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On fibred biset functors with fibres of order prime and four

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ABSTRACT

This note has two purposes: First, to present a counterexample to a conjecture parametrizing the simple modules over Green biset functors, appearing in an author's previous article. This parametrization fails for the monomial Burnside ring over a cyclic group of order four. Second, to classify the simple modules for the monomial Burnside ring over a group of prime order, for which the above-mentioned parametrization holds.

Introduction

This note presents a counterexample to a conjecture appearing in [5], parametrizing the simple modules over a Green biset functor. The conjecture generalized the classification of simple biset functors, as well as the classification of simple modules over Green functors appearing in Bouc [2]. It relied on the assumption that for a simple module over a Green biset functor its minimal groups should be isomorphic, which we will see is not generally true.

For a better understanding of this note, the reader is invited to take a look at [5], where he can acquaint himself with the context of modules over Green biset functors.

Given a Green biset functor A, defined in a class of groups $\mathcal{Z}$ closed under subquotients and direct products, and over a commutative ring with identity R, one can define the category $\mathcal{P}_A$. The objects of $\mathcal{P}_A$ are the groups in $\mathcal{Z}$, and given two groups G and H in $\mathcal{Z}$, the set $Hom_{\mathcal{P}_A}(G,H)$ is $A(H\times G)$. Composition in $\mathcal{P}_A$ is given through the product $\times$ of the definition of a Green biset functor, that is, given α in $A(G\times H)$ and β in $A(H\times K)$, the product $\alpha\circ\beta$ is defined as

$$A\big(\mathrm{Def}^{G\times\Delta(H)\times K}_{G\times K}\circ\mathrm{Res}^{G\times H\times H\times K}_{G\times\Delta(H)\times K}\big)(\alpha\times\beta).$$

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The identity element in $A(G\times G)$ is $A(\mathrm{Ind}_{\Delta(G)}^{G\times G}\circ\mathrm{Inf}_1^{\Delta(G)})(\varepsilon_A)$, where $\varepsilon_A\in A(1)$ is the identity element of the definition of a Green biset functor. Even if this product may seem a bit strange, in many cases the category $\mathcal{P}_A$ is already known and has been studied. For example, if A is the Burnside ring functor, $\mathcal{P}_A$ is the biset category defined in $\mathcal{Z}$. It is proved in [5] that for any Green biset functor A, the category of A-modules is equivalent to the category of R-linear functors from $\mathcal{P}_A$ to R-Mod, and it is through this equivalence that they are studied.

In Section 2 of [5], we defined $I_A(G)$ for a group G in $\mathcal{Z}$ as the submodule of $A(G\times G)$ generated by elements which can be factored through $\circ$ by groups in $\mathcal{Z}$ of order smaller than $|G|$. We denote by $\hat{A}(G)$ the quotient $A(G\times G)/I_A(G)$. Conjecture 2.16 in [5] stated that the isomorphism classes of simple A-modules were in one-to-one correspondence with the equivalence classes of couples (H,V) where H is a group in $\mathcal{Z}$ such that $\hat{A}(H)\neq 0$ and V is a simple $\hat{A}(H)$-module. Two couples (H,V) and (G,W) are related if H and G are isomorphic and V and W are isomorphic as $\hat{A}(H)$-modules (the $\hat{A}(H)$-action on W is defined in Section 4 of [5]). The correspondence assigned to the class of a simple A-module S, the class of the couple (H,V) where H is a minimal group for S and $V=S(H)$. We will see in Section 2 that for the monomial Burnside ring over a cyclic group of order four and with coefficients in a field, we can find a simple module which has two non-isomorphic minimal groups.

For a finite abelian group C and a finite group G, the monomial Burnside ring of G with coefficients in C is a particular case of the ring of monomial representations introduced by Dress [4]. Fibred biset functors were defined by Boltje and Coşkun as functors from the category in which the morphisms from a group G to a group H is the monomial Burnside ring of $H\times G$, they called these morphisms fibred bisets. This category is precisely $\mathcal{P}_A$ when A is the monomial Burnside ring functor, and so fibred biset functors coincide with A-modules for this functor. Boltje and Coşkun also considered the case in which C may be an infinite abelian group, but we shall not consider this case. Unfortunately, there is no published material on the subject, I thank Laurence Barker and Olcay Coşkun for sharing this with me.

Another important element in this note will be the Yoneda–Dress construction of the Burnside ring functor B at C, denoted by B_C. It assigns to a finite group G the Burnside ring $B(G\times C)$, and it is a Green biset functor. Since the monomial Burnside ring of G with coefficients in C is a subgroup of $B_C(G)$, we will denote it by $B_C^1(G)$. We will see that there are various similarities between B_C and B_C^1.

1. Definitions

All groups in this note will be finite.

R will denote a commutative ring with identity.

Given a group G, we will denote its center by $Z(G)$. The Burnside ring of G will be denoted by $B(G)$, and $RB(G)$ if it has coefficients in R.

Definition 1. Let C be an abelian group and G be any group. A finite C-free $(G\times C)$-set is called a C-fibred G-set.

A C-orbit of a C-fibred G-set is called a fibre.

The monomial Burnside ring for G with coefficients in C, denoted by $B_C^1(G)$, is the abelian subgroup of $B(G\times C)$ generated by the C-fibred G-sets. We write $RB_C^1(G)$ if we are taking coefficients in R.

If X is a C-fibred G-set, denote by $[X]$ its set of fibres. Then G acts on $[X]$ and X is $(G\times C)$-transitive if and only if $[X]$ is G-transitive. In this case, $[X]$ is isomorphic as G-set to G/D for some $D\leqslant G$ and we can define a group homomorphism $\delta: D\to C$ such that if D is the stabilizer of the orbit Cx, then $ax=\delta(a)x$ for all $a\in D$. The subgroup D and the morphism δ determine X, since $Stab_{G\times C}(x)$ is equal to $\{(a,\delta(a)^{-1})\mid a\in D\}$.

Notation 2. Given $D \leqslant G$ and $\delta : D \to C$ a group homomorphism, we will write D_δ for $\{(a, \delta(a)^{-1}) \mid a \in D\}$ and $C_\delta G/D$ for the C-fibred G-set $(G \times C)/D_\delta$. We will write CG/D if δ is the trivial morphism. The morphism δ is called a C-subcharacter of G.

The C-subcharacters of G admit an action of G by conjugation ${}^g(D,\delta) = ({}^gD, {}^g\delta)$ and with this action we have:

Remark 3. (See 2.2 in Barker [1].) As an abelian group

$$B_C^1(G) = \bigoplus_{(D,\delta)} \mathbb{Z}[C_\delta G/D]$$

where (D,δ) runs over a set of representatives of the G-classes of C-subcharacters of G.

The following notations are explained in more detail in Bouc [3]. Given U an (H,G)-biset and V a (K,H)-biset, the composition of V and U is denoted by $V \times_H U$. With this composition we know that if H and G are groups and $L \leqslant H \times G$, then the corresponding element in $RB(H \times G)$ satisfies the Bouc decomposition (2.3.26 in [3]):

$$\mathrm{Ind}_D^H \times_D \mathrm{Inf}_{D/C}^D \times_{D/C} \mathrm{Iso}(f) \times_{B/A} \mathrm{Def}_{B/A}^B \times_B \mathrm{Res}_B^G$$

with $C \trianglelefteq D \leqslant H$, $A \trianglelefteq B \leqslant G$ and $f : B/A \to D/C$ an isomorphism.

Notation 4. As it is done in [5], we will write B_C for the Yoneda–Dress construction of the Burnside ring functor B at C.

The functor B_C is defined as follows. In objects, it sends a group G to $B(G \times C)$. In arrows, for a (G,H)-biset X, the map $B_C(X) : B_C(H) \to B_C(G)$ is the linear extension of the correspondence $T \mapsto X \times_H T$, where T is an $(H \times C)$-set and $X \times_H T$ has the natural action of $(G \times C)$-set coming from the action of C on T.

We will denote by T_{C-f} the subset of elements of T in which C acts freely. Clearly, it is an H-set.

Lemma 5. *Assigning to each group G the $\mathbb{Z}$-module $B_C^1(G)$ defines a Green biset functor.*

Proof. We first prove it is a biset functor.

Let G and H be groups and X be a finite (G,H)-biset. Let T be a C-fibred H-set. We define $B_C^1(X)(T) = (B_C(X)(T))_{C-f}$.

To prove that composition is associative, let Z be a (K,G)-biset. We must show

$$\big((Z \times_G X) \times_H T\big)_{C-f} \cong \big(Z \times_G (X \times_H T)_{C-f}\big)_{C-f}.$$

We claim that the right-hand side of this isomorphism is equal to $(Z \times_G (X \times_H T))_{C-f}$. To prove it, we prove that in general, if W is a $(G \times C)$-set, then $(Z \times_G W_{C-f})_{C-f}$ is equal to $(Z \times_G W)_{C-f}$. Let $[z,w]$ be an element in $(Z \times_G W)_{C-f}$. The element $[z,w]$ is an orbit for which any representative has the form (zg^{-1}, gw) with $g \in G$. To prove that gw is in W_{C-f}, suppose $cgw = gw$. Then, $[z,w] = [z,cw]$ and this is equal to $c[z,w]$, so $c = 1$. The other inclusion is obvious.

It remains then to prove

$$\big((Z \times_G X) \times_H T\big)_{C-f} \cong \big(Z \times_G (X \times_H T)\big)_{C-f},$$

as $(K \times C)$-sets, which holds because B_C is a biset functor.

Next we prove it is a Green biset functor.

Following Dress [4], we define the product

$$B^1_C(G) \times B^1_C(H) \to B^1_C(G \times H)$$

on the C-fibred G-set T and the C-fibred H-set Y as the set of C-orbits of $T \times Y$ with respect to the action $c(t, y) = (ct, c^{-1}y)$. The orbit of (t, y) is denoted by $t \otimes y$. We extend this product by linearity and denote it by $T \otimes Y$. The action of C in $t \otimes y$ is given by $ct \otimes y$ and so it is easy to see that C acts freely on $T \otimes Y$. The identity element in $B^1_C(1)$ is the class of C. It is not hard to see that this product is associative and respects the identity element. To prove it is functorial, take X a (K, H)-biset and Z an (L, G)-biset. We must show that

$$(Z \times_G T)_{C-f} \otimes (X \times_H Y)_{C-f} \cong \big((Z \times X) \times_{G\times H} (T \otimes Y)\big)_{C-f}$$

as $(K \times L \times C)$-sets. We can prove this in two steps: First, it is easy to observe that for any C-sets N and M, the product $M_{C-f} \otimes N_{C-f}$ is isomorphic as C-set to $(M \otimes N)_{C-f}$. Then it remains to prove

$$(Z \times_G T) \otimes (X \times_H Y) \cong (Z \times X) \times_{H\times G} (T \otimes Y)$$

as $(K \times L \times C)$-sets. If $[z, t] \otimes [x, y]$ is an element on the left-hand side, then sending it to $[(z, x), t \otimes y]$ defines the desired isomorphism of $(K \times L \times C)$-sets. □

2. Fibred biset functors

The category $\mathcal{P}_{RB^1_C}$, mentioned in the introduction and defined in Section 4 of [5], has for objects the class of all finite groups; the set of morphisms from G to H is the abelian group $RB^1_C(H \times G)$ and composition is given in the following way: If $T \in RB^1_C(G \times H)$ and $Y \in RB^1_C(H \times K)$, then $T \circ Y$ is given by restricting $T \otimes Y$ to $G \times \Delta(H) \times K$ and then deflating the result to $G \times K$. The identity element in $RB^1_C(G \times G)$ is the class of $C(G \times G)/\Delta(G)$. As it is done in Section 4.2 of [5], composition $\circ$ can be obtained by first taking the orbits of $T \times Y$ under the $(H \times C)$-action given by

$$(h, c)(t, y) = \big((h, c)t, \big(h, c^{-1}\big)y\big),$$

and then choosing the orbits in which C acts freely.

Definition 6. From Proposition 2.11 in [5], the category of RB^1_C-modules is equivalent to the category of R-linear functors from $\mathcal{P}_{RB^1_C}$ to R-Mod. These functors are called fibred biset functors.

Notation 7. Let E be a subgroup of $H \times K \times C$. We will write $p_1(E)$, $p_2(E)$ and $p_3(E)$ for the projections of E in H, K and C respectively; $p_{1,2}(E)$ will denote the projection over $H \times K$, and in the same way we define the other possible combinations of indices. We write $k_1(E)$ for $\{h \in p_1(E) \mid (h, 1, 1) \in E\}$. Similarly, we define $k_2(E)$, $k_3(E)$ and $k_{i,j}(E)$ for all possible combinations of i and j.

The following formula was already known to Boltje and Coşkun. Here we prove it as an explicit expression of composition $\circ$ in the category $\mathcal{P}_{RB^1_C}$. The proof follows the lines of Lemma 4.5 in [5].

The definition of the product $*$ can be found in Notation 2.3.19 of [3].

Lemma 8. *Let $X = [C_\nu(G \times H)/V] \in RB^1_C(G \times H)$ and $Y = [C_\mu(H \times K)/U] \in RB^1_C(H \times K)$ be two transitive elements. Then the composition $X \circ Y \in RB^1_C(G \times K)$ in the category $\mathcal{P}_{RB^1_C}$ is isomorphic to*

$$\bigsqcup_{h\in S} C_{\nu\mu^h}(G \times K)/\big(V * {}^{(h,1)}U\big).$$

The notation is as follows: Let $[p_2(V)\setminus H/p_1(U)]$ *be a set of representatives of the double cosets of* $p_2(V)$ *and* $p_1(U)$ *in* H*, then* S *is the subset of elements* h *in* $[p_2(V)\setminus H/p_1(U)]$ *such that* $\nu(1,h')\mu(h'^h,1)=1$ *for all* h' *in* $k_2(V)\cap {}^hk_1(U)$*; by* $\nu\mu^h$ *we mean the morphism from* $V*{}^{(h,1)}U$ *to* C *defined by* $\nu\mu^h(g,k)=\nu(g,h_1)\mu(h_1^h,k)$ *when* h_1 *is an element in* H *such that* (g,h_1) *in* V *and* (h_1,k) *in* ${}^{(h,1)}U$.

Proof. Notice that $\nu\mu^h$ is a function if and only if $\nu(1,h')\mu(h'^h,1)=1$ for all $h'\in k_2(V)\cap {}^hk_1(U)$.

Let W be the $(G\times K\times C)$-set obtained by taking the orbits of $X\times Y$ under the action of $H\times C$

$$(h,c)(x,y)=\big((h,c)x,\big(h,c^{-1}\big)y\big),$$

for all $c\in C$, $h\in H$, $x\in X$, $y\in Y$.

Now let $[(g,h,c)V_\nu,(h',k,c')U_\mu]$ be an element in W. Then its orbit under the action of $G\times K\times C$ is equal to the orbit of $[(1,1,1)V_\nu,(h^{-1}h',1,1)U_\nu]$. From this it is not hard to see that the orbits of W are indexed by $[p_2(V)\setminus H/p_1(U)]$. To find the orbits in which C acts freely, suppose $c\in C$ fixes $[(1,1,1)V_\nu,(h,1,1)U_\mu]$. This means there exists $(h',c')\in H\times C$ such that

$$(1,1,c)V_\nu=\big(h',1,c'\big)V_\nu \quad \text{and} \quad (h,1,1)U_\mu=\big(h'h,1,c'^{-1}\big)U_\mu.$$

Hence $\nu(h',1)=c'^{-1}c$ and $\mu(h^{-1}h'h,1)=c'$. So that, c is equal to $\mu(h^{-1}h'h,1)\nu(h',1)$, which gives us the condition on the set S.

The fact that the stabilizer on $G\times K\times C$ of $[(1,1,1)V_\nu,(h,1,1)U_\mu]$ is the subgroup $(V*{}^{(h,1)}U)_{\nu\mu^h}$ follows as in the previous paragraph. $\square$

The following lemma and corollary state for RB_C^1 analogous results proved for RB_C in [5].

Lemma 9. *Let* $X=C_\delta(G\times H)/D$ *be a transitive element in* $RB_C^1(G\times H)$*. Denote by* e *the natural transformation from* RB *to* RB_C^1 *defined in a* G*-set* X *by* $e_G(X)=X\times C$*. Consider* $E=p_1(D)$*,* $E'=E/k_1(D_\delta)$*,* $F=p_2(D)$*,* $F'=F/k_2(D_\delta)$*. Then* X *can be decomposed in* $\mathcal{P}_{RB_C^1}$ *as*

$$e_{G\times E'}\big(\mathrm{Ind}_E^G\times_E \mathrm{Inf}_{E'}^E\big)\circ\beta_1 \quad \textit{and as} \quad \beta_2\circ e_{F'\times H}\big(\mathrm{Def}_{F'}^F\times_F \mathrm{Res}_F^H\big)$$

for some $\beta_1\in RB_C^1(E'\times H)$*,* $\beta_2\in RB_C^1(G\times F')$*.*

Proof. We will only prove the existence of the first decomposition, since the proof of the second one follows by analogy.

Observe that $e_{G\times E'}(\mathrm{Ind}_E^G\times_E\mathrm{Inf}_{E'}^E)$ is the C-fibred $(G\times E')$-set $C(G\times E')/U$ where $U=\{(g,gk_1(V_\delta))\mid g\in E\}$.

Consider the isomorphism σ from $p_1(D)/k_1(D)$ to $p_2(D)/k_2(D)$, existing by Goursat's Lemma 2.3.25 in [3]. Define β_1 as $C_\omega(E'\times H)/W$ where

$$W=\big\{\big(gk_1(D_\delta),h\big)\;\big|\;\text{if}\;\sigma\big(gk_1(D)\big)=hk_2(D)\big\}$$

and $\omega: W\to C$ by $\omega(gk_1(D_\delta),h)=\delta(g,h)$. That W is a group follows from $k_1(D_\delta)\leqslant k_1(D)$. The extension of δ to W is well defined, since it is not hard to see that $k_1(D_\delta)$ is equal to $k_1(Ker(\delta))$. Also, since $p_2(U)=p_1(W)=E'$ and $k_2(U)=1$, by the previous lemma, $e_{G\times E'}(\mathrm{Ind}_E^G\times_E\mathrm{Inf}_{E'}^E)\circ\beta_1$ is isomorphic to $C_\delta(G\times H)/(U*W)$. Finally, $U*W=\{(g,h)\mid\sigma(gk_1(D))=hk_2(D)\}$, and by Goursat's Lemma, this is equal to D. $\square$

This decomposition leads us to the same conclusions we obtained from Lemma 4.8 of [5] for RB_C. That is, if G and H have the same order n and $C_\delta(G\times H)/D$ does not factor through $\circ$ by a group of order smaller than n, then we must have $p_1(D) = G$, $p_2(D) = H$, $k_1(D_\delta) = 1$ and $k_2(D_\delta) = 1$. In particular, Corollary 4.9 of the same reference is also valid, so we have:

Corollary 10. *Let C be a group of prime order and S be a simple RB_C^1-module. If H and K are two minimal groups for S, then they are isomorphic.*

We will be back to the classification of simple RB_C^1-modules for C of prime order in the last section of the article. Now, we will find the counterexample mentioned in the introduction.

2.1. The counterexample

In Section 2 of [5], given a Green biset functor A defined in a class of groups $\mathcal{Z}$, we defined $I_A(G)$ as the submodule of $A(G \times G)$ generated by elements of the form $a \circ b$, where a is in $A(G \times K)$, b is in $A(K \times G)$ and K is a group in $\mathcal{Z}$ of order smaller than $|G|$. We denote by $\hat{A}(G)$ the quotient $A(G \times G)/I_A(G)$. From Section 4 of [5], we also know that if V is a simple $\hat{A}(G)$-module, we can construct a simple A-module that has G as a minimal group. This A-module is defined as the quotient $L_{G,V}/J_{G,V}$, where $L_{G,V}$ is defined as $A(D \times G) \otimes_{A(G\times G)} V$ for $D \in \mathcal{Z}$ and $L_{G,V}(a)(x \otimes v) = (a \circ x) \otimes v$ for $a \in A(D' \times D)$. The subfunctor $J_{G,V}$ is defined as

$$J_{G,V}(G) = \left\{ \sum_{i=1}^{n} x_i \otimes n_i \;\middle|\; \sum_{i=1}^{n} (y \circ x_i) \cdot n_i = 0\ \forall y \in A(G \times D) \right\}.$$

To construct the counterexample we will take coefficients in a field k. We will find a group C and a simple kB_C^1-module S which has two non-isomorphic minimal groups.

Lemma 11. *Let C be a cyclic group and G and H be groups. Suppose that $D \leqslant G \times H$ is such that $p_1(D) = G$ and $p_2(D) = H$. Let $\delta : D \to C$ be a morphism of groups. We will write $D^o = \{(h, g) \mid (g, h) \in D\}$ and define $\delta^o : D^o \to C$ as $\delta^o(h, g) = \delta(g, h)^{-1}$. If $X = C_\delta(G \times H)/D$ and $X^o = C_{\delta^o}(H \times G)/D^o$, then $X \circ X^o$ is an idempotent in $B_C^1(G \times G)$.*

Proof. Since $\delta(1, h)\delta^o(h, 1) = 1$ for all $h \in k_2(D)$, by Lemma 8 the composition $X \circ X^o$ is equal to $W = C_{\delta'}(G \times G)/D'$. Here, $D' = D * D^o$ and if $(g_1, g_2) \in D'$ with $h \in H$ being such that $(g_1, h) \in D$ and $(h, g_2) \in D^o$, then $\delta'(g_1, g_2) = \delta(g_1, h)\delta^o(h, g_2)$. From this it is not hard to see that $D' = \{(g_1, g_2) \mid g_1 g_2^{-1} \in k_1(D)\}$ and $\delta'(g_1, g_2) = \delta(g_1 g_2^{-1}, 1)$.

Observe that $k_1(D') = k_2(D') = k_1(D)$ and clearly, $\delta'(1, g)\delta'(g, 1) = 1$ for all $g \in k_1(D)$. In the same way, if $g_1, g_2 \in G$ are such that there exists $g \in G$ with $(g_1, g) \in D'$ and $(g, g_2) \in D'$ then $\delta'(g_1, g)\delta'(g, g_2) = \delta(g_1 g_2^{-1}, 1)$. Finally, $p_1(D') = G$ since $gg^{-1} \in k_1(D)$ for all $g \in G$, and it is easy to see that $D' * D' = D'$. So, Lemma 8 gives us $W \circ W = W$. □

If now we find two non-isomorphic groups G and H having the same order, and a transitive element $X = C_\delta(G \times H)/D$ in $kB_C^1(G \times H)$ with $p_1(D) = G$, $p_2(D) = H$ and such that the class of $W = X \circ X^o$ is different from zero in $k\hat{B}_C^1(G)$, then we can construct a simple kB_C^1-module S which has G and H as minimal groups. By the previous lemma, W will be an idempotent in $k\hat{B}_C^1(G)$, so we can find V a simple $k\hat{B}_C^1(G)$-module such that there exists $v \in V$ with $(X \circ X^o)v \neq 0$. From the definition of $S = S_{G,V}$, this implies $S_{G,V}(H) \neq 0$.

Example 12. Let $C = \langle c \rangle$ be a group of order 4, G the quaternion group

$$\langle x, y \mid x^4 = 1,\ yxy^{-1} = x^{-1},\ x^2 = y^2 \rangle$$

and H the dihedral group of order 8

$$\langle a, b \mid a^4 = b^2 = 1,\ bab^{-1} = a^{-1} \rangle.$$

Consider the subgroup of $G \times H$ generated by (x, a) and (y, b), call it D. The subgroup of D generated by (x^{-1}, a) is a normal subgroup of order 4, and the quotient D/D_1 is isomorphic to C in such a way that we can define a morphism $\delta : D \to C$ sending (x, a) to c^2 and (y, b) to c^{-1}. It is easy to observe that $p_1(D) = G$, $p_2(D) = H$, $k_1(D) = \langle x^2 \rangle$ and $k_2(D) = \langle a^2 \rangle$. By the previous lemma, we have that if $X = C_\delta(G \times H)/D$, then $W = X \circ X^o$ is an idempotent in $kB_C^1(G \times G)$. We will see now that the class of W in $k\hat{B}_C^1(G)$ is different from 0.

Let $D' = D * D^o$ and $\delta' : D' \to C$ be the morphism obtained from δ as in the previous lemma. Suppose that W is in $I_{kB_C^1}(G)$. Since W is a transitive $(G \times G \times C)$-set, this implies that there exists K a group of order smaller than 8, $U \leqslant G \times K$ and $V \leqslant K \times G$ such that $D' = U * V$ (the conjugate of a group of the form $U * V$ has again this form, so we can suppose $D' = U * V$), and group homomorphisms $\mu : U \to C$ and $\nu : V \to C$ such that $\delta' = \mu\nu$ in the sense of Lemma 8.

Now, using point 2 of Lemma 2.3.22 in [3] and the fact that $p_1(D') = p_1(D)$ and $k_1(D') = k_1(D)$, we have that $p_1(U) = G$ and that $k_1(U)$ can only have order one or two. Since $p_1(U)/k_1(U)$ is isomorphic to $p_2(U)/k_2(U)$ and the latter must have order smaller than 8, we obtain that $k_1(U)$ has order two. This in turn implies that $p_2(U)/k_2(U)$ has order 4, and since $|p_2(U)| < 8$, we have $k_2(U) = 1$. Hence, U is isomorphic to G. Also, since $k_1(U) = k_1(D')$, we have $\mu(x^2, 1) = \delta(x^2, 1)$. Now, $\delta(x^2, 1) \neq 1$, but all morphisms from G to C send x^2 to 1, a contradiction.

2.2. *Simple fibred biset functors with fibre of prime order*

From now on C will be a group of prime order p.

From Corollary 10, we have that Conjecture 2.16 of [5] holds for the functor RB_C^1, the proof is a particular case of Proposition 4.2 in [5]. We will state this result after describing the structure of the algebra $R\hat{B}_C^1(G)$ for a group G.

We will see that if $C_\delta(G \times G)/D$ is a transitive C-fibred $(G \times G)$-set the class of which is different from 0 in $R\hat{B}_C^1(G)$, then D can only be of the form $\{(\sigma(g), g) \mid g \in G\}$ for σ an automorphism of G, or of the form $\{(\omega(g)\zeta(c), g) \mid (g, c) \in G \times C\}$ for ω an automorphism of G and $\zeta : C \to Z(G) \cap \Phi(G)$ an injective morphism of groups where $\Phi(G)$ is the Frattini subgroup of G. In the first case δ will be any morphism from G to C. In the second case δ will assign c^{-1} to the couple $(\omega(g)\zeta(c), g)$, this is well defined since ζ is injective. Of course, the second case can only occur if p divides $|Z(G)|$.

If p does not divide $|Z(G)|$, we will prove that $R\hat{B}_C^1(G)$ is isomorphic to the group algebra $R\hat{G}$ where $\hat{G} = Hom(G, C) \rtimes Out(G)$. If p divides $|Z(G)|$, we will consider Y_G the set of injective morphisms $\zeta : C \to Z(G) \cap \Phi(G)$ and then define $\mathcal{Y}_G = Out(G) \times Y_G$. The R-module $R\mathcal{Y}_G$ forms an R-algebra with the product

$$(\omega, \zeta) \circ (\alpha, \chi) = \begin{cases} (\omega\alpha, \omega\chi) & \text{if } \zeta = \omega\chi, \\ 0 & \text{otherwise} \end{cases}$$

for elements (ω, ζ) and (α, χ) in $\mathcal{Y}_G$. The algebra $R\mathcal{Y}_G$ can also be made into an $(R\hat{G}, R\hat{G})$-bimodule. We could give the definitions of the actions now, and prove directly that $R\mathcal{Y}_G$ is indeed an $(R\hat{G}, R\hat{G})$-bimodule. Nonetheless, the nature of these actions is given by the structure of $R\hat{B}_C^1(G)$, so they are best understood in the proof of the following lemma. The R-module $R\mathcal{Y}_G \oplus R\hat{G}$ forms then an R-algebra.

Now suppose that G and H are two groups such that there exists an isomorphism $\varphi : G \to H$. If (t, σ) is a generator of $R\hat{G}$, then identifying $\varphi\sigma\varphi^{-1}$ with its class in $Out(H)$ we have that $(t\varphi^{-1}, \varphi\sigma\varphi^{-1})$ is in $R\hat{H}$. On the other hand, if (ω, ζ) is a generator in $R\mathcal{Y}_G$, then $(\varphi\omega\varphi^{-1}, \varphi|_{Z(G)}\zeta)$ is also in $R\mathcal{Y}_H$.

Notation 13. Let $\mathcal{H}(G)$ be the group algebra $R\hat{G}$ if p does not divide $|Z(G)|$ and $R\mathcal{Y}_G \oplus R\hat{G}$ in the other case.

We will write $\mathcal{S}eed$ for the set of equivalence classes of couples (G, V) where G is a group and V is a simple $\mathcal{H}(G)$-module. Two couples (G, V) and (H, W) are related if G and H are isomorphic, through an isomorphism $\varphi : G \to H$, and V is isomorphic to ${}^{\varphi}W$ as $\mathcal{H}(G)$-modules. Here ${}^{\varphi}W$ denotes the $\mathcal{H}(G)$-module with action given through the elements defined in the previous paragraph.

With these observations, Proposition 4.2 in [5] can be written as follows.

Proposition 14. *Let $\mathcal{S}$ be the set of isomorphism classes of simple RB^1_C-modules. Then the elements of $\mathcal{S}$ are in one-to-one correspondence with the elements of $\mathcal{S}eed$ in the following way: Given S a simple RB^1_C-module we associate to its isomorphism class the equivalence class of (G, V) where G is a minimal group of S and $V = S(G)$. Given the class of a couple (G, V), we associate the isomorphism class of the functor $S_{G,V}$ defined in the previous section.*

It only remains to see that the algebra $R\hat{B}^1_C(G)$ is isomorphic to $\mathcal{H}(G)$.

Lemma 15.

i) *If p does not divide $|Z(G)|$, then $R\hat{B}^1_C(G)$ is isomorphic to the group algebra $R\hat{G}$.*
ii) *If p divides $|Z(G)|$, then $R\hat{B}^1_C(G)$ is isomorphic to $R\mathcal{Y}_G \oplus R\hat{G}$ as R-algebras.*

Proof. Let $C_\delta(G \times G)/D$ be a transitive C-fibred $(G \times G)$-set the class of which is different from 0 in $R\hat{B}^1_C(G)$. From Lemma 9 we have that D_δ must satisfy $p_1(D_\delta) = p_2(D_\delta) = G$ and $k_1(D_\delta) = k_2(D_\delta) = 1$. Also, since δ is a function, we have that $k_3(D_\delta) = 1$. Goursat's Lemma then implies that D_δ is isomorphic to $p_{2,3}(D_\delta)$, also isomorphic to $p_{1,3}(D_\delta)$. Since C has prime order, we have two choices for $p_{2,3}(D_\delta)$, either it is of the form $G \times C$ or of the form $\{(g, t(g)) \mid g \in G,\ t : G \to C\}$, for some group homomorphism t.

By Goursat's Lemma, if $p_{2,3}(D_\delta)$ is equal to $G \times C$, then

$$D_\delta = \bigl\{\bigl(\alpha(g, c), g, c\bigr) \bigm| (g, c) \in G \times C,\ \alpha : G \times C \twoheadrightarrow G\bigr\}$$

with α an epimorphism of groups. Since $k_2(D_\delta) = k_3(D_\delta) = 1$, we have that $\alpha(g, c) = \omega(g)\zeta(c)$ with ω an automorphism of G and ζ and injective morphism from C to $Z(G)$. In particular, if p does not divide the order of $Z(G)$, then this case cannot occur.

Suppose that $p_{2,3}(D_\delta) = \{(g, t(g)) \mid g \in G,\ t : G \to C\}$, for a group homomorphism t. Goursat's Lemma implies that there exists σ an automorphism of G such that $D_\delta = \{(\sigma(g), g, t(g)) \mid g \in G\}$. Hence $D = \Delta_\sigma(G)$ and $\delta(g_1, g_2) = t(g_2^{-1})$. We will then replace δ by t and write $X_{t,\sigma}$ for $C_\delta(G \times G)/D$ in this case. The isomorphism classes of these elements in $R\hat{B}^1_C(G)$ form an R-basis for it, since Lemma 2.3.22 in [3] and Goursat's Lemma imply that $\Delta_\sigma(G)$ cannot be written as $M * N$ for any $M \leqslant G \times K$ and $N \leqslant K \times G$ with K of order smaller than $|G|$. Let us see that we have a bijective correspondence between the basic elements $[X_{t,\sigma}]$ of $R\hat{B}^1_C(G)$ and $Hom(G, C) \rtimes Out(G)$. Any representative of the isomorphism class of $X_{t,\sigma}$ is of the form $X_{tc_2^{-1}, c_1\sigma c_2^{-1}}$ where c_1 denotes the conjugation by some $g_1 \in G$ and c_2^{-1} denotes the conjugation by some $g_2^{-1} \in G$. Since C is abelian, tc_2^{-1} is equal to t, and the class of σ in $Out(G)$ is the same as the class of $c_1\sigma c_2^{-1}$. On the other hand, if we take σc_g any representative of the class of an automorphism σ in $Out(G)$, then $X_{t,\sigma} \cong X_{t,\sigma c_g}$.

It remains to see that this bijection is a morphism of rings. Using Lemma 8 it is easy to see that

$$X_{t_1,\sigma_1} \circ X_{t_2,\sigma_2} = X_{(t_1 \circ \sigma_2)t_2, \sigma_1\sigma_2}$$

and the product in $\hat{G}$ is precisely $(t_1, \sigma_1)(t_2, \sigma_2) = ((t_1 \circ \sigma_2)t_2, \sigma_1\sigma_2)$.

This proves point i). From now on, we suppose that p divides $|Z(G)|$.

As we said before, if p divides $|Z(G)|$, then we can consider the case of C-fibred $(G \times G)$-sets $C_\delta(G \times G)/D$ such that $p_{2,3}(D_\delta) = G \times C$. In this case, D_δ equals

$$\{(\omega(g)\zeta(c), g, c) \mid (g, c) \in G \times C\}$$

where ω is an automorphism of G and ζ is an injective morphism from C to $Z(G)$. We will prove that the class of $C_\delta(G \times G)/D$ in $R\hat{B}^1_C(G)$ is different from 0 if and only if $Im\,\zeta \subseteq Z(G) \cap \Phi(G)$, and we will write $Y_{\omega,\zeta}$ for $C_\delta(G \times G)/D$ in this case. The claim will be proved in two steps, first let us prove that the class of $Y_{\omega,\zeta}$ in $R\hat{B}^1_C(G)$ is different from 0 if and only if $\mu|_{Z(G)} \circ \zeta = 1$ for every group homomorphism $\mu : G \to C$. Using Lemma 2.3.22 of [3] it is easy to see that $D = \{(\omega(g)\zeta(c), g) \mid (g, c) \in G \times C\}$ is equal to $M * N$ for some $M \leqslant G \times K$ and $N \leqslant K \times G$ with K a group of order smaller than $|G|$ if and only if K has order $|G|/p$ and M and N are isomorphic to G. Suppose now that there exist $\mu : G \to C$ and $\nu : G \to C$ such that $\delta(g_1, g_2) = \mu(g_1)\nu(g_2)$, then in particular for every $c \in C$, $\delta(\zeta(c), 1) = c^{-1} = \mu\zeta(c)$. Conversely, if there exists $\mu : G \to C$ such that $\mu|_{Z(G)} \circ \zeta \neq 1$, then we can find $\mu' : G \to C$ such that $\mu'\zeta(c) = c^{-1}$ for all $c \neq 1$, and define $\nu : G \to C$ as $\nu(g) = \mu'\omega(g^{-1})$. So we have $\mu'(\omega(g)\zeta(c))\nu(g) = c^{-1}$ which is equal to $\delta(\omega(g)\zeta(c), g)$.

Now we prove that for $\zeta : C \hookrightarrow Z(G)$, we have $Im\,\zeta \subseteq \Phi(G)$ if and only if $\mu|_{Z(G)} \circ \zeta = 1$ for every group homomorphism $\mu : G \to C$ (thanks to the referee for this observation). Suppose $Im\,\zeta \subseteq \Phi(G)$ and let $\mu : G \to C$ be a morphism of groups. If there exists $c \in C$ such that $\mu\zeta(c) \neq 1$ then $Ker\,\mu$ is a normal subgroup of G of index p and so it is maximal. But clearly $\zeta(c) \notin Ker\,\mu$, which is a contradiction. Now suppose that for all $\mu : G \to C$ we have $\mu \circ \zeta|_{Z(G)} \neq 1$. Let M be a maximal subgroup of G and c be a non-trivial element of $Im\,\zeta = C'$. If $c \notin M$, then $C' \cap M = 1$, and since $C' \leqslant Z(G)$, we have that $C'M$ is a subgroup of G. Since M is maximal, $G = C'M$. But this means that there exists $\mu : G \to C$ such that $\mu(c) \neq 1$, a contradiction.

In a similar way as it is done in point i), we have a bijective correspondence between the isomorphism classes of elements $Y_{\omega,\zeta}$ in $R\hat{B}^1_C(G)$ and $R\mathcal{Y}_G$. This establishes an isomorphism of R-modules between $R\hat{B}^1_C(G)$ and $R\mathcal{Y}_G \oplus R\hat{G}$. Now we describe the algebra structure. The following calculations are made using Lemma 8, Lemma 9 and Lemma 2.3.22 in [3].

The composition of elements $Y_{\omega,\zeta}$ is given by

$$Y_{\omega,\zeta} \circ Y_{\alpha,\chi} = \begin{cases} Y_{\omega\alpha,\omega\chi} & \text{if } \zeta = \omega\chi, \\ 0 & \text{otherwise.} \end{cases}$$

The product $X_{t,\sigma} \circ Y_{\omega,\zeta}$ is different from 0 if and only if $t\zeta(c)c \neq 1$ for all $c \neq 1$. Then, if we let Id_C be the identity morphism of C, we have that $(t\zeta)Id_C$ defines an automorphism on C, which we will call r. Given $g \in G$ there exists only one $c_g \in C$ such that $t\omega(g) = r(c_g)$ and sending g to $\omega(g)\zeta(c_g)$ defines an automorphism on G, which we will call s. We have

$$X_{t,\sigma} \circ Y_{\omega,\zeta} = \begin{cases} Y_{\sigma s,\sigma\zeta r^{-1}} & \text{if } r = (t\zeta)Id_C \text{ is an automorphism,} \\ 0 & \text{otherwise.} \end{cases}$$

Using this formula on the indices defines a left action of $R\hat{G}$ on $R\mathcal{Y}_G$. On the other hand, $Y_{\omega,\zeta} \circ X_{t,\sigma}$ is different from 0 if and only if $\omega\sigma(g) \neq \zeta t(g)$ for all $g \in G$, $g \neq 1$. Then sending $g \in G$ to $\omega\sigma(g)\zeta t(g)$ defines an automorphism in G and we have

$$Y_{\omega,\zeta} \circ X_{t,\sigma} = \begin{cases} Y_{(\omega\sigma)\zeta t,\zeta} & \text{if } (\omega\sigma)\zeta t \text{ is an automorphism,} \\ 0 & \text{otherwise.} \end{cases}$$

With this we have the right action of $R\hat{G}$ on $R\mathcal{Y}_G$. It can be proved directly that with these actions $R\mathcal{Y}_G \oplus R\hat{G}$ is an R-algebra, and it is clearly isomorphic to $R\hat{B}^1_C(G)$. □

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