NUMERICAL LINEAR ALGEBRA WITH APPLICATIONS
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Computation of H_∞ controllers for infinite dimensional plants using numerical linear algebra[‡]

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SUMMARY

The mixed sensitivity minimization problem is revisited for a class of single-input-single-output unstable infinite dimensional plants with low order weights. It is shown that H_∞ controllers can be computed from the singularity conditions of a parameterized matrix whose dimension is the same as the order of the sensitivity weight. The result is applied to the design of H_∞ controllers with integral action. Connections with the so-called Hamiltonian approach are also established. Copyright © 2012 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Weighted sensitivity minimization for time delay systems was the first H_∞ control problem solved for infinite dimensional systems, [1–3]. The methods used in [2, 3] were extended to cover a larger class of distributed parameter systems in [4–9]. Another type of H_∞ control problem studied for delay systems was robust stabilization in the gap metric, [10, 11]. These are examples of the so-called one-block problems. Typically, the problem is turned into a Nehari problem, and its solution is obtained by the computation of the singular values of the associated Hankel operator. For the solution of the mixed sensitivity minimization (two-block) problem for single-input-single-output unstable infinite dimensional systems, first computational procedures were given in [12–14]. In these papers, the optimal performance level and the corresponding controller are obtained by studying a "Hankel+Toeplitz", or a "skew-Toeplitz" operator, [15–17]. However, with the exception of [7, 10] (both of them deal with one-block problems) *explicit* formula for the controller could not be given in the previous cited papers. One needed to follow a complicated substitutions and transformations to arrive at the controller from the singular vectors of the related operators. In [18], an explicit formula is obtained, for the first time, for the optimal controller in the mixed sensitivity minimization problem involving infinite dimensional plants and finite dimensional weights. The derivation of this controller was carried out by using the AAK theory, [19], and by observations leading to simplifications, see also [20, 21]. Computations involve a spectral factorization (depending only on the weights) and solution of a set of $2(n_1 + \ell)$ linear equations with the same number of unknowns, where n_1 is the order of the sensitivity weight and ℓ is the number of unstable poles of the plants. Later, it was shown that the mixed sensitivity minimization can be solved using a dual approach of [18] for a class of plants with infinitely many unstable poles, [22, 23]. The largest class of infinite dimensional plants covered by the method of [18] and controller implementation issues have been discussed in [24].

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Besides these *direct* frequency domain methods mentioned earlier, there are also approximation based H_∞ controller design for infinite dimensional systems, see, for example, [25–28]. They are mainly relying on state-space methods, see [29] and references therein. For systems with time delays, there are alternative methods of H_∞ controller design exploiting the special nature of the time delay operator; see the list of references in [30].

In this paper, the formula given in [18] is revisited. It is shown that under certain mild assumptions, the number of equations to be solved can be reduced to n_1. In this sense, the new set of equations can be seen as the extension of the Zhou–Khargonekar formula, [3], to the two-block problems involving possibly unstable plants. For stable plants, connections between the skew-Toeplitz method, [20], and the Zhou–Khargonekar formula, [3], were demonstrated in [31, 32].

The paper is organized as follows. The controller formula of [18] is given in the next section. Conditions under which the reductions in the number of equations can be performed are discussed in Section 3. Application of this main result to the design of H_∞ controllers with integral action can be found in Section 4. The paper ends with some concluding remarks.

2. TOKER–ÖZBAY FORMULA

In this paper, an infinite dimensional plant is considered, it is represented by the transfer function $P(s)$, where s is the Laplace transform variable, that is P is an irrational function of the complex variable s. Given two weighting functions $W_1(s)$ and $W_2(s)$, the mixed sensitivity minimization problem is to find

$$\gamma_{\text{opt}} := \inf_{C \in \mathscr{C}(P)} \left\| \begin{bmatrix} W_1(1+PC)^{-1} \\ W_2 PC(1+PC)^{-1} \end{bmatrix} \right\|_\infty , \tag{1}$$

where $\mathscr{C}(P)$ is the set of all controllers $C(s)$ for which the feedback system formed by C and P is stable. Feedback system stability is equivalent to having the closed loop system transfer functions $S := (1+PC)^{-1}$, CS, and PC in $\mathscr{H}_\infty$. The optimal controller solving Equation (1) is denoted by C_{opt}. Typically $W_1(s)$ is a low order low-pass filter representing the class of reference signals to be tracked and $W_2(s)$ is an improper low order high-pass filter representing a bound on the multiplicative plant uncertainty; for detailed discussions on weight selections and connections with robust control problems, see [33–36].

The plant is assumed to have finitely many poles in $\mathbb{C}_+$ and no poles on the Im-axis. In this case, $P(s)$ can be factored as

$$P(s) = \frac{M_n(s) N_o(s)}{M_d(s)}, \tag{2}$$

where M_n is an inner (all-pass) function, $N_o(s)$ is an outer (minimum phase) function, and $M_d(s)$ is a rational inner function. Let $\alpha_1, \ldots, \alpha_\ell \in \mathbb{C}_+$ be the zeros of $M_d(s)$, that is, unstable poles of the plant. For simplicity of the notation, it is assumed that $\alpha_1, \ldots, \alpha_\ell$ are distinct.

Because W_1 is rational, it can be written as $W_1(s) = nW_1(s)/dW_1(s)$, for two coprime polynomials nW_1 and dW_1; it is assumed that $\deg(nW_1) \leqslant \deg(dW_1) =: n_1 \geqslant 1$. Define

$$E_\gamma(s) := \left(\frac{W_1(-s) W_1(s)}{\gamma^2} - 1 \right) \tag{3}$$

and let $\beta_1, \ldots, \beta_{2n_1}$ be the zeros of $E_\gamma(s)$, enumerated in such a way that $-\beta_{n_1+k} = \beta_k \in \overline{\mathbb{C}}_+$, for $k = 1, \ldots, n_1$. Note that each β_k is dependent on $\gamma > 0$, which is a candidate for γ_{opt}. We assume that for $\gamma = \gamma_{\text{opt}}$, the zeros of E_γ are distinct. Note that this condition is satisfied generically (if not, a small perturbation in the problem data changes γ_{opt} that moves the locations of $\beta_1, \ldots, \beta_{n_1}$).

Now, define a rational function that depends on $\gamma > 0$ and the weights W_1 and W_2,

$$F_\gamma(s) := \gamma \frac{dW_1(-s)}{nW_1(s)} G_\gamma(s), \tag{4}$$

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where $G_\gamma \in \mathscr{H}_\infty$ is an outer function determined from the spectral factorization

$$G_\gamma(-s)G_\gamma(s) = \left(1 + \frac{W_2(-s)W_2(s)}{W_1(-s)W_1(s)} - \frac{W_2(-s)W_2(s)}{\gamma^2}\right)^{-1}. \tag{5}$$

With the above definitions, the optimal controller can be expressed as

$$C_{\text{opt}}(s) = E_\gamma(s)M_d(s)\frac{F_\gamma(s)L(s)}{1 + M_n(s)F_\gamma(s)L(s)}N_o^{-1}(s), \tag{6}$$

where $\gamma = \gamma_{\text{opt}}$ and $L(s)$ is a transfer function of the form

$$L(s) = \frac{[1\ s \ldots\ s^{n-1}]\Psi_2}{[1\ s \ldots\ s^{n-1}]\Psi_1}, \quad n := n_1 + \ell, \tag{7}$$

where the coefficient vectors

$$\Psi_1 = [\psi_{10}\ \ldots\ \psi_{1(n-1)}]^{\mathrm{T}} \text{ and } \Psi_2 = [\psi_{20}\ \ldots\ \psi_{2(n-1)}]^{\mathrm{T}} \tag{8}$$

are to be determined from the interpolation conditions given in [18]. These interpolation conditions can be expressed in the matrix form. In order to do this, we need to first define some specific matrices.

Let $\mathfrak{J}_k$ be the $k \times k$ diagonal matrix, $k \geqslant 1$, whose ith diagonal entry is $(-1)^{i+1}$. For a given vector $\mathbf{x} = [x_1, \ldots, x_k]^{\mathrm{T}} \in \mathbb{C}^k$ with $x_i \neq x_j$ for $i \neq j$, and a positive integer $m \geqslant 1$, we define the associated Vandermonde matrix of size $k \times m$ as

$$\mathscr{V}_{\mathbf{x}}^m := \begin{bmatrix} 1 & x_1 & \ldots & x_1^{m-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_k & \ldots & x_k^{m-1} \end{bmatrix}. \tag{9}$$

Similarly, define $\mathscr{V}_\alpha^n$ and $\mathscr{V}_\beta^n$ for the vectors $\alpha = [\alpha_1, \ldots, \alpha_\ell]^{\mathrm{T}}$ and $\beta = [\beta_1, \ldots, \beta_{n_1}]^{\mathrm{T}}$, respectively, and form the square matrix

$$\mathscr{V}_n := \begin{bmatrix} \mathscr{V}_\alpha^n \\ \mathscr{V}_\beta^n \end{bmatrix}.$$

Define the diagonal matrices

$$\begin{aligned} \mathscr{D}_\ell &= \text{diag}\{M_n(\alpha_1)F_\gamma(\alpha_1), \ldots, M_n(\alpha_\ell)F_\gamma(\alpha_\ell)\} \\ \mathscr{D}_{n_1} &= \text{diag}\{M_n(\beta_1)F_\gamma(\beta_1), \ldots, M_n(\beta_{n_1})F_\gamma(\beta_{n_1})\} \\ \mathscr{D}_n &= \text{block diag}\{\mathscr{D}_\ell, \mathscr{D}_{n_1}\}. \end{aligned}$$

In [18], it has been shown that γ_{opt} is the largest γ for which the set of linear equations

$$0 = \mathscr{V}_n\Psi_1 + \mathscr{D}_n\mathscr{V}_n\Psi_2 \tag{10}$$

$$0 = \mathscr{D}_n\mathscr{V}_n\mathfrak{J}_n\Psi_1 + \mathscr{V}_n\mathfrak{J}_n\Psi_2 \tag{11}$$

has a non-trivial solution Ψ_1, Ψ_2. First set of conditions, (10), lead to

$$\Psi_1 = -(\mathscr{V}_n)^{-1}\mathscr{D}_n\mathscr{V}_n\Psi_2. \tag{12}$$

Also note that if we set

$$\Psi_1 = \pm\mathfrak{J}_n\Psi_2 \tag{13}$$

in Equation (10), we obtain Equation (11). Therefore, Equations (12) and (13) can replace Equations (10) and (11) provided that the sign in Equation (13) is determined. With Equations (12) and (13), we have

$$L(s) = -\frac{[1\ s\ \ldots\ s^{n-1}]\Psi_2}{[1\ s\ \ldots\ s^{n-1}](\mathscr{V}_n)^{-1}\mathscr{D}_n\mathscr{V}_n\Psi_2} = \pm\frac{[1\ s\ \ldots\ s^{n-1}]\Psi_2}{[1\ s\ \ldots\ s^{n-1}]\mathfrak{J}_n\Psi_2}, \tag{14}$$

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which leads to

$$L(0) = -\frac{[1\ 0\ \ldots\ 0]\Psi_2}{[1\ 0\ \ldots\ 0](\mathscr{V}_n)^{-1}\mathscr{D}_n\mathscr{V}_n\Psi_2} = \pm 1. \tag{15}$$

Also note that $|L(j\omega)| = 1$ for all $\omega \in \mathbb{R}$.

Now, for the computation of Ψ_2, let us first define

$$\mathfrak{J}_n\Psi_2 =: \Phi = [\Phi_1^{\mathrm{T}}\ \ \Phi_2^{\mathrm{T}}]^{\mathrm{T}} \quad \text{with} \quad \Phi_1 = [\phi_0, \ldots, \phi_{\ell-1}]^{\mathrm{T}},\ \ \Phi_2 = [\phi_\ell, \ldots, \phi_{n-1}]^{\mathrm{T}} \tag{16}$$

and transform the Equation (10) into the form

$$\mathscr{R}_\gamma \Phi = 0, \tag{17}$$

where

$$\mathscr{R}_\gamma := \begin{bmatrix} \mathscr{V}_\alpha^\ell & \mathscr{D}_\alpha \mathscr{V}_\alpha^{n_1} \\ \mathscr{V}_\beta^\ell & \mathscr{D}_\beta \mathscr{V}_\beta^{n_1} \end{bmatrix} \pm \begin{bmatrix} \mathscr{D}_\ell & \mathbf{0} \\ \mathbf{0} & \mathscr{D}_{n_1} \end{bmatrix} \begin{bmatrix} \mathscr{V}_\alpha^\ell & \mathscr{D}_\alpha \mathscr{V}_\alpha^{n_1} \\ \mathscr{V}_\beta^\ell & \mathscr{D}_\beta \mathscr{V}_\beta^{n_1} \end{bmatrix} \mathfrak{J}_n, \tag{18}$$

with

$$\begin{aligned} \mathscr{D}_\alpha &= \operatorname{diag}\{\alpha_1^\ell, \ldots, \alpha_\ell^\ell\} \\ \mathscr{D}_\beta &= \operatorname{diag}\{\beta_1^\ell, \ldots, \beta_{n_1}^\ell\}. \end{aligned}$$

Thus, γ_{opt} is the largest γ that makes the matrix $\mathscr{R}_\gamma$ singular with the $+$ or $-$ sign in Equation (18). The corresponding Φ determines the sign via Equation (15) and hence C_{opt}, Equation (6), is obtained via Equations (14) and (16).

3. REMARKS ON THE SET OF LINEAR EQUATIONS DEFINING C_{OPT}

In Equation (17), there are $n = \ell + n_1$ equations. For the first set of ℓ equations, note that interpolation points $\alpha_1, \ldots \alpha_\ell$ are fixed and, hence, the only dependence on γ is in F_γ. Typically, the weights W_1 and W_2 are low order, hence, F_γ is low order and be computed easily (explicit computation of its coefficients in terms of γ is possible). Motivated by this observation, we separate the equations in Equation (17) into two pieces:

$$(I \pm \mathscr{F}_\ell \mathfrak{J}_\ell)\Phi_1 + (\mathscr{V}_\alpha^\ell)^{-1}\mathscr{D}_\alpha(\mathscr{V}_\alpha^{n_1} \pm \mathscr{D}_\ell \mathscr{V}_\alpha^{n_1}(-1)^\ell \mathfrak{J}_{n_1})\Phi_2 = 0 \tag{19}$$

$$(\mathscr{V}_\beta^{n_1})^{-1}\mathscr{D}_\beta^{-1}(\mathscr{V}_\beta^\ell \pm \mathscr{D}_{n_1}\mathscr{V}_\beta^\ell \mathfrak{J}_\ell)\Phi_1 + (I \pm \mathscr{F}_{n_1}(-1)^\ell \mathfrak{J}_{n_1})\Phi_2 = 0, \tag{20}$$

where

$$\mathscr{F}_\ell = (\mathscr{V}_\alpha^\ell)^{-1}\mathscr{D}_\ell \mathscr{V}_\alpha^\ell \tag{21}$$

$$\mathscr{F}_{n_1} = (\mathscr{V}_\beta^{n_1})^{-1}\mathscr{D}_{n_1}\mathscr{V}_\beta^{n_1}. \tag{22}$$

Define the canonical matrix

$$A_d = \begin{bmatrix} 0 & \cdots & 0 & -a_0 \\ 1 & & & -a_1 \\ & \ddots & & \vdots \\ & & 1 & -a_{\ell-1} \end{bmatrix}, \tag{23}$$

where $a_0, \ldots, a_{\ell-1}$ are determined from the identity

$$\prod_{j=1}^{\ell}(s - \alpha_j) =: s^\ell + a_{\ell-1}s^{\ell-1} + \cdots + a_0.$$

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Note that A_d is the "A-matrix" of the observable canonical realization of $1/M_d(s)$. Its eigenvalues are $\alpha_1,\ldots,\alpha_\ell$ with the corresponding left eigenvectors being the rows of $\mathscr{V}_\alpha^\ell$. So,

$$\mathscr{F}_\ell = (\mathscr{V}_\alpha^\ell)^{-1}\mathscr{D}_\ell \mathscr{V}_\alpha^\ell = M_n(A_d)F_\gamma(A_d). \tag{24}$$

Now, assume that $(I \pm \mathscr{F}_\ell \mathfrak{J}_\ell)$ is non-singular for $\gamma = \gamma_{\text{opt}}$. Then, from Equation (19), we have

$$\Phi_1 = -(I \pm \mathscr{F}_\ell \mathfrak{J}_\ell)^{-1}(\mathscr{V}_\alpha^\ell)^{-1}\mathscr{D}_\alpha(\mathscr{V}_\alpha^{n_1} \pm \mathscr{D}_\ell \mathscr{V}_\alpha^{n_1}(-1)^\ell \mathfrak{J}_{n_1})\Phi_2. \tag{25}$$

Substituting Equation (25) into Equation (20), we obtain n_1 set of equations from which the sign of $L(s)$, γ_{opt}, and Φ_2 are obtained:

$$\mathscr{P}_\gamma \Phi_2 = 0, \tag{26}$$

where

$$\begin{aligned}\mathscr{P}_\gamma \;:=\; & -(\mathscr{V}_\beta^{n_1})^{-1}\mathscr{D}_\beta^{-1}(\mathscr{V}_\beta^\ell \pm \mathscr{D}_{n_1}\mathscr{V}_\beta^\ell \mathfrak{J}_\ell)(I \pm \mathscr{F}_\ell \mathfrak{J}_\ell)^{-1}(\mathscr{V}_\alpha^\ell)^{-1}\mathscr{D}_\alpha(\mathscr{V}_\alpha^{n_1} \pm \mathscr{D}_\ell \mathscr{V}_\alpha^{n_1}(-1)^\ell \mathfrak{J}_{n_1}) \\ & +(I \pm \mathscr{F}_{n_1}(-1)^\ell \mathfrak{J}_{n_1}).\end{aligned} \tag{27}$$

The optimal mixed sensitivity level γ_{opt} is the largest γ for which there exists a non-zero Φ_2 satisfying Equation (26). In other words, γ_{opt} is the largest γ that makes the smallest singular value of $\mathscr{P}_\gamma$ equal to zero. Thus, the size of the matrix, $\mathscr{P}_\gamma$, for which the SVD is to be taken, is reduced to n_1, provided that the inverse $(I \pm M_n(A_d)F_\gamma(A_d))^{-1}$ can be computed easily as a function of γ, see Section 4 for an example, where first order weights are considered.

3.1. *The case where $W_1(s)$ is of first order*

We have seen that if the matrix $(I \pm \mathscr{F}_\ell \mathfrak{J}_\ell)$ is invertible, where $\mathscr{F}_\ell$ is given by Equation (24), then the optimal controller can be obtained by studying singularities of the matrix $\mathscr{P}_\gamma$, whose size is $n_1 \times n_1$, where n_1 is the degree of the sensitivity weight, W_1. Typically, n_1 is a small integer. In fact, as in the example of Section 4, in many interesting problems $n_1 = 1$, so Equation (27) is a scalar function of γ.

Let us examine the components of Equation (27) for $n_1 = 1$. First, note that in this case, we have

$$(\mathscr{V}_\beta^{n_1})^{-1} = 1, \quad \mathscr{D}_\beta^{-1} = \beta_1^{-\ell}, \quad \mathscr{V}_\beta^\ell = [1, \beta_1, \ldots, \beta_1^{\ell-1}], \quad \mathfrak{J}_{n_1} = 1,$$

and

$$\mathscr{F}_{n_1} = \mathscr{D}_{n_1} = M_n(\beta_1)F_\gamma(\beta_1), \quad \mathscr{D}_\alpha \mathscr{V}_\alpha^{n_1} = [\alpha_1^\ell, \ldots, \alpha_\ell^\ell]^{\mathrm{T}}.$$

Moreover, for $n_1 = 1$, the vector $(\mathscr{V}_\alpha^\ell)^{-1}\mathscr{D}_\alpha \mathscr{V}_\alpha^{n_1}$ can be computed as

$$(\mathscr{V}_\alpha^\ell)^{-1}\mathscr{D}_\alpha \mathscr{V}_\alpha^{n_1} = \mathbf{a},$$

where $\mathbf{a}$ is the last column of A_d, Equation (23), that is,

$$\mathbf{a} := -[a_0, \ldots, a_{\ell-1}]^{\mathrm{T}}. \tag{28}$$

Let us define the vector

$$\mathbf{b} := -\beta_1^{-\ell}[1, \beta_1, \ldots, \beta_1^{\ell-1}]. \tag{29}$$

Then, for the case $n_1 = 1$, the matrix Equation (27) becomes a scalar:

$$\begin{aligned}\mathscr{P}_\gamma = \; & \mathbf{b}(I \pm M_n(\beta_1)F_\gamma(\beta_1)\mathfrak{J}_\ell)(I \pm M_n(A_d)F_\gamma(A_d)\mathfrak{J}_\ell)^{-1}(I \pm M_n(A_d)F_\gamma(A_d)(-1)^\ell)\mathbf{a} \\ & +(1 \pm M_n(\beta_1)F_\gamma(\beta_1)(-1)^\ell).\end{aligned} \tag{30}$$

Note that in Equation (30), the terms $M_n(A_d)$, $\mathfrak{J}_\ell$, and $\mathbf{a}$ are independent of γ. The coefficients of $F_\gamma(A_d)$ depend on γ. When $n_1 = 1$, the roots of E_γ, that is, β_1 and $\beta_2 = -\beta_1$ can be computed explicitly in terms of γ. So, the vector $\mathbf{b}$ and scalars $M_n(\beta_1)$ and $F_\gamma(\beta_1)$ can be evaluated numerically.

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3.2. Remarks on the interpolation conditions

Another point to be noted is that by definition, Equation (14), we have $L(-s) = 1/L(s)$. Because M_n is an inner function, we also have $M_n(-s) = 1/M_n(s)$. Recall that F_γ is defined as Equation (4) where G_γ is determined from the spectral factorization Equation (5). These two equations imply that

$$F_\gamma(-s)F_\gamma(s) = \left(\left(\frac{W_1(-s)W_1(s)}{\gamma^2} - 1\right)\left(1 - \frac{W_2(-s)W_2(s)}{\gamma^2}\right) + 1\right)^{-1}.$$

Hence, for each β_k, a zero of $E_\gamma(s) = \left(\frac{W_1(-s)W_1(s)}{\gamma^2} - 1\right)$, we have

$$F_\gamma(-\beta_k) = 1/F_\gamma(\beta_k).$$

Thus, in addition to the interpolation conditions Equation (17), $L(s)$ satisfies

$$1 + M_n(-\beta_k)F_\gamma(-\beta_k)L(-\beta_k) = 0 \quad \forall k = 1, \ldots, n_1. \tag{31}$$

This means that the function

$$\frac{1 + M_n(s)F_\gamma(s)L(s)}{M_d(s)E_\gamma(s)}$$

has no poles at the zeros of M_d and E_γ.

Let $W_1(s) = C_1(sI - A_1)^{-1}B_1$ be a minimal realization (we consider a strictly proper weight for simplicity of the notation, for general case see [21]). Then E_γ^{-1} has a minimal realization in the form

$$E_\gamma^{-1}(s) = C_\gamma(sI - A_\gamma)^{-1}B_\gamma - 1,$$

where

$$A_\gamma = \begin{bmatrix} A_1 & B_1B_1^{\mathrm{T}}/\gamma \\ -C_1^{\mathrm{T}}C_1/\gamma & -A_1^{\mathrm{T}} \end{bmatrix} \quad B_\gamma = \begin{bmatrix} -B_1/\sqrt{\gamma} \\ 0 \end{bmatrix} \quad C_\gamma = \begin{bmatrix} 0 \\ B_1/\sqrt{\gamma} \end{bmatrix}^{\mathrm{T}}.$$

The zeros of $E_\gamma(s)$, namely, $\beta_1, \ldots, \beta_{2n_1}$ are the eigenvalues of the Hamiltonian matrix A_γ. Because we assumed that these eigenvalues are distinct and enumerated in such a way that $\beta_k = -\beta_{n_1+k} \in \overline{\mathbb{C}}_+$ for $k = 1, \ldots, n_1$, we can find a $2n_1 \times 2n_1$ invertible matrix T_2 such that

$$A_\gamma = T_2 \begin{bmatrix} \Lambda_\gamma^+ & 0 \\ 0 & -\Lambda_\gamma^+ \end{bmatrix} T_2^{-1}$$

where Λ_γ^+ is the diagonal matrix whose diagonal entries are $\beta_1, \ldots, \beta_{n_1}$.

Appending (31) to (17), after some matrix manipulations, we obtain (recall the notation $n := n_1 + \ell$)

$$\left(I_n \pm \begin{bmatrix} I_n & 0_{n_1} \end{bmatrix} \begin{bmatrix} M_n(A_d)F_\gamma(A_d) & 0 \\ 0 & M_n(A_\gamma)F_\gamma(A_\gamma) \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \mathfrak{I}_n Q_1^{-1}\right)\widehat{\Phi} = 0, \tag{32}$$

where I_n is the $n \times n$ identity matrix, 0_{n_1} is the $n_1 \times n_1$ matrix whose entries are 0 and

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} := \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} \mathcal{V}_\alpha^n \\ \mathcal{V}_\beta^n \\ \mathcal{V}_\beta^n \mathfrak{I}_n \end{bmatrix}, \widehat{\Phi} := Q_1\Phi, \tag{33}$$

with T_1 being the invertible matrix that satisfies $A_d = T_1\Lambda_\alpha T_1^{-1}$, where Λ_α is the diagonal matrix whose entries are $\alpha_1, \ldots, \alpha_\ell$; the partitioning in Equation (33) is such that Q_1 is an $n \times n$ square matrix, and Q_2 is an $n_1 \times n$ matrix.

Equation (32) shows the extension of [32] where mixed sensitivity minimization was considered for stable plants. In the stable case $\ell = 0$, and Q_1 and Q_2 are square matrices of dimensions $n_1 \times n_1$. In that case, $M_n(A_\gamma)F_\gamma(A_\gamma)$, together with Q_1 and Q_2 determine γ_{opt} and the corresponding Φ.

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4. EXAMPLE: DESIGN OF H_∞ CONTROLLERS WITH INTEGRAL ACTION

In this section, we examine the controller structure for a specific choice of weights:

$$W_1(s) = \frac{1}{s}, \quad W_2(s) = ks, \tag{34}$$

where $k > 0$ represents the relative importance of the multiplicative uncertainty with respect to the tracking performance under step-like reference inputs [33, 35]. With Equation (34), the functions $E_\gamma(s)$ and $F_\gamma(s)$ are computed as

$$E_\gamma(s) = \frac{1+\gamma^2 s^2}{-\gamma^2 s^2}, \quad F_\gamma(s) = \frac{-\gamma s}{ks^2 + k_\gamma s + 1}, \quad \text{where} \quad k_\gamma = \sqrt{2k - \frac{k^2}{\gamma^2}}. \tag{35}$$

It can be shown that, [20], for the weights in Equation (34), we have $\gamma_{\text{opt}} > \sqrt{k/2}$, independent of the plant. Therefore, the search for γ_{opt} is conducted for the values of γ that makes k_γ real and positive.

The discussion of Section 3.1, in particular Equation (30), requires computation of $M_n(A_d)$ and $F_\gamma(A_d)$ for the given plant parameters A_d (the "A-matrix" of the observable canonical realization of M_d) and M_n. Once A_d is given, we compute

$$F_\gamma(A_d) = -\gamma\left(kA_d + A_d^{-1} + k_\gamma I\right)^{-1}.$$

With the above E_γ and F_γ, the optimal controller is in the form

$$C_{\text{opt}}(s) = \left(\frac{1}{\gamma s}\right)\left(\frac{M_d(s)(1+\gamma^2 s^2)L(s)}{(ks^2 + k_\gamma s + 1) - \gamma s M_n(s) L(s)}\right) N_o^{-1}(s). \tag{36}$$

Because $|L(j\omega)| = 1$ and $|M_d(j\omega)| = 1$ for all $\omega \in \mathbb{R}$, we have that $M_d(0) \neq 0$ and $L(0) \neq 0$. Furthermore, when the plant $P(s)$ does not have a pole at the origin, we have $N_o^{-1}(0) \neq 0$. Hence, the controller Equation (36) contains an integral action due to the term $1/(\gamma s)$.

Note that with Equation (34), we have $n_1 = 1$ and from Equation (35), $\beta_1 = j/\gamma$. In particular, when the plant to be controlled is stable, we have $\ell = 0$. In this case, $L(s) = \pm 1$, and γ_{opt} must be such that for $\gamma = \gamma_{\text{opt}}$, we have

$$X(\gamma) := \left(1 - \frac{k}{\gamma^2}\left(1 - j\sqrt{2\frac{\gamma^2}{k} - 1}\right)\right) \mp jM_n(j/\gamma) = 0. \tag{37}$$

The equality Equation (37) is equivalent to $\mathscr{P}_\gamma = 0$, where $\mathscr{P}_\gamma$ is defined in Equation (30); because $\ell = 0$, in this case, the first term in Equation (30) multiplying $\mathbf{b}$ is absent.

For the numerical example with $k = 1$ and

$$M_n(s) = e^{-0.25s}\left(\frac{1-2e^{-s}}{2-e^{-s}}\right),$$

the function $X(\gamma)$ versus γ is shown in Figure 1 for $L(s) = +1$ and $L(s) = -1$. The largest γ that satisfies $X(\gamma) = 0$ is $\gamma_{\text{opt}} = 2.82$ for $L(s) = -1$; and this gives the optimal controller

$$C_{\text{opt}}(s) = \left(\frac{1}{2.82s}\right)\frac{-(1+7.95s^2)}{(s^2 + 3.51s + 1) + 2.82sM_n(s)} N_o^{-1}(s),$$

where N_o is the outer part of the stable plant $P = M_n N_o$.

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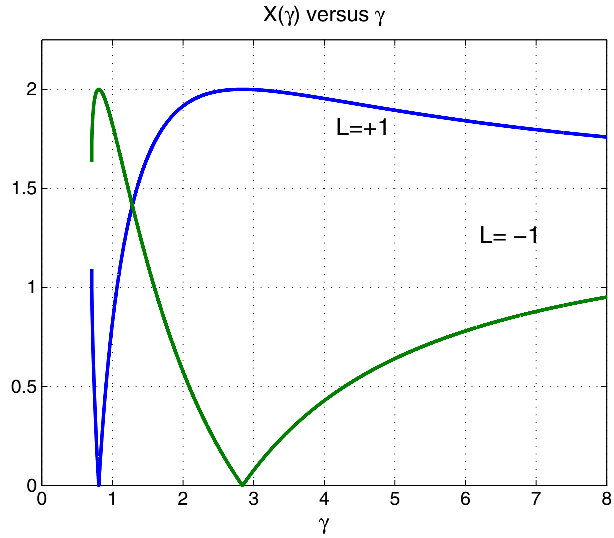

Figure 1. $X(\gamma)$ versus h.

5. CONCLUSIONS

In this paper, we have revisited the H_∞ optimal controller formula derived in [18] for the mixed sensitivity minimization problem involving infinite dimensional plants with finitely many poles in $\mathbb{C}_+$. We have seen that the $2(n_1+\ell)$ equations, (10) and (11), of [18] can be reduced to a set of n_1 equations, (26). Solution of these equations involve a search of finding the largest value of γ for which the matrix $\mathscr{P}_\gamma$, defined in Equation (27), becomes singular.

In the particular case where W_1 is first order (i.e., $n_1 = 1$), we have a scalar equation, (30), whose largest zero as a function of γ gives the optimal performance level γ_{opt} and defines the optimal controller C_{opt}. Moreover, with specific first order weights $W_1(s) = 1/s$ and $W_2(s) = ks$, we have illustrated the structure of an integral action H_∞ controller, Equation (36).

Finally, Equations (32) and (33) can be considered as an extension of the Zhou–Khargonekar formula (computation of γ_{opt} in the sensitivity minimization problem from a Hamiltonian matrix for stable plants), [3], to the mixed sensitivity problem for unstable plants, such an extension for stable plants was carried out earlier in [32].

REFERENCES

1. Flamm DS, Mitter SK. H^∞ sensitivity minimization for delay systems. *Systems and Control Letters* 1987; **9**:17–24.
2. Foias C, Tannenbaum A, Zames G. Weighted sensitivity minimization for delay systems. *IEEE Transactions on Automatic Control* 1986; **31**:763–766.
3. Zhou K, Khargonekar PP. On the weighted sensitivity minimization problem for delay systems. *Systems and Control Letters* 1987; **8**:307–312.
4. Foias C, Tannenbaum A, Zames G. Some explicit formulae for the singular values of a certain Hankel operators with factorizable symbol. *SIAM Journal on Mathematical Analysis* 1988; **19**:1081–1091.
5. Kashima K, Yamamoto Y. A new characterization of invariant subspaces of H^2 and applications to the optimal sensitivity problem. *Systems and Control Letters* 2005; **54**:539–545.
6. Lypchuk T, Smith MC, Tannenbaum A. Weighted sensitivity minimization: general plants in H_∞ and rational weights. *Linear Algebra and its Applications* 1988; **109**:71–90.
7. Özbay H. A simpler formula for the singular values of a certain Hankel operator. *Systems and Control Letters* 1990; **15**:381–390.
8. Smith MC. Singular values and vectors of a class of Hankel operators. *Systems and Control Letters* 1989; **12**:301–308.
9. Yamamoto Y, Hirata K, Tannenbaum A. Some remarks on Hamiltonians and the infinite-dimensional one block H^∞ problem. *Systems and Control Letters* 1996; **29**:111–117.
10. Dym H, Georgiou TT, Smith MC. Explicit formulas for optimally robust controllers for delay systems. *IEEE Transactions on Automatic Control* 1995; **40**:656–669.

DOI: 10.1002/nla

11. Georgiou TT, Smith MC. Robust stabilization in the gap metric: controller design for distributed plants. *IEEE Transactions on Automatic Control* 1992; **37**:1133–1143.
12. Flamm DS, Yang H. Optimal mixed sensitivity for SISO distributed plants. *IEEE Transactions on Automatic Control* 1994; **39**:1150–1165.
13. Özbay H. H^∞ optimal controller design for a class of distributed parameter systems. *International Journal of Control* 1993; **58**:739–782.
14. Özbay H, Smith MC, Tannenbaum A. Mixed sensitivity optimization for a class of unstable infinite dimensional systems. *Linear Algebra and its Applications* 1993; **178**:43–83.
15. Bercovici H, Foias C, Tannenbaum A. On skew Toeplitz operators. *Operator Theory: Advances and Applications* 1988; **32**:21–43.
16. Zames G, Mitter SK. A note on essential spectrum and norms of mixed Hankel – Toeplitz operators. *Systems and Control Letters* 1988; **10**:159–165.
17. Özbay H, Tannenbaum A. A skew Toeplitz approach to the H^∞ optimal control of multivariable distributed systems. *SIAM Journal on Control and Optimization* 1990; **28**:653–670.
18. Toker O, Özbay H. H_∞ optimal and suboptimal controllers for infinite dimensional SISO plants. *IEEE Transactions on Automatic Control* 1995; **40**:751–755.
19. Adamjan VM, Arov DZ, Krein MG. Analytic properties of Schmidt pairs for a Hankel operator and the generalized Shur–Takagi problem. *Mathematics of the USSR–Sbornik* 1971; **15**:31–73.
20. Foias C, Özbay H, Tannenbaum A. *Robust Control of Infinite Dimensional Systems: Frequency Domain Methods*, LNCIS No. 209. Springer-Verlag: London, 1996.
21. Gu C, Toker O, Özbay H. On the two-block H_∞ problem for a class of unstable distributed systems. *Linear Algebra and its Applications* 1996; **234**:227–244.
22. Gumussoy S, Özbay H. On the mixed sensitivity minimization for systems with infinitely many unstable modes. *Systems and Control Letters* 2004; **53**:211–216.
23. Kashima K, Yamamoto Y. On standard H_∞ control problems for systems with infinitely many unstable poles. *Systems and Control Letters* 2008; **57**:309–314.
24. Gumussoy S, Özbay H. Remarks on H^∞ controller design for SISO plants with time delays. *Preprints of the 5th IFAC Symposium on Robust Control Design (ROCOND'06)*, Toulouse, France, July 2006.
25. Curtain RF, Zhou Y. A weighted mixed-sensitivity H_∞-control design for irrational transfer matrices. *IEEE Transactions on Automatic Control* 1996; **41**:1312–1321.
26. Morris KA. H_∞-output feedback of infinite-dimensional systems via approximation. *Systems and Control Letters* 2001; **44**:211–217.
27. Rodriguez A, Dahleh MA. Weighted H_∞ optimization for stable infinite dimensional systems using finite dimensional techniques. *Proceedings of the 29th IEEE Conference on Decision and Control*, Honolulu HI, December 1990; 1814–1819.
28. Xiao M, Başar T. Finite-dimensional compensators for the H_∞-optimal control of infinite-dimensional systems via a Galerkin-type approximation. *SIAM Journal on Control and Optimization* 1999; **37**:1614–1647.
29. Curtain RF, Zwart H. *An Introduction to Infinite-Dimensional Linear Systems Theory*. Springer-Verlag: New York, 1995.
30. Meinsma G, Mirkin L. H_∞ control of systems with multiple I/O delays via decomposition to adobe problems. *IEEE Transactions on Automatic Control* 2005; **50**:199–211.
31. Hirata K, Yamamoto Y, Tannenbaum AR. A Hamiltonian-based solution to the two-block H_∞ problem for general plants in H_∞ and rational weights. *Systems and Control Letters* 2000; **40**:83–95.
32. Kashima K, Özbay H, Yamamoto Y. A Hamiltonian-based solution to the mixed sensitivity optimization problem for stable pseudorational plants. *Systems and Control Letters* 2005; **54**:1063–1068.
33. Doyle JC, Francis BA, Tannenbaum AR. *Feedback Control Theory*. Macmillan Publishing Co.: New York, 1992.
34. Morris KA. *Introduction to Feedback Control*. Harcourt - Academic Press: San Diego, 2001.
35. Özbay H. *Introduction to Feedback Control Theory*. CRC Press LLC: Boca Raton FL, 2000.
36. Zhou K, Doyle JC, Glover K. *Robust and Optimal Control*. Prentice-Hall: Upper Saddle River, New Jersey, 1996.

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