = \bar{H}_x + \psi'\bar{H}_\psi + \psi''\bar{H}_{\psi'} + \cdots + \psi^{(k+1)}\bar{H}_{\psi^{(k)}}$ into (25) yields

$$\bar{H}_x + \psi'\bar{H}_\psi + \psi''\bar{H}_{\psi'} + \cdots + \psi^{(k+1)}\bar{H}_{\psi^{(k)}} = H_x + \psi' H_\psi + \psi'' H_{\psi'} + \cdots + \psi^{(k+1)} H_{\psi^{(k)}}$$
$$+ e^{\bar{H}} + e^{-\bar{H}} - e^{H} - e^{-H}. \tag{40}$$

Evidently, $H_{\psi^{(k)}} = \bar{H}_{\psi^{(k)}}$ and consequently H can be represented as

$$H = h(x, \psi, \psi', \ldots, \psi^{(k)}) + r(x, \psi, \psi', \ldots, \psi^{(k-1)}, P_n, P_{n+1}, \ldots, P_{n+m}). \tag{41}$$

Substitute H found in (41) in the relation $t_x - 2\cosh t = C(x)$ obtained from the n-integral for (25). We have

$$\begin{aligned} &h_x + r_x + (h_\psi + r_\psi)\psi' + (h_{\psi'} + r_{\psi'})\psi'' \\ &\quad + \cdots (h_{\psi^{(k-1)}} + r_{\psi^{(k-1)}})\psi^{(k)} + h_{\psi^{(k)}}\psi^{(k+1)} - 2\cosh(h + r) = C(x). \end{aligned} \tag{42}$$

Differentiate it with respect to $P_{n+m} := z$:

$$r_{xz} + r_{\psi z}\psi' + r_{\psi' z}\psi'' + \cdots + r_{\psi^{k-1}z}\psi^{(k)} - 2r_z\cosh(h + r) = 0. \tag{43}$$

We find h+r from the last equation

$$h + r = \operatorname{arccosh}\left(\frac{r_{xz} + r_{\psi z}\psi' + \cdots + r_{\psi^{k-1}z}\psi^{(k)}}{2r_z}\right). \tag{44}$$

Denote $r_{xz} + r_{\psi z}\psi' + \cdots + r_{\psi^{k-2}z}\psi^{(k-1)} = A(x, \psi, \psi', \psi^{(k-1)}, P_n, \ldots, P_{n+m})$. Differentiate (44) with respect to $z = P_{n+m}$ and set

$$r_z = \frac{\left(\frac{A + r_{\psi^{(k-1)}z}\psi^{(k)}}{2r_z}\right)_z}{\sqrt{\left(\frac{A + r_{\psi^{(k-1)}z}\psi^{(k)}}{2r_z}\right)^2 - 1}}, \tag{45}$$

where A and r do not depend on $\psi^{(k)}$. Let us denote $p := r_z$, $y := \psi^{(k-1)}$ and $\xi := \psi^{(k)}$; then

$$p^2\left(\frac{A + p_y\xi}{2p}\right)^2 - p^2 = \left(\frac{A + p_y\xi}{2p}\right)_z = \left(\frac{(A_z + p_{yz}\xi)p - p_z(A + p_y\xi)}{2p^2}\right)^2 \tag{46}$$

or

$$p^4(A + p_y\xi)^2 - 4p^6 = ((A_z p - p_z A) + (p_{yz}p - p_y p_z)\xi)^2. \tag{47}$$

The comparison of the coefficients in (47) gives rise to three equalities: $p^4 p_y^2 = (p_{yz}p - p_y p_z)^2$, $p^4 A p_y = (A_z p - p_z A)(p_{yz}p - p_y p_z)$ and $p^4 A^2 - 4p^6 = (A_z p - p_z A)^2$, that are consistent only if $p = r_z = r_{P_{n+m}} = 0$. The condition $H_{P_{n+m}} = 0$ contradicts our assumption that H essentially depends on P_{n+m}. (Note that if $H_{P_k} = 0$ for all $k \in \mathbb{Z}$, then we have a trivial solution $t = H(x)$ for equation (25).) Therefore, in case (*d*) the solution cannot be represented in form (30). □

One can use the n-integral $I = t_x - C_5\,\mathrm{e}^{\alpha t} - C_6\,\mathrm{e}^{-\alpha t}$, solve the equation $I = C(x)$ which with the help of the substitution $u = \mathrm{e}^{\alpha t}$ can be brought to the Riccati equation, and see that equation (25) admits a general solution in a more complicated form (31).

5. Conclusions

Darboux-integrable semi-discrete chains are studied. The structures of their integrals are described. It is proved that if the chain admits an n-integral of order k, then it also admits an n-integral linearly depending on the highest order variable $t_{[k]}$. Similarly, if the chain admits an x-integral $F(x, t_{-k}, t_{-k+1}, \ldots t_m)$, then there is an x-integral $F^0(x, t_{-k}, t_{-k+1}, \ldots t_m)$ solving the equation

$$\frac{\partial^2 F^0}{\partial t_{-k}\partial t_m} = 0.$$

The previously found list of the Darboux-integrable chains of a particular form $t_{1x} = t_x + d(t, t_1)$ is studied in detail. The tables of multiplication for the corresponding characteristic algebras are given, and the explicit formulas for general solutions are constructed.

The problem of complete classification of Darboux-integrable chains (1) is still open. Another important open problem is connected with the systems of discrete equations: find Darboux-integrable discrete versions of the exponential-type hyperbolic systems corresponding to the Cartan matrices of semi-simple Lie algebras.

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