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Structure theory of central simple $\mathbb{Z}_d$-graded algebras

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Abstract

This paper investigates the structure theory of $\mathbb{Z}_d$- central simple graded algebras and gives the complete decomposition into building block algebras. The results are also applied to generalized Clifford algebras, which are motivating examples of $\mathbb{Z}_d$-central simple graded algebras.

Key Words: Central simple graded algebra, primitive d^{th} root of unity, ω-Clifford algebra

1. Introduction

Central simple $\mathbb{Z}_2$-graded algebras were introduced and studied by Wall in [10]. Brauer equivalence classes of central simple $\mathbb{Z}_2$-graded algebras form an abelian group that is usually called the Brauer-Wall group. Knus [7] has generalized the results of Wall, replacing the grading group $\mathbb{Z}_2$ by a finite abelian group G. His notion of central simple graded algebra not only depends on the base field F and the grading group G, but also on the choice of a symmetric bilinear map $G \times G \to F^*$. In this paper, we are interested in the case where G is the cyclic group of order d, and F contains a primitive d-th root ω of 1. Symmetric bilinear maps $G \times G \to F^*$ are then in bijective correspondence with the $d^{-\text{th}}$ root of 1, and we choose the bilinear 1 map corresponding to ω. The corresponding Brauer group has been determined completely in the case where d is a primary number, see [1, Prop. 3.9]. Central simple $\mathbb{Z}_d$-graded algebras have also been considered in [9]. Other results on the Brauer group of Knus may be found in [2] and [3]. We would like to point out that the construction due to Knus is only the first in a long list of generalizations that appeared in the literature, where one considers algebras with actions and/or grading by groups and even Hopf algebra, and a survey can be found in [4]. Most results in the literature focus on the Brauer group, that is, central simple graded algebras are studied up to Brauer equivalence. In this note, we develop a decomposition theory for the algebras themselves. At some places, there is a partial overlap with the results in the literature, but, on the other hand, our methods are elementary and self-contained.

Throughout this paper, F will stand for a field containing ω, a primitive d^{th} root of unity and $d \geq 2$, a fixed integer. By an algebra we shall mean a finite dimensional associative algebra with identity over the field F. By a *graded algebra* A, we shall mean a $\mathbb{Z}_d$-graded algebra,

$$A = A_0 \oplus A_1 \oplus \cdots \oplus A_{d-1},$$

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where the suffices are integers mod d, such that $A_iA_j \subseteq A_{i+j}$ for all $i,j \in \mathbb{Z}_d$. For any $k \in \mathbb{Z}_d$, each $a \in A_k$ is said to be *homogeneous of degree* k and written $\partial a = k$; the set $\bigcup_{k=0}^{d-1} A_k$ of all homogeneous elements is denoted by $\mathcal{H}(A)$. A *graded subspace* V of a graded algebra is a subspace that can be expressed as $V = \sum_{i=0}^{d-1}(V \cap A_i)$. A *subalgebra* (respectively an *ideal*) of A is said to be *graded* if it is graded as a subspace. For example, for any $H \subset \mathcal{H}(A)$ the centralizer $C_A(H)$ is a graded subalgebra and the ideal $\langle H \rangle$ generated by H is a graded ideal. In particular the center $Z(A)$ is a graded subalgebra. When A has no proper graded ideals, it is called *a simple graded algebra* (SGA). A map $\varphi : A \to B$ is called *a graded homomorphism* if φ is a homomorphism such that $\varphi(A_k) \subset B_k$ for all $k \in \mathbb{Z}_d$. For any graded algebra the unique algebra homomorphism for which $\phi(h) = \omega^{\partial h} h$ where $h \in \mathcal{H}(A)$ is a graded automorphism and it is called *the main automorphism (associated with ω).* The *graded center* $\hat{Z}(A)$ of the graded algebra is defined as the subalgebra spanned by homogeneous elements $c \in \mathcal{H}(A)$ such that $ch = \omega^{\partial c \partial h} hc$ for all $h \in \mathcal{H}(A)$. When $\hat{Z}(A) = F$ the graded algebra is called a *central graded algebra* (CGA). Our main concern will be *central simple graded algebras* ($CSGA$)s. In the next section, we shall establish some results related with graded tensor products of $CSGA$s and describe some examples that we are going to use as building block algebras in the structure theorems of the last section.

2. Building block algebras and their combinations

To begin with, we give an elementary proposition to set up a grading on a given algebra.

Proposition 2.1 *Let A be an algebra, ϕ be an algebra automorphism of A and let*

$$A_k = \{a \in A | \phi(a) = \omega^k a\}; \quad k = 0, 1, \ldots, d-1.$$

If $A = \sum_{k=0}^{d-1} A_k$, then A becomes a $\mathbb{Z}_d$-graded algebra with homogeneous components A_k. Further in the case where ϕ is an inner automorphism determined by z, the subalgebra A_0 is the centralizer of z in A and $\hat{Z}(A) = (Z(A))_0$.

Proof. The subsets A_k are the eigenspaces of ϕ belonging to the eigenvalues ω^k and hence $A = \sum_{k=0}^{d-1} A_k$ is a direct sum. Considering ϕ as a ring homomorphism we see that $A_kA_l \subseteq A_{k+l}$. As for the last statement we note first that if
$\phi(a) = z^{-1}az$ for all $a \in A$, then $a \in A_0$ if and only if $az = za$, that is $A_0 = C_A(z)$, in particular $z \epsilon A_0$. Further, c is a homogeneous element contained in $\hat{Z}(A)$ only if $cz = \omega^{\partial c \partial z} zc = zc$. This implies that $c \in A_0 \cap Z(A)$, that is to say, $\hat{Z}(A) = Z(A)_0$. □

Corollary 2.2 *Every inner automorphism ϕ_Ω of $\mathbb{M}_n(F)$ determined by a diagonal $n \times n$ matrix Ω whose diagonal entries are the d^{th} roots of unity in F induces a grading on $\mathbb{M}_n(F)$ for which $\mathbb{M}_n(F)$ is a $CSGA$. The same is also true for $\mathbb{M}_n(F)$ and the matrix*

$$\Omega = \begin{bmatrix} 0 & 0 & \cdots & 0 & a \\ \omega^{-1} & 0 & \cdots & 0 & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & \omega^{-d+1} & 0 \end{bmatrix}$$

for any $a \in F$.

Proof. For the usual matrix units E_{ij} we have $\Omega^{-1}(E_{ij})\Omega = \omega^{-l_i+l_j}(E_{ij})$ where ω^{l_i} and ω^{l_j} are the i^{th} and the j^{th} diagonal entries of Ω, respectively. Since $\mathbb{M}_n(F) = \sum_{i,j} FE_{ij}$, by the previous proposition ϕ_Ω induces a grading on $\mathbb{M}_n(F)$ whose k^{th} homogeneous component is $\sum_{l_j-l_i=k} FE_{ij}$ and graded center is $\hat{Z}(\mathbb{M}_n(F))$. Since the matrix algebra is simple as an ungraded algebra, it is a $CSGA$. The last statement follows from the equality $\Omega = E_1^{-1}E_2$ where

$$E_1 = \begin{bmatrix} 1 & & & \\ & \omega & & 0 \\ & & \ddots & \\ & 0 & & \omega^{d-1} \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & a \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

For (as algebra) $\mathbb{M}_d(F)$ is generated by E_1, E_2 satisfying $E_1E_2 = \omega E_2E_1$, consequently $\Omega^{-1}E_1\Omega = \omega E_1$ and $\Omega^{-1}E_2\Omega = \omega E_2$. □

The above proposition and its corollary allow us to exhibit building block algebras that will be used to describe all $CSGA$s:

(1) For any algebra A, $\phi=$ identity gives the grading, $A_0 = A$, $A_1 = \cdots = A_{d-1} = 0$. This grading is said to be *trivial* and the algebra A with this *trivial grading* is denoted by (A). If A is central simple as an ungraded algebra then (A) becomes a $CSGA$.

(2) If D is a central division algebra over F which contains an element z for which $D = \sum_{k=0}^{d-1} D_k$ where $D_k = \{a \in D | z^{-1}az = \omega^k a\}$, then D is a $CSGA$.

(3) For any nonzero element $a \in F$ the factor algebra K of $F[x]$ by the ideal generated by $x^d - a$ is a graded algebra corresponding to the F-automorphism ϕ for which $\phi(x) = \omega x$. It is a $CSGA$ when $a \neq 0$. When $a = 0$ the algebra is neither graded central nor graded simple.

(4) The graded algebra $\mathbb{M}_n(F)$ obtained from the inner automorphism associated with

$$\Omega = \begin{bmatrix} 1 & & & \\ & \omega & & 0 \\ & & \ddots & \\ & 0 & & \omega^{n-1} \end{bmatrix}$$

has homogeneous elements M of degree k whose (i, j)-entry is $M_{ij} = 0$ when $j - i$ is not congruent to $k \pmod d$. This grading is the generalized form of the *checker-board grading* of $\mathbb{M}_n(F)$ and this graded algebra is denoted by $\hat{\mathbb{M}}_n(F)$. This can be extended to $\hat{\mathbb{M}}_n(A)$ by means of the identification $\mathbb{M}_n(A) = \mathbb{M}_n(F) \otimes A$ for any graded algebra A.

(5) When $n = d$, the matrix algebra $\mathbb{M}_d(F)$ with grading associated with

$$\Omega = \begin{bmatrix} 0 & 0 & \cdots & 0 & a \\ \omega^{-1} & 0 & \cdots & 0 & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & \omega^{-d+1} & 0 \end{bmatrix}$$

is graded isomorphic to the generalized quaternion algebra $(\frac{a,1}{F})_\omega$ (see [6]).

After constructing CSGA building blocks, we establish certain results to combine them conveniently, by means of graded tensor products, to produce new $CSGA$s. The graded tensor product $A\widehat{\otimes}B$ of the graded algebras A and B is the same as $A \otimes B$ as a vector space and it has multiplication given by

$$(a \otimes b)(c \otimes d) = \omega^{\partial b \partial c}(ac) \otimes (bd)$$

for $a, c \in \mathcal{H}(\mathcal{A})$ and $b, d \in \mathcal{H}(\mathcal{B})$.

Proposition 2.3 *If A and B are central graded algebras so is $A\widehat{\otimes}B$.*

Proof. Let $\gamma = \sum_{i=1}^m a_i \otimes b_i$ be a homogeneous generator of $\hat{Z}(A\widehat{\otimes}B)$ and suppose that the b_i are linearly independent. Then all the degrees $\partial a_i + \partial b_i$ are equal and

$$\gamma(a \otimes b) = \omega^{\partial\gamma\partial(a\hat{\otimes}b)}(a \otimes b)\gamma$$

for all $a \in \mathcal{H}(\mathcal{A}), b \in \mathcal{H}(\mathcal{B})$. Taking $b = 1$ we obtain

$$\sum_{i=1}^m \omega^{\partial b_i \partial a} a_i a \otimes b_i = \sum_{i=1}^m \omega^{(\partial a_i + \partial b_i)\partial a} a a_i \otimes b_i$$

for all $a \in \mathcal{H}(\mathcal{A})$.

Linear independence of the b_i implies that $a_i \in \hat{Z}(A)$ for all i. Since A is assumed to be central we deduce that $a_i \in F$ for all i and hence $\gamma = 1 \otimes b'$ where $b' = a_1 b_1 + \cdots + a_m b_m$. Substituting this γ into the above equality with $a = 1$, it follows that

$$(1 \otimes b')(1 \otimes b) = \omega^{\partial b' \partial b}(1 \otimes b)(1 \otimes b')$$

for all $b \in \mathcal{H}(\mathcal{B})$ implying that $1 \otimes bb' = \omega^{\partial b' \partial b} 1 \otimes b'b$ and hence $b' \in \hat{Z}(B) = F$. Therefore $\gamma \in F(1 \otimes 1)$, that is $\hat{Z}(A\widehat{\otimes}B) = F$ as asserted. □

We remark that the following result already appears in [7, Prop. 2.1], but that the proof presented here is different, and that it is given for the sake of completeness.

Theorem 2.4 *If A is a CSGA and B is a SGA, then $A\widehat{\otimes}B$ is a SGA. In particular if A and B are both CSGAs, then so is $A\widehat{\otimes}B$.*

Proof. The second statement follows at once from the first one by Proposition 2.3. so that it is sufficient to prove the first statement. To this end take a nonzero graded ideal I of $A\widehat{\otimes}B$. It contains a nonzero homogeneous element. Pick up one of these that has the form $\alpha = \sum_{i=1}^{r} a_i \otimes b_i$ with $a_i \in \mathcal{H}(\mathcal{A})$ $b_i \in \mathcal{H}(\mathcal{B})$ where r is as small as possible. We note that the a_i (respectively the b_i) are linearly independent and $\partial a_i + \partial b_i$ is independent of i. Since A is a $CSGA$, the graded ideal generated by a_1 is A and, hence, there exist $a_k^{'}, a_k^{''} \in \mathcal{H}(A)$ such that $1 = \sum_{k=1}^{m} a_k^{'} a_1 a_k^{''}$ with $\partial a_k^{'} + \partial a_1 + \partial\, a_k^{''} \equiv 0 \pmod d$ and this yields an element

$$\alpha^{'} = \sum_{k=1}^{m} \omega^{-\partial b_1 \partial a_k^{''}} (a_k^{'} \otimes 1)\ \alpha(a_k^{''} \otimes 1) = 1 \otimes b_1 + \sum_{k=1}^{m}\sum_{i=2}^{r} \omega^{-\partial b_1 \partial a_k^{''} + \partial b_i \partial a_k^{''}} (a_k^{'} a_i a_k^{''} \otimes b_i).$$

We observe that this element of I is of the form

$$\tilde{\alpha} = 1 \otimes b_1 + \sum_{i=2}^{r} \tilde{a_i} \otimes b_i$$

where the $\tilde{a}_i$ are homogeneous and $\partial \tilde{a}_i + \partial b_i = \partial b_1$ for all i. Since the b_i are linearly independent we have $\tilde{\alpha} \neq 0$. The same process applied to b_1 yields the element

$$\alpha_0 = 1 \otimes 1 + \sum_{i=2}^{r} \tilde{a_i} \otimes \tilde{b_i}$$

of I where the $\tilde{a_i}$ and the $\tilde{b_i}$ are homogeneous such that $\partial \tilde{a_i} + \partial\, \tilde{b_i} = 0$ and $\{1, \tilde{a_1}, \ldots, \tilde{a_r}\}$ is linearly independent. Now for each $a \epsilon \mathcal{H}(\mathcal{A})$ we obtain

$$(a \otimes 1)\alpha_0 - \alpha_0(a \otimes 1) = \sum_{i=2}^{r} (a\tilde{a}_i - \omega^{\partial \tilde{b}_i \partial a}\ \tilde{a_i}\ a) \otimes \tilde{b_i}$$

contained in I. The minimality of r forces

$$a\tilde{a}_i - \omega^{\partial \tilde{b}_i \partial a}\ \tilde{a_i}\ a = 0 \quad , \quad i = 2, \ldots, r,$$

and it follows from $\partial \tilde{a_i} + \partial\, \tilde{b_i} = 0$ that

$$\tilde{a_i}\ a = \omega^{\partial \tilde{a_i} \partial a}\ a\tilde{a}_i \quad , \quad i = 2, \ldots, r,$$

that is to say, $\tilde{a_i} \in \hat{Z}(A) = F$, contradicting linear independence of $\{1, \tilde{a_2}, \ldots, \tilde{a_r}\}$ unless $r = 1$. Therefore $r = 1$ and $\alpha_0 = 1 \otimes 1$ is contained in I. Thus we proved that any nonzero graded ideal of $A\widehat{\otimes}B$ contains its identity element, that is $A\widehat{\otimes}B$ is a SGA. □

Theorem 2.5 *Let A and B be finite dimensional graded algebras. If there exists an invertible element $z \in A$ such that $az = \omega^{\partial a} za$ for all homogeneous elements $a \in A$, then $A\widehat{\otimes}B$ and $A \otimes B$ are isomorphic. Further, if $z \in A_0$ this isomorphism is a graded isomorphism.*

Proof. We use the universal property of ordinary tensor product. To this end we construct homomorphisms $f : A \mapsto A\widehat{\otimes}B$ and $g : B \mapsto A\widehat{\otimes}B$ so that
$f(a)g(b) = g(b)f(a)$ for all $a \in A$ and $b \in B$ from which the existence of a homomorphism

$$h : A \otimes B \mapsto A\widehat{\otimes}B$$

satisfying $h\ i_A = f$ and $h\ i_B = g$ is deduced:

$$\begin{array}{ccccc} A & \overset{i_A}{\to} & A \otimes B & \overset{i_A}{\leftarrow} & B. \\ & f\searrow & \downarrow h & g\swarrow & \\ & & A\widehat{\otimes}B & & \end{array}$$

In fact, define $f(a) = a \otimes 1$ for all $a \in A$ and

$$g(b) = g(b_0 + b_1 + \cdots + b_{d-1}) = 1 \otimes b_0 + z \otimes b_1 + \cdots + z^{d-1} \otimes b_{d-1}$$

for $b_k \in B_k$ as the required homomorphisms. For $a_i \in A_i$ and $b_k \in B_k$ we see that

$$f(a_i)g(b_k) = (a_i \otimes 1)(z^k \otimes b_k) = a_i z^k \otimes b_k = \omega^{ik} z^k a_i \otimes b_k = (z^k \otimes b_k)(a_i \otimes 1) = g(b_k)f(a_i),$$

which implies our requirement $f(a)g(b) = g(b)f(a)$ for all $a \in A$ and $b \in B$. Thus the ordinary homomorphism $h : A\otimes B \mapsto A\widehat{\otimes}B$ is established. This homomorphism is surjective since $a_i \otimes b_k = h(a_i z^{-k})$ for all $a_i \in A_i$ and $b_k \in B_k$. Comparing dimensions we see that h is an isomorphism. Finally in the case $z \in A_0$, the maps f and g above become graded maps and we have

$$\partial h(a_i \otimes b_k) = \partial(f(a_i)g(b_k)) = \partial(a_i \otimes b_k),$$

and hence h is a graded isomorphism. □

Corollary 2.6 *If A is trivially graded, for any graded algebra B, then there exists a graded algebra isomorphism*

$$\hat{\mathbb{M}}_r(A)\hat{\otimes}B \cong \hat{\mathbb{M}}_r(A) \otimes B.$$

Proof. By Corollary 2.2., the homogeneous elements of $\hat{\mathbb{M}}_r(A)$ are given by the property $M\Omega = \omega^{\partial M}\Omega M$. Therefore assumptions of Theorem 2.5 are satisfied and the graded isomorphism is deduced. □

Corollary 2.7 *If A is a graded algebra, then there are graded algebra isomorphisms*

(a) $\hat{\mathbb{M}}_r(F)\hat{\otimes}A \cong \hat{\mathbb{M}}_r(F) \otimes A \cong \hat{\mathbb{M}}_r(A)$

(b) $\hat{\mathbb{M}}_r(F)\hat{\otimes}\hat{\mathbb{M}}_s(A) \cong \hat{\mathbb{M}}_r(F) \otimes \hat{\mathbb{M}}_s(A) \cong \hat{\mathbb{M}}_{rs}(A)$.

Proof. We use the fact that F is trivially graded.

(a) The first isomorphism follows from the above corollary and the second one from the definition of $\hat{\mathbb{M}}_n(A)$.

(b) The first isomorphism follows from **(a)** and the second one from the definition of the grading on $\hat{\mathbb{M}}_n(A)$. □

3. Classification of $CSGA$s

In this section we classify $CSGA$s and we shall use this classification to determine the structure of generalized Clifford algebras. This classification will allow us to develop the structure theory of $\mathbb{Z}_d$-central simple graded algebras in the last section. First we state 2 lemmas:

Lemma 3.1 *Let d be any positive integer and let A be a Z_d-SGA. If u_k is any nonzero homogeneous element in A_k, then for each $t = 0, \ldots, d-1$ we have*

$$A_t = \sum_{r+k+\ell \equiv t \pmod d} A_r u_k A_\ell.$$

Proof. Consider the graded subspace

$$L = \sum_{t=0}^{d-1} \sum_{r+k+\ell \equiv t \pmod d} A_r u_k A_\ell.$$

It is a nonzero graded ideal of A. Since A is graded simple it is equal to $A = \sum_{t=0}^{d-1} A_t$ and the result follows. □

Lemma 3.2 *Let A be a simple Z_d graded algebra with $A_k \neq 0$ for some $k \geq 1$. Then $A_0 = \sum_{k=1}^{d-1} A_k A_{d-k}$.*

Proof. It is obtained at once by considering the graded ideal

$$A = \sum_{k=1}^{d-1} A_k A_{d-k} + A_1 + \ldots + A_{d-1},$$

which is nonzero by the assumption $A_k \neq 0$ for some $k \geq 1$. □

Now we are in a position to establish the crucial result in our investigation.

Theorem 3.3 *Let A be a $CSGA$ which is not simple as an ungraded algebra. Then $Z(A) \cap A_0 = F$ and A has a central homogeneous element u of degree $m \not\equiv 0 \pmod d$ such that*

(i) $u^d \in \dot{F}$

(ii) *for each $k = mq + r$ we have $A_k = A_r u^q$.*

Proof. Since A is not simple, it has a proper ideal J. This ideal cannot contain a nonzero homogeneous element because otherwise it would contain a nonzero homogeneous ideal and hence A would not be graded simple. So J contains nonzero nonhomogeneous elements. Pick up one with *least number of homogeneous components,* say $j = j_1 + j_2 + \cdots + j_r \in J$ where $0 \neq j_k \in \mathcal{H}(A), r \geq 2$ with distinct degrees. By Lemma 3.1. we have

$$A_0 = \sum_{k+l+\partial j_1 \equiv 0 (\mathrm{mod}\ d)} A_k j_1 A_l,$$

which implies

$$1 = \sum_{k+l \equiv -\partial j_1 (\mathrm{mod}\ d)} a_k j_1 a_l; \quad a_i \in A_i.$$

Therefore J contains a nonzero element

$$u = \sum_{k+l=-\partial j_1} a_k j a_l = \sum_{s=1}^{r} \sum_{k+l=-\partial j_1} a_k j_s a_l = 1 + u_2 + \cdots + u_r,$$

with nonzero homogeneous components $1, u_2, \ldots, u_r$ of distinct degrees. In particular, $\partial u_2 \neq 0$. We claim first that u_2 is central in A. In fact since for each $a \in \mathcal{H}(A)$ the element

$$au - ua = (au_2 - u_2 a) + \cdots + (au_r - u_r a)$$

is contained in J with less than r homogeneous components, it follows that

$$au_2 - u_2 a = \ldots = au_r - u_r a = 0.$$

Secondly, u_2 is not nilpotent, for $u_2^m = 0,\ m > 0$ would imply that

$$u_2^{m-1} u = u_2^{m-1} + u_2^m + u_2^{m-1} u_3 + \cdots + u_2^{m-1} u_r$$

is contained in J with fewer than r homogeneous components and the minimality of r would give $u_2^{m-1} = 0$ and eventually $u_2 = 0$. Therefore we obtain

$$0 \neq u_2^d \in Z(A) \cap A_0 \subset \hat{Z}(A) = F,$$

which establishes $Z(A) \cap A_0 = F$, $u_2^d \in \dot{F}$ and also u_2 is invertible. Clearly invertible elements in $Z(A) \cap \mathcal{H}(A)$ form a multiplicative group G and the degree function ∂ from this group into the additive group $\mathbb{Z}_d$ is a nontrivial homomorphism since $\partial u_2 \neq 0$. Now let ∂u be a generator of Im ∂; then for each $k = q\partial u + r \in \mathbb{Z}_d$ we have by Lemma 3.1., that

$$A_k = \sum_{s+t=r} A_s u^q A_t = (\sum_{s+t=r} A_s A_t) u^q = A_r u^q$$

and the proof is completed. □

The following is an immediate consequence of the theorem.

Corollary 3.4 *Let A be a CSGA. Then $Z(A) = F$ if and only if A is central simple as an ungraded algebra. Further if A has a homogeneous element $z \in Z(A)$ of degree 1, then*

(a) $A = A_0[z] = A_0 \oplus A_0 z \oplus \cdots \oplus A_0 z^{d-1}$,

(b) $Z(A) = F \oplus Fz \oplus \cdots \oplus Fz^{d-1}$,

(c) A_0 *is central simple.*

It follows from **(b)** that $z^d = a \in F$, so that $Z(A) \cong F[X]/(X^d - a)$.

This result can be applied to ω-Clifford algebras $C = (a_1, \cdots, a_n)^d_\omega$ defined by generators $e_1, \cdots, e_n$ and relations

$$e_i^d = a_i \quad \text{for} \quad i = 1, \cdots, n \quad ; \quad e_j e_i = \omega e_i e_j \quad \text{for} \quad j > i,$$

where $a_1, \cdots, a_n$ are nonzero elements of the base field F and ω is a specified primitive d^{th} root of unity. For this purpose (see [6]) it is enough to determine the center. Now, an element

$$z = \sum a_{k_1 \cdots k_n} e_1^{k_1} \cdots e_n^{k_n}$$

of degree $k \equiv k_1 + \cdots + k_n \pmod{d}$ in C is central if and only if $ze_i = e_i z$ for all $i = 1, \cdots, n$. This shows that $a_{k_1 \cdots k_n} \neq 0$ only if

$$\begin{array}{rl}
k_1 + \cdots + k_{n-1} + k_n & \equiv k \pmod{d} \\
k_2 + \cdots + k_{n-1} + k_n & \equiv 0 \pmod{d} \\
-k_1 + k_3 + \cdots + k_{n-1} + k_n & \equiv 0 \pmod{d} \\
\cdots\cdots & \\
-k_1 - k_2 - \cdots - k_{n-1} + k_n & \equiv 0 \pmod{d} \\
-k_1 - k_2 \cdots - k_{n-1} & \equiv 0 \pmod{d}.
\end{array}$$

Hence $k_i \equiv k$ if i is even and $k_i \equiv -k$ if i is odd.Thus $Z(C) \neq F$ if and only if n is odd and if $Z(C) = F$ if n is even.

Corollary 3.4. divides $CSGA$s into 2 classes. Regarding the cases in the above motivating example of Clifford algebras, a $CSGA$ is said to be of *even type* if it is central; equivalently, if it is simple as an ungraded algebra, it is said to be of *odd type*.

4. Structure of $\mathbb{Z}_d$-*CSGA*s

To begin with we state the following theorem, which allows us to introduce even and odd type central simple graded algebras.

Theorem 4.1. *Let A be a $CSGA$, then it has a central homogeneous element z such that its degree $\partial z = m$ is a divisor of d and $Z(A)$, the center of A, is of the form*

$$Z(A) = F \oplus Fz \oplus \cdots \oplus Fz^{t-1},$$

where $t = d/m$.

Proof. We first note that

$$Z(A) \cap A_0 = \hat{Z}(A)_0 = F.$$

Now let C be the set of nonzero central elements of A. Then for any $c \in C$ the ideal cA is a nonzero graded ideal and hence $cA = A$, that is, c is invertible. This shows that C is a multiplicative group and the degree function $\partial : C \to \mathbb{Z}_d$ is a homomorphism. Let m be the degree of the generator of the cyclic group C and let $z \in C$ be such that $\partial z = m$; then each $c \in C$ is of the form

$$c = cz^{-\partial c} z^{\partial c} \text{ where } cz^{-\partial c} \in \ Z(A) \cap A_0 = F,$$

that is $c \in Fz^{\partial c}$. Since $Z(A)$ is a graded subalgebra of A and
$z^t = z^{d/m} \in Z(A) \cap A_0 = F$, we deduce that $Z(A) = F \oplus Fz \oplus \cdots \oplus Fz^{t-1}$. The last statement follows from

Corollary 3.4. □

For the central element z in the above theorem, 2 extreme cases, $\partial z = m = 1$ or d, are of utmost importance because it turns out that for generalized Clifford algebras of vector spaces, which are motivating examples of $CSGA$s, this degree is either 1 or d, according to the vector spaces under consideration being odd or even dimensional. The same situation occurs in the case that d is a prime number. Taking this into account we can give the following definition.

Definition 4.2 *A central simple graded algebra A is said to be of even type if its center is F, and it is said to be of odd type if its center is*

$$Z(A) = F[z] = F \oplus Fz \oplus \cdots \oplus Fz^{d-1},$$

where z is a homogeneous element of degree 1.
This definition comes from Wall's paper [10] when $d = 2$; in Wall's situation, every central simple graded algebra is either even or odd, and this is obviously not true in our case.

Now our aim is to describe $CSGA$s of odd or even type by decomposing them into building block $CSGA$s.

Proposition 4.3 *If A is a $CSGA$ with a nontrivial grading such that A_0 is central simple as ungraded algebra, then A cannot be of even type.*

Proof. Suppose A_0 is central simple and $Z(A) = F$. Then A is central simple by Corollary 3.4. We know that $C := C_A(A_0)$ is graded subalgebra of A. By the double centralizer theorem C is also central simple and there is an (ungraded) isomorphism

$$\varphi : A_0 \otimes C \to A$$

so that $\varphi(a_0 \otimes c) = a_0 c$ for all $a_0 \in A_0$ and $c \in C$. Therefore $C \neq C_0$, $A_k = A_0 C_k$ and

$$F = Z(A_0) = C_A(A_0) \cap A_0 = C_0.$$

It follows from this that a homogeneous element h of C is either invertible or nilpotent, for $h^d \in C_0 = F$. In the case C_k has no invertible elements $C_k = 0$, because otherwise for any $0 \neq v \in C_k$ we would get

$$C = CvC = \sum_{s,t} C_s v C_t,$$

and hence

$$F = C_0 = \sum_{s=0}^{d-1} C_s v C_{d-s-k}.$$

This implies that $C_s v C_{d-s-k} \neq 0$ for some s, say $avb = \alpha \in \dot{F}$. As we indicated above, a and b are either invertible or nilpotent; if one of them is nilpotent, say $a^m = 0$, then

$$0 = a^m vb = \alpha a^{m-1}$$

implies $a^{m-1} = 0$, and eventually $a = 0$ which is impossible. That is to say a and b are both invertible, so that v is also invertible, contradicting our assumption. Now let U be the group of invertible homogeneous elements of C and let $u \in U$ be of smallest degree $r \neq 0$. This r is a divisor of d, for the degree map ∂ is a homomorphism from U to $\mathbb{Z}_d$, and further for any multiple $l = qr$ we have $C_l = (C_l u^{-q})u^q \subset C_0 u^q = Fu^q$; therefore $C_l = Fu^q$ and $C = F + Fu + \cdots + Fu^{\frac{d}{r}-1}$. This contradicts the fact that C is a central F-algebra and completes the proof. □

Theorem 4.4 *Let A be a CSGA of odd type. Then*

(i) *A_0 is central simple as an ungraded algebra;*

(ii) *$A = A_0[z] = A_0 \oplus A_0 z \oplus \cdots \oplus A_0 z^{d-1}$ and*
$C_A(A_0) = F[z] = F \oplus Fz \oplus \cdots \oplus Fz^{d-1}$ for some central homogeneous element z of degree 1 such that $z^d \in \dot{F}$, which is uniquely determined up to a scalar multiple with these properties;

(iii) *There are graded isomorphisms*

$$A \cong (A_0) \hat{\otimes} F\langle \sqrt[d]{a} \rangle \cong (A_0) \otimes F\langle \sqrt[d]{a} \rangle,$$

where $F\langle \sqrt[d]{a} \rangle$ stands for the graded algebra $F[x]/\langle x^d - a \rangle$;

(iv)(a) *If $x^d - a$ is irreducible over F, then A is central simple over the field $F(\sqrt[d]{a})$,*

(b) *If $x^d - a$ has a root in F, then*

$$Z(A) \cong \underbrace{F \times \cdots \times F}_{d-\text{copies}} \text{ and } A \cong \underbrace{A_0 \times \cdots \times A_0}_{d-\text{copies}}.$$

Proof. **(i)** and **(ii)** Since A is of odd type , $Z(A) = F + Fz + \cdots + Fz^{d-1}$. Now for any homogeneous element h of A we have $h = (hz^{-\partial h})z^{\partial h} \in A_0$ with $(hz^{-\partial h}) \in (A_0)$ so that

$$A = A_0[z] = A_0 + A_0 z + \cdots + A_0 z^{d-1},$$

showing also that $Z(A) = C_A(A_0)$, and hence

$$F = \widehat{Z}(A) = A_0 \cap Z(A) = Z(A_0),$$

that is to say, A_0 is central. As for simplicity, take a nonzero ideal I of A_0 and form

$$J = I + Iz + \cdots + Iz^{d-1}.$$

This J is a nonzero graded ideal of A and hence it must be equal to A; consequently $I = A_0$. This proves **(i)**. To prove the remaining uniqueness part of **(ii)**, we take another element $z_1 \in Z(A) \cap A_1$ with $z_1^d = b \in \dot{F}$. We have $z_1 \in Fz$ and hence $z_1 = cz$ for some $c \in F$. Therefore

$$b = z_1^d = c^d z^d = c^d a,$$

that is $ba^{-1} = c^d \in \dot{F}^d$.

(iii) The homomorphism from $F[x]$ to $Z(A) = F[z]$ mapping to x to z yields the trivial graded isomorphism

$$Z(A) \cong F\langle \sqrt[d]{a} \rangle \cong F[x]/\langle x^d - a \rangle.$$

Since A_0 and $Z(A) = C_A(A_0) = F \oplus Fz \oplus \cdots \oplus Fz^{d-1}$ commute and $A = A_0[z] = A_0 Z(A)$, we obtain the homomorphism

$$A \cong A_0 \otimes Z(A)$$

as an ungraded isomorphism. Since A_0 is trivially graded, this isomorphism yields the graded isomorphisms

$$A \cong (A_0) \otimes Z(A) \cong (A_0) \widehat{\otimes} Z(A).$$

(iv)(a) If the polynomial $x^d - a$ is irreducible then the ring $F, \langle \sqrt[d]{a} \rangle$ is a field denoted by $F(\sqrt[d]{a})$ and $A \cong A_0 \otimes F(\sqrt[d]{a})$. Since A_0 is central simple F-algebra and $F(\sqrt[d]{a})$ is a simple F-algebra with center

$$Z(A) \cong F \otimes F(\sqrt[d]{a}) \cong F(\sqrt[d]{a}),$$

A is a central simple algebra over $F(\sqrt[d]{a})$.

(b) In the case $a = c^d \in \dot{F}^d$, we have

$$x^d - a = (x - c)(x - \omega c) \cdots (x - \omega^{d-1} c)$$

and hence

$$Z(A) = F[z] \cong F[x]/\langle x^d - a \rangle \cong \underbrace{F \times \cdots \times F}_{d-\text{copies}}$$

which implies

$$A \cong A_0 \otimes Z(A) \cong \underbrace{A_0 \times \cdots \times A_0}_{d-\text{copies}}.$$

□

To handle the even case we first give a lemma that might be interesting by itself.

Lemma 4.5 *Let $x^d - a$ be an irreducible polynomial over F and let D be a central division F-algebra which has no subfields isomorphic with $F(\sqrt[d]{a})$. Then $E = F(\sqrt[d]{a}) \otimes D$ is a central division algebra over $F(\sqrt[d]{a})$.*

Proof. We prove the assertion by induction on the number of prime divisors of d. Let $d = p_1 p_2 \cdots p_m$ where $p_1, p_2, \ldots, p_m$ are prime numbers. To begin with we suppose $m = 1$, say $d = p$. Since $x^p - a$ is irreducible, $B \cong F[x]/\langle x^p - a \rangle$ is a field and $E = B \otimes_F D$ is central simple algebra over, B, and as such we can write $E \cong \mathbb{M}_r(S)$ where S is a central division algebra over B. To complete the proof it suffices to show that $r = 1$. For this purpose, let $s = \dim_B S$ and $t = \dim_F D = \dim_B E$ and let M be the irreducible right E-module then $t = \dim_B E = r^2 s$. Also we have

$$\dim_B M = \dim_S M \dim_B S = rs = \frac{t}{r},$$

which gives

$$\dim_F M = \dim_B M \ \dim_F B = \frac{pt}{r}.$$

On the other hand , $\dim_F M$ is a multiple of $t = \dim_F D$ so that

$$\dim_F M = \dim_F D \ \dim_D M.$$

Thus r must be either 1 or p. If $r = p$, then $t = \dim_F M$ implies that $M \cong D$ as a left D-module. But $z \otimes 1 \in B \otimes D$ commutes with $1 \otimes D$ so the left multiplication map f by $1 \otimes z$ is a D-linear map on M. This shows that $f \in \mathrm{End}_D M \cong \mathrm{End}_D(D) = D^{op}$. Since $z^p = a$ we have $f^p = a$ and D^{op} contains a root of $x^p - a$. This contradicts the assumption of our lemma. Therefore $r = 1$ and hence $T \cong S$ is a central division algebra over B. Suppose now that our assumption holds in the case where d has less than m prime factors. Letting

$$d_k = p_{k+1}p_{k+2}\cdots p_m, \alpha = \sqrt[d]{a} \ \text{ and } \ \alpha_k = \alpha^{d_k}$$

we form the tower

$$F = F_0 \subset F_1 \subset \cdots \subset F_m,$$

where $F_k = F(\alpha_k)$. An easy degree argument shows that each F_k is the splitting field of $x^{p_k} - \alpha_k$ over F_{k-1} of degree p_k and that F_k is the splitting field of $x^{p_1p_2\cdots p_k} - a$ of degree $p_1p_2\cdots p_k$. Therefore $x^{p_1p_2\cdots p_k} - a$ is irreducible over F and by the induction hypothesis $E_k = F_k \otimes_F D$ is a central division algebra over F_k for each $k < m$. In particular, E_{m-1} is a central division algebra over F_{m-1}, the polynomial $x^{p_m} - a$ is irreducible over F_{m-1}, and we proved above that $F_m \otimes_{F_{m-1}} E_{m-1}$ is a central division algebra over $F_m = F(\sqrt[d]{a})$. Since

$$F_m \otimes_{F_{m-1}} E_{m-1} = F_m \otimes_{F_{m-1}} (F_{m-1} \otimes_F D) \cong F_m \otimes_F D$$

the proof is completed. □

In the following $\widetilde{\mathbb{M}}_n(D)$ will denote the graded algebra $\mathbb{M}_n(D)$ by assigning a matrix degree i, if all its degree have i.

Theorem 4.6 *Let A be a CSGA over F of even type with a nontrivial grading and let D be a central division algebra over F such that $A \cong \mathbb{M}_n(D)$ as ungraded algebras and characteristic of F does not divide d. Then*

$$Z(A_0) = C_A(A_0) = F \oplus Fz \oplus \cdots \oplus Fz^{d-1}$$

for some $z \in Z(A_0)$ with $z^d = c \in \dot{F}$ and the following statements hold:

(i) *If $c \in F^d$, then there is a graded space $V = \bigoplus_{i=0}^{p-1} V_i$ such that*

(a) *$A \cong EndV\widehat{\otimes}(D)$ (as graded algebras),*

(b) *$A_0 \cong \mathbb{M}_{r_0}(D) \times \cdots \times \mathbb{M}_{r_{d-1}}(D)$ where $r_i = dimV_i, i = 0, \ldots, d-1$,*

(c) *$Z(A_0) \cong \underbrace{F \times \cdots \times F}_{d-\text{copies}}$.*

(ii) *If $x^d - c$ is irreducible over F and D has a subfield isomorphic with $F(\sqrt[d]{c}) \cong Z(A_0)$, then there exists a grading on D such that*

(a) *$A \cong \widetilde{\mathbb{M}}_n(D) \cong \widetilde{\mathbb{M}}_n(F)\widehat{\otimes}D$ (as graded algebras),*

(b) *$A_0 \cong \mathbb{M}_n(D_0)$,*

(c) *A_0 is central simple over $Z(A_0)$.*

(iii) *If $x^d - c$ is irreducible over F but D has no subfields isomorphic to $F(\sqrt[d]{c}) \cong Z(A_0)$, then*

(a) *$n = dm$ and $A \cong (\mathbb{M}_m(D))\widehat{\otimes}(a,1)_\omega^{(d)}$ (as graded algebras),*

(b) *$A_0 \cong \mathbb{M}_m(D) \otimes F(\sqrt[d]{c})$,*

(c) A_0 *is central simple over* $Z(A_0)$.

Proof. We note first of all that since A is an even type $CSGA$, it is central simple as an ungraded algebra and hence by the Noether-Skolem theorem the main automorphism ϕ is an inner automorphism determined by an invertible element z of A. Writing

$$\phi(a) = z^{-1}az \quad for \quad a \in A$$

and using the fact that ϕ^d is identity we obtain $z^d \in Z(A) = F$. Thus $z^d \in \dot{F}$. Further for an element $a = a_0 + a_1 + \cdots + a_{d-1}$ in A with $a_k \in A_k$ we have $a \in C_A(z)$ if and only if $\phi(a) = a$ and this is the case if and only if $a = a_0$. Thus we obtain

$$F[z] \subset Z(A_0) \quad \text{and} \quad A_0 = C_A(z)$$

and consequently

$$Z(A_0) = C_A(A_0) \cap A_0 = C_A(A_0) \cap C_A(z) = C_A(A_0),$$

since $z \in C_A(z) = A_0$.

(i) Assume $c = b^d$ for some $b \in F$. Then replacing z by $\frac{z}{b}$ we may assume without loss of generality that $z^d = 1$. Letting

$$\epsilon_i = \frac{1}{d}\sum_{l=0}^{d-1}(\omega^i z)^l;\ i = 0, \ldots, d-1$$

and using

$$\sum_{k=0}^{d-1}\omega^{ik} = \begin{cases} d & \text{if} \quad i = 0 \\ 0 & \text{if} \quad 1 \le i \le d-1, \end{cases}$$

we see that $\epsilon_0, \ldots, \epsilon_{d-1}$ are orthogonal idempotents of A. They also satisfy

$$\sum_{i=0}^{d-1}\epsilon_i = 1 \quad \text{and} \quad \sum_{j=0}^{d-1}\omega^{-j}\epsilon_j = z.$$

If we fix an isomorphism $\varphi : A \to \mathbb{M}_n(D)$ and write $E_i = \varphi(\epsilon_i)$ we obtain orthogonal idempotents $E_0, E_1, \ldots, E_{d-1}$ in $\mathbb{M}_n(D)$. It is well known (see for example [5], p. 62) that there exists an invertible matrix $P \in \mathbb{M}_n(D)$ such that the matrices $P^{-1}E_iP$ are orthogonal idempotents:

$$P^{-1}E_iP = \operatorname{diag}(0, \ldots, 0;\ \underbrace{1, 1, \ldots, 1;}_{r_i-\text{copies}}\ 0, \ldots, 0)$$

for which

$$\sum_{i=0}^{d-1}P^{-1}E_iP = I \quad \text{and} \quad r_0 + r_1 + \cdots + r_{d-1} = n.$$

Thus considering the composition θ of the map φ and the inner automorphism of $\mathbb{M}_n(D)$ determined by P we get the image $\theta(z) = \Omega$ as the diagonal matrix whose diagonal blocks are $I_{r_0}, \omega^{-1}I_{r_1}, \ldots, \omega^{-d+1}I_{r_k}$ where I_{r_k} stands for the $r_k \times r_k$ identity matrix. Since we have

$$A_k = \{a \in A |\ \phi(a) = \omega^k a\} = \{a \in A |\ az = \omega^k za\},$$

the isomorphism $\theta : A \to \mathbb{M}_n(D)$ satisfies

$$\theta(A_k) = \{M \in \mathbb{M}_n(D) | M\Omega = \omega^k \Omega M\}.$$

Using the building block algebra $\mathrm{End}(V)$ constructed by Corollary 2.2. we can say that θ provides a graded isomorphism between A and $\mathbb{M}_n(D) \cong \mathrm{End}(V)\widehat{\otimes}(D)$ where V is a graded space with homogeneous components V_i of dimension r_i and that

$$A_0 \cong \mathbb{M}_{r_0}(D) \times \cdots \times \mathbb{M}_{r_{d-1}}(D) \quad \text{and} \quad Z(A_0) \cong \underbrace{F \times \cdots \times F}_{d-\text{copies}}.$$

As for the cases where $x^d - c$ is irreducible over F and the subalgebra $F[z]$ is a field, and hence applying the double centralizer theorem to the central simple F-algebra A, and is a simple subalgebra $F[z]$, we see that $A_0 = C_A(z) = C_A(F[z])$ is simple and its center is $Z(A_0) = C_A(A_0) = F[z]$.

(ii) In the case $Z(A_0) = F[z]$ is isomorphic to a subfield of the matrix algebra $\mathbb{M}_n(D)$, it contains a scalar matrix ζI with $\zeta \in D$ such that $F[\zeta] \cong F[z] \cong F[\varphi(z)]$ where φ is the fixed isomorphism between A and $\mathbb{M}_n(D)$. Thus by the Noether-Skolem theorem there is an inner isomorphism ψ of $\mathbb{M}_n(D)$. Let $\theta = \psi\varphi$, we have $\theta(z) = \zeta I$. Since $A_k = \{a \in A | az = \omega^k za\}$, θ becomes a graded isomorphism with respect to the grading of $\mathbb{M}_n(D)$ induced by the inner automorphism associated with ζI according to Corollary 2.2. In this grading of $\mathbb{M}_n(D)$, a matrix M is homogeneous of degree k if and only if $M(\zeta I) = \omega^k(\zeta I)M$ and, equivalently, $M_{ij}\zeta = \omega^k \zeta M_{ij}$ for all i, j. This shows that $(\mathbb{M}_n(D))_k = \mathbb{M}_n(D_k)$; $k = 0, \ldots, d-1$ if D is regarded as the graded algebra with grading determined by the inner automorphism associated with $\zeta \in D$. In particular, we have $A_0 \cong (\mathbb{M}_n(D))_0 = \mathbb{M}_n(D_0)$, a central simple algebra over its center $Z(A_0)$.

(iii) In the case $Z(A_0) = F[z]$ is a field that cannot be embedded into D, the irreducible polynomial $x^d - c$ has no root in the central division algebra D. It follows from Lemma 4.5. that $E = F[z] \otimes_F D$ is a central division algebra over $F[z]$. Let V be an irreducible right module over the central simple algebra $A \cong \mathbb{M}_n(D)$. Then by the Wedderburn-Artin Theorem, $D^{op} \cong \mathrm{End}_A(V)$, V is a left vector space over D naturally and right vector space over D^{op} and also over E^{op} with the right action given by $\nu(\sum b_i \otimes d_i) = \sum d_i \nu b_i$ where $d_i \in D, b_i \in F[z]$. Letting $m = \dim_E V$, this yields

$$n = \dim_D V = \dim_D E = \dim_F(F[z]) \dim_E V = dm$$

and therefore

$$\mathbb{M}_n(D) = \mathbb{M}_{dm}(D) \cong \mathbb{M}_d(F) \otimes \mathbb{M}_m(D).$$

This gives an isomorphism φ from A to $\mathbb{M}_d(F) \otimes \mathbb{M}_m(D)$. On the other hand the latter contains an element $\zeta \otimes 1$ where

$$\zeta = \begin{pmatrix} 0 & 0 & \cdots & 0 & c \\ \omega^{-1} & 0 & \cdots & 0 & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \ldots & 0 & \omega^{-d+1} & 0 \end{pmatrix}$$

such that $(\zeta \otimes 1)^d = c$. It follows from $\mathbb{M}_d(F) \otimes \mathbb{M}_m(D)$ containing 2 isomorphic subfields $F(\varphi(z))$ and $F(\zeta \otimes 1)$ that the isomorphism between them can be extended to an inner automorphism by the Noether-Skolem theorem.

Considering this inner automorphism with φ above, we obtain an isomorphism θ from A onto $\mathbb{M}_d(F)\otimes\mathbb{M}_m(D)$ such that $\theta(z)=\zeta\otimes 1$ and it becomes a graded isomorphism if the gradings are determined by z and $\zeta\otimes 1$, respectively. The grading of A is known to be determined by z, so it is enough to give a grading to the second algebra by $\zeta\otimes 1$. It is easy to verify that with this grading, the algebra becomes graded isomorphic with $(a,1)_\omega^{(d)}\otimes(\mathbb{M}_m(D))$ since the second factor is trivially graded. The rest is obvious. □

5. Application to generalized Clifford algebras

As an immediate consequence of Theorem 4.4. and 4.6. we can give a complete description of generalized Clifford algebras. In fact, let $V=Fe_1\oplus Fe_2\oplus\cdots\oplus Fe_n$ be a vector space with an ordered basis $\{e_1,\ldots,e_n\}$ and let $C(V)=(a_1,\ldots,a_n)_\omega^{(d)}$ be the generalized Clifford algebra associated with V, that is, the algebra generated by $e_1,\ldots,e_n$ subject to the relations

$$e_i^d=a_i \quad \textit{and} \quad e_je_i=\omega e_ie_j \quad \text{when} \quad j>i \quad \text{where} \quad a_1,a_2,\ldots,a_n\in F.$$

We have

$$\begin{aligned}(e_1^{l_1}e_2^{l_2}\cdots e_n^{l_n})(e_1^{k_1}e_2^{k_2}\cdots e_n^{k_n}) &= \omega^{\sum_{i<j}^{k_il_j}}e_1^{k_1+l_1}e_2^{k_2+l_2}\cdots e_n^{k_n+l_n}\\ &= \omega^{\sum_{i<j}^{k_il_j-l_ik_j}}(e_1^{k_1}e_2^{k_2}\cdots e_n^{k_n})(e_1^{l_1}e_2^{l_2}\cdots e_n^{l_n}),\end{aligned}$$

which yields that $C(V)$ is a $CSGA$ with $\partial e_i=1$, and that
$z=e_1^{(-1)^0}e_2^{(-1)^1}\cdots e_n^{(-1)^{n-1}}$ is a central element of degree 1 or 0 according as $n=\dim(V)$ is odd or even, respectively. It satisfies the polynomial x^d-a with
$a=(-1)^{m(d-1)}a_1^{(-1)^0}a_2^{(-1)^1}\cdots a_n^{(-1)^{n-1}}$ where m is the integral part of $\frac{n}{2}$. Consequently Theorem 4.4. and 4.6. yield the following structure theorems.

Theorem 5.1 *Let $C=C(V)$ be the generalized Clifford algebra of a vector space V of odd dimension $n=2m+1$ and let $C_0=C_0(V)$ be its subalgebra consisting of homogeneous elements of degree zero. Then*

(i) *C_0 is central simple as an ungraded algebra;*

(ii) $C=C_0[z]=C_0\oplus C_0z\oplus\cdots\oplus C_0z^{d-1}$ and $C_A(C_0)=F[z]=F\oplus Fz\oplus\cdots\oplus Fz^{d-1}$ where $z=e_1e_2^{-1}\cdots e_{n-1}^{-1}e_n$ and $z^d=a=(-1)^{m(d-1)}a_1a_2^{-1}\cdots a_{n-1}^{-1}a_n$*;*

(iii) *There are graded algebra isomorphisms*

$$C\cong(C_0)\widehat{\otimes}F\langle\sqrt[d]{a}\rangle\cong(C_0)\otimes F\langle\sqrt[d]{a}\rangle$$

where $F\langle\sqrt[d]{a}\rangle$ stands for the graded algebra $F[x]/\langle x^d-a\rangle$;

(iv)(a) *If x^d-a is irreducible over F, then C is central simple over the field $F(\sqrt[d]{a})$;*

(b) *If x^d-a has a root in F then*

$$Z(C)\cong\underbrace{F\times\cdots\times F}_{d-\text{copies}} \quad \textit{and} \quad C\cong\underbrace{C_0\times\cdots\times C_0}_{d-\text{copies}}.$$

Proof. It follows from Theorem 4.4. and the facts indicated just above. □

Theorem 5.2 *Let $C = C(V)$ be the generalized Clifford algebra of a vector space V of even dimension $n = 2m$ and let $C_0 = C_0(V)$ be its subalgebra consisting of homogeneous elements of degree 0. Then C is a central simple algebra over F, say $C \cong M_t(D)$ (as ungraded algebras) for some central division algebra D over F, $t = d^s$, and*

$$Z(C_0) = F \oplus Fz \oplus \cdots \oplus Fz^{d-1},$$

where

$$z = e_1 e_2^{-1} \cdots e_{n-1} e_n^{-1} \text{ and } z^d = a = (-1)^{m(d-1)} a_1 a_2^{-1} \cdots a_{n-1} a_n^{-1}$$

and the following statements hold:

(i) *If $a \in F^d$ then we have*

(a) *$C \cong \widehat{\mathbb{M}}_t(D)$ (as graded algebras) where t is a power of d;*

(b) *$C_0 \cong M_d(D) \times \cdots \times M_d(D)$ where $r = \frac{t}{d}$;*

(c) *$Z(C_0) \cong \underbrace{F \times \cdots \times F}_{d-\text{copies}}$.*

(ii) *If $x^d - a$ is irreducible over F and D has a subfield isomorphic with $F(\sqrt[d]{a}) \cong Z(A_0)$, then there exists a grading on D such that*

(a) *$C \cong \widehat{\mathbb{M}}_n(D) \cong \widehat{\mathbb{M}}_n(F) \widehat{\otimes} D$ (as graded algebras),*

(b) *$C_0 \cong M_n(D_0)$,*

(c) *C_0 is central simple over $F(\sqrt[d]{a})$.*

(iii) *If $x^d - a$ is irreducible over F but D has no subfields isomorphic to $F(\sqrt[d]{a}) \cong Z(A_0)$, then*

(a) *$n = dm$ and $C \cong (M_m(D)) \widehat{\otimes} (a, 1)_\omega^{(d)}$ (as graded algebras),*

(b) *$C_0 \cong M_m(D) \otimes F(\sqrt[d]{a})$,*

(c) *C_0 is central simple over $F(\sqrt[d]{a})$.*

Proof. All we have to show is that $t = d^s$ and that the assertion **(i)–(b)** holds. The rest will follow from Theorem 4.6. and the facts indicated just above. The first of these is obtained at once from $\dim(C) = t^2 \dim(D) = d^n$ (see [6]). As for the second one, letting $a = b^d, b \in \dot{F}$, and $z_1 = \frac{z}{b}$ we see from the proof of Theorem 4.6.(i) that the elements

$$\epsilon_i = \frac{1}{d} \sum_{l=0}^{d-1} (\omega^i z^l)^l; \quad i = 0, \ldots, d-1$$

are central orthogonal idempotents such that $\epsilon_0 + \cdots + \epsilon_{d-1} = 1$ and hence the algebra C_0 is semi-simple with simple components $C_0 \epsilon_i$. On the other hand for each fixed pair i, j, the linear map φ on V sending e_1 to $\omega^{j-i} e_1$ and fixing all other e_k extends to a graded automorphism of C and this automorphism of C_0 sending to ϵ_i to ϵ_j; thus it gives an isomorphism between $C_0 \epsilon_i$ and $C_0 \epsilon_j$. Now by Theorem 4.6.(i) we have

$$C_0 \cong M_{r_0}(D) \times \cdots \times M_{r_{d-1}}(D)$$

and we have just proven that the simple components $M_{r_i}(D)$ are isomorphic, so that r_i are equal; namely, they are equal to $r = \frac{t}{d}$. □

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