Math. Z. (2012) 270:939–959
DOI 10.1007/s00209-010-0833-z

Mathematische Zeitschrift

Fusion systems and constructing free actions on products of spheres

Özgün Ünlü · Ergün Yalçın

Received: 2 July 2010 / Accepted: 22 November 2010 / Published online: 15 January 2011

Abstract We show that every rank two p-group acts freely and smoothly on a product of two spheres. This follows from a more general construction: given a smooth action of a finite group G on a manifold M, we construct a smooth free action on $M \times \mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_k}$ when the set of isotropy subgroups of the G-action on M can be associated to a fusion system satisfying certain properties. Another consequence of this construction is that if G is an (almost) extra-special p-group of rank r, then it acts freely and smoothly on a product of r spheres.

Mathematics Subject Classification (2000) Primary 57S25; Secondary 20D20

1 Introduction

In [25], Smith proved that if a finite group G acts freely on a sphere, then G has no subgroup isomorphic to the elementary abelian group $\mathbb{Z}/p \times \mathbb{Z}/p$ for any prime number p. Later in [21], Milnor showed that there are other restrictions on such a G, more precisely, he proved that if G acts freely on a sphere $\mathbb{S}^n$, then G has no subgroup isomorphic to the dihedral group D_{2p} of order $2p$ for any odd prime number p.

Conversely, Madsen–Thomas–Wall [19] proved that Smith's condition together with Milnor's condition is enough to ensure the existence of a free smooth action on a sphere $\mathbb{S}^n$ for some $n \geq 1$. The existence proof of Madsen–Thomas–Wall used surgery theory and exploited some natural constructions of free group actions on spheres. Specifically, they considered the unit spheres of linear representations of subgroups to show that certain surgery obstructions vanish.

The first author is partially supported by TÜBİTAK-TBAG/109T384 and the second author is partially supported by TÜBA-GEBİP/2005-16.

Ö. Ünlü · E. Yalçın (✉)
Department of Mathematics, Bilkent University, 06800 Ankara, Turkey
e-mail: yalcine@fen.bilkent.edu.tr

Ö. Ünlü
e-mail: unluo@fen.bilkent.edu.tr

As a generalization of the above problem, we are interested in the problem of characterizing those finite groups which can act freely and smoothly on a product of two spheres. Heller [13] found a restriction on the existence of such actions similar to Smith's condition: if a finite group G acts freely on a product of two spheres, then G has no subgroup isomorphic to the elementary abelian group $\mathbb{Z}/p \times \mathbb{Z}/p \times \mathbb{Z}/p$. The maximum rank of elementary abelian subgroups $(\mathbb{Z}/p)^k \leq G$ is called the rank of G. So, Heller's result says that if a finite group G acts freely on a product of two spheres, then G must have $\mathrm{rk}(G) \leq 2$. So far, no condition analogous to Milnor's condition is found for the existence of smooth actions and it appears as if to prove a converse, we need more constructions of natural actions.

As a first attempt to construct free actions on products of two spheres, one can take a product of two unit spheres $\mathbb{S}(V_1) \times \mathbb{S}(V_2)$ where V_1 and V_2 are linear representations of the group. However, it is not hard to see that for many groups of rank 2, it is not possible to find two linear spheres such that the action on their product is free. For example, when p is an odd prime, the extraspecial p-group order p^3 and exponent p does not act freely on a product of two linear spheres although this group has rank equal to two.

Another natural construction is to take a representation with small fixity and consider the Stiefel manifolds associated to this representation. The fixity of a G-representation V is the maximum dimension of fixed subspaces V^g over all nontrivial elements g in G. If G has an n-dimensional complex representation of fixity 1, then G acts freely on the Stiefel manifold $V_{n,2}(\mathbb{C}) \simeq U(n)/U(n-2)$. This space is the total space of a sphere bundle over a sphere, and taking fiber joins, one obtains a free action on a product of two spheres. This method was used by Adem et al. [1] to show that for $p \geq 5$, every rank two p-group acts freely and smoothly on a product of two spheres. However, there are examples of rank two 2-groups and 3-groups which have no representation with fixity 1, so this method is not enough to construct free actions of rank two p-groups on products of two spheres for all primes p.

A more general idea for constructing free actions on a product of two spheres is to start with a representation sphere $\mathbb{S}(V)$ and construct a G-equivariant sphere bundle over it so that the action on the total space is free and the bundle is non-equivariantly trivial. In the homotopy category, a similar idea was used by Adem and Smith [2] to show that many rank two finite groups can act freely on a finite complex homotopy equivalent to a product of two spheres. In particular, they showed that every rank two p-group acts freely on a finite CW-complex homotopy equivalent to a product of two spheres.

In this paper, we prove the following:

Theorem 1.1 *A finite p-group G acts freely and smoothly on a product of two spheres if and only if* $\mathrm{rk}(G) \leq 2$.

The proof uses another method of construction of free actions on products of spheres which was introduced by Ünlü [27] in his thesis. The method uses a theorem of Lück-Oliver [18, Thm. 2.6] on constructions of equivariant vector bundles over a finite dimensional G–CW-complex. We now describe briefly the main idea of the Lück-Oliver construction: Let X be a finite dimensional G–CW-complex and $\mathcal{H}$ be the family of isotropy subgroups of X. Given a compatible family of unitary representations $\rho_H : H \to U(n)$ where $H \in \mathcal{H}$, one would like to construct a G-equivariant vector bundle over X so that the representation over a point with isotropy H is isomorphic to $(\rho_H)^{\oplus k}$ for some k. Lück and Oliver [18] shows that this can be done if there is a finite group Γ which satisfies the following two conditions:

(i) There is a family of maps $\{\alpha_H : H \to \Gamma \mid H \in \mathcal{H}\}$ which is compatible in the sense that if $c_g : H \to K$ is a map induced by conjugation with $g \in G$, then there is a $\gamma \in \Gamma$ such that the following diagram commutes:

$$\begin{tikzcd} H \arrow[r, "\alpha_H"] \arrow[d, "c_g"] & \Gamma \arrow[d, "c_\gamma"] \\ K \arrow[r, "\alpha_K"] & \Gamma \end{tikzcd}$$

(ii) Γ has a representation $\rho : \Gamma \to U(n)$ such that $\rho_H = \rho \circ \alpha_H$ for all $H \in \mathcal{H}$.

In [27], Ünlü showed that when all the groups in the family $\mathcal{H}$ are cyclic p-groups, there is a finite group Γ satisfying the above conditions for a family of representations ρ_H such that H action on the unit sphere $\mathbb{S}(\rho_H)$ is free. As a result of this, Ünlü [27] was able to show that when p is odd, every rank two p-group acts freely and smoothly on a product of two spheres. When $p = 2$, the groups one has to deal with are rank one 2-groups and these can be cyclic or generalized quaternion. It turns out that maps between subgroups of quaternion groups have a much richer structure, so the method given in does not extend directly to families of rank one 2-groups.

In this paper, we find a systemic way of constructing a finite group Γ satisfying the above conditions (i) and (ii) for some suitable representation families. We first choose a finite group S and map all subgroups in the family $\mathcal{H}$ into S via some maps $\iota_H : H \to S$. Then, we study the fusion system on S that comes from the conjugations in G and different choices of mappings ι_H (see [7] or [24] for a definition of a fusion system). Although the fusion systems that arise in this way are not necessarily saturated, we use some of the machinery developed for studying saturated fusion systems. In particular, we use a theorem of Park [23] to find Γ as the automorphism group of an S–S-biset.

This method of finding a finite group Γ works for more general groups then the families formed by rank one 2-groups. For example, for a family $\mathcal{H}$ formed by elementary abelian p-groups, we can easily find a finite group Γ by choosing an appropriate S–S-biset. Moreover, this process can be recursively continued to obtain the following theorem.

Theorem 1.2 *Let G be a finite group acting smoothly on a manifold M. If all the isotropy subgroups of M are elementary abelian groups with rank $\leq k$, then G acts freely and smoothly on $M \times \mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_k}$ for some positive integers $n_1, \ldots, n_k$.*

As a corollary, we obtain the following:

Corollary 1.3 *Let G be an (almost) extraspecial p-group of rank r. Then, G acts freely and smoothly on a product of r spheres.*

The paper is organized as follows: Sects. 2 and 3 are preliminary sections on equivariant principal bundles and equivariant obstruction theory. In Sect. 4, we review the work of Lück and Oliver [18] on constructions of equivariant bundles and prove Theorem 4.3 which is a slightly different version of Theorem 2.7 in [18]. Then, in Sect. 5, we introduce a method for constructing finite groups Γ satisfying the properties explained above. This is done using a theorem of Park [23] on bisets associated to fusion systems. Finally, in Sect. 6, we prove our main theorems, Theorems 1.1 and 1.2.

2 Equivariant principal bundles

In this section, we introduce the basic definitions of equivariant bundle theory. We refer the reader to [16,17], and [18] for more details.

Let G be a compact Lie group. A relative G–CW-complex (X, A) is a pair of G-spaces together with a G-invariant filtration

$$A = X^{(-1)} \subseteq X^{(0)} \subseteq X^{(1)} \subseteq \cdots \subseteq X^{(n)} \subseteq \cdots \subseteq \bigcup_{n \geq -1} X^{(n)} = X$$

such that A is a Hausdorff space, X carries the colimit topology with respect to this filtration and for all $n \geq 0$, the space $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching equivariant n-dimensional cells, i.e., there exists a G-pushout diagram as follows

$$\begin{tikzcd} \coprod_{\sigma \in I_n} G/H_\sigma \times \mathbb{S}^{n-1} \arrow[r] \arrow[d] & X^{(n-1)} \arrow[d] \\ \coprod_{\sigma \in I_n} G/H_\sigma \times \mathbb{D}^n \arrow[r] & X^{(n)} \end{tikzcd}$$

where I_n is an index set, H_σ is a subgroup of G for $\sigma \in I_n$, and $n \geq 0$. Elements of I_n are called equivariant n-cells and for $\sigma \in I_n$, the map $G/H_\sigma \times \mathbb{S}^{n-1} \to X^{(n-1)}$ is called the attaching map and the map $G/H_\sigma \times \mathbb{D}^n \to X^{(n)}$ is called the characteristics map of the cell. Here we consider $\mathbb{S}^{-1} = \emptyset$ and $\mathbb{D}^0 = \{\text{a point}\}$. The space $X^{(n)}$ is called the n-skeleton of (X, A) for $n \geq -1$. For more details about G-CW-complexes, see Sect. II.1-2 in [9] and Sect. I.1-2 in [16].

We now give the definition for the classifying space of a group relative to a family.

Definition 2.1 Let $\mathcal{H}$ be a family of closed subgroups of G closed under conjugation. Define $E_{\mathcal{H}}(G)$ as the realization of the nerve of the category $\mathcal{E}_{\mathcal{H}}(G)$ whose objects are pairs $(G/H, xH)$ where $H \in \mathcal{H}$ and $x \in G$ and morphisms from $(G/H, xH)$ to $(G/K, yK)$ are the G-maps from G/H to G/K which sends xH to yK.

We can consider the space $E_{\mathcal{H}}(G)$ as a G-CW-complex with the G-action induced by $g(G/H, xH) = (G/H, gxH)$ on the objects of the category $\mathcal{E}_{\mathcal{H}}(G)$. For any $H \in \mathcal{H}$, the space $E_{\mathcal{H}}(G)^H$ is the realization of the nerve of the full subcategory of $\mathcal{E}_{\mathcal{H}}(G)$ with objects $(G/K, xK)$ where $H \leq K^x$. The object $(G/H, H)$ is an initial object in this subcategory. Hence $E_{\mathcal{H}}(G)^H$ is contractible for any $H \in \mathcal{H}$ and we get the following classifying property of $E_{\mathcal{H}}(G)$.

Proposition 2.2 [16, Prop 2.3] *Let (X, A) be a relative G–CW-complex such that $G_x \in \mathcal{H}$ for all $x \in X$. Then, any G-map from A to $E_{\mathcal{H}}(G)$ extends to a G-map from X to $E_{\mathcal{H}}(G)$ and any two such extensions are G-homotopic relative to A.*

Let G be a finite group and Γ be a compact Lie group. A G-equivariant Γ-bundle over a left G-space X is a Γ-principal bundle $p : E \to X$ where E is a left G-space, p is a G-equivariant map, and the right action of Γ on E and the left action of G on E commute. Let $\mathrm{Bdl}_{G,\Gamma}(X)$ denote the isomorphism classes of G-equivariant Γ-bundles over X.

Let $\mathrm{Or}_{\mathcal{H}}(G)$ denote the orbit category whose objects are orbits G/H where $H \in \mathcal{H}$ and morphisms are G-maps from G/H to G/K. Assume that we are given an element

$$\mathbf{A} = (p_H) \in \varprojlim_{(G/H) \in \mathrm{Or}_{\mathcal{H}}(G)} \mathrm{Bdl}_{G,\Gamma}(G/H) \subseteq \prod_{H \in \mathcal{H}} \mathrm{Bdl}_{G,\Gamma}(G/H)$$

where a G-map from G/H to G/K induces a function from $\mathrm{Bdl}_{G,\Gamma}(G/K)$ to $\mathrm{Bdl}_{G,\Gamma}(G/H)$ by pullbacks. A G-equivariant $\mathbf{A}$-bundle over a left G-space X is a G-equivariant Γ-bundle

$p : E \to X$ such that for any $H \in \mathcal{H}$ and any G-equivariant map $i : G/H \to X$, the pullback $i^*(p)$ is isomorphic to p_H in $\mathrm{Bdl}_{G,\Gamma}(G/H)$. Let $\mathrm{Bdl}_{G,\mathbf{A}}(X)$ denote the isomorphism classes of G-equivariant $\mathbf{A}$-bundles over X.

Lemma 2.3 *We have*

$$\mathrm{Bdl}_{G,\Gamma}(G/H) \cong \mathrm{Rep}(H,\Gamma) := \mathrm{Hom}(H,\Gamma)/\,\mathrm{Inn}(\Gamma)$$

where $\mathrm{Hom}(H,\Gamma)$ *is the set of homomorphisms from H to* Γ *and* $\mathrm{Inn}(\Gamma)$ *is the group of inner automorphisms of* Γ *and the action of* $\mathrm{Inn}(\Gamma)$ *on* $\mathrm{Hom}(H,\Gamma)$ *is given by composition.*

Proof For a G-equivariant Γ-bundle p_H over the G-space G/H, let $E(p_H)$ denote the total space of the bundle p_H. Take a point $x \in p_H^{-1}(H) \subseteq E(p_H)$. Since $G \times \Gamma$ acts transitively on $E(p_H)$, we have $E(p_H) = (G \times \Gamma)/(G \times \Gamma)_x$ and $(G \times \Gamma)_x \cap (1 \times \Gamma) = \{1\}$. So, by Goursat's lemma,

$$(G \times \Gamma)_x = \Delta(\alpha_x) := \{(h, \alpha_x(h)) \mid h \in H\}$$

where the homomorphism $\alpha_x : H \to \Gamma$ is defined by the equation $hx(\alpha_x(h))^{-1} = x$ for $h \in H$. Let f be a bundle isomorphism from p_H to another G-equivariant Γ-bundle q_H over the G-space G/H. Take $y \in q_H^{-1}(H) \subseteq E(q_H)$ and define $\alpha_y : H \to \Gamma$ as above. Then there exists $\gamma_{x,y} \in \Gamma$ such that $f(x) = y\gamma_{x,y}$. So, for all $h \in H$, we have $\alpha_y(h) = \gamma_{x,y}\alpha_x(h)\gamma_{x,y}^{-1}$. Hence, up to composition with an inner automorphism of Γ, there exists a unique map $\alpha_H : H \to \Gamma$ such that $E(p_H) \cong G \times_H \Gamma$ where the action of H on $G \times \Gamma$ is given by $h(g,\gamma) = (gh^{-1}, \alpha_H(h)\gamma)$. □

We can view the family $\mathcal{H}$ as a category where the elements of $\mathcal{H}$ are the objects of the category and morphisms are compositions of conjugations in G with inclusions. A morphism in the category $\mathrm{Or}_{\mathcal{H}}(G)$ is a G-map from G/H to G/K and can be written in the form $\hat{a} : G/H \to G/K$ where $\hat{a}(gH) = ga^{-1}K$ for $a \in G$ such that $aHa^{-1} \le K$. Now the map induced by $\hat{a}$ from $\mathrm{Bdl}_{G,\Gamma}(G/K)$ to $\mathrm{Bdl}_{G,\Gamma}(G/H)$ by pullbacks is equivalent to the map from $\mathrm{Rep}(K,\Gamma)$ to $\mathrm{Rep}(H,\Gamma)$ induced by conjugation $c_a : H \to K$ given by $c_a(h) = aha^{-1}$. Hence we can consider

$$\mathbf{A} = (p_H) \in \varprojlim_{(G/H)\in \mathrm{Or}_{\mathcal{H}}(G)} \mathrm{Bdl}_{G,\Gamma}(G/H)$$

as an element

$$\mathbf{A} = (\alpha_H) \in \varprojlim_{H \in \mathcal{H}} \mathrm{Rep}(H,\Gamma) \subseteq \prod_{H \in \mathcal{H}} \mathrm{Rep}(H,\Gamma).$$

A family of representations $\alpha_H : H \to \Gamma$ is called a *compatible family* of representations if it is an element of a limit as above. We now describe the classifying space for G-equivariant $\mathbf{A}$-bundles.

Definition 2.4 Let $\mathbf{A} = (\alpha_H)$ be as above and let $\mathcal{H}_{\mathbf{A}}$ be the family of subgroups $W \le G \times \Gamma$ such that $W = \Delta(\alpha_H)$ for some representation α_H in $\mathbf{A}$. Define

$$E_{\mathcal{H}}(G,\mathbf{A}) = E_{\mathcal{H}_{\mathbf{A}}}(G \times \Gamma) \;\text{ and }\; B_{\mathcal{H}}(G,\mathbf{A}) = E_{\mathcal{H}_{\mathbf{A}}}(G,\mathbf{A})/(\{1\} \times \Gamma).$$

Note that the G-equivariant Γ-principal bundle $E_{\mathcal{H}}(G,\mathbf{A}) \to B_{\mathcal{H}}(G,\mathbf{A})$ is indeed a G-equivariant $\mathbf{A}$-bundle. This is because for any $x \in E_{\mathcal{H}}(G,\mathbf{A})$, we have $x\Gamma \in B_{\mathcal{H}}(G,\mathbf{A})$ and the pullback of the bundle $E_{\mathcal{H}}(G,\mathbf{A}) \to B_{\mathcal{H}}(G,\mathbf{A})$ by the natural inclusion of

$G/G_{x\Gamma} \to B_{\mathcal{H}}(G, \mathbf{A})$ is isomorphic to the bundle $(G \times \Gamma)/(G \times \Gamma)_x \to G/G_{x\Gamma}$. We also know that $(G \times \Gamma)_x \in \mathcal{H}_{\mathbf{A}}$ hence $(G \times \Gamma)_x = \Delta(\alpha_H)$ for some α_H in $\mathbf{A}$. In particular, $H = G_{x\Gamma}$. Hence the pullback of the bundle $E_{\mathcal{H}}(G, \mathbf{A}) \to B_{\mathcal{H}}(G, \mathbf{A})$ by the natural inclusion of $G/G_{x\Gamma} \to B_{\mathcal{H}}(G, \mathbf{A})$ is isomorphic to $p_{G_{x\Gamma}}$ in $\mathbf{A}$. We have the following:

Proposition 2.5 [18, Lemma 2.4] *Let* $\mathbf{A} = (\alpha_H : H \to \Gamma)$ *be a compatible family of representations. Then, the following hold:*

(i) *The bundle* $E_{\mathcal{H}}(G, \mathbf{A}) \to B_{\mathcal{H}}(G, \mathbf{A})$ *is the universal* G*-equivariant* $\mathbf{A}$*-bundle: If* X *is a* G*-CW-complex such that* $G_x \in \mathcal{H}$ *for all* $x \in X$*, then the map defined by pullbacks* $[X, B_{\mathcal{H}}(G, \mathbf{A})]_G \to \mathrm{Bdl}_{G,\mathbf{A}}(X)$ *is a bijection.*
(ii) *For all* $H \in \mathcal{H}$*, we have* $B_{\mathcal{H}}(G, \mathbf{A})^H \simeq BC_\Gamma(\alpha_H)$ *where* $C_\Gamma(\alpha_H)$ *denotes the centralizer of the image of* α_H *in* Γ.

Proof For the first statement observe that if $E \to X$ is a G-equivariant $\mathbf{A}$-bundle, then by construction there is a $(G \times \Gamma)$-map from E to $E_{\mathcal{H}}(G, \mathbf{A})$. Since both spaces have free Γ-action, taking orbit spaces we get a G-map $X \to B_{\mathcal{H}}(G, \mathbf{A})$ where the bundle $E \to X$ is the pullback bundle via this map.

To prove the second statement, let $\alpha_H : H \to \Gamma$ be a representation and let $C = \{1\} \times C_\Gamma(\alpha_H)$. Then C acts freely on the contractible space $E_{\mathcal{H}}(G, \mathbf{A})^{\Delta(\alpha_H)}$ and

$$E_{\mathcal{H}}(G, \mathbf{A})^{\Delta(\alpha_H)}/C \cong B_{\mathcal{H}}(G, \mathbf{A})^H$$

where the homeomorphism is given by $f(xC) = x\Gamma$ for $x \in E_{\mathcal{H}}(G, \mathbf{A})^{\Delta(\alpha_H)}$. To see that f is a homeomorphism, first note that f is well-defined and the image of f is in $B_{\mathcal{H}}(G, \mathbf{A})^H$. Now, take $x, y \in E_{\mathcal{H}}(G, \mathbf{A})$ such that $f(xC) = f(yC)$. Then, $x = y\gamma$ for some $\gamma \in \Gamma$. Since $hx = x\alpha_H(h)$ and $hy = y\alpha_H(h)$ for all $h \in H$, we get $\alpha_H(h)\gamma = \gamma\alpha_H(h)$ for all $h \in H$. Thus $\gamma \in C_\Gamma(\alpha_H)$ and $xC = yC$. This proves that f is one-to-one. To show that f is onto, let $x\Gamma \in B_{\mathcal{H}}(G, \mathbf{A})^H$. Then $H \le G_{x\Gamma}$ and there exists $\beta : G_{x\Gamma} \to \Gamma$ in $\mathbf{A}$ such that $(G \times \Gamma)_x = \Delta(\beta)$. Since the family of maps in $\mathbf{A}$ are compatible, there is a $\gamma \in \Gamma$ such that $c_\gamma \circ \beta|_H = \alpha_H$. Then $\Delta(\alpha_H) \le (G \times \Gamma)_{x\gamma}$. This means that $x\gamma \in E_{\mathcal{H}}(G, \mathbf{A})^{\Delta(\alpha_H)}$ and applying f to it, we get $f(x\gamma C) = x\Gamma$. So, f is onto. Hence we conclude that $B_{\mathcal{H}}(G, \mathbf{A})^H$ is homotopy equivalent to $BC_\Gamma(\alpha_H)$. □

3 Equivariant obstruction theory

In this section, we fix our notation for Bredon cohomology and state the main theorem of the equivariant obstruction theory that will be used in the next section. We refer the reader to [5,16], and [18] for more details.

Let G be a finite group and $\mathcal{H}$ be a family of subgroups closed under conjugation. As before we denote the orbit category of G relative to the family $\mathcal{H}$ by $\mathrm{Or}_{\mathcal{H}}(G)$. Let (X, A) be a relative G–CW-complex whose all isotropy groups are in $\mathcal{H}$. A coefficient system for Bredon cohomology is a contravariant functor $M : \mathrm{Or}_{\mathcal{H}}(G) \to \mathcal{Ab}$ where $\mathcal{Ab}$ denotes the category of abelian groups and group homomorphisms between them. A coefficient system is sometimes called a $\mathbb{Z}\,\mathrm{Or}_{\mathcal{H}}(G)$-module with the usual convention of modules over a small category. So, morphisms between $\mathbb{Z}\,\mathrm{Or}_{\mathcal{H}}(G)$-modules are given by a natural transformation of functors. Notice that the $\mathbb{Z}\,\mathrm{Or}_{\mathcal{H}}(G)$-module category is an abelian category, so the usual constructions of modules over a ring are available to do homological algebra. To simplify the notation, we call a $\mathbb{Z}\,\mathrm{Or}_{\mathcal{H}}(G)$-module, a $\mathbb{Z}\mathrm{O}_{\mathcal{H}}$-module.

Now, let us fix some notation for some of the $\mathbb{Z}O_{\mathcal{H}}$-modules that we will be considering. For example, consider the contravariant functor $\pi_n(X^?, A^?) : \mathrm{Or}_{\mathcal{H}}(G) \to \mathcal{A}b$ given by $\pi_n(X^?, A^?)(G/H) = \pi_n(X^H, A^H)$ for any object G/H in $\mathrm{Or}_{\mathcal{H}}(G)$ and a morphism $\hat{a} : G/H \to G/K$ in $\mathrm{Or}_{\mathcal{H}}(G)$ defined by $gH \to ga^{-1}K$ is sent to the morphism from $\pi_n(X^K, A^K)$ to $\pi_n(X^H, A^H)$ induced by left multiplication $x \to a^{-1}x$ considered as a map from (X^K, A^K) to (X^H, A^H).

Similarly, we set $C_n(X^?, A^?; M) : \mathrm{Or}_{\mathcal{H}}(G) \to \mathcal{A}b$ as

$$C_n(X^?, A^?)(G/H) = C_n(X^H, A^H; \mathbb{Z})$$

and morphisms defined in a similar way as above. The boundary maps of the chain complexes $C_*(X^H, A^H; \mathbb{Z})$ commute with conjugation and restriction maps, so when we put them together, we obtain a chain complex of $\mathbb{Z}O_{\mathcal{H}}$-modules

$$\cdots \longrightarrow C_2(X^?, A^?) \xrightarrow{\partial_1} C_1(X^?, A^?) \xrightarrow{\partial_0} C_0(X^?, A^?) \longrightarrow 0.$$

We define $H_n(X^?, A^?)$ as the cohomology of this chain complex. Note that the $\mathbb{Z}O_{\mathcal{H}}$-module $H_n(X^?, A^?; M) : \mathrm{Or}_{\mathcal{H}}(G) \to \mathcal{A}b$ satisfies

$$H_n(X^?, A^?)(G/H) = H_n(X^H, A^H; \mathbb{Z}).$$

Definition 3.1 Let (X, A) be a relative G-CW-complex and M be a $\mathbb{Z}O_{\mathcal{H}}$-module. The Bredon cohomology $H^*_G(X, A; M)$ of the pair (X, A) with coefficients in M is defined as the cohomology of the cochain complex

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}O_{\mathcal{H}}}(C_0(X^?, A^?), M) \xrightarrow{\delta^0} \mathrm{Hom}_{\mathbb{Z}O_{\mathcal{H}}}(C_1(X^?, A^?), M) \xrightarrow{\delta^1} \cdots$$

Bredon cohomology is useful to describe obstructions for extending equivariant maps. Let (X, A) be a relative G-CW-complex and Y be a G-space such that for all $H \leq G$ the invariant space Y^H is an $(n-1)$-simple space. Assume $f : X^{(n)} \to Y$ is a G-equivariant map. Then we define an element c_f in $\mathrm{Hom}_{\mathbb{Z}O_{\mathcal{H}}}(C_n(X^?, A^?), \pi_{n-1}(Y^?))$ for $H \in \mathcal{H}$ as follows: For every $H \in \mathcal{H}$, the homomorphism $c_f(H)$ is the map

$$c_f(H) : C_n(X^H, A^H) \to \pi_{n-1}(Y^H)$$

which takes $\sigma \in C_n(X^H, A^H)$ to the homotopy class of the map $f \circ \phi_\sigma : \mathbb{S}^{n-1} \to Y^H$ where ϕ_σ is the attaching map of the cell σ in the following pushout diagram:

$$\begin{array}{ccc} \partial(\sigma) & \xrightarrow{\phi_\sigma} & X^{(n-1)} \\ \downarrow & & \downarrow \\ \sigma & \longrightarrow & X^{(n)} \end{array}$$

The cochain c_f is a cocyle by Proposition II.1.1 in [5]. Hence we can define $\mathrm{obs}(f) = [c_f] \in H^n_G(X, A; \pi_{n-1}(Y^?))$. The cohomology class $\mathrm{obs}(f)$ is the obstruction to extending $f|_{X^{(n-1)}}$ to $X^{(n+1)}$. More precisely:

Proposition 3.2 *Let (X, A) be a relative G-CW-complex and Y be a G-space such that for all $H \leq G$, the invariant space Y^H is a simple space. Let $f : X^{(n)} \to Y$ be a G-equivariant map. Then $f|_{X^{(n-1)}}$ can be extended to an equivariant map from $X^{(n+1)}$ to Y if and only if* $\mathrm{obs}(f) = 0$ *in* $H^{n+1}_G(X, A; \pi_n(Y^?))$.

Proof See Proposition II.1.2 in [5]. □

Note that the category of $\mathbb{Z}\mathrm{O}_{\mathcal{H}}$-modules has enough injectives (see [5, p. 24]). Hence for any $\mathbb{Z}\mathrm{O}_{\mathcal{H}}$-module M, there exists an injective resolution

$$0 \longrightarrow M \xrightarrow{\epsilon} I^0 \xrightarrow{\rho^0} I^1 \xrightarrow{\rho^1} \cdots$$

For a $\mathbb{Z}\mathrm{O}_{\mathcal{H}}$-module N, we define the ext-group $\mathrm{Ext}^n_{\mathbb{Z}\mathrm{O}_{\mathcal{H}}}(N, M)$ as the cohomology of the cochain complex

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}\mathrm{O}_{\mathcal{H}}}(N, I^0) \xrightarrow{(\rho^0)_*} \mathrm{Hom}_{\mathbb{Z}\mathrm{O}_{\mathcal{H}}}(N, I^1) \xrightarrow{(\rho^1)_*} \cdots$$

Note that since we already know that the $\mathbb{Z}\mathrm{O}_{\mathcal{H}}$-module category has enough projectives, one can also calculate the above ext-groups using a projective resolutions of N.

The following proposition is used in the next section. We include a proof of it here for the convenience of the reader. The proof is given by standard homological algebra and can be found in the literature (see [5, Chap. 1, 10.4] or [20, Chap. 1, Thm 6.2]).

Proposition 3.3 *Let (X, A) be a G-CW-complex and $\mathcal{H}$ be a family of subgroups of G closed under conjugation such that for all $x \in X$, the isotropy subgroup G_x is in the family $\mathcal{H}$. Then for any $\mathbb{Z}\mathrm{O}_{\mathcal{H}}$-module M,*

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathbb{Z}\mathrm{O}_{\mathcal{H}}}(H_q(X^?, A^?), M) \implies H_G^{p+q}(X, A; M).$$

Proof Let $(C_*(X^?, A^?), \partial)$ denote the chain complex of (X, A) and let

$$0 \longrightarrow M \xrightarrow{\epsilon} I^0 \xrightarrow{\rho^0} I^1 \xrightarrow{\rho^1} I^2 \xrightarrow{\rho^2} \cdots$$

be an injective resolution of M as a $\mathbb{Z}\mathrm{O}_{\mathcal{H}}$-module. Define a double complex

$$D^{p,q} = \mathrm{Hom}_{\mathbb{Z}\mathrm{O}_{\mathcal{H}}}(C_q(X^?, A^?), I^p)$$

where $d_1 : D^{p,q} \to D^{p+1,q}$ is given by $d_1(f) = \rho^p \circ f$ and $d_2 : D^{p,q} \to D^{p,q+1}$ is given by $d_2(f) = (-1)^p f \circ \partial_{q+1}$ for $f \in D^{p,q}$. Now the spectral sequence of this double complex is in the form

$$E_2^{p,q} = H^p\left(H^q\left(D^{*,*}, d_2\right), d_1\right) \implies H^{p+q}(\mathrm{Tot}(D^{*,*}), d_1 + d_2)$$

where $\mathrm{Tot}(D^{*,*})$ is the total complex of the double complex $D^{*,*}$ (see p. 108 in [4]).

Since I^p is injective for all $p \geq 0$, we have

$$H^q\left(D^{p,*}, d_2\right) = H^q\left(\mathrm{Hom}_{\mathbb{Z}\mathrm{O}_{\mathcal{H}}}(C_*(X^?, A^?), I^p), d_2\right) = \mathrm{Hom}_{\mathbb{Z}\mathrm{O}_{\mathcal{H}}}(H_q(X^?, A^?), I^p).$$

Using this and the definition of ext-groups, we obtain

$$E_2^{p,q} = H^p\left(\mathrm{Hom}_{\mathbb{Z}\mathrm{O}_{\mathcal{H}}}(H_q(X^?, A^?), I^*), d_1\right) = \mathrm{Ext}^p_{\mathbb{Z}\mathrm{O}_{\mathcal{H}}}(H_q(X^?, A^?), M)).$$

Since $C_q(X^?, A^?)$ is projective as a $\mathbb{Z}\mathrm{O}_{\mathcal{H}}$-module for all $q \geq 0$, the following two cochain complexes are chain homotopy equivalent

$$(\mathrm{Tot}(D^{*,*}), d_1 + d_2) \simeq (\mathrm{Hom}_{\mathbb{Z}\mathrm{O}_{\mathcal{H}}}(C_*(X^?, A^?), M), d_2)$$

(see p. 45 in [3]). Hence

$$H^{p+q}(\mathrm{Tot}(D^{*,*}), d_1 + d_2) = H_G^{p+q}(X, A; M).$$

Therefore the spectral sequence for the double complex $D^{*,*}$ gives a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathbb{Z}\mathrm{O}_{\mathcal{H}}}(H_q(X^?, A^?), M)) \implies H_G^{p+q}(X, A; M).$$

□

4 Construction of equivariant bundles

The main theorem of this section is a slightly different version of a theorem of Lück and Oliver [18, Thm 2.7] on construction of equivariant bundles. This is the theorem that was mentioned in the introduction and it is the starting point of our construction of free actions on products of spheres.

Let Υ_k be a family of topological groups indexed by the positive integers. Given two maps $f, g : \Upsilon_k \to \Upsilon_m$, the product $[f] \cdot [g]$ of homotopy classes $[f]$ and $[g]$ in $[\Upsilon_k, \Upsilon_m]$ is defined as the homotopy class of the composition

$$\Upsilon_k \xrightarrow{\Delta} \Upsilon_k \times \Upsilon_k \xrightarrow{f \times g} \Upsilon_m \times \Upsilon_m \xrightarrow{\mu} \Upsilon_m$$

where Δ denotes the diagonal map and μ is the multiplication in Υ_m.

For each k, let i_k and j_k be injective homomorphisms from Υ_k to Υ_{k+1}. For every $m > k$, let

$$i_{k,m}, j_{k,m} : \Upsilon_k \to \Upsilon_m$$

denote the compositions $i_{m-1} \circ i_{m-2} \circ \cdots \circ i_k$ and $j_{m-1} \circ j_{m-2} \circ \cdots \circ j_k$, respectively.

Definition 4.1 We call a sequence of triples $\{(\Upsilon_k, i_k, j_k)\}_{k=1}^{\infty}$ an r-powering tower if for every $k \geq 1$, the centralizer of every finite subgroup of Υ_k is a path connected group, and for every $m > k$, we have

$$[i_{k,m}] = \underbrace{[j_{k,m}] \cdot [j_{k,m}] \cdot \ldots \cdot [j_{k,m}]}_{r^{(m-k)}-\text{many}}.$$

The main example of a powering tower is the following:

Example 4.2 For $k \geq 1$, let $\Upsilon_k = U(nr^{k-1})$ and i_k and j_k be the inclusions from $U(nr^{k-1})$ to $U(nr^k)$ given by

$$i_k(A) = \begin{bmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{bmatrix} \quad \text{and} \quad j_k(A) = \begin{bmatrix} A & & & \\ & I & & \\ & & \ddots & \\ & & & I \end{bmatrix}.$$

The centralizer of a finite group in $\Upsilon_k = U(nr^{k-1})$ for $k \geq 1$ is isomorphic to a product $\prod_i U(m_i)$ of unitary groups, hence it is path connected. Let $H_s : [0, 1] \to U(nr^k)$ be a path with the following end points:

$$H_s(0) = \begin{bmatrix} A & & & \\ & I & & \\ & & \ddots & \\ & & & I \end{bmatrix} \quad \text{and} \quad H_s(1) = \begin{bmatrix} I & & & & & & \\ & \ddots & & & & & \\ & & I & & & & \\ & & & A & & & \\ & & & & I & & \\ & & & & & \ddots & \\ & & & & & & I \end{bmatrix} \begin{matrix} \\ \\ \\ \longleftarrow s^{\text{th}} \text{ position} \\ \\ \\ \\ \end{matrix}$$

Now $\prod_{s=1}^{r} H_s$ is a path from $j_k(A)^r$ to $i_k(A)$, so we get $i_k \simeq (j_k)^r$ for all $k \geq 1$. Applying this recursively, we obtain $i_{k,m} \simeq (j_{k,m})^{r^{m-k}}$ for every $m > k$. Hence (Υ_k, i_k, j_k) is an r-powering tower. □

In our applications, the only r-powering tower we consider is the tower given in the above example. So, one can read the rest of this section with this particular tower in mind. The reason we keep the exposition more general is that we believe this more general set up can be useful for constructing equivariant fibre bundles with fibres homeomorphic to a product of spheres.

Now we give our main construction.

Theorem 4.3 (Compare to Theorem 2.7 in [18]). *Let G be a finite group and $\mathcal{H}$ be a family of subgroups of G closed under conjugation. Suppose that Γ is a finite group and*

$$\mathbf{A} = (\alpha_H) \in \lim_{\substack{\longleftarrow \\ H \in \mathcal{H}}} \operatorname{Rep}(H, \Gamma).$$

Let $\{(\Upsilon_k, i_k, j_k)\}_{k=1}^{\infty}$ be a $|\Gamma|$-powering tower. Then, for any representation $\rho : \Gamma \to \Upsilon_1$ and for any $d \geq 1$, there exist an $m \geq 1$ and a G-equivariant $(i_{1,m} \circ \rho)_*(\mathbf{A})$-bundle

$$\Upsilon_m \to E \to E_{\mathcal{H}} G^{(d)}$$

which is (non-equivariantly) trivial as an Υ_m-principal bundle.

Proof Let Z be the mapping cylinder of the (unique up to homotopy) map

$$B_{\mathcal{H}}(G, \mathbf{A}) \to E_{\mathcal{H}} G$$

and let B denote $B_{\mathcal{H}}(G, \mathbf{A})$ in Z. Let

$$\mathbf{A}_m = (i_{1,m} \circ \rho)_*(\mathbf{A}) \quad \text{and} \quad B_m = B_{\mathcal{H}}(G, \mathbf{A}_m)$$

for every $m \geq 1$, and let

$$f : B \to B_1, \quad I_{k,m} : B_k \to B_m, \quad \text{and} \quad J_{k,m} : B_k \to B_m$$

be the maps induced, respectively, by ρ, $i_{k,m}$, and $j_{k,m}$ for every $1 \leq k < m$. For any $H \in \mathcal{H}$, we have $B_1^H \simeq BC_{\Upsilon_1}(\rho \circ \alpha_H)$ by Proposition 2.5, so B_1^H is simply connected. Therefore, we can extend f to a G-map $f_2 : Z^{(2)} \to B_1$. Assume that we have a G-map

$$f_n : Z^{(n)} \to B_k$$

for some $n \geq 2$ where $k \geq 1$. For every $m > k$, let the elements

$$\operatorname{obs}(I_{k,m} \circ f_n),\ \operatorname{obs}(J_{k,m} \circ f_n) \in H_G^{n+1}(Z, B; \pi_n(B_m^?))$$

be the obstructions to extending the restrictions $I_{k,m} \circ f_n|_{Z^{(n-1)}}$ and $J_{k,m} \circ f_n|_{Z^{(n-1)}}$ to G-maps from $Z^{(n+1)}$ to B_m as in Proposition 3.2. Since $\{(\Upsilon_k, i_k, j_k)\}_{k=1}^{\infty}$ is a $|\Gamma|$-powering tower, we have

$$\operatorname{obs}(I_{k,m} \circ f_n) = |\Gamma|^{m-k} \operatorname{obs}(J_{k,m} \circ f_n)$$

for every $m > k$. Here we use the fact that for a Lie group $\mathcal{G}$, the map $\pi_n(\mu) : \pi_n(\mathcal{G}) \times \pi_n(\mathcal{G}) \to \pi_n(\mathcal{G})$ induced by multiplication $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ coincides with the usual group operation in $\pi_n(\mathcal{G})$ (see [26, p. 44, Cor 10]).

By Proposition 3.3, there is a cohomology spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^p_{\mathbb{Z}O_{\mathcal{H}}}(H_q(Z^?, B^?), \pi_n(B_m^?)) \implies H_G^{p+q}(Z, B; \pi_n(B_m^?)).$$

Note that for every $H \in \mathcal{H}$, we have $H_q(Z^H, B^H) \cong \widetilde{H}_{q-1}(B^H) \cong \widetilde{H}_{q-1}(BC_\Gamma(\alpha_H))$ by Proposition 2.5. So, $|\Gamma|$ annihilates $H_q(Z^H, B^H)$ for every $H \in \mathcal{H}$. Therefore, we can find an $m > k$ such that

$$\mathrm{obs}(I_{k,m} \circ f_n) = 0.$$

This implies that for any $d \geq 1$, there exists an $m > k$ such that there is a G-map

$$Z^{(d+1)} \xrightarrow{f_{d+1}} B_m.$$

The pullback of the bundle $E_{\mathcal{H}}(G, \mathbf{A}_m) \to B_m$ by the composition map

$$E_{\mathcal{H}}G^{(d+1)} \longrightarrow Z^{(d+1)} \xrightarrow{f_{d+1}} B_m.$$

is a G-equivariant $(i_{1,m} \circ \rho)_*(\mathbf{A})$-bundle

$$\xi_{d+1} : \Gamma_m \to E \to E_{\mathcal{H}}G^{(d+1)}.$$

If $\{1\} \in \mathcal{H}$, then $E_{\mathcal{H}}G$ is contractible and we obtain a trivial Υ_m-principal bundle when we pullback the bundle ξ_{d+1} to a bundle ξ_d over $E_{\mathcal{H}}G^{(d)}$ by the inclusion map. If $\mathcal{H}$ does not include $\{1\}$, then we can extend $\mathcal{H}$ to a larger family $\mathcal{H}'$ which is defined by

$$\mathcal{H}' = \{K \leq H \mid H \in \mathcal{H}\}.$$

We can also extend the compatible family of representations $\mathbf{A}$ to a compatible family of representation $\mathbf{A}'$ for $\mathcal{H}'$ by taking the restrictions of representations in $\mathbf{A}$. Then, by the above argument, there is an $m \geq 1$ and a G-equivariant $(i_{1,m} \circ \rho)_*(\mathbf{A})$-bundle

$$\xi'_d : \Upsilon_m \to E \to E_{\mathcal{H}'}G^{(d)}$$

which is trivial as an Υ_m-principal bundle. Since there is a G-map $E_{\mathcal{H}}G^{(d)} \to E_{\mathcal{H}'}G^{(d)}$, we can consider the pullback of ξ'_d to a bundle ξ_d over $E_{\mathcal{H}}G^{(d)}$. The bundle ξ_d has the desired properties. □

Corollary 4.4 *Let G be a finite group and M be a finite dimensional smooth manifold with a smooth G-action. Let $\mathcal{H}$ denote the family of isotropy subgroups of the G action on M. Let Γ be a finite group and*

$$\mathbf{A} = (\alpha_H) \in \lim_{\substack{\longleftarrow \\ H \in \mathcal{H}}} \mathrm{Rep}(H, \Gamma)$$

be a family of compatible representations. Then, for every $\rho : \Gamma \to U(n)$, there exist positive integers N and k, and a smooth G-action on $M \times \mathbb{S}^N$ such that for every $x \in M$, the G_x-action on the sphere $\{x\} \times \mathbb{S}^N$ is diffeomorphic to the linear G-action on $\mathbb{S}(V^{\oplus k})$ where $V = \rho \circ \alpha_{G_x}$.

Proof Let $\{(\Upsilon_k, i_k, j_k)\}_{k=1}^{\infty}$ be the $|\Gamma|$-powering tower described in Example 4.2. Then, by Theorem 4.3, for any $d \geq 1$, there exist an $m > 1$ and a G-equivariant $(i_{1,m} \circ \rho)_*(\mathbf{A})$-bundle

$$\Upsilon_m \to \overline{E} \to E_{\mathcal{H}}G^{(d)}$$

which is trivial as an Υ_m-principal bundle. Consider the vector bundle

$$\mathbb{C}^s \to \overline{E} \times_{\Upsilon_m} \mathbb{C}^s \xrightarrow{\pi} E_{\mathcal{H}}G^{(d)}$$

where $s = n|\Gamma|^{m-1}$. Choose d larger than the dimension of M. Since the isotropy subgroups of the G-action on M are all in $\mathcal{H}$, there is a G-map $f : M \to E_{\mathcal{H}}G^{(d)}$ which is unique up to homotopy. Consider the following pullback

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"] & \overline{E} \times_{\Upsilon_m} \mathbb{C}^s \arrow[d, "\pi"] \\ M \arrow[r, "f"] & E_{\mathcal{H}}G^{(d)}. \end{tikzcd}$$

The bundle $E \xrightarrow{p} M$ is a topological bundle, so the total space is not necessarily a smooth manifold. To get a smooth total space, we need to replace the bundle $E \xrightarrow{p} M$ with a smooth bundle. This is done by considering a smooth universal bundle which is constructed as follows: Let V be the direct sum of infinitely many copies of the regular representation of G over the real numbers $\mathbb{R}$. Let $BO(2s, V)$ denote the G-space of $2s$-planes in V and $EO(2s, V)$ denote the G-space whose points are pairs (W, w) where W is a $2s$-plane in V and $w \in W$. The map $EO(2s, V) \to BO(2s, V)$ defined by $(W, w) \to W$ gives a G-equivariant vector bundle which is the universal bundle of $2s$-dimensional G-equivariant vector bundles. So we can consider p as a pullback

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"] & EO(2s, V) \arrow[d] \\ M \arrow[r, "h"] & BO(2s, V) \end{tikzcd}$$

for some map $h : M \to BO(2s, V)$. In fact, since M is a finite dimensional manifold, the same is true if we replace V with a direct sum of q copies of the regular representation for a large q (see Proposition III.9.3 in [22]).

Note that h is G-homotopic to a smooth G-map (see Theorem VI.4.2 in [6]), so there is a smooth G-equivariant vector bundle $p' : E' \to M$ topologically equivalent to the G-equivariant vector bundle $p : E \to M$. For every $x \in M$, the G_x-action on $\mathbb{S}((p')^{-1}(x))$ is the same as the G_x-action on $\mathbb{S}(p^{-1}(x))$ which is given by the linear G_x-action on $\mathbb{S}((\rho \circ \alpha_{G_x})^{\oplus k})$ where k is some positive integer.

The bundle $p : E \to M$ has a (nonequivariant) topological trivialization, so does $p' : E' \to M$. Now a continuous trivialization can be replaced by a smooth trivialization leading to a diffeomorphism $\mathbb{S}(E') \approx M \times \mathbb{S}^N$ where $\mathbb{S}(E')$ is the total space of the corresponding sphere bundle and $N = 2s - 1$. This is explained in detail in Chap. 4 of [14] (see also Proposition 6.20 in [15]). Note that the differential structure on the product $M \times \mathbb{S}^N$ is the product differential structure and $\mathbb{S}^N$ denotes the standard sphere, not an exotic one. □

Remark 4.5 The dimension of the sphere in the above corollary is usually very big and it depends on the dimension of M. So, this construction is not very useful for constructing free actions on products of two equal dimensional spheres. It is an interesting problem to classify all rank two finite groups which can act freely and smoothly on $\mathbb{S}^n \times \mathbb{S}^n$ for some n. See [10,11], and [12] for more details on this problem.

5 Embedding fusion systems

A key ingredient in the construction of an equivariant vector bundle is the existence of a finite group Γ and a family of compatible representations $\mathbf{A} = (\alpha_H : H \to \Gamma)$. The compatibility of representations (α_H) means that for each map $c_g : H \to K$ induced by conjugation $c_g(h) = ghg^{-1}$, there exists a $\gamma \in \Gamma$ such that the following diagram commutes:

$$\begin{tikzcd} H \arrow[r, "\alpha_H"] \arrow[d, "c_g"] & \Gamma \arrow[d, "c_\gamma"] \\ K \arrow[r, "\alpha_K"] & \Gamma \end{tikzcd} \tag{1}$$

To find Γ and a family of compatible representations, we use an intermediate finite group S and define Γ in terms of S and a fusion system on S. More precisely, we assume that there is a finite group S and a family of maps $\iota_H : H \to S$ such that the diagram (1) above comes from a diagram of the following form:

$$\begin{tikzcd} H \arrow[r, "\iota_H"] \arrow[d, "c_g"] \arrow[rrr, bend left, "\alpha_H"] & \iota_H(H) \arrow[r, hook] \arrow[d, tail, "f"] & S \arrow[r, hook, "\iota"] & \Gamma \arrow[d, "c_\gamma"] \\ K \arrow[r, "\iota_K"] \arrow[rrr, bend right, "\alpha_K"] & \iota_K(K) \arrow[r, hook] & S \arrow[r, hook, "\iota"] & \Gamma \end{tikzcd} \tag{2}$$

In general, the monomorphisms $f : \iota_H(H) \to \iota_K(K)$ that complete these diagrams do not have to exist, but we assume that they always exist. In fact, in our applications the maps ι_H are always injective, so we can take f as the composition $\iota_K \circ c_g \circ \iota_H^{-1}$. Note that the monomorphisms $f : \iota_H(H) \to \iota_K(K)$ do not only depend on the conjugations c_g, but also depend on different choices of maps ι_H. These monomorphisms between subgroups of S satisfy certain properties and the best way to study them is via the theory of abstract fusion systems. We now introduce the terminology of fusion systems.

Definition 5.1 Let S be a finite group. A *fusion system* $\mathcal{F}$ on S is a category whose objects are subgroups of S and whose morphisms are injective group homomorphisms where the composition of morphisms in $\mathcal{F}$ is the usual composition of group homomorphisms and where for every $P, Q \le S$, the morphism set $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ satisfies the following:

(i) $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q)$ where $\mathrm{Hom}_S(P, Q)$ is the set of all conjugation homomorphisms induced by elements in S.
(ii) For every morphism φ in $\mathrm{Hom}_{\mathcal{F}}(P, Q)$, the induced group isomorphism $P \to \varphi(P)$ and its inverse are also morphisms in $\mathcal{F}$.

An obvious example of a fusion system is the fusion system $\mathcal{F}_S(G)$ where G is a finite group, S a subgroup of G, and the set of morphisms $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ is defined as the set of all maps induced by conjugations by elements of G. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two fusion systems on a group S, then we write $\mathcal{F}_1 \subseteq \mathcal{F}_2$ to mean that all morphisms in $\mathcal{F}_1$ are also morphisms in $\mathcal{F}_2$. We have the following:

Lemma 5.2 *Let G be a finite group and $\mathcal{H}$ be a family of subgroups of G. Let S be a finite group and $\{\iota_H : H \to S \mid H \in \mathcal{H}\}$ be a family of maps. Suppose that $\mathcal{F}$ is a fusion system on*

S such that for every map $c_g : H \to K$ induced by conjugation, there is a monomorphism f in $\mathcal{F}$ such that the following diagram commutes

$$\begin{tikzcd} H \arrow[r, "\iota_H"] \arrow[d, "c_g"] & \iota_H(H) \arrow[d, tail, "f"] \\ K \arrow[r, "\iota_K"] & \iota_K(K). \end{tikzcd}$$

If Γ is a finite group which includes S as a subgroup and satisfies $\mathcal{F} \subseteq \mathcal{F}_S(\Gamma)$, then the family of maps (α_H), where α_H is defined as the composition

$$\alpha_H : H \xrightarrow{\iota_H} S \hookrightarrow \Gamma$$

for all $H \in \mathcal{H}$, is a compatible family.

Given a fusion system on S, a good way to find a finite group Γ satisfying $\mathcal{F} \subseteq \mathcal{F}_S(\Gamma)$ is to use certain S–S-bisets. Before we explain this construction, we first introduce some terminology about bisets.

An S–S-biset Ω is a non-empty set where S acts both from right and from left in such a way that for all $s, s' \in S$ and $x \in \Omega$, we have $(sx)s' = s(xs')$. Let Ω be an S–S-biset, Q be a subgroup of S, and $\varphi : Q \to S$ be a monomorphism. Then, we write ${}_Q\Omega$ to denote the Q–S-biset obtained from Ω by restricting the left S-action to Q and we write ${}_\varphi\Omega$ to denote the Q–S-biset obtained from Ω where the left Q-action is induced by φ.

We now discuss the construction of the finite group Γ for a given biset. This construction is the same as the construction given by Park in [23] for saturated fusion systems on p-groups. Let S be a finite group and Ω be an S–S-biset. Let

$$\Gamma_\Omega = \{f : \Omega \to \Omega \mid f(xs) = f(x)s \text{ for all } s \in S, x \in \Omega\}$$

denote the group of automorphisms of Ω preserving the right S-action. Define $\iota : S \to \Gamma_\Omega$ as the homomorphism satisfying $\iota(s)(x) = sx$ for all $x \in \Omega$. If the left S-action on Ω is free and Ω is non-empty, then ι is a monomorphism, hence in that case we can consider S as a subgroup of Γ_Ω.

Lemma 5.3 (Theorem 3, [23]). *Let Ω be an S-S-biset with a free left S-action and let Q be a subgroup of S and $\varphi : Q \to S$ be a monomorphism. Then, ${}_\varphi\Omega$ and ${}_Q\Omega$ are isomorphic as Q-S-bisets if and only if φ is a morphism in the fusion system $\mathcal{F}_S(\Gamma_\Omega)$.*

Proof Let $\eta : {}_Q\Omega \to {}_\varphi\Omega$ be a function. Note that η is a Q–S-biset isomorphism if and only if η is an element in Γ_Ω and the conjugation c_η restricted to Q is equal to $\varphi : Q \to S$. This is because

$$c_\eta(q)(x) = \eta(q\eta^{-1}(x)) = \varphi(q)\eta(\eta^{-1}(x)) = \varphi(q)(x)$$

for all $q \in Q$ and $x \in \Omega$. □

We make the following definition for the situation considered in Lemma 5.3.

Definition 5.4 Let $\mathcal{F}$ be a fusion system on a finite group S. Then, a left free S-S-biset Ω is called *left $\mathcal{F}$-stable* if for every subgroup $Q \le S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$, the Q-S-bisets ${}_Q\Omega$ and ${}_\varphi\Omega$ are isomorphic.

Hence, by Lemma 5.3, we have the following.

Theorem 5.5 *Let $\mathcal{F}$ be a fusion system on a finite group S. If Ω is a left $\mathcal{F}$-stable S-S-biset, then $\mathcal{F} \subseteq \mathcal{F}_S(\Gamma_\Omega)$.*

This theorem together with Lemma 5.2 gives an explicit way to construct a finite group Γ and a compatible family of representations $(\alpha_H : H \to \Gamma)$. Note that if Ω is also free as a right S-set, then the group Γ can be described in a simple way as follows: If $|\Omega/S| = n$, then Γ is the wreath product $S \wr \Sigma_n := (S \times \cdots \times S) \rtimes \Sigma_n$ where the product of S's is n-fold and the symmetry group Σ_n acts on the product by permuting the coordinates. The fusion data is encoded in the way S is embedded in Γ. In general, the image of $\iota : S \to \Gamma$ is not in the product $S \times \cdots \times S$ (see [23] for more details).

For our constructions, we also need to find a representation of Γ such that its restriction via the maps α_H is in a desired form. For this, we again use S as an intermediate step, start with a representation V of S and obtain a representation of Γ in terms of V.

Definition 5.6 Let V be a left $\mathbb{C}S$-module and let Ω be a S-S-biset. Then we define $\mathbb{C}\Gamma_\Omega$-module $\widetilde{V}$ as the module

$$\widetilde{V} = \mathbb{C}\Omega \otimes_{\mathbb{C}S} V$$

where $\mathbb{C}\Omega$ is the permutation $\mathbb{C}S$-$\mathbb{C}S$-bimodule with basis given by Ω. The left $\mathbb{C}\Gamma_\Omega$-action on $\mathbb{C}\Omega$ is given by evaluation of the bijections in Γ_Ω at the elements of Ω and $\widetilde{V}$ is considered as a left $\mathbb{C}\Gamma_\Omega$-module via this action.

Note that every transitive S–S-biset is of the form $S \times_\Delta S$ for some $\Delta \leq S \times S$, where $S \times_\Delta S$ is the equivalence class of pairs (s_1, s_2) where $(s_1t_1, s_2) \sim (s_1, t_2s_2)$ if and only if $(t_1, t_2) \in \Delta$. The left and right actions are given by usual left and right multiplication in S. An S-S-biset is called bifree if both left and right S actions are free. It is clear from the above description that a transitive bifree S-S-biset $S \times_\Delta S$ has the property that $\Delta \cap (S \times 1) = 1$ and $\Delta \cap (1 \times S) = 1$. Applying Goursat's theorem, we obtain that Δ is a graph of an injective map $\varphi : Q \to S$ where $Q \leq S$. In this case we denote Δ by

$$\Delta(\varphi) = \{(s, \varphi(s)) \mid s \in Q\}.$$

So, a bifree S-S-biset is a disjoint union of bisets of the form $S \times_{\Delta(\varphi)} S$ where $\varphi : Q \to S$ is a monomorphism.

Definition 5.7 Let Ω be a finite bifree S–S-biset. Then we define the isotropy of Ω as the family

$$\mathrm{Isot}(\Omega) = \left\{\varphi : Q \to S \;\middle|\; S \times_{\Delta(\varphi)} S \text{ is isomorphic to a transitive summand of } \Omega\right\}.$$

It is known that every transitive biset can be written as a product of five basic bisets (see Lemma 2.3.26 in [8]). Since Ω is bifree, only three of these basic bisets, namely restriction, isogation, and induction, are needed to write the transitive summands of Ω as a composition of basic bisets. This gives us the following calculation:

Proposition 5.8 *Let V be a left $\mathbb{C}S$-module and Ω be a bifree S-S-biset. Let $\widetilde{V}$ be the $\mathbb{C}\Gamma_\Omega$-module constructed as above. Then, for $H \leq S$, the $\mathbb{C}H$-module $\mathrm{Res}^{\Gamma_\Omega}_H \widetilde{V}$ is a direct sum of modules in the form*

$$\mathrm{Ind}^H_{H \cap Q^x} \mathrm{Iso}^*(\varphi \circ c_x) \, \mathrm{Res}^S_{\varphi({}^xH \cap Q)} V$$

where $x \in S$ and $\varphi : Q \to S$ is in $\mathrm{Isot}(\Omega)$.

Proof By writing the transitive summands of Ω as a composition of the three basic bisets, we can express

$$\operatorname{Res}^{\Gamma_\Omega}_S \widetilde{V} = \mathbb{C}\Omega \otimes_{\mathbb{C}S} V$$

as a direct sum of $\mathbb{C}S$-modules in the form

$$\operatorname{Ind}^S_Q \operatorname{Iso}^*(\varphi) \operatorname{Res}^S_{\varphi(Q)} V$$

where $\varphi : Q \to S$ is in $\operatorname{Isot}(\Omega)$. Note that $\operatorname{Iso}^*(\varphi)$ is the contravariant isogation defined by $\operatorname{Iso}^*(\varphi)(M) = \varphi^*(M)$ where M is a $\varphi(Q)$-module.

Let H be a subgroup of S. Then, the $\mathbb{C}H$-module $\operatorname{Res}^{\Gamma_\Omega}_H \widetilde{V}$ is a direct sum of $\mathbb{C}H$-modules in the form

$$\operatorname{Res}^S_H \operatorname{Ind}^S_Q \operatorname{Iso}^*(\varphi) \operatorname{Res}^S_{\varphi(Q)} V.$$

Using the Mackey decomposition formula, we can decompose $\operatorname{Res}^S_H \operatorname{Ind}^S_Q$ further. We obtain a direct sum with summands of the form

$$\operatorname{Ind}^H_{H\cap Q^x} \operatorname{Iso}^*(c_x) \operatorname{Res}^Q_{{}^xH\cap Q} \operatorname{Iso}^*(\varphi) \operatorname{Res}^S_{\varphi(Q)} V$$

which is isomorphic to

$$\operatorname{Ind}^H_{H\cap Q^x} \operatorname{Iso}^*(\varphi \circ c_x) \operatorname{Res}^S_{\varphi({}^xH\cap Q)} V.$$

This completes the proof. □

This proposition shows that if we want to use this method of construction of a finite group Γ using a left $\mathcal{F}$-stable biset Ω, we need to put some restrictions on the isotropy subgroups of Ω. The existence of left $\mathcal{F}$-stable bisets with certain restrictions on their isotropy subgroups is an interesting problem and we plan to discuss this in a future paper. For the main theorems of this paper, it is possible to avoid this discussion by finding specific bisets with desired properties using ad hoc methods. These bisets will be described in the next section.

6 Constructions of free actions on products of spheres

In this section, we prove our main theorems, Theorems 1.1 and 1.2, stated in the introduction. We will first prove Theorem 1.1 which states that a p-group G acts freely and smoothly on a product of two spheres if and only if $\operatorname{rk}(G) \leq 2$. We start with a well-known lemma which is often used as a starting point for constructing free actions.

Lemma 6.1 *Let G be a p-group with* $\operatorname{rk} G = r$. *If* $\operatorname{rk} Z(G) = k$, *then G acts smoothly on a product of k spheres with isotropy subgroups having rank at most $r - k$.*

Proof Let the center of G be of the form $Z(G) \cong \mathbb{Z}/p^{n_1} \times \cdots \times \mathbb{Z}/p^{n_k}$ with generators $a_1, \ldots, a_k$. For $j \in \{1, 2, \ldots, k\}$, let $\chi_j : Z(G) \to \mathbb{C}$ denote the one-dimensional representation of $Z(G)$ defined by $a_j \mapsto e^{2\pi i/p^{n_j}}$, and $a_{j'} \mapsto 1$ for $j' \neq j$. Let $\theta_j = \operatorname{Ind}^G_{Z(G)}(\chi_j)$. Define $M = \mathbb{S}(\theta_1) \times \cdots \times \mathbb{S}(\theta_k)$ with the diagonal G-action. Note that $Z(G)$ acts freely on M, so if H is an isotropy subgroup of G, then we must have $H \cap Z(G) = \{1\}$. Thus, $HZ(G) \cong H \times Z(G)$ is a subgroup of G. This proves that $k + \operatorname{rk} H \leq r$. □

The above lemma, in particular, says that if $\operatorname{rk} G = r$ and $\operatorname{rk} Z(G) = r - 1$, then G acts smoothly on a product of $(r-1)$-many spheres with rank one isotropy subgroups. When p is odd, all rank one p-groups are cyclic. In the case of 2-groups, in addition to cyclic groups, we also have the family of generalized quaternions Q_{2^n} where $n \geq 3$. In either case, given a finite collection of rank one p-groups, we can find a rank one p-group into which all other rank one p-groups can be embedded. In the proof of Theorem 1.1, S will be this large rank one p-group into which all isotropy subgroups can be embedded. So, when p is odd, S will be a cyclic group of order p^N and when $p = 2$, it will be a quaternion group Q_{2^N} where N is a large enough positive integer.

As a fusion system on S we will always consider the fusion system $\mathcal{F}$ where all the monomorphisms between subgroups of S are in $\mathcal{F}$. For this S and $\mathcal{F}$, we construct left $\mathcal{F}$-stable bisets with reasonable isotropy structures. We construct these bisets using a more general lemma. Before we state this lemma, we introduce a definition.

Definition 6.2 Let $\mathcal{F}$ be a fusion system on a finite group S. Then we say K is an $\mathcal{F}$-*characteristic* subgroup of S if for any subgroup $L \leq K$ and for any morphism $\varphi : L \to S$ in $\mathcal{F}$, there exists a morphism $\widetilde{\varphi} : K \to K$ in $\mathcal{F}$ such that $\widetilde{\varphi}(l) = \varphi(l)$ for all $l \in L$.

Now, we have the following:

Lemma 6.3 *Let $\mathcal{F}$ be a fusion system on a finite group S and K be an abelian $\mathcal{F}$-characteristic subgroup of S. Assume that Ω is the S-S-biset defined as follows*

$$\Omega = \coprod_{\varphi \in \operatorname{Aut}_{\mathcal{F}}(K)} S \times_{\Delta(\varphi)} S.$$

Then the S-S-biset Ω is left $\mathcal{F}$-stable.

Proof We first prove that for any $\mathcal{F}$-morphism $\psi : K \to S$, the K–S-bisets ${}_K\Omega$ and ${}_\psi\Omega$ are isomorphic. For this, let $\{s_1, \ldots, s_m\}$ be a set of coset representatives for K so that $S = \amalg_i s_i K$. Using this decomposition, we can write

$$\Omega = \coprod_{\varphi \in \operatorname{Aut}_{\mathcal{F}}(K)} \coprod_{i=1}^{m} E_{i,\varphi}$$

where $E_{i,\varphi} = \{[s_i, s] \mid [s_i, s] \in S \times_{\Delta(\varphi)} S\}$. Define $\theta : {}_K\Omega \to {}_\psi\Omega$ as the map which takes $[s_i, s] \in E_{i,\varphi}$ to $[s_i, s] \in E_{i,\varphi'}$ where

$$\varphi' = \varphi \circ c_{s_i}^{-1} \circ \psi^{-1} \circ c_{s_i}.$$

Note that since K is $\mathcal{F}$-characteristic, we have $\psi(K) = K = c_{s_i}(K)$, so φ' is also in $\operatorname{Aut}_{\mathcal{F}}(K)$. It is straight forward to check that θ is a bijection and satisfies $\theta(kx) = \psi(k)\theta(x)$ for every $k \in K$ and $x \in \Omega$.

Now take any subgroup $Q \leq S$ and $\psi \in \operatorname{Hom}_{\mathcal{F}}(Q, S)$. We want to show that Q–S-bisets ${}_Q\Omega$ and ${}_\psi\Omega$ are isomorphic. We can think of a Q-S-biset as a left $(Q \times S)$-set by defining the left $(Q \times S)$-action by $(q, s)x = qxs^{-1}$ for all $q \in Q$ and $s \in S$. This allows us to apply the usual theory of left sets to bisets. In particular, to show that ${}_Q\Omega$ and ${}_\psi\Omega$ are isomorphic, it is enough to show that for every $H \leq Q \times S$, the number of fixed points of left H- and H_ψ-actions on Ω are equal where $H_\psi = \{(\psi(x), y) \mid (x, y) \in H\}$.

Take any subgroup $H \leq Q \times S$. If H is not a group in the form $\Delta(\theta)$, where $\theta : L \to S$ is a $\mathcal{F}$-morphism and L is a subgroup of K, then $|\Omega^H| = 0 = |\Omega^{H_\psi}|$. If H is a group in the form $\Delta(\theta)$ where $\theta : L \to S$ is a $\mathcal{F}$-morphism and L is a subgroup of K, then there exists a morphism $\widetilde{\psi} : K \to K$ in $\mathcal{F}$ such that $\widetilde{\psi}(l) = \psi(l)$ for all $l \in L$. This implies that $|\Omega^H| = |\Omega^{H_\psi}|$ because K–S-bisets ${}_K\Omega$ and ${}_{\widetilde{\psi}}\Omega$ are isomorphic. From these we can conclude that Q–S-bisets ${}_Q\Omega$ and ${}_\psi\Omega$ are isomorphic. □

Lemma 6.3 is used to show that the bisets given in the following two examples are left $\mathcal{F}$-stable.

Example 6.4 Let p be a prime number and S be the cyclic group of order p^N where $N > 1$. Let $\mathcal{F}$ be the fusion system on S such that all monomorphisms between the subgroups of S are morphisms in $\mathcal{F}$. Define

$$\Omega = \coprod_{\varphi \in \mathrm{Aut}(S)} S \times_{\Delta(\varphi)} S.$$

By Lemma 6.3, Ω is left $\mathcal{F}$-stable. So, by Theorem 5.5, $\mathcal{F} \subseteq \mathcal{F}_S(\Gamma_\Omega)$. This implies that if a finite group G acts on a space with cyclic p-group isotropy, then its isotropy subgroups can be embedded in Γ_Ω in a compatible way. To obtain the maps $\alpha_H : H \to \Gamma_\Omega$, we first choose a family of injective maps $\iota_H : H \to S$ for all isotropy subgroups H, then we apply Lemma 5.2 to conclude that the compositions $\alpha_H = \iota \circ \iota_H$ form a compatible family of maps. Here $\iota : S \to \Gamma_\Omega$ is the canonical inclusion defined in Sect. 5 which satisfies $\iota(s)(x) = sx$ for all $s \in S$ and $x \in \Omega$.

As a representation V of S, we can take the one dimensional complex representation given by multiplication with the p^N-th root of unity. Then, for every $H \leq S$, the representation $\mathrm{Res}^{\Gamma}_H \widetilde{V}$ is isomorphic to the direct sum $\oplus_\varphi \mathrm{Res}^S_H \varphi^*(V)$, hence H acts freely on $\mathbb{S}(\widetilde{V})$. □

Remark 6.5 Note that the finite group Γ_Ω that is constructed in the above example is exactly the same as the construction given in Sect. 4.2 of [27]. To see this, note that the group Γ_Ω constructed above can be expressed as wreath product $S \wr \Sigma_n$ where $n = |\mathrm{Aut}(S)|$. We can write a specific group isomorphism $\Gamma_\Omega \to S \wr \Sigma_n$ as follows: Observe that there is a S-S-biset isomorphism between $S \times_{\Delta(\varphi)} S$ and the S-S-biset ${}_\varphi S$ where the left S action on ${}_\varphi S$ is via the automorphism φ. This isomorphism is given by the map $\theta : S \times_{\Delta(\varphi)} S \to {}_\varphi S$ defined by $\theta([(s_1, s_2]) = \varphi(s_1)s_2$. So, we have

$$\Omega \cong \coprod_{\varphi \in \mathrm{Aut}(S)} {}_\varphi S.$$

Giving an ordering for the elements of $\mathrm{Aut}(S)$, we can write $\mathrm{Aut}(S) = \{\varphi_1, \dots, \varphi_n\}$. Now we define a map from Γ_Ω to the wreath product $S \wr \Sigma_n := (S \times \cdots \times S) \rtimes \Sigma_n$ by sending an automorphism $f : \Omega \to \Omega$ to the element $(f(e_1), \dots, f(e_n); \sigma)$ where e_i denotes the identity element of the i-th component in the above disjoint union and σ is the permutation of the components induced by the automorphism f. This map induces an isomorphism and under this isomorphism the embedding $\iota : S \to \Gamma_\Omega$ becomes the embedding $S \to S \wr \Sigma_n$ defined by $\iota(s) = (\varphi_1(s), \dots, \varphi_n(s); \mathrm{id})$. One can easily check that the representation of Γ_Ω is also the same as the one given in Sect. 4.2 of [27]. □

Example 6.6 Let S be the generalized quaternion group Q_{2^N} of order 2^N where $N \geq 3$. Let $\mathcal{F}$ be the fusion system on S such that all monomorphisms between the subgroups of S are morphisms in $\mathcal{F}$. Define

$$\Omega = S \times_{\Delta(\mathrm{id}_{C_2})} S$$

where C_2 is the unique cyclic group of order 2 in S. Since C_2 is a $\mathcal{F}$-characteristic subgroup and $\mathrm{Aut}(C_2) = \{\mathrm{id}_{C_2}\}$, by Lemma 6.3, we can conclude that Ω is left $\mathcal{F}$-stable, and hence

$\mathcal{F} \subseteq \mathcal{F}_S(\Gamma_\Omega)$. Let G be a finite group acting on a space X and let $\mathcal{H}$ denote the family of isotropy subgroups of G-action on X. If every element in $\mathcal{H}$ is a rank one 2-group, then we can choose a large N and embed every element $H \in \mathcal{H}$ into $S = Q_{2^N}$ via some embedding $\iota_H : H \to S$. Since the fusion system $\mathcal{F}$ includes all possible monomorphisms between subgroups of S, the condition in Lemma 5.2 holds. So, the isotropy subgroups $H \in \mathcal{H}$ can be embedded in Γ_Ω in a compatible way.

If V is a representation of S, then for any $H \leq S$, the representation $\mathrm{Res}^S_H \widetilde{V}$ is isomorphic to a multiple of the representation

$$\mathrm{Ind}^H_{C_2} \mathrm{Res}^S_{C_2} V.$$

So, if we choose V with the property that C_2 acts freely on $\mathbb{S}(V)$, then $\mathrm{Res}^S_H \widetilde{V}$ also has the same property. □

The finite group Γ_Ω constructed in the above example can also be expressed as a wreath product $S \wr \Sigma_n$ where $n = |\Omega/S| = |S : C_2|$. But in this case, the image of $\iota : S \to \Gamma_\Omega$ is not in the subgroup $S \times \cdots \times S$. Under the natural projection $\pi : \Gamma_\Omega \to \Sigma_n$, the element $\pi(\iota(s))$, where $s \in S$, corresponds to the permutation induced by the s action on the coset set S/C_2.

Now we are ready to prove the following.

Theorem 6.7 *Let G be a finite group acting smoothly on a manifold M so that the isotropy subgroup G_x for every point $x \in M$ is a rank one p-group. Then, there exists a positive integer N such that G acts freely and smoothly on $M \times \mathbb{S}^N$.*

Proof Let $\mathcal{H}$ denote the family of isotropy subgroups of the G-action on M. By Examples 6.4 and 6.6, we know that there exists a finite group Γ and a family of compatible representations

$$\mathbf{A} = (\alpha_H) \in \varprojlim_{H \in \mathcal{H}} \mathrm{Rep}(H, \Gamma).$$

In these examples we also showed that there is a representation $\rho : \Gamma \to U(n)$ such that the composition $\rho \circ \alpha_H : H \to U(n)$ is a free representation for every $H \in \mathcal{H}$. Hence, by Corollary 4.4, G acts freely and smoothly on $M \times \mathbb{S}^N$ for some positive integer N. □

As an immediate corollary, we obtain the following.

Theorem 6.8 *Let G be a p-group with* $\mathrm{rk}\, G = r$. *If* $\mathrm{rk}\, Z(G) \geq r - 1$, *then G acts freely and smoothly on a product of r spheres.*

Proof This follows from Lemma 6.1 and Theorem 6.7. □

Now Theorem 1.1 follows as a special case.

Proof of Theorem 1.1 It is proved in [13] that $(\mathbb{Z}/p)^3$ does not act freely on a product of two spheres. Hence it is enough to construct free actions of p-groups which has $\mathrm{rk}(G) \leq 2$. Note that every finite p-group has a nontrivial center, so the existence of such actions follows from Theorem 6.8. □

In the rest of the section, we prove Theorem 1.2. The proof is similar to the above proof. We first consider the following example.

Example 6.9 Let p be a prime number and S be the elementary abelian p-group of order p^N for some $N \geq 1$. Let $\mathcal{F}$ be the fusion system on S such that all monomorphisms between the subgroups of S are morphisms in the fusion system S. Define

$$\Omega = \coprod_{\varphi \in \mathrm{Aut}(S)} S \times_{\Delta(\varphi)} S.$$

Note that Ω is left $\mathcal{F}$-stable by Lemma 6.3 and hence, by Theorem 5.5, we have $\mathcal{F} \subseteq \mathcal{F}_S(\Gamma)$. If G is a finite group acting on a space with elementary abelian isotropy subgroups, then we can find a compatible family of representations $\alpha_H : H \to \Gamma$ by first choosing embeddings $\iota_H : H \to S$ and then by applying Lemma 5.2.

In the following application, we can take the representation V of S as the augmented regular representation $V = \mathbb{C}G - \mathbb{C}$, and then construct $\widetilde{V}$ in the usual way. Note that for any isotropy subgroup H, the representation $\alpha_H^*(\widetilde{V})$ is isomorphic to the direct sum

$$\bigoplus_{\varphi \in \mathrm{Aut}(S)} (\iota_H)^* \varphi^*(V)$$

which is isomorphic to $(\iota_H)^*(V^{\oplus n})$ where $n = |\operatorname{Aut}(S)|$. Note that we can choose S so that when H is an isotropy subgroup of maximal rank the embedding $\iota_H : H \to S$ is an isomorphism. The action of an isotropy subgroup H on $\mathbb{S}(\widetilde{V})$ will have no fixed points if H has maximal rank.

Proof of Theorem 1.2 Let $\mathcal{H}$ denote the family of isotropy subgroups of G action on M. By Example 6.9, there is a finite group Γ and a family of compatible representations

$$\mathbf{A} = (\alpha_H) \in \lim_{\substack{\longleftarrow \\ H \in \mathcal{H}}} \mathrm{Rep}(H, \Gamma)$$

together with a representation $\rho : \Gamma \to U(n)$ such that for every $H \in \mathcal{H}$ of maximal rank, the representation $\rho_H = \rho \circ \alpha_H : H \to U(n)$ has the property that H acts on $\mathbb{S}(\rho_H)$ without fixed points. By Corollary 4.4, there is a smooth action on $M \times \mathbb{S}^{n_1}$ for some positive integer n_1 such that for every $G_x \in \mathcal{H}$ of maximal rank, G_x action on $\{x\} \times \mathbb{S}^{n_1}$ is without fixed points. So, the isotropy subgroups of G action on $M \times \mathbb{S}^{n_1}$ has rank $\leq k - 1$. Repeating the argument recursively, we can conclude that G acts freely and smoothly on $M \times \mathbb{S}^{n_1} \times \cdots \mathbb{S}^{n_k}$ for some positive integers $n_1, \ldots, n_k$. □

The proof of Corollary 1.3 follows easily from Theorem 1.2. To see this, observe that if G is an (almost) extraspecial p-group of rank r, then every subgroup which intersects trivially with the center is an elementary abelian subgroup with rank less than or equal to $r - 1$. This is because the Frattini subgroup of G is included in the center $Z(G)$ of G and that $Z(G)$ is cyclic. Let a be a central element of order p in G. Let χ be the one-dimensional representation of $\langle a \rangle$ defined by $a \mapsto e^{2\pi i/p}$, and define $\theta = \mathrm{Ind}_{\langle a \rangle}^{G}(\chi)$. Then, G action on $M = \mathbb{S}(\theta)$ has all its isotropy groups elementary abelian with rank less than or equal to $r - 1$. So, the result follows from Theorem 1.2.

Note that Theorem 1.2 applies to a larger class of groups than (almost) extra-special p-groups. For example, if G is a p-group such that the elements of order p in the Frattini subgroup $\Phi(G)$ of G are all central, then the action constructed in Lemma 6.1 will satisfy the assumptions of Theorem 1.2, so we can obtain free smooth actions of these groups on r many spheres where r is the rank of the group. A particular example of such a group would be a p-group G which is a central extension of two elementary abelian p-groups.

References

1. Adem, A., Davis, J.F., Ünlü, O.: Fixity and free group actions on products of spheres. Comment. Math. Helv. **79**, 758–778 (2004)
2. Adem, A., Smith, J.H.: Periodic complexes and group actions. Ann. Math. **154**(2), 407–435 (2001)
3. Benson, D.J.: Representations and cohomology I: basic representation theory of finite groups and associative algebras. In: Cambridge Studies in Advanced Mathematics, vol. 30. Cambridge University Press, Cambridge (1998)
4. Benson, D.J.: Representations and cohomology II: cohomology of groups and modules. In: Cambridge Studies in Advanced Mathematics, vol. 31. Cambridge University Press, Cambridge (1998)
5. Bredon, G.: Equivariant Cohomology Theories. Lecture Notes in Mathematics, vol. 34. Springer, Berlin (1967)
6. Bredon, G.: Introduction to Compact Transformation Groups. Academic Press, Burlington (1972)
7. Broto, C., Levi, R., Oliver, R.: The homotopy theory of fusion systems. J. Am. Math. Soc. **16**(4), 779–856 (2003)
8. Bouc, S.: Biset Functors for Finite Groups. Lecture Notes in Mathematics, vol. 1990. Springer, Berlin (2010)
9. Tom Dieck, T.: Transformation Groups. Studies in Mathematics, vol. 8. de Gruyter, Berlin (1987)
10. Hambleton, I.: Some examples of free actions on products of spheres. Topology **45**, 735–749 (2006)
11. Hambleton, I., Ünlü, O.: Examples of free actions on products of spheres. Q. J. Math. **60**, 461–474 (2009)
12. Hambleton, I., Ünlü, O.: Free actions of finite groups on $\mathbb{S}^n \times \mathbb{S}^n$. Trans. Am. Math. Soc. **362**, 3289–3317 (2010)
13. Heller, A.: A note on spaces with operators. Ill. J. Math. **3**, 98–100 (1959)
14. Hirsch, M.W.: Differential Topology. Springer, New York (1976)
15. Lee, J.M.: Introduction to Smooth Manifolds. Graduate Texts in Mathematics, vol. 218. Springer, New York (2003)
16. Lück, W.: Transformation Groups and Algebraic K-Theory. Lecture Notes in Mathematics, vol. 1408 (1989)
17. Lück, W.: Survey on classifying spaces for families of subgroups, Infinite groups: geometric, combinatorial and dynamical aspects, vol. 248, pp. 269–322, Progr. Math. Birkhäuser, Basel (2005)
18. Lück, W., Oliver, R.: The completion theorem in K-theory for proper actions of a discrete group. Topology **40**(3), 585–616 (2001)
19. Madsen, I., Thomas, C.B., Wall, C.T.C.: The topological spherical space form problem II. Topology **15**, 375–382 (1978)
20. May, J.P. et al.: Equivariant homotopy and cohomology theory, CBMS regional conference series in mathematics, vol. 91. American Mathematical Society, Providence, RI (1996)
21. Milnor, J.: Groups which act on $\mathbb{S}^n$ without fixed points. Am. J. Math. **79**, 623–630 (1957)
22. Osborn, H.: Vector Bundles, vol. 1. Foundations and Stiefel-Whitney Classes. Academic Press, New York (1982)
23. Park, S.: Realizing a fusion system by a single group. Arch. Math. **94**, 405–410 (2010)
24. Ragnarsson, K., Stancu, R.: Saturated fusion systems as idempotents in the double Burnside ring (preprint)
25. Smith, P.A.: Permutable periodic transformations. Proc. Natl. Acad. Sci. **30**, 105–108 (1944)
26. Spanier, E.H.: Algebraic Topology. Springer, New York (1966)
27. Ünlü, Ö.: Doctoral dissertation, UW-Madison (2004)