ORDER QUANTITY AND PRICING DECISIONS IN LINEAR COST INVENTORY SYSTEMS

A THESIS
SUBMITTED TO THE DEPARTMENT OF INDUSTRIAL ENGINEERING
AND THE INSTITUTE OF ENGINEERING AND SCIENCE
OF BİLKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

By
L. Hakan Polatoğlu
January 1993

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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Abstract

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January 1993

The primary concern of this study is to reveal the fundamental characteristics of the linear cost inventory model where price is a decision variable in addition to procurement quantity. In this context, the optimal solution must not only strike a balance between leftovers and shortages, but also simultaneously search for the best pricing alternative within the low price high demand and high price low demand tradeoff. To some extent, this problem has been studied in the literature. However, it seems that, there is a need to improve the model in order to understand the decision process better. To this end, optimal decisions must be characterised under a more general problem setting than it has been assumed in the existing models. In this study, we employ such a general model.

The overall decision problem can be formulated under a dynamic programming structure. It follows that, the single period model is the basis of this periodic decision model. For this reason, we concentrate first on this problem. Having characterised the optimal solution to this basic model we extend the decision model to account for the multi-period setting.

It is established with the results of this study that the decision problem in question is understood better. It is found that the characteristics of the optimal decision under the proposed model can be substantially different from the properties of the optimal solution of the corresponding classical model where there is no pricing decision. The primary reason for this is the fact that when there is a shortage in any period, the price that is set in this period could affect the future revenue which must be accounted in the overall decision problem. That is, in a general model, price is an information which has an economic value that is transferred from one period to another just like transfering inventories or backlogs to future periods.

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Chapter 1

Introduction and Literature Review

Reorder point, order quantity inventory models are essentially short term planning models. By assumption, the ordering policy does not change the demand pattern or the price structure in the market place during the planning horizon. This assumption is approximated in a perfectly competitive market where there is no pricing decision to make for the individual vendor. However, there may be incentives for the vendor to increase inventories and wait until the most profitable point in time, if the price is expected to rise in the future; or to clear inventories, if the price is expected to decline. Under imperfect competition, the individual vendor excercises a degree of monopoly power in the market. He may set a price for his product but then he faces a demand level, governed by some probability distribution, the expected value of which is decreasing in price. In this context, in addition to the procurement decision, the vendor is confronted by a simultaneous pricing decision.

The simplest model for the study of optimal procurement and pricing decisions is a single-product, periodic review pure inventory model. The planning horizon is divided into review periods which are linked by period ending inventory levels. The vendor is assumed to have full information about costs and demand distributions that are applicable to all periods of the horizon. At the beginning of a review period, given the inventory position (on hand plus on order minus backorders), his problem is to determine the procurement and pricing policies which jointly maximize the expected present value of total profit during the planning horizon.

It has been a common practice in demand modeling to express random demand as a combination of expected demand and a random term. The former has some form of price dependency while the latter is price independent. A number of special cases of this model have been studied in the literature. These differ, essentially, in the way the demand process is represented. In the *additive model*, $X_n(p) = \overline{X}_n(p) + \varepsilon_n$ where $X_n(p)$ is the demand during period n when the price is p, $\overline{X}_n(p) = E[X_n(p)]$ and ε_n, $n = 1, 2, \ldots$ are independent random variables with $E[\varepsilon_n] = 0$. In most studies, it is also assumed that $\overline{X}(p)$ is nonincreasing in p and, without loss of generality, $X_n(p) = X(p)$, $n = 1, 2, \ldots$. In the *multiplicative model*, $X_n(p) = \overline{X}_n(p) \cdot \varepsilon_n$ where $E[\varepsilon_n] = 1$. In the *riskless model*, $X_n(p) = \overline{X}_n(p)$ so that demand in any period is represented by its expected value. This latter case serves both as a first order approximation and as a benchmark for the probabilistic versions of the model.

Whitin [15] appears to have been the first to link price theory and inventory control in a one-period model. Demonstrating that a higher profit level could be achieved for the proposed model, compared to the *newsboy problem*, he claimed that decision making would be improved by taking price as a control variable.

Mills [7] formalized Whitin's intuitive approach by studying a one-period inventory model (no holding or shortage costs) with additive demand. He showed that under demand uncertainty the optimal price is less than the optimal *riskless price*. Mills [7, 8] also studied the multi-period (infinite horizon) model for which the optimal price was found to be less than that of the one-period model. In addition, he demonstrated that the difference between the optimal starting stock and the expected demand evaluated at the optimal price is greater for the multi-period model.

Later, Karlin and Carr [3] provided a more general inventory model. For both static (one-period model with unit holding and shortage costs) and dynamic (infinite horizon multi-period lost-sales model without holding or shortage costs) cases they studied the optimal decision variables under additive and multiplicative demand, and derived the necessary conditions for optimality. They showed, under reasonable assumptions, that the optimal price is greater (less) than the *riskless price* for the multiplicative (additive) demand for both static and dynamic models.

Nevins [9] provided an empirical study of a special infinite horizon multi-period lost-sales model. He employed the multiplicative demand model with a linear expected demand function under the additional assumptions of a nondecreasing quadratic procurement cost function, a constant unit inventory holding cost and no shortage cost. For various problem data, he observed that there exists a stochastic equilibrium in which expected demand evaluated at the optimal price equals to the optimal procurement, and there is a tendency that equilibrium inventory level is preserved. However, it appears that Nevins' definition of equilibrium inventory is erroneous. Expected sales rather than expected demand should be employed in this definition.

Zabel [18] attempted to provide analytical support to Nevins' empirical findings. For the one-period model with multiplicative demand, he demonstrated, under some restrictive conditions, the existence of the equilibrium inventory level. Following Nevins' definition, he showed that the equilibrium inventory level decreases as holding cost is increased as observed by Nevins. Zabel also stated the conditions that guarantee the existence and uniqueness of the optimal solution. In a later paper [19], he showed that stronger conditions are needed to guarantee a unique optimal price for the first period of a two-period problem. In addition to the multiplicative demand, Zabel [19] also considered an additive demand model which is slightly different from Mills' [7] definition. For this model, he demonstrated that under some restrictive assumptions the optimal values of the decision variables at each period are unique. Moreover, comparing additive and multiplicative demand cases, Zabel concluded that the former tends to yield lower prices and higher inventory levels than the latter. The major source of this characteristic difference is seen as the variance of demand. For the additive model, the variance is constant and for the multiplicative model it is a decreasing function of price. Therefore, higher prices in the latter model are less risky.

Thowsen [13] formulated a finite horizon multi-period model under additive demand which incorporates partial backlogging. He derived sufficient conditions under which the optimal procurement is determined by a single critical number policy. He showed that these conditions are satisfied for the case with linear expected demand function and a PF_2 distribution for the random term.

Young [16] represented the random demand as a combination of the additive and multiplicative models. For the one-period problem, he stated the sufficient conditions under which the optimal starting stock level is unique. Comparing the results with those of the riskless model, he also showed that, if the coefficient of variation of demand is nonincreasing in price, then the riskless revenue exceeds the marginal procurement cost at optimality. The converse is true if the variance of demand is nondecreasing in price. Moreover, correcting Zabel's [18] definition of equilibrium, Young demonstrated the existence of an equilibrium inventory level under his assumptions. In addition, Young [17] also studied the infinite horizon multi-period lost-sales problem under his demand model. Assuming that the unsold inventory at the end of each period has an economic value that is equal to the present worth of its procurement cost, he showed that the periods could be separated from each other and the optimal solution could be obtained from the analysis of one-period model.

It appears that Mills' [8] and Karlin and Carr's [3] approaches establish the conceptual framework of the general inventory model. Nevins' [9], Zabel's [18, 19], Thowsen's [13] and Young's [16, 17] studies, however, concentrate mostly on the existence and uniqueness of the optimal solutions for various special cases of the general model. It is demonstrated by these studies that seriously restrictive assumptions on the form of the expected demand function, on the demand distribution or on the structure of the expected loss function are needed to provide analytical results on the cited issues. In this regard, the existing studies fail to provide a complete understanding of the form of the optimal policies due to analytical intractability.

The above mentioned demand models have been used traditionally as a convenient tool to isolate the effects of uncertainty in the context of the *theory of the firm*. The disadvantage of this representation, however, is the structural restrictions it brings into the model. For instance, the additive model is restricted by a price-independent (constant) variance. Also it allows negative demand unless the price values are bounded from above. The multiplicative model implies the curious restriction that the demand equals to the product of its expected value and a random term. As a result of this, variance of demand is the square of its expected value times the variance of the random term. Therefore, variance decreases at a rate faster than expected value and it approaches to zero at high prices.

We believe that there is a need to study the model under general demand uncertainty. It is essential to reveal the fundamental properties of the model independent of the demand pattern. Especially, uniqueness conditions for optimality must be studied in a more general setting. In this study, we attempt to develop and analyze the model under a general demand uncertainty.

In the classical multi-period inventory model, the proportion of the shortage which is backlogged to the next period is determined by the partial backlogging function. In our model, on the other hand, backlogging needs additional consideration due to the pricing decision. This fact is often ignored by the existing models either by assuming a lost-sales model or by making simplifying assumptions about the forgone revenue due to shortages. In our model, however, we introduce a special relationship (bargaining) between the vendor and the customer over the price that is charged for the backlogs.

In what follows, we introduce the single period model in chapter 2. Then, in chapter 3, we study the multi-period model. Chapter 4 provides some numerical examples on the theoretical issues which are discussed in the first three chapters. Finally, in chapter 5 we conclude our findings.

Chapter 2

Single Period Model

In this chapter, we study the optimal procurement and pricing decisions in a single product one-period pure inventory system. We view this model as a building block of the multi-period model and attempt to establish its characteristics to this end.

2.1 Basic Model and Assumptions

In this model, the vendor is to make the best procurement and pricing decisions to maximize his profit prior to the beginning of the period. Inventory level before ordering is i. The amount procured, if any, is $q-i$. A random demand $X(p)$ occurs during the period and at the end of the period the inventory level is reduced to $q-X(p)$. We consider the case where $i \geq 0$. For $i < 0$, the one-period problem is initiated with an unknown history. That is, the following questions can not be accounted for unless we make assumptions: (1) What fraction of the backlog do we have to satisfy? (2) At what price should we sell that fraction? (3) Do we deduct the backlog from the actual demand or not? These questions will be referred to later in the multi-period model.

We assume that inventory costs are proportional to the period ending inventory level. We denote the unit holding, shortage and procurement costs by h, s and c, respectively. We also denote the fixed ordering cost by $\mathcal{K}$. In addition, we assume that, the price is bounded from below and above by P_ℓ and P_u, respectively, which are the price floor and price ceiling in a regulatory environment. If there are no price regulations, then we consider the price range of $(0, \infty)$. We also assume that $P_u > c$ so that it is possible to make profit by retailing.

It follows from the discussion in [6] that a way of incorporating price and uncertainty in demand is through an implicit relationship of the type:

$$\mathcal{F}(X, p, \varepsilon) = 0,$$

where ε is a random term with a known probability distribution. Assuming that $\mathcal{F}$ has continuous partial derivatives we may express the random demand as:

$$X = \mathcal{X}(p, \varepsilon). \tag{2.1}$$

Note that the additive and the multiplicative demand models are special forms of (2.1).

We assume that demand distribution, $F(x;p)$, is defined over $x \in (-\infty, \infty)$ and $p \in [P_\ell, P_u]$ such that for all $p \in [P_\ell, P_u]$ we have $F(X_1(p);p) = 0$ and $F(X_2(p);p) = 1$, where $X_1(p)$ and $X_2(p)$ are the lower and upper bounds on $X(p)$, respectively, which are differentiable functions of p and $0 \leq X_1(p) \leq X_2(p) < \infty$. We shall restrict our analysis only to the continuous demand case, bearing in mind that a similar one exists otherwise.

We assume that the expected demand exists (finite), and it is determined from

$$\overline{X}(p) = \int_{X_1(p)}^{X_2(p)} x \cdot f(x;p) \cdot dx \;\; = \;\; \int_0^\infty [1 - F(x;p)] \cdot dx, \tag{2.2}$$

where $f(x;p)$ is the demand density function. We assume that $\overline{X}(p)$ is a monotone decreasing function of p on $(0,\infty)$ (if p is confined to $[P_\ell, P_u]$, then we extend $\overline{X}(p)$ on $(0, P_\ell)$ and (P_u, ∞) by appropriate functions to satisfy the requirements without loss of generality). Moreover, we require that $\overline{X}(p)$ is $o(1/p)$ as $p \to 0^+$ and $p \to \infty$. This implies that the function $p \cdot \overline{X}(p)$ starts at zero, first increases and eventually dies away. This function, which is denoted by $R(p)$, is called the *riskless total revenue* by Mills [8]. $R(p)$ is a positive valued, finite and differentiable function, which plays an important role in model development. It is shown in Appendix A that $R(p)$ is pseudoconcave on $(0,\infty)$ when $\overline{X}(p)$ is either a concave or convex decreasing function; it is also indicated that $R(p)$ is not pseudoconcave for all monotone decreasing $\overline{X}(p)$ functions. We assume that $R(p)$ is unimodal; hence, there exists a unique finite price which maximizes $R(p)$.

It is intuitive that, in a "fair" market, the probability that demand is less than the given level x, $F(x;p)$, increases as the price increases. That is,

$$\frac{\partial F(x;p)}{\partial p} > 0 \quad \forall x \in (X_1(p), X_2(p)). \tag{2.3}$$

It is worthwhile to note that condition (2.3) is sufficient for the requirement that $\overline{X}(p)$ is a decreasing function of p:

$$\frac{\partial F(x;p)}{\partial p} > 0 \;\Rightarrow\; \frac{d\overline{X}(p)}{dp} = -\int_0^\infty \frac{\partial F(x;p)}{\partial p} \cdot dx \;\; < 0. \tag{2.4}$$

2.2 Mathematical Model

In this section we develop and analyze the mathematical model under probabilistic demand for the determination of the optimal price and the beginning inventory level.

2.2.1 Optimization Problem

Considering the representations introduced in Section 2, the profit function can be expressed as:

$$\Pi(p,q) = M(p,q) - \mathcal{K}\cdot\delta(q-i), \tag{2.5}$$

where $\delta(\cdot)$ is the *Heavyside* function and

$$M(p,q) = \begin{cases} p\cdot q - c\cdot(q-i) - s\cdot(X(p)-q), & q \le X(p) \le X_2(p), \\ p\cdot X(p) - c\cdot(q-i) - h\cdot(q-X(p)), & X_1(p) \le X(p) \le q, \end{cases} \tag{2.6}$$

is the pseudo-profit function. We can write the expected profit as:

$$\overline{\Pi}(p,q) = E[\Pi(p,q)] = \overline{M}(p,q) - \mathcal{K}\cdot\delta(q-i), \tag{2.7}$$

where

$$\overline{M}(p,q) = E[M(p,q)] = p\cdot\overline{X}(p) - c\cdot(q-i) - L(p,q). \tag{2.8}$$

The first term in (2.8) is the *riskless total revenue* function. The second term is the procurement cost. The last term is the *expected loss* function which is given by

$$\begin{aligned} L(p,q) &= h\cdot\int_{X_1(p)}^{q} (q-x)\cdot f(x;p)\cdot dx \;+\; (p+s)\cdot\int_{q}^{X_2(p)} (x-q)\cdot f(x;p)\cdot dx \\ &= (p+s)\cdot[\overline{X}(p) - q] + (p+s+h)\cdot\Theta(p,q), \end{aligned} \tag{2.9}$$

where $\Theta(p,q)$ is the *expected leftovers* [1], i.e.,

$$\Theta(p,q) = \int_{X_1(p)}^{q} (q-x)\cdot f(x;p)\cdot dx \;=\; \int_{X_1(p)}^{q} F(x;p)\cdot dx. \tag{2.10}$$

We assume that $\Theta(p,q)$ is differentiable in p for $q \ge 0$. Also, we observe that $\Theta(p,q)$ satisfies

$$\Theta(p,q) \ge \max\{0, q - \overline{X}(p)\}, \tag{2.11}$$

and it is a convex, non-decreasing and differentiable function of q for a given p. Moreover, condition (2.3) implies that

$$\frac{\partial\Theta(p,q)}{\partial p} = \int_{X_1(p)}^{q} \frac{\partial F(q;p)}{\partial p}\cdot dx \;> 0,$$

[1] An alternative representation of $\Theta(p,q)$ is

$$\Theta(p,q) = [\Delta_q(p) + q - \overline{X}(p)]/2,$$

where $\Delta_q(p)$ is the *total expected deviation* of demand from q at a price level of p which is defined as:

$$\Delta_q(p) = \int_{X_1(p)}^{X_2(p)} |x-q|\cdot f(x;p)\cdot dx > 0.$$

for all $q \in (X_1(p), X_2(p))$.

From (2.8) and (2.9) it follows that

$$\overline{M}(p,q) = p\cdot[q - \Theta(p,q)] - c\cdot(q-i) - h\cdot\Theta(p,q) - s\cdot[\overline{X}(p) - (q - \Theta(p,q))]. \tag{2.12}$$

Therefore, $\overline{M}(p,q)$ is the expected net revenue, less the procurement cost, less the expected holding cost, and less the expected shortage cost. At the expense of loosing intuition about its terms, we shall refer to $\overline{M}(p,q)$ in what follows in the following form :

$$\overline{M}(p,q) = (p+s-c)\cdot q - s\cdot\overline{X}(p) - (p+s+h)\cdot\Theta(p,q) + c\cdot i. \tag{2.13}$$

It is clear that, $\overline{M}(p,q)$ is continuous in p on $[P_\ell, P_u]$ and in q on $[0,\infty)$.

Now, the optimization problem becomes

$$\overline{\Pi}(p^\star, q^\star) = \max_{p,q}\{\overline{\Pi}(p,q) : \ q \in [i,\infty), \ p \in [P_\ell, P_u]\}, \tag{2.14}$$

where $p^\star$ and $q^\star$ are the optimal values of the decision variables p and q. For this problem we define the suboptimal function

$$\overline{M}^\star(q) = \max\{\overline{M}(p,q) : \ p \in [P_\ell, P_u]\} = \overline{M}(p_q, q), \tag{2.15}$$

where p_q is the maximizer. Therefore, $\overline{M}^\star(q)$ traces the *best price* trajectory over the q range. Moreover, since $\overline{M}(p,q)$ is continuous in p and q, it follows from the *Envelope Theorem* that $\overline{M}^\star(q)$ is a continuous function of q (see Appendix B for a proof).

In analyzing (2.14) and (2.15), we need to consider first and second degree partial derivatives of $\overline{M}(p,q)$ with respect to p and q, which are given by

$$\frac{\partial\overline{M}(p,q)}{\partial p} = q - s\cdot\frac{d\overline{X}(p)}{dp} - \Theta(p,q) - (p+s+h)\cdot\frac{\partial\Theta(p,q)}{\partial p}, \tag{2.16}$$

$$\frac{\partial^2\overline{M}(p,q)}{\partial p^2} = -s\cdot\frac{d^2\overline{X}(p)}{dp^2} - 2\cdot\frac{\partial\Theta(p,q)}{\partial p} - (p+s+h)\cdot\frac{\partial^2\Theta(p,q)}{\partial p^2}, \tag{2.17}$$

$$\frac{\partial^2\overline{M}(p,q)}{\partial p\partial q} = 1 - F(q;p) - (p+s+h)\cdot\frac{\partial F(q;p)}{\partial p},$$

$$\frac{\partial\overline{M}(p,q)}{\partial q} = (p+s-c) - (p+s+h)\cdot F(q;p), \tag{2.18}$$

$$\frac{\partial^2\overline{M}(p,q)}{\partial q^2} = -(p+s+h)\cdot f(q;p) \ \leq 0. \tag{2.19}$$

From (2.19) we conclude that $\overline{M}(p,q)$ is q-concave on $(0,\infty)$, which refers to the *newsboy problem* setting. On the other hand, (2.16) implies that p_q is independent of the procurement cost. In other words, the vendor is to maximize his expected profit given that he starts the period with q units. The price dependence of $\overline{M}(p,q)$, however, is not clear from (2.16) or (2.17).

There is a critical question about the existence of p_q if the price limits are abolished, that is when $p \in (0,\infty)$. Since,

$$\lim_{p\to 0} \overline{X}(p) = \infty \quad \text{and} \quad \lim_{p\to\infty} \overline{X}(p) = 0,$$

$X_1(p)$ and $X_2(p)$ must satisfy:

$$\lim_{p\to 0} X_1(p) = \lim_{p\to 0} X_2(p) = \infty,$$

and

$$\lim_{p\to\infty} X_1(p) = \lim_{p\to\infty} X_2(p) = 0.$$

Under this setting, it is true that $\forall q \in (0,\infty)\ \exists p_1, p_2 \in (0,\infty)$ such that $X_2(p_2) \le q \le X_1(p_1)$. Therefore, from (2.16) we obtain

$$\frac{\partial \overline{M}(p,q)}{\partial p}|_{p\le p_1} = q - s\cdot\frac{d\overline{X}(p)}{dp} > 0,$$

and

$$\begin{aligned}\frac{\partial \overline{M}(p,q)}{\partial p}|_{p\ge p_2} &= q - s\cdot\frac{d\overline{X}(p)}{dp} - q + \overline{X}(p) + (p+s+h)\cdot\frac{d\overline{X}(p)}{dp} \\ &= \overline{X}(p) + (p+h)\cdot\frac{d\overline{X}(p)}{dp}.\end{aligned}$$

Moreover, it follows from Corollary A1 in Appendix A that for $p > P_h$, $\overline{X}(p) + (p+h)\cdot d\overline{X}(p)/dp < 0$. Thus, we have

$$\frac{\partial \overline{M}(p,q)}{\partial p}|_{p\ge max\{p_2,P_h\}} < 0,$$

and there exists a solution p_q if there were no price limits.

If p_q is independent of q (a boundary point solution or a constant), then it follows from (2.19) that $\overline{M}^\star(q)$ is concave at that q. However, if $p_q \in (P_\ell, P_u)$, then it must satisfy the first order condition $\partial\overline{M}(p,q)/\partial p|_{p_q} = 0$ and the second order condition $\partial^2\overline{M}(p,q)/\partial p^2|_{p_q} < 0$, for a given q. Since $\overline{M}(p,q)$ has continuous partial derivatives, we can perform implicit differentiation on the first order condition to obtain

$$\frac{dp_q}{dq} = \frac{1 - F(q;p_q) - (p_q+s+h)\cdot\partial F(q;p)/\partial p|_{p_q}}{-\partial^2\overline{M}(p,q)/\partial p^2|_{p_q}}, \tag{2.20}$$

in which the denominator is always positive. Depending on the value of p_q and the price dependency of $F(\cdot;p)$ function, however, the numerator can be positive or negative. Thus, the sign of dp_q/dq is not clear.

Since dp_q/dq exists, we can write the first derivative of $\overline{M}^\star(q)$ as

$$\frac{d\overline{M}^\star(q)}{dq} = \frac{\partial\overline{M}(p_q,q)}{\partial q} + \frac{\partial\overline{M}(p,q)}{\partial p}\Big|_{p_q}\cdot\frac{dp_q}{dq}. \tag{2.21}$$

If $p_q \in (P_\ell, P_u)$, then $\partial\overline{M}(p,q)/\partial p|_{p_q} = 0$ otherwise $dp_q/dq = 0$. Therefore, in all combinations of right-hand and left-hand derivatives the second term in (2.21) vanishes. Consequently, we get

$$\frac{d\overline{M}^\star(q)}{dq} = (p_q + s - c) - (p_q + s + h)\cdot F(q;p_q). \tag{2.22}$$

In order to interpret (2.22) we rewrite it as follows:

$$\frac{d\overline{M}^\star(q)}{dq} = (p_q + s)\cdot[1 - F(q;p_q)] - h\cdot F(q;p_q) - c. \tag{2.23}$$

If the vendor administers his profit maximizing price as he starts with a stock size of q, then $F(q;p_q)$ represents the probability that there will be no shortage. It follows from (2.23) that $\overline{M}^\star(q)$ increases in q at a rate of (p_q+s) if there is a shortage with probability $[1-F(q;p_q)]$ and decreases at a rate of h with probability $F(q;p_q)$ when there is no shortage. In addition to these two possibilities, $\overline{M}^\star(q)$ decreases at a rate of c due to the procurement cost. Thus the vendor can increase his profit by stocking more given that he is short. When he is short any increase in q will pay him p_q for the sale of a unit and s for not being short of that unit. Intuitively, the vendor should follow a pricing strategy which will simultaneously minimize $F(q;p_q)$ and keep p_q as high as possible. There is a tradeoff, however, since $F(q;p_q)$ increases in p_q.

2.2.2 Existence Problem

Intuitively, $\overline{M}^\star(q)$ must have a peak on $[0,\infty)$. However, the existence of this point or, if it exists, its location are not immediately clear. In the following analysis, we shall identify two separate regions of q in which $\overline{M}^\star(q)$ is monotone, then we shall prove the existence of its peak.

Lemma 1. $\forall q \in [0, X_1(P_u)]$, *$\overline{M}^\star(q)$ is a linear increasing function of q and $p_q = P_u$.*

Proof. $\forall q \in [0, X_1(P_u)]$ we have $F(q;p_q) = 0$. Therefore, from (2.10), $\Theta(p_q,q) = 0$ and from (2.13) we obtain :

$$\begin{aligned}\overline{M}^\star(q) &= \max\{(p+s-c)\cdot q - s\cdot\overline{X}(p) + c\cdot i :\ p\in[P_\ell,P_u]\} \\ &= (P_u+s-c)\cdot q - s\cdot\overline{X}(P_u) + c\cdot i, \end{aligned} \tag{2.24}$$

which is a linear increasing function of q and $p_q = P_u$.

Lemma 1 indicates that, if we are sure that demand will exceed our stock, i.e. if $q \leq X_1(P_u)$, then we should charge the customers at the highest rate because we not only reduce shortages in this way but we also incur the maximum unit profit.

If $X_1(P_u) = 0$, then the region indicated in Lemma 1 disappears and we loose the information about the slope of $\overline{M}^\star(q)$ at $q = 0$. To account for this possibility, considering (2.22) and the fact that $0 \leq F(q; p_q) \leq 1$ we obtain :

$$-(h + c) \leq \frac{d\overline{M}^\star(q)}{dq} \leq (p_q + s - c), \tag{2.25}$$

which gives the lower and upper limits of the rate of change of expected profit with respect to the beginning inventory level. It is now clear from (2.24) and (2.25) that at $q = 0$, $\overline{M}^\star(q)$ increases at the maximum rate of $P_u + s - c$.

Lemma 2. *$\forall q \in [X_2(P_\ell), \infty)$, $\overline{M}^\star(q)$ is a linear decreasing function of q and p_q is a constant.*

Proof. For $q > X_2(P_\ell)$ we have $F(q; p_q) = 1$. Therefore, from (2.10), $\Theta(p_q, q) = q - \overline{X}(p_q)$ and from (2.13) we obtain

$$\begin{aligned} \overline{M}^\star(q) &= \max\{(p + h)\cdot\overline{X}(p) : \ p \in [P_\ell, P_u]\} - (c + h)\cdot q + c\cdot i \\ &= (\bar{P}_h + h)\cdot\overline{X}(\bar{P}_h) - (c + h)\cdot q + c\cdot i, \end{aligned} \tag{2.26}$$

where $\bar{P}_h = \min\{\max\{P_h, P_\ell\}, P_u\}$ and P_h is the maximizer of the pseudoconcave function $(p + h)\cdot\overline{X}(p)$.

We now establish the existence of $\tilde{q}$, where $\tilde{q} = \max\{\overline{M}^\star(q) : \ q \in [0, \infty)\}$.

Theorem 1. *$\exists \tilde{q} \in (X_1(P_u), X_2(P_\ell))$ such that $\overline{M}^\star(q) \leq \overline{M}^\star(\tilde{q})$ $\forall q \in [0, \infty)$.*

Proof. By Lemma 1, $\overline{M}^\star(q)$ is a linear increasing function of q on $[0, X_1(P_u)]$ with a slope of $(P_u + s - c) > 0$. By Lemma 2, $\overline{M}^\star(q)$ is a linear decreasing function of q on $[X_2(P_\ell), \infty)$ with a slope of $-(c + h) < 0$. From (2.25), $(P_u + s - c)$ and $-(c + h)$ are the largest and the smallest possible slopes of $\overline{M}^\star(q)$, respectively. The proof follows.

Therefore, $\tilde{q}$ must satisfy the first order optimality condition on $\overline{M}^\star(q)$ which can be obtained from (2.22) as :

$$F(q; p_q) = \frac{p_q + s - c}{p_q + s + h}. \tag{2.27}$$

The right hand side of (2.27), RHS, is a concave increasing function of p_q. It becomes negative for $p_q < c - s$. It follows from (2.22) that, for those p_q values $\overline{M}^\star(q)$ is decreasing, thus $\tilde{q}$ can not be realized at any price level less than $c - s$. Alternatively, for $p_q \geq c - s$, RHS attains values between 0 and 1, and we always have a solution for q given such RHS.

In his pioneering work [15], Whitin brings an intuitive approach to condition (2.27) for a similar decision problem. First, he introduces two conflicting factors: expected profit and expected loss. According to his construct, the expected profit from adding an additional unit to inventory is equal to unit profit times the probability of selling that unit, plus the avoidance of goodwill loss per unit times the same probability, i.e., $[1 - F(q; p_q)]\cdot(p_q - c) + [1 - F(q; p_q)]\cdot s$. On the other hand, the expected loss resulting from adding the extra unit is equal to the probability of not selling the unit during the period multiplied by the unit loss from liquidation, i.e., $F(q;p_q)\cdot(h+c)$. He then argues that if profits are to be maximized, then the expected profit obtainable through stocking an additional unit must be equal to the expected loss, that is:

$$[1-F(q;p_q)]\cdot(p_q+s-c)=F(q;p_q)\cdot(h+c),$$

which is equivalent to (2.27).

It is possible to construct an upper bound on $\tilde{q}$ by employing the *Markov Inequality* and condition (2.27). To this end, defining $\tilde{p}=p_{\tilde{q}}$ we write

$$\frac{\tilde{p}+s-c}{\tilde{p}+s+h}=F(\tilde{q};\tilde{p})\geq 1-\frac{\overline{X}(\tilde{p})}{\tilde{q}}\Rightarrow \tilde{q}\leq\frac{(\tilde{p}+s+h)\cdot\overline{X}(\tilde{p})}{h+c},$$

which implies

$$\tilde{q}\leq\frac{1}{h+c}\cdot\max\{(p+s+h)\cdot\overline{X}(p):\ p\in[P_\ell,P_u]\},\tag{2.28}$$

where the maximization problem can be solved for a given $\overline{X}(p)$ function and the data. *Markov Inequality* usually yields weak bounds, nevertheless, (2.28) can be useful especially in numerical procedures.

2.2.3 Unimodality

Unimodality of $\overline{M}^\star(q)$ enables us to identify an (σ,Σ) type policy which may be employed in determining the optimal q. Moreover, in the multi-period extension of the theory, this becomes an important issue related to the dynamic decision problem.

If $p_q\in(P_\ell,P_u)$, then differentiating (2.22) with respect to q we obtain

$$\frac{d^2\overline{M}^\star(q)}{dq^2}=\frac{dp_q}{dq}\cdot[1-F(q;p_q)]-(p_q+s+h)\cdot\frac{dF(q;p_q)}{dq}.\tag{2.29}$$

Noting that

$$\frac{dF(q;p_q)}{dq}=f(q;p_q)+\frac{\partial F(q;p_q)}{\partial p}|_{p_q}\cdot\frac{dp_q}{dq},\tag{2.30}$$

we rewrite (2.29) as

$$\frac{d^2\overline{M}^\star(q)}{dq^2}=-\frac{\partial^2\overline{M}(p,q)}{\partial p^2}|_{p_q}\cdot(\frac{dp_q}{dq})^2-(p_q+s+h)\cdot f(q;p_q).\tag{2.31}$$

First term in (2.31) is always positive and the second is always negative. However, their relative magnitudes are not clear. Thus, convexity of $\overline{M}^\star(q)$ is not evident from (2.31).

Note that, $F(q;p_q)$ is a function of q only, where $F(q;p_q)=0$ for $0\leq q\leq X_1(P_u)$ and $F(q;p_q)=1$ for $X_2(P_\ell)\leq q$. Therefore, $F(q;p_q)$ has to rise from 0 to 1 between minimum and maximum possible demand values. Meanwhile, it is clear from Lemma 1 and 2 that p_q should decrease from P_u to $\tilde{P}_h$. If these changes occur monotonically, then there will be a unique first order q, which satisfies (2.27). That is, if $dF(q;p_q)/dq\geq 0$ and $dp_q/dq\leq 0$, then from (2.29) it follows that $\overline{M}^{\star}(q)$ is concave. However, we can state a weaker condition by noting that, it is sufficient to have $dp_q/dq \leq 0$ at $q = \tilde{q}$, provided that $dF(q;p_q)/dq \geq 0 \ \forall q$. That is,

$$\frac{dF(q;p_q)}{dq} \geq 0 \text{ and } \frac{dp_q}{dq}\Big|_{\tilde{q}} \leq 0 \ \Rightarrow \ \overline{M}^{\star}(q) \text{ is unimodal.} \tag{2.32}$$

Moreover, from (2.20) and (2.27) we obtain

$$\frac{dp_q}{dq}\Big|_{\tilde{q}} \leq 0 \ \Leftrightarrow \ \frac{\partial F(q;p)}{\partial p}\Big|_{\tilde{p},\tilde{q}} \geq \frac{h+c}{(\tilde{p}+s+h)^2}, \tag{2.33}$$

and we can employ (2.33) in (2.32). On the other hand, we realize that for unimodality of $\overline{M}^{\star}(q)$ it is necessary and sufficient to have

$$\frac{d^2\overline{M}^{\star}(q)}{dq^2}\Big|_{\tilde{q}} \leq 0. \tag{2.34}$$

Unimodality of $\overline{M}^{\star}(q)$ means once the expected profit of the vendor starts declining at some starting stock level ($\tilde{q}$), then he will not be able to avoid this fall by procuring more and incurring the best price. In this case, demand being sensitive to price responds to the vendor's profitability. This concept can be related to the degree of monopoly power of the vendor (Mills [8] also mentions this connection without any further detail), however, this is beyond our interest and we leave that discussion open.

2.2.4 Optimal Solution

If $\overline{M}^{\star}(q)$ is unimodal, then from (2.14) it follows that $q^\star$ can be determined by an $(\sigma, \mathcal{S})$ type policy operating on $\overline{M}^{\star}(q)$, where $\mathcal{S} = \tilde{q}$ and $\sigma = \min\{q : \ \overline{M}^{\star}(q) = \overline{M}^{\star}(\mathcal{S}) - \mathcal{K}\}$. Consequently, the decision rule is $q^\star = \mathcal{S}$ if $i < \sigma$ otherwise $q^\star = i$, and $p^\star = \operatorname{argmax}\{\overline{M}(p,q^\star) : p \in [P_\ell, P_u]\}$.

2.3 Special Cases

In this section, first we consider the deterministic demand model (the *riskless* model introduced by Mills [7]) and establish its relation to the probabilistic model. Then, we analyze the additive and the multiplicative models. We provide the relationships that exist between the optimal prices of these models. Finally, under linear expected demand ($\overline{X}(p) = a - b\cdot p$, where $a, b > 0$ and $c < P_u < a/b$), we prove the unimodality of $\overline{M}^{\star}(q)$ for uniformly distributed additive ε and for exponentially distributed multiplicative ε.

2.3.1 Deterministic Model

In this part, we use the subscript "r" to denote the functions and variables of the riskless model. If there is no uncertainty in demand, then we have $X(p) = \overline{X}(p)$. Under this specialization, leftovers are given by $\Theta_r(p,q) = \max\{0, q - \overline{X}(p)\}$, which is a continuous function. It is, however, non-differentiable at the trajectory given by $q = \overline{X}(p)$.

In the following discussion, first we prove that $M_r^\star(q)$ is unimodal, then we determine the optimal values of the decision variables, and finally we compare the deterministic and probabilistic profit functions.

Theorem 2. *$M_r^\star(q)$ is quasiconcave in q on $[0,\infty)$.*

Proof. For $q \leq \overline{X}(P_u)$ we have $\Theta_r(p,q) = 0$. Thus, from Lemma 1 it follows that $M_r^\star(q)$ is a linear increasing function of q and $p_q = P_u$.

For $\overline{X}(P_u) \leq q$ we define $\bar{p}$ such that $\overline{X}(\bar{p}) = \min\{q, \overline{X}(P_\ell)\}$. Therefore,

$$\Theta_r(p,q) = \begin{cases} 0 & , \quad P_\ell \leq p \leq \bar{p} \\ q - \overline{X}(p) & , \quad \bar{p} \leq p \leq P_u. \end{cases} \tag{2.35}$$

Under this setting, by Lemma 1 we have

$$\text{argmax}\{M_r(p,q) : \; P_\ell \leq p \leq \bar{p}\} = \bar{p},$$

which implies that

$$M_r^\star(q) = \max\{M_r(p,q) : \; \bar{p} \leq p \leq P_u\},$$

where M_r can be obtained from (2.13) and (2.35) as:

$$M_r(p,q) = (p+h)\cdot\overline{X}(p) - (c+h)\cdot q + c\cdot i.$$

We note that $(p+h)\cdot\overline{X}(p)$ is increasing on $[P_\ell, \bar{P}_h]$ and decreasing on $[\bar{P}_h, P_u]$. Moreover, $q \leq \overline{X}(\bar{P}_h) \;\Leftrightarrow\; \bar{p} \geq \bar{P}_h$.

It follows from the above discussion that

$$M_r^\star(q) = \begin{cases} (P_u + s - c)\cdot q - s\cdot\overline{X}(P_u) + c\cdot i & , \quad q \leq \overline{X}(P_u), \\ (\bar{p} - c)\cdot q + c\cdot i & , \quad \overline{X}(P_u) \leq q \leq \overline{X}(\bar{P}_h), \\ -(c+h)\cdot q + (\bar{P}_h + h)\cdot\overline{X}(\bar{P}_h) + c\cdot i & , \quad \overline{X}(\bar{P}_h) \leq q. \end{cases} \tag{2.36}$$

Corollary A3 in Appendix A indicates that $(\bar{p} - c)\cdot q$ is a pseudoconcave function of q on $(\overline{X}(P_u), \overline{X}(P_\ell))$. Thus, the result follows from (2.36).

From (2.36) it is also clear that

$$\begin{aligned} M_r(\tilde{p}_r, \tilde{q}_r) &= max\{M_r^\star(q) : \; 0 \leq q < \infty\} \\ &= max\{(\bar{p} - c)\cdot q : \; \overline{X}(P_u) \leq q \leq \overline{X}(\bar{P}_h)\} + c\cdot i, \\ &= max\{(p - c)\cdot\overline{X}(p) : \; \bar{P}_h \leq \bar{p} \leq P_u\} + c\cdot i. \end{aligned} \tag{2.37}$$

The maximand in (2.37) is the *riskless profit* function, which is maximized at P_c. According to Corollary A2 we have $P_h < P_c$ which implies that $\bar{P}_h \leq \bar{P}_c$, where $\bar{P}_c = \min\{\max\{P_\ell, P_c\}, P_u\}$. Therefore, the maximizer in (2.37) is $\bar{P}_c$, and we have $\tilde{p}_r = \bar{P}_c$ and $\tilde{q}_r = \overline{X}(\bar{P}_c)$. Since M_r is unimodal, the optimal procurement quantity is determined by an $(\sigma, \ddagger)$ policy, where

$$\ddagger = \overline{X}(\bar{P}_c),$$

and

$$\sigma = min\{q : \ M_r^\star(q) = M_r^\star(\ddagger) - \mathcal{K}\}.$$

It is intuitive that $\tilde{q}_r = \overline{X}(\tilde{p}_r)$, that is we procure up to as much as the demand so that we would not pay any penalty for shortages or leftovers. If $\sigma < i$, however, then it is optimal not to order ($q_r^\star = i$) and under our general setting, $q_r^\star$ need not be equal to $\overline{X}(p_r^\star)$. For this reason, it is interesting to note that, although the demand is deterministic, under the optimal strategy there can be shortages or leftovers.

It also follows from (2.36) that $p_r^\star \in [\bar{P}_h, P_u]$. Thus, if $P_\ell < P_h$, then P_h can be considered as a lower limit on price that is determined by the expected demand in the market and the cost of carrying inventories. It is indicated in Corollary A2' that as h gets larger P_h gets smaller. Therefore, greater inventory costs enable the vendor to set lower prices in order to maximize his profit. To be more precise, if the vendor has more stocks than $\overline{X}(P_h)$, i.e. $i > \overline{X}(P_h)$, then he administers a price of P_h and sells all of his stock. Note that this is a short-term planning decision. For a better business strategy he has to take into account the future beyond one-period.

We have $\Theta(p, q) \geq \Theta_r(p, q)$ from (2.11). Thus, it follows from (2.13) that $\overline{M}(p, q) \leq M_r(p, q)$ which implies $\overline{\Pi}(p, q) \leq \Pi_r(p, q)$. Also, comparing $\overline{M}^\star(q)$ and $M_r^\star(q)$ we conclude that $\overline{M}^\star(q)$ remains below the quasiconcave function $M_r^\star(q)$ and approaches it at both tails. Therefore, we make the same or more profit in deterministic demand case than we expect (mathematically) in probabilistic case, which is intuitive.

2.3.2 Additive Model

Let $G(\cdot)$ be the distribution of ε, then we have

$$\begin{aligned}
x \in [X_1(p), X_2(p)] \ &\Leftrightarrow \ \epsilon \in [X_1(p) - \overline{X}(p), X_2(p) - \overline{X}(p)], \\
F(x; p) &= G(x - \overline{X}(p)), \\
f(x; p) &= g(x - \overline{X}(p)),
\end{aligned}$$

$$\Theta(p, q) = \int_{X_1(p) - \overline{X}(p)}^{q - \overline{X}(p)} G(\epsilon) \cdot d\epsilon,$$

$$\begin{aligned}
\frac{\partial F(x;p)}{\partial p} &= -\frac{d\overline{X}(p)}{dp}\cdot f(x;p), \qquad (2.38)\\
\frac{\partial \Theta(p,q)}{\partial p} &= -\frac{d\overline{X}(p)}{dp}\cdot F(q;p),\\
\frac{\partial^2 \Theta(p,q)}{\partial p^2} &= -\frac{d^2\overline{X}(p)}{dp^2}\cdot F(q;p) + \left(\frac{d\overline{X}(p)}{dp}\right)^2\cdot f(q;p).
\end{aligned}$$

Under these observations, (2.16) and (2.17) are given by:

$$\frac{\partial \overline{M}(p,q)}{\partial p} = q - s\cdot\frac{d\overline{X}(p)}{dp} - \Theta(p,q) + (p+s+h)\cdot\frac{d\overline{X}(p)}{dp}\cdot F(q;p) = 0, \qquad (2.39)$$

and

$$\begin{aligned}
\frac{\partial^2 \overline{M}(p,q)}{\partial p^2} = -s\cdot\frac{d^2\overline{X}(p)}{dp^2} + 2\cdot\frac{d\overline{X}(p)}{dp}\cdot F(q;p) + (p+s+h)\cdot\frac{d^2\overline{X}(p)}{dp^2}\cdot F(q;p)\\
-(p+s+h)\cdot f(q;p)\cdot\left(\frac{d\overline{X}(p)}{dp}\right)^2. \qquad (2.40)
\end{aligned}$$

It is worthwhile to note that (2.38) together with (2.4) imply that

$$\frac{d\overline{X}(p)}{dp} < 0 \quad \Leftrightarrow \quad \frac{\partial F(x;p)}{\partial p} > 0. \qquad (2.41)$$

It is clear that if $p_q \in (P_\ell, P_u)$, then it must satisfy the first order condition $\partial\overline{M}(p,q)/\partial p|_{p_q} = 0$. Evaluating this condition for $q = \tilde{q}$ and considering (2.27) we obtain:

$$\frac{\partial \overline{M}(p,\tilde{q})}{\partial p}|_{\tilde{p}} = \tilde{q} - \Theta(\tilde{p},\tilde{q}) + (\tilde{p}-c)\cdot\frac{d\overline{X}(p)}{dp}|_{\tilde{p}} = 0, \qquad (2.42)$$

which implies $\tilde{p} \geq c$. Moreover, adding and subtracting $\overline{X}(\tilde{p})$ in (2.42) we get:

$$\tilde{q} - \Theta(\tilde{p},\tilde{q}) - \overline{X}(\tilde{p}) + \{\overline{X}(p) + (p-c)\cdot\frac{d\overline{X}(p)}{dp}\}|_{\tilde{p}} = 0. \qquad (2.43)$$

By definition, $\Theta(p,q) \geq q - \overline{X}(p)$. Therefore, the expression in the brackets, which is the derivative of the riskless profit function, evaluated at $\tilde{p}$ must be positive. Thus, we conclude that

$$c \leq \tilde{p} \leq P_c. \qquad (2.44)$$

This result was first proved by Mills [7] for a simple model. Karlin and Carr [3] showed that the same conclusion is true for the model we are studying by a different approach.

Next, we shall discuss the conditions leading to unimodality of $\overline{M}^\star(q)$. Considering (2.30) and (2.20), the sufficient condition (2.32) can be written as:

$$\frac{dF(q;p_q)}{dq} \geq 0 \quad \Leftrightarrow \quad 1 - F(q;p_q) + \frac{2\cdot s}{p_q+s+h}$$

$$- \frac{q - \Theta(p_q, q)}{(p_q + s + h)\cdot(d\overline{X}(p)/dp)^2|_{p_q}} \cdot \{2\cdot\frac{d\overline{X}(p)}{dp} + (p + s + h)\cdot\frac{d^2\overline{X}(p)}{dp^2}\}|_{p_q} \geq 0, \quad (2.45)$$

and

$$\frac{dp_q}{dq}|_{\tilde{q}} \leq 0 \;\Leftrightarrow\; f(\tilde{q};\tilde{p}) \geq \frac{-(h + c)}{(\tilde{p} + s + h)^2\cdot d\overline{X}(p)/dp|_{\tilde{p}}}. \quad (2.46)$$

Note that in (2.45) the sum of first three terms is positive. Thus, if the expression in the brackets is negative (this is true when $\overline{X}(p)$ is linear or concave), then that condition is satisfied. On the other hand, the necessary and sufficient condition (2.34) is given by:

$$(\frac{h + c}{\tilde{p} + s + h})^2 + (\tilde{p} + s + h)\cdot f(\tilde{q};\tilde{p})\cdot\{2\cdot\frac{d\overline{X}(p)}{dp} + (p - c)\cdot\frac{d^2\overline{X}(p)}{dp^2}\}|_{\tilde{p}} \leq 0, \quad (2.47)$$

which implies that the second derivative of the *riskless revenue* function evaluated at $\tilde{p}$ must be negative.

For a given set of problem specifications, unimodality can be verified by testing the validity of the above cited conditions. For example, suppose that the expected demand function is linear, where $\overline{X}(p) = a - b\cdot p$, $a, b > 0$ and $p \in [0, a/b]$ with $c < a/b$. In addition, to prevent negative demand let us assume that $P_u < a/b$ such that $X = \overline{X}(P_u) + \epsilon \geq 0 \;\; \forall\epsilon$. Since $d^2\overline{X}(p)/dp^2 = 0$, it follows from (2.40) that $\partial^2\overline{M}(p,q)/\partial p^2 < 0$ (this observation is essential in achieving better numerical computation performance) and from (2.45) that $dF(q;p_q)/dq \geq 0$. Moreover, (2.46) can be writen as:

$$\frac{dp_q}{dq}|_{\tilde{q}} \leq 0 \;\Leftrightarrow\; f(\tilde{q};\tilde{p}) \geq \frac{h + c}{b\cdot(\tilde{p} + s + h)^2}, \quad (2.48)$$

and (2.47) reduces to

$$\overline{M}^{\star}(q) \text{ is unimodal } \Leftrightarrow f(\tilde{q};\tilde{p}) \geq \frac{(h + c)^2}{2\cdot b\cdot(\tilde{p} + s + h)^3}. \quad (2.49)$$

Clearly, (2.49) is weaker than (2.48). Furthermore, it can be deduced from (2.44) that if

$$f(\tilde{q};\tilde{p}) \geq (h + c)^2/[2\cdot b\cdot(c + s + h)^3], \quad (2.50)$$

then (2.49) will hold.

For a given distribution and the data the conditions (2.48), (2.49) or (2.50) can be tested. For instance, if ϵ has a uniform distribution on $[-\lambda, \lambda]$, then for all $q \in (X_1(p), X_2(p))$:

$$\begin{aligned} f(q;p) &= \frac{1}{2\cdot\lambda}, \\ F(q;p) &= \frac{q - a + b\cdot p + \lambda}{2\cdot\lambda}, && (2.51) \\ \Theta(p,q) &= \frac{(q - a + b\cdot p + \lambda)^2}{4\cdot\lambda}. && (2.52) \end{aligned}$$

From (2.49) we obtain the condition for unimodality as

$$\lambda \leq b\cdot(\tilde{p}+s+h)^3/(h+c)^2, \tag{2.53}$$

or from (2.50) the sufficient condition as

$$\lambda \leq b\cdot(c+s+h)^3/(h+c)^2. \tag{2.54}$$

Under the proposed special case, we can view λ as a measure of demand uncertainty and b as a measure of sensitivity of demand of price changes. From (2.53) we conclude that the less the uncertainty and/or the more the demand sensitivity are, the more unimodality will be favored.

An alternative approach is to solve $\tilde{p}$ and $\tilde{q}$ from (2.27) simultaneously under the hypothesis that $\overline{M}^{\star}(q)$ is unimodal. To this end, we rewrite (2.27) and (2.42), respectively, as:

$$F(\tilde{q};\tilde{p}) = \frac{\tilde{p}+s-c}{\tilde{p}+s+h} = \frac{\tilde{q}-a+b\cdot\tilde{p}+\lambda}{2\cdot\lambda}, \tag{2.55}$$

and

$$\tilde{q}-\Theta(\tilde{p},\tilde{q})-b\cdot(\tilde{p}-c)=0. \tag{2.56}$$

Using (2.52) we solve for $\tilde{q}$:

$$\tilde{q} = \lambda\cdot(\frac{\tilde{p}+s-c}{\tilde{p}+s+h})^2 + b\cdot(\tilde{p}-c). \tag{2.57}$$

Next, substituting (2.57) in (2.55) we get:

$$\frac{\tilde{p}+s-c}{\tilde{p}+s+h} = 1-\sqrt{\frac{a+b\cdot c-2\cdot b\cdot\tilde{p}}{\lambda}}. \tag{2.58}$$

Since under the additive model $\tilde{p} < P_c$ and for a linear expected demand function $P_c = (a+b\cdot c)/2b$, the term in the square root is always defined. After manipulations we rewrite (2.58) as

$$2\cdot(\tilde{p}+s+h)^2\cdot(P_c-\tilde{p})-\lambda\cdot(h+c)^2/b=0, \tag{2.59}$$

which is a polynomial having a local maximum at $[2\cdot P_c-(h+s)]/3$. It follows that this function has at least one and at most two positive roots. In addition, one of the positive roots is always located in the interval $([2\cdot P_c-(h+s)]/3, P_c)$. Since the third critical point, on the feasible price range, to make a local minimum does not exist, we conclude that $\overline{M}^{\star}(q)$ is unimodal.

2.3.3 Multiplicative Model

Let $G(\cdot)$ be the distribution of ϵ, then we have

$$\begin{aligned} x\in[X_1(p),X_2(p)] &\Leftrightarrow \epsilon\in[X_1(p)/\overline{X}(p), X_2/\overline{X}(p)],\\ F(x;p) &= G(x/\overline{X}(p)), \end{aligned}$$

$$
\begin{aligned}
f(x;p) &= g(x/\overline{X}(p))/\overline{X}(p), \\
\Theta(p,q) &= \int_{X_1(p)/\overline{X}(p)}^{q/\overline{X}(p)} G(\epsilon)\cdot d\epsilon, \\
\frac{\partial F(x;p)}{\partial p} &= -\frac{d\overline{X}(p)}{dp}\cdot\frac{x}{\overline{X}(p)}\cdot f(x;p), \qquad (2.60)\\
\frac{\partial\Theta(p,q)}{\partial p} &= -\frac{d\overline{X}(p)}{dp}\cdot\frac{q\cdot F(q;p)-\Theta(p,q)}{\overline{X}(p)}, \\
\frac{\partial^2\Theta(p,q)}{\partial p^2} &= -\frac{d^2\overline{X}(p)}{dp^2}\cdot\frac{q\cdot F(q;p)-\Theta(p,q)}{\overline{X}(p)} + \left(\frac{d\overline{X}(p)}{dp}\cdot\frac{q}{\overline{X}(p)}\right)^2\cdot f(q;p).
\end{aligned}
$$

Under these observations (2.16) and (2.17) are given by:

$$
\frac{\partial\overline{M}(p,q)}{\partial p} = q - s\cdot\frac{d\overline{X}(p)}{dp} - \Theta(p,q) + (p+s+h)\cdot\frac{d\overline{X}(p)}{dp}\cdot\frac{q\cdot F(q;p)-\Theta(p,q)}{\overline{X}(p)}, \qquad (2.61)
$$

and

$$
\begin{aligned}
\frac{\partial^2\overline{M}(p,q)}{\partial p^2} = -s\cdot\frac{d^2\overline{X}(p)}{dp^2} &+ \frac{q\cdot F(q;p)-\Theta(p,q)}{\overline{X}(p)}\cdot\left(2\cdot\frac{d\overline{X}(p)}{dp} + \frac{d^2\overline{X}(p)}{dp^2}\cdot(p+s+h)\right) \\
&- (p+s+h)\cdot f(q;p)\cdot\left(\frac{d\overline{X}(p)}{dp}\cdot\frac{q}{\overline{X}(p)}\right)^2. \qquad (2.62)
\end{aligned}
$$

Clearly condition (2.41) also holds for the multiplicative model.

If $p_q \in (P_\ell, P_u)$, then it must satisfy the first order condition $\partial\overline{M}(p,q)/\partial p|_{p_q} = 0$. Evaluating (2.16) at p_q, setting it equal to zero and arranging terms we get

$$
\begin{aligned}
\frac{\partial\overline{M}(p,q)}{\partial p}|_{p_q} = q\cdot[1-F(q;p_q)] &+ \frac{q\cdot F(q;p_q)-\Theta(p_q,q)}{\overline{X}(p_q)}\cdot\{\overline{X}(p)+(p+h)\cdot\frac{d\overline{X}(p)}{dp}\}|_{p_q} \\
&- s\cdot\frac{d\overline{X}(p)}{dp}|_{p_q}\cdot\frac{\overline{X}(p_q)-q\cdot F(q;p_q)+\Theta(p_q,q)}{\overline{X}(p_q)} = 0. \qquad (2.63)
\end{aligned}
$$

Since $\Theta(p,q) \geq q - \overline{X}(p)$, we have $\Theta(p,q) + \overline{X}(p) - q\cdot F(q;p) \geq 0$. Thus, the first and the third terms in (2.63) are positive. Moreover, we note that $q\cdot F(q;p) - \Theta(p,q) \geq 0$. Therefore, (2.63) implies that

$$
\{\overline{X}(p)+(p+h)\cdot\frac{d\overline{X}(p)}{dp}\}|_{p_q} \leq 0 \;\Leftrightarrow\; p_q \geq \bar{P}_h.
$$

Furthermore, evaluating (2.63) at $\tilde{q}$ and rearranging the terms we obtain

$$
\frac{\tilde{q}}{\overline{X}(\tilde{p})}\cdot\{\overline{X}(p)+(p-c)\cdot\frac{d\overline{X}(p)}{dp}\}|_{\tilde{p}} - \frac{\Theta(\tilde{p},\tilde{q})}{\overline{X}(\tilde{p})}\cdot\{\overline{X}(p)+(p+h)\cdot\frac{d\overline{X}(p)}{dp}\}|_{\tilde{p}}
$$

$$- s \cdot \frac{d\overline{X}(p)}{dp}|_{\tilde{p}} \cdot \frac{\overline{X}(\tilde{p}) - \tilde{q} + \Theta(\tilde{p}, \tilde{q})}{\overline{X}(\tilde{p})} = 0. \tag{2.64}$$

The second term is positive, since $\tilde{p} \geq \bar{P}_h$, and so is the third term. Therefore, we must have

$$\{\overline{X}(p) + (p - c) \cdot \frac{d\overline{X}(p)}{dp}\}|_{\tilde{p}} \leq 0 \;\Leftrightarrow\; \tilde{p} \geq P_c > c.$$

This result is the same as Karlin and Carr's [3] conclusion, which was proved by a different approach than ours.

Considering (2.30) and (2.20), the sufficient conditions for unimodality of $\overline{M}^{\star}(q)$, (2.32), can be written, respectively, as:

$$\frac{dF(q;p_q)}{dq} \geq 0 \;\Leftrightarrow\; 1 - F(q;p_q) + \frac{2 \cdot s \cdot \overline{X}(p_q)}{(p_q + s + h) \cdot q}$$

$$- \frac{[q - \Theta(p,q)] \cdot \overline{X}(p)}{(p + s + h) \cdot q \cdot (d\overline{X}(p)/dp)^2}|_{p_q} \cdot \{2 \cdot \frac{d\overline{X}(p)}{dp} + (p + s + h) \cdot \frac{d^2\overline{X}(p)}{dp^2}\}|_{p_q} \geq 0, \tag{2.65}$$

and

$$\frac{dp_q}{dq}|_{\tilde{q}} \leq 0 \;\Leftrightarrow\; f(\tilde{q};\tilde{p}) \geq \frac{-\overline{X}(\tilde{p}) \cdot (h + c)}{\tilde{q} \cdot (\tilde{p} + s + h)^2 \cdot d\overline{X}(p)/dp|_{\tilde{p}}}. \tag{2.66}$$

The sum of the first three terms in (2.65) is positive. Thus, if the expression in the brackets is negative (that is true when $\overline{X}(p)$ is linear or concave), then that condition is satisfied. On the other hand, the necessary and sufficient condition (2.34) is given by:

$$\left(\frac{h + c}{\tilde{p} + s + h}\right)^2 + (\tilde{p} + s + h) \cdot f(\tilde{q};\tilde{p}) \cdot [\tilde{q} - \Theta(\tilde{p}, \tilde{q})] \cdot \left\{\frac{2 \cdot d\overline{X}(p)/dp}{\overline{X}(p)} - \frac{d^2\overline{X}(p)/dp^2}{d\overline{X}(p)/dp}\right\}|_{\tilde{p}} \leq 0, \tag{2.67}$$

which implies that the expected demand must be "normal" at $\tilde{p}$ or, equivalently, expected marginal revenue to be decreasing at $\tilde{p}$.

The above cited conditions can be tested for a given set of problem specifications. For instance, under a linear expected demand assumption, which is described in the previous section, it follows from (2.62) that $\partial^2 \overline{M}(p,q)/\partial p^2 < 0$ and from (2.65) that $dF(q;p_q)/dq \geq 0$. Moreover, (2.66) can be written as:

$$\frac{dp_q}{dq}|_{\tilde{q}} \leq 0 \;\Leftrightarrow\; f(\tilde{q};\tilde{p}) \geq \frac{(h + c) \cdot \overline{X}(\tilde{p})}{b \cdot \tilde{q} \cdot (\tilde{p} + s + h)^2}, \tag{2.68}$$

and (2.67) reduces to

$$\overline{M}^{\star}(q) \text{ is unimodal } \;\Leftrightarrow\; f(\tilde{q};\tilde{p}) \geq \frac{(h + c)^2 \cdot \overline{X}(\tilde{p})}{2 \cdot b \cdot [\tilde{q} - \Theta(\tilde{p}, \tilde{q})] \cdot (\tilde{p} + s + h)^3}. \tag{2.69}$$

If ϵ is exponential, for example, then for all $q \in (0, \infty)$:

$$\begin{aligned} f(q;p) &= \frac{e^{-q/\overline{X}(p)}}{\overline{X}(p)}, \\ F(q;p) &= 1 - e^{-q/\overline{X}(p)}, \\ \Theta(p,q) &= q - \overline{X}(p) \cdot F(q;p). \end{aligned}$$

Using these relationships and (2.27), the unimodality condition (2.69) could be written as:

$$2 \cdot \tilde{p}^2 + (3 \cdot h + 4 \cdot s - c) \cdot \tilde{p} + 2 \cdot (s + h) \cdot (s - c) - \frac{a}{b} \cdot (h + c) \geq 0. \tag{2.70}$$

The quadratic form in (2.70) has a critical point at $-(3 \cdot h + 4 \cdot s - c)/4$ which is less than c, hence, it is also less than P_c. Since $\tilde{p} \geq P_c$, if

$$2 \cdot P_c^2 + (3 \cdot h + 4 \cdot s - c) \cdot P_c + 2 \cdot (s + h) \cdot (s - c) - \frac{a}{b} \cdot (h + c) \geq 0, \tag{2.71}$$

then (2.70) will hold. After necessary manipulations, (2.71) reduces to

$$(\frac{a}{b})^2 + (h + 4 \cdot s - c) \cdot \frac{a}{b} + (4 \cdot s^2 + 4 \cdot h \cdot s - h \cdot c) \geq 0, \tag{2.72}$$

where the critical point of the quadratic form is $-(h + 4 \cdot s - c)/2$ which is less than c. Since $a/b > c$ under linear expected demand assumption, condition (2.72) can be rewritten as:

$$a/b \geq \frac{1}{2} \cdot [\sqrt{(h + c) \cdot (h + c - 8 \cdot s)} - (h + 4 \cdot s - c)]. \tag{2.73}$$

We note that (2.72) holds when $h + c < 8 \cdot s$ (i.e. when the expression in the square root is negative). Otherwise, we observe that

$$a/b \geq c - 2 \cdot s \geq \frac{1}{2} \cdot [\sqrt{(h + c) \cdot (h + c - 8 \cdot s)} - (h + 4 \cdot s - c)],$$

which holds by the natural assumption that $a/b > c$. Therefore, $\overline{M}^*(q)$ is unimodal for exponential multiplicative demand model. Zabel [18] arrived at the same conclusion, under some restrictions, for the case where $s = 0$.

Since pricing decision affects the period ending inventory level, the analysis of the multi-period model does not trivially follow from the analysis of the one-period model. In the next chapter we shall dwell on this issue.

Chapter 3

Multi-Period Model

In this chapter, we extend the planning horizon more than one period and try to characterize the optimal procurement and pricing decisions. In this regard, before getting into the mathematical model we shall first describe the multi-period setting.

We assume that the planning horizon is divided into N review periods, which are indexed by n. The last period, n=1, is the end of the planning horizon. If there are any shortages in this period, they will be lost. Also, we assume that there is no salvage value for the leftovers. At the beginning of each period, the vendor decides how much to order, $q_n - i_n$, and what price to administer, p_n, until the next decision point. i_n is the beginning inventory level before ordering and q_n is the beginning inventory level after ordering in period n. With these decisions, the vendor tries to maximize the mathematical expectation of the sum of current period's profit and the discounted profit of the remaining periods, which is denoted by Π_n. We assume that $i_N \geq 0$, so that the decision problem is not initiated with an unaccountable debt. For simplicity we assume constant unit holding, shortage, procurement costs and a fixed ordering cost, which are denoted by h, s, c and $\mathcal{K}$, respectively. We also consider a common discount factor for each period and denote it by α. Furthermore, we assume that procurement leadtime is negligibly short compared to the length of a period and all payments realize without any significant delay or additional cost.

Let us consider period n, where $1 \leq n < N$. It is clear that i_n depends on p_{n+1}, q_{n+1} and the backlogging rule. Therefore, in any period, except the last one, the pricing decision can not be made independent of the future periods. Moreover, it follows from the analysis of the one-period model that procurement quantity and pricing decisions could not be analytically decoupled. Hence, the overall optimization problem, that is the determination of optimal procurement quantity and price for all periods, does not follow directly from the classical multi-period model. In other words, since price is a decision variable which is a factor that affects demand, we need to extend the analysis of the classical multi-period model which employs price only as a unit revenue.

A conceptual complication arises in relation to unsatisfied customers when there is a shortage in any intermediate period. In the classical model, it is customary to assume a backlogging rule which allows all customers to wait another period (full backlogging), some customers to wait another period (partial backlogging) or all customers to quit (lost sales). When there is a pricing decision, however, the willingness of a customer to wait one more period may be contingent upon price. That is, there might be a bargain between an unsatisfied customer and the vendor for their mutual benefit. Therefore, it is likely that such a customer-vendor interaction will affect the optimal solution. To study this, we could employ various backlogging rules in our model. For instance, we may assume that the vendor issues a "rain check" for customers who are willing to wait, provided that they could pay the current price in the future. Under the multi-period model that we are studying, we may assume that, if there is a shortage, then, upon mutual agreement, the customers are to wait until their demand is satisfied regardless of the price; wait as long as they could pay the current period's price at any time in the future; wait only one period at any price; wait only one period at the current period's price; or, we may assume that the vendor does not allow backlogging. It is intuitive to expect and it will be clear in the following sections that a backlogging rule induces a special structure into the model. Since a variety of different backlogging rules can be employed, it is difficult to generalize possible vendor-customer relationships without making further assumptions. In fact, this generalization will not be argued in this study. Instead, we shall be analysing the model under three different backlogging rules to demonstrate the characteristic differences between them. Not to complicate the analysis further, it will be assumed that the vendor-customer relationship is homogeneous; that is, there is no difference between the customers, and the vendor is not practicing any price discrimination. It is also assumed that the backlogs are cleared before satisfying the current demand in any period.

3.1 Mathematical Model

Under the proposed assumptions, the expected n-period profit can be expressed as a backward dynamic programming recursion:

$$\overline{\Pi}_n(i_n, p_N, p_{N-1}, \ldots, p_n, q_n) = \overline{M}_n(i_n, p_N, p_{N-1}, \ldots, p_n, q_n) - \mathcal{K}\cdot\delta(q_n - i_n), \tag{3.1}$$

where $\overline{M}_n$ is the expected n-period pseudo-profit function (i.e., the expected profit regardless of the ordering cost) which will be defined later. $\overline{\Pi}_n$ is expressed not only as a function of current period's decision variables, p_n and q_n, but also in terms of all previous pricing decision variables which might be employed by a backlogging rule in general. We adopt the convension that if the backlogging rule does not require a subset of the price variables p_N through p_{n+1}, then those will be simply dropped from the notation. For instance, according to the fifth backlogging rule none of p_N through p_{n+1} are needed. It is also clear that the decision variables p_N through p_{n+1} are needed only when $q_n < 0$. That is, for any $p'_N, p'_{N-1}, \ldots, p'_{n+1} \in [P_\ell, P_u]$ it follows that

$$q_n \geq 0 \Rightarrow \overline{M}_n(i_n, p'_N, p'_{N-1}, \ldots, p'_{n+1}, p_n, q_n) = \overline{M}_n(i_n, p_N, p_{N-1}, \ldots, p_{n+1}, p_n, q_n), \qquad (3.2)$$

$\forall p_N, p_{N-1}, \ldots, p_{n+1} \in [P_\ell, P_u]$.

Furthermore, we introduce the following notation:

$$\begin{aligned} \overline{\Pi}^{\star}_n(i_n, p_N, p_{N-1}, \ldots, p_{n+1}, q_n) &= max\{\overline{\Pi}_n(i_n, p_N, p_{N-1}, \ldots, p_n, q_n) : \ p_n \in [P_\ell, P_u]\} \\ &= \overline{M}^{\star}_n(i_n, p_N, p_{N-1}, \ldots, p_{n+1}, q_n) - \mathcal{K} \cdot \delta(q_n - i_n), \end{aligned} \qquad (3.3)$$

and

$$\overline{\Pi}^{\star\star}_n(i_n, p_N, p_{N-1}, \ldots, p_{n+1}) = max\{\overline{\Pi}^{\star}_n(i_n, p_N, p_{N-1}, \ldots, p_{n+1}, q_n) : \ q_n \geq i_n\}, \qquad (3.4)$$

where

$$\overline{M}^{\star}_n(i_n, p_N, p_{N-1}, \ldots, p_{n+1}, q_n) = max\{\overline{M}_n(i_n, p_N, p_{N-1}, \ldots, p_n, q_n) : \ p_n \in [P_\ell, P_u]\}. \qquad (3.5)$$

Therefore, the overall optimization problem is to determine the optimal decision variables $p^{\star}_n$ and $q^{\star}_n$ for all n, which jointly maximize $\overline{\Pi}_N$ for a given i_N.

Since there is no cost of pricing, intuitively, the vendor must reconsider pricing at every decision epoch, because this can only improve his objective. However, the same argument does not hold for the procurement decision, because there is an ordering cost. If pricing decision is ignored, then the classical inventory theory indicates that $q^{\star}_n$ is given by an (σ_n, Σ_n) policy. With the addition of pricing decision, however, we intuitively expect to have a different optimal control policy, which might inherit an (σ_n, Σ_n) type policy for the determination of $q^{\star}_n$. If such a policy exists, then it would operate on the $\overline{M}^{\star}_n$ function which must satisfy the separation property defined by:

$$\overline{M}^{\star}_n(i_n, p_N, p_{N-1}, \ldots, p_{n+1}, q_n) = \overline{M}^{\star}_n(0, p_N, p_{N-1}, \ldots, p_{n+1}, q_n) + m_n(i_n), \qquad (3.6)$$

where m_n is a continuous function. Thus, it is essential to study the characteristics of this function which leads us to an optimal control scheme. To this end, we include here the definition of a class of functions, which is introduced by Porteus [11], that will be referred to and extended later.

Definition 1. *$\mathcal{C}(\gamma, \mathcal{K})$ is a set of univariate continuous functions which are:*
(a) increasing on $(-\infty, \gamma]$;

(c) $\mathcal{K}$-decreasing on $[\gamma, \infty)$,
such that they
(d) have a finite maximizer on $(0, \infty)$.
It is clear that if a function belongs to $\mathcal{C}(\gamma, \mathcal{K})$, then it is quasi-$\mathcal{K}$-concave .

Suppose that $\overline{M}_n^\star$ satisfies condition (3.6), then we can define a critical on-hand inventory level often referred to as "order-up-to level":

$$\Sigma_n = argsup\{\overline{M}_n^\star(0, p_N, p_{N-1}, \ldots, p_{n+1}, q_n) : \ 0 < q_n < \infty\}. \tag{3.7}$$

It follows from (3.2) that Σ_n is independent of $p_N, p_{N-1}, \ldots,$ or p_{n+1}. Also, it is intuitive to expect that Σ_n is finite. Moreover, if $\overline{M}_n^\star$ is independent of $p_N, p_{N-1}, \ldots, p_{n+1}$ and $\overline{M}_n^\star(0, q_n) \in \mathcal{C}(\gamma, \mathcal{K})$ for some $\gamma \in \mathcal{R}^+$, then there exists a "reorder level" σ_n, which is defined by

$$\sigma_n = min\{\sigma : \ \overline{M}_n^\star(0, \sigma) = \overline{M}_n^\star(0, \Sigma_n) - \mathcal{K}\}, \tag{3.8}$$

and $q_n^\star$ is obtained from

$$q_n^\star = i_n + (\Sigma_n - i_n)\cdot\delta(\sigma_n - i_n). \tag{3.9}$$

According to this (σ_n, Σ_n) policy we rewrite (3.4) as:

$$\overline{\Pi}_n^{\star\star}(i_n) = \begin{cases} \overline{M}_n^\star(i_n, \Sigma_n) - \mathcal{K} & , \quad i_n \leq \sigma_n, \\ \overline{M}_n^\star(i_n, i_n) & , \quad \sigma_n < i_n. \end{cases} \tag{3.10}$$

On the other hand, if $\overline{M}_n^\star$ satisfies (3.6) but depends on $p_N, p_{N-1}, \ldots,$ or p_{n+1}, then it is clear that the solution for σ_n, given by (3.8), can depend on a subset of these price variables. In this case an (σ_n, Σ_n) type policy is not optimal in general, and we might consider a new criterion for the determination of $q_n^\star$. To this end, we shall introduce a set of regularity conditions imposed on $\overline{M}_n^\star$ under which an optimal control scheme can still be devised. These conditions establish a class of functions characterized by the following

Definition 2. *$\mathcal{C}'(\gamma, \mathcal{K})$ is a set of univariate continuous functions which are:*
(a) quasiconvex on $(-\infty, 0]$;
(b) increasing on $[0, \gamma]$;
(c) $\mathcal{K}$-decreasing on $[\gamma, \infty)$,
such that they
(d) have a finite maximizer on $(0, \infty)$.
It is clear that $\mathcal{C}(\gamma, \mathcal{K}) \subset \mathcal{C}'(\gamma, \mathcal{K})$.

Suppose that there exists $\gamma \in \mathcal{R}^+$ for all $p_N, p_{N-1}, \ldots, p_{n+1} \in [P_\ell, P_u]$ such that $\overline{M}_n^\star(0, p_N, p_{N-1}, \ldots, p_{n+1}, q_n) \in \mathcal{C}'(\gamma, \mathcal{K})$, then $q_n^\star$ is obtained from

$$\begin{aligned} q_n^\star \ = \ & i_n + (\Sigma_n - i_n)\cdot\delta(\Sigma_n - i_n)\cdot\delta\Big(\overline{M}_n^\star(i_n, p_N, p_{N-1}, \ldots, p_{n+1}, \Sigma_n) - \mathcal{K} \\ & - \overline{M}_n^\star(i_n, p_N, p_{N-1}, \ldots, p_{n+1}, i_n)\Big) \end{aligned}$$

$$= i_n + (\Sigma_n - i_n)\cdot\delta(\Sigma_n - i_n)\cdot\delta\left(\overline{M}^\star_n(0, p_N, p_{N-1}, \ldots, p_{n+1}, \Sigma_n) - \mathcal{K} - \overline{M}^\star_n(0, p_N, p_{N-1}, \ldots, p_{n+1}, i_n)\right) \quad (3.11)$$

According to this policy we rewrite (3.4) as:

$$\overline{\Pi}^{\star\star}_n(i_n, p_N, p_{N-1}, \ldots, p_{n+1}) = m_n(i_n) + \begin{cases} max\{\overline{M}^\star_n(0, p_N, p_{N-1}, \ldots, p_{n+1}, i_n), \overline{M}^\star_n(0, p_N, p_{N-1}, \ldots, p_{n+1}, \Sigma_n) - \mathcal{K}\} & , \quad i_n < \Sigma_n, \\ \overline{M}^\star_n(0, p_N, p_{N-1}, \ldots, p_{n+1}, i_n) & , \quad \Sigma_n \leq i_n. \end{cases} \quad (3.12)$$

Comparing the policies given by (3.9) and (3.11) we observe that there is a reorder point in the former but not in the latter. In principle, the latter also functions like an (σ_n, Σ_n) policy, but there is not a single level of critical inventory that triggers the ordering mechanism. That is, the previous pricing decisions as well as the beginning inventory level must be taken into account in reorder decisions. It is also worth mentioning that, the price which maximizes $\overline{M}_n$ for $q_n = i_n$ is needed for the latter, whereas for the former it is needed only when $\sigma_n < i_n$. This implies that more computational work is required under the latter policy.

3.2 Special Cases

In this section, we introduce three special backlogging rules and establish the pseudo-profit function under each of these characterizations.

3.2.1 Case I

Suppose that the vendor does not allow backlogging. Under this rule the pseudo-profit function for $n \geq 1$ is expressed as:

$$M_n(i_n, p_n, q_n) = -c\cdot(q_n - i_n) + \begin{cases} \alpha\cdot\overline{\Pi}^{\star\star}_{n-1}(0) + p_n\cdot q_n - s\cdot(X(p_n) - q_n) & , \quad 0 \leq q_n \leq X(p_n), \\ \alpha\cdot\overline{\Pi}^{\star\star}_{n-1}(q_n - X(p_n)) + p_n\cdot X(p_n) - h\cdot(q_n - X(p_n)) & , \quad X(p_n) \leq q_n, \end{cases} \quad (3.13)$$

where $X(p_n)$ is the random demand in period n at a price level of p_n, $\overline{\Pi}^{\star\star}_0 = 0$ and s represents the unit penalty when the demand is lost. Note that (3.13) satisfies

$$M_n(i_n, p_n, q_n) = M_n(0, p_n, q_n) + c\cdot i_n, \quad (3.14)$$

which implies that the separation property given by (3.6) holds for $n = 1, \ldots, N$ with $m_n(i_n) = c\cdot i_n$.

3.2.2 Case II

Suppose that if there is shortage, then the customers are willing to wait only one period and pay the current period's price. Under this backlogging rule the pseudo-profit function for $n > 1$ is expressed as:

$$M_n(i_n, p_{n+1}, p_n, q_n) = -c\cdot(q_n - i_n)$$

$$+ \begin{cases} \alpha\cdot\overline{\Pi}^{**}_{n-1}(-X(p_n), p_n) + \alpha\cdot p_n\cdot X(p_n) + p_{n+1}\cdot q_n + s\cdot q_n & , \quad q_n \leq 0, \\ \alpha\cdot\overline{\Pi}^{**}_{n-1}(q_n - X(p_n), p_n) + p_n\cdot q_n + \alpha\cdot p_n\cdot(X(p_n) - q_n) & , \quad 0 \leq q_n \leq X(p_n), \\ \alpha\cdot\overline{\Pi}^{**}_{n-1}(q_n - X(p_n), p_n) + p_n\cdot X(p_n) - h\cdot(q_n - X(p_n)) & , \quad X(p_n) \leq q_n, \end{cases} \tag{3.15}$$

where s represents the unit penalty when the backlogged demand is lost. Note that the model does not assign any specific penalty for a shortage if it occurs for the first time. Since $i_N \geq 0$, the q_n range of $(-\infty, 0)$ is ignored for $n = N$. Moreover, we let $p_{N+1} = 0$.

Suppose $i_n < 0$, that is there is a backlog from period $n + 1$. If the vendor decides not to procure anything or to procure some but not enough to cover the whole backlog, that is $q_n < 0$, then two things will occur according to the backlogging rule: (i) all of the demand in period n will be backlogged to period $n - 1$ and (ii) the unsatisfied portion of the backlog from period $n+1$ will be lost. To account for (i) we add $\alpha{\cdot}p_n{\cdot}X(p_n)$ as a revenue to the profit function. That is, the vendor promises to supply $X(p_n)$ at p_n the next period and the customers are willing to wait one more period. Since the payment takes place in period $n - 1$, we must discount it by α to period n. Note that with this formulation we add the revenue, which will be collected in the next period into the current period, ahead of time. However, it is possible that the vendor may find it more profitable not to satisfy all of the backlog or some portion of it in period $n - 1$. If this happens, then we must deduct the revenue that corresponds to unsatisfied portion of the backlog from period $n - 1$'s revenue. Thus, in period n we must consider an analogous deduction for period $n + 1$ to account for (ii). This amount is given by $p_{n+1}\cdot q_n$, where p_{n+1} is the price promised to the customers in period $n + 1$ and q_n is the amount of the backlog from period $n + 1$ which is not satisfied in period n. Note that since $q_n \leq 0$, $p_{n+1}\cdot q_n$ represents a negative cash flow (loss). Moreover, since the backlog from period n to $n - 1$ is $X(p_n)$ the discounted $n - 1$ period profit is given by $\alpha\cdot\overline{\Pi}^{**}_{n-1}(-X(p_n), p_n)$.

If $0 \leq q_n \leq X(p_n)$, then the vendor decides to satisfy all of the back orders, if any, and some portion of the demand in period n. Thus, we add the discounted revenue $\alpha \cdot p_n \cdot (X(p_n) - q_n)$ to period n's profit supposing that the backlogged amount $(X(p_n) - q_n)$ will be satisfied in the next period at a price of p_n.

The last period, i.e. $n = 1$, is a lost sales model. Therefore, we have

$$M_1(i_1, p_2, p_1, q_1) = -c\cdot(q_1 - i_1) + \begin{cases} p_2\cdot q_1 - s\cdot(X(p_1) - q_1) & , \quad q_1 \le 0, \\ p_1\cdot q_1 - s\cdot(X(p_1) - q_1) & , \quad 0 \le q_1 \le X(p_1), \\ p_1\cdot X(p_1) - h\cdot(q_1 - X(p_1)) & , \quad X(p_1) \le q_1, \end{cases} \quad (3.16)$$

where $p_2\cdot q_1$ represents the lost revenue due to not meeting the demand which is unsatisfied in period 2 and carried into period 1.

Finally, considering (3.15) and (3.16) we identify a functional simplification:

$$M_n(i_n, p_{n+1}, p_n, q_n) = M_n(0, p_{n+1}, p_n, q_n) + c\cdot i_n, \quad (3.17)$$

which implies that the separation property given by (3.6) holds for $n = 1, \ldots, N$ with $m_n(i_n) = c\cdot i_n$.

3.2.3 Case III

Suppose that if there is shortage, then the customers agree to wait one more period and pay that period's price. Under this backlogging rule the pseudo-profit function for $n > 1$ is expressed as:

$$M_n(i_n, p_n, q_n) = -c\cdot(q_n - i_n) - p_n\cdot i_n\cdot\delta(-i_n)$$

$$+ \begin{cases} \alpha\cdot\overline{\Pi}^{**}_{n-1}(-X(p_n)) + p_n\cdot q_n + s\cdot q_n & , \quad q_n \le 0, \\ \alpha\cdot\overline{\Pi}^{**}_{n-1}(q_n - X(p_n)) + p_n\cdot q_n & , \quad 0 \le q_n \le X(p_n), \\ \alpha\cdot\overline{\Pi}^{**}_{n-1}(q_n - X(p_n)) + p_n\cdot X(p_n) - h\cdot(q_n - X(p_n)) & , \quad X(p_n) \le q_n, \end{cases} \quad (3.18)$$

where s represents the unit penalty when the backlogged demand is lost. Since $i_N \ge 0$, the q_n range of $(-\infty, 0)$ is ignored for $n = N$.

Suppose $i_n < 0$, that is there is a backlog from period $n + 1$, then the revenue in period n is equal to $p_n\cdot(q_n - i_n)$. On the other hand, if $i_n > 0$, then the revenue is $p_n\cdot q_n$ for $q_n \le X(p_n)$ and it is $p_n\cdot X(p_n)$ for $X(p_n) \le q_n$. Since the term $p_n\cdot i_n$ appears only when $i_n < 0$, we represent it by $p_n\cdot i_n\cdot\delta(-i_n)$.

The last period, i.e. $n = 1$, is a lost sales model. Therefore, we have

$$M_1(i_1, p_1, q_1) = -c\cdot(q_1 - i_1) - p_1\cdot i_1\cdot\delta(-i_1) + \begin{cases} p_1\cdot q_1 - s\cdot(X(p_1) - q_1) & , \quad q_1 \le X(p_1), \\ p_1\cdot X(p_1) - h\cdot(q_1 - X(p_1)) & , \quad X(p_1) \le q_1. \end{cases} \quad (3.19)$$

In the next two sections we shall study these special cases separately under deterministic and probabilistic demand models to characterize possible optimal decision policies.

3.3 Deterministic Demand

If there is no uncertainty, then the relationship between demand and price in each period is characterized by the $\overline{X}(p_n)$ function. That is, we replace $X(p_n)$ by $\overline{X}(p_n)$, M by $\overline{M}$ and Π by $\overline{\Pi}$. In what follows we shall analyse each special case under deterministic demand assumption.

3.3.1 Special Case I

We shall prove that, under special case I, $\overline{M}_n^{\star} \in \mathcal{C}(\overline{X}(\bar{P}_c), \mathcal{K})$ such that the optimal procurement policy is given by (3.9). To this end, we shall follow an induction proof. First, we shall prove that $\overline{M}_1^{\star} \in \mathcal{C}(\overline{X}(\bar{P}_c), \mathcal{K})$, which will be the basic step. Then, we shall assume that that $\overline{M}_{n-1}^{\star} \in \mathcal{C}(\overline{X}(\bar{P}_c), \mathcal{K})$ such that $\overline{\Pi}_{n-1}^{\star\star}$ is given by (3.10). Finally, we shall demonstrate that $\overline{M}_n^{\star} \in \mathcal{C}(\overline{X}(\bar{P}_c), \mathcal{K})$, which will complete the proof.

Lemma 3. $\overline{M}_1^{\star}(i_1, q_1) \in \mathcal{C}(\overline{X}(\bar{P}_c), \mathcal{K})$.

Proof. The proof follows from Theorem 2, which indicates that $\overline{M}_1^{\star}$ is a quasiconcave function of q_1 on $[0, \infty)$ and $\Sigma_1 = \overline{X}(\bar{P}_c)$.

Therefore, $q_1^{\star}$ is obtained from (3.9) and $\overline{\Pi}_1^{\star\star}$ is given by (3.10).

We now consider $\overline{M}_n^{\star}$. We represent this function as:

$$\overline{M}_n^{\star}(i_n, q_n) = max\{\overline{M}_n^{(1)}(i_n, q_n) \ , \ \overline{M}_n^{(2)}(i_n, q_n)\}, \tag{3.20}$$

where

$$\overline{M}_n^{(1)}(i_n, q_n) = max\{\overline{M}_n(i_n, p_n, q_n) : \ P_\ell \leq p_n \leq \bar{p}\} \tag{3.21}$$

$$\overline{M}_n^{(2)}(i_n, q_n) = max\{\overline{M}_n(i_n, p_n, q_n) : \ \bar{p} \leq p_n \leq P_u\}, \tag{3.22}$$

and $\bar{p}$ is such that:

$$\overline{X}(\bar{p}) = max\{min\{q_n, \overline{X}(P_\ell)\}, \overline{X}(P_u)\}. \tag{3.23}$$

Hence, $\overline{M}_n^{(1)}$ and $\overline{M}_n^{(2)}$ are of two complementary subproblems which are related to each other through $\bar{p}$, which in turn depends on q_n. Note that if $p_n \in [P_\ell, \bar{p}]$ (or $p_n \in [\bar{p}, P_u]$), then $q_n \leq$ (or $\geq$) $\overline{X}(p_n)$. Therefore, $\overline{M}_n^{(1)}$ (or $\overline{M}_n^{(2)}$) represents the pseudo-profit under a pricing policy that keeps the period ending inventory level at or below (or above) zero.

Since q_n is not defined on $(-\infty, 0)$, condition (a) of the definition of $\mathcal{C}(\gamma, \mathcal{K})$ is irrelevant here. We now establish the validity of condition (b) of the same definition:

Lemma 4. $\overline{M}_n^{\star}(i_n, q_n)$ *is an increasing function of* q_n *on* $[0, \overline{X}(\bar{P}_c)]$ *for* $N \geq n \geq 2$.

Proof. We shall demonstrate that $\overline{M}_n^{(1)}$ and $\overline{M}_n^{(2)}$ are increasing functions of q_n on $[0, \overline{X}(\bar{P}_c)]$, which implies that $\overline{M}_n^{\star}$, given by (3.20), is also an increasing function of q_n. From (3.13) and (3.21) it follows that

$$\overline{M}_n^{(1)}(i_n, q_n) = max\{\alpha \cdot \overline{\Pi}_{n-1}^{\star\star}(0) + p_n \cdot q_n - s \cdot \overline{X}(p_n) + (s-c) \cdot q_n + c \cdot i_n : \ P_\ell \leq p_n \leq \bar{p}\},$$

where the maximand $p_n \cdot q_n - s \cdot \overline{X}(p_n)$ is an increasing function of p_n. Hence, we have

$$
\begin{aligned}
\overline{M}_n^{(1)}(i_n, q_n) &= \alpha \cdot \overline{\Pi}_{n-1}^{\star\star}(0) + (\bar{p} + s - c) \cdot q_n - s \cdot \overline{X}(\bar{p}) + c \cdot i_n \\
&= \alpha \cdot \overline{\Pi}_{n-1}^{\star\star}(0) + c \cdot i_n + \begin{cases} (P_u + s - c) \cdot q_n - s \cdot \overline{X}(P_u), & q_n \le \overline{X}(P_u), \\ (\bar{p} - c) \cdot \overline{X}(\bar{p}), & \overline{X}(P_u) \le q_n. \end{cases}
\end{aligned} \tag{3.24}
$$

Since $q_n \le \overline{X}(\bar{P}_c)$, it follows from (3.23) that either $\bar{p} = P_u$ or $\overline{X}(\bar{p}) = q_n$. If the former holds, then it trivially follows from (3.24) that $\overline{M}_n^{(1)}$ is a linear increasing function of q_n. Otherwise, that is if $\overline{X}(\bar{p}) = q_n$, then it is an increasing function of q_n [1].

Next we consider the subproblem on $\overline{M}_n^{(2)}$. Recalling the assumption that $\overline{M}_{n-1}^{\star} \in \mathcal{C}\ (\overline{X}(\bar{P}_c), \mathcal{K})$, the expression (3.22) will be:

$$
\begin{aligned}
\overline{M}_n^{(2)}(i_n, q_n) &= max\{\alpha \cdot \overline{\Pi}_{n-1}^{\star\star}(q_n - \overline{X}(p_n)) - (c + h) \cdot q_n \\
&\qquad + (p_n + h) \cdot \overline{X}(p_n) + c \cdot i_n :\ \bar{p} \le p_n \le P_u\} \qquad (3.25) \\
&= max\{\alpha \cdot max\{\overline{M}_{n-1}^{\star}(q_n - \overline{X}(p_n), q_n - \overline{X}(p_n))\ , \\
&\qquad \overline{M}_{n-1}^{\star}(q_n - \overline{X}(p_n), \not{\mathcal{L}}_{n-1}) - \mathcal{K}\} - (c + h) \cdot q_n \\
&\qquad + (p_n + h) \cdot \overline{X}(p_n) + c \cdot i_n :\ \bar{p} \le p_n \le P_u\} \\
&= max\{\overline{M}_n^{(21)}(i_n, q_n)\ ,\ \overline{M}_n^{(22)}(i_n, q_n)\},
\end{aligned} \tag{3.26}
$$

where $\overline{M}_n^{(21)}$ and $\overline{M}_n^{(22)}$ are defined by:

$$
\begin{aligned}
\overline{M}_n^{(21)}(i_n, q_n) &= max\{\alpha \cdot \overline{M}_{n-1}^{\star}(q_n - \overline{X}(p_n), q_n - \overline{X}(p_n)) - (c + h) \cdot q_n \\
&\qquad + (p_n + h) \cdot \overline{X}(p_n) + c \cdot i_n :\ \bar{p} \le p_n \le P_u\} \\
&= max\{\alpha \cdot \overline{M}_{n-1}^{\star}(0, q_n - \overline{X}(p_n)) - [(1 - \alpha) \cdot c + h] \cdot q_n \\
&\qquad + (p_n + h - \alpha \cdot c) \cdot \overline{X}(p_n) + c \cdot i_n :\ \bar{p} \le p_n \le P_u\},
\end{aligned} \tag{3.27}
$$

and

$$
\begin{aligned}
\overline{M}_n^{(22)}(i_n, q_n) &= max\{\alpha \cdot (\overline{M}_{n-1}^{\star}(q_n - \overline{X}(p_n), \not{\mathcal{L}}_{n-1}) - \mathcal{K}) - (c + h) \cdot q_n \\
&\qquad + (p_n + h) \cdot \overline{X}(p_n) + c \cdot i_n :\ \bar{p} \le p_n \le P_u\} \\
&= max\{\alpha \cdot (\overline{M}_{n-1}^{\star}(0, \not{\mathcal{L}}_{n-1}) - \mathcal{K}) - [(1 - \alpha) \cdot c + h] \cdot q_n \\
&\qquad + (p_n + h - \alpha \cdot c) \cdot \overline{X}(p_n) + c \cdot i_n :\ \bar{p} \le p_n \le P_u\}.
\end{aligned} \tag{3.28}
$$

---

[1] Let $\phi(q_n) = (\bar{p} - c) \cdot q_n$, where $q_n = \overline{X}(\bar{p})$. Then, $\frac{d\phi(q_n)}{dq_n} = \frac{d[(\bar{p} - c) \cdot \overline{X}(\bar{p})]}{d\bar{p}} \cdot \frac{d\bar{p}}{dq_n} \ge (\le) 0 \Leftrightarrow \bar{p} \ge (\le) \bar{P}_c$.

It follows from (3.23) that for $q_n \in [0, \overline{X}(P_u)]$ we have $\bar{p} = P_u$, which implies that $\overline{M}_n^{(2)}$, given by (3.25), disappears and from (3.20) we obtain $\overline{M}_n^{\star} = \overline{M}_n^{(1)}$. Therefore, for the purpose of the proof, we shall restrict the analysis of $\overline{M}_n^{(2)}$ on the q_n range of $[\overline{X}(P_u), \overline{X}(\bar{P}_c)]$, which implies that $\bar{P}_c \leq \bar{p}$. Moreover, considering the maximands in (3.27) and (3.28) we note that the maximizer of the function $(p + h - \alpha \cdot c) \cdot \overline{X}(p)$ on $[P_\ell, P_u]$ is $\bar{P}_{hc}$, where $\bar{P}_{hc} \leq \bar{P}_c$. Hence, this function is decreasing on $[\bar{p}, P_u]$. In addition, since $q_n - \overline{X}(p_n) \geq 0$ on $[\bar{p}, P_u]$, the maximizer in (3.27), say p'_q, must satisfy

$$0 \leq q_n - \overline{X}(p'_q) \leq \rlap{\,|}C_{n-1}. \tag{3.29}$$

Furthermore, for any q_n we choose q'_n with $\overline{X}(P_u) \leq q_n < q'_n \leq \overline{X}(\bar{P}_c)$ in order to demonstrate that $\overline{M}_n^{(21)}(i_n, q_n) < \overline{M}_n^{(21)}(i_n, q'_n)$. Also, we define $\bar{p}'$ by

$$\overline{X}(\bar{p}') = q'_n, \tag{3.30}$$

which implies that $\bar{P}_c \leq \bar{p}' \leq P_u$. Moreover, we identify p' which satisfies

$$q_n - \overline{X}(p'_q) = q'_n - \overline{X}(p'), \tag{3.31}$$

that implies $p' < p'_q$. Therefore, from (3.29), (3.30) and (3.31) it follows that

$$q'_n - \overline{X}(p') = q_n - \overline{X}(p'_q) \geq 0 = q'_n - \overline{X}(\bar{p}'),$$

which implies

$$\bar{P}_c \leq \bar{p}' \leq p' < p'_q \leq P_u. \tag{3.32}$$

Thus, substituting (3.31) in (3.27) and considering (3.32) we get

$$\begin{aligned}
\overline{M}_n^{(21)}(i_n, q_n) &= \alpha \cdot \overline{M}_{n-1}^{\star}(0, q_n - \overline{X}(p'_q)) - [(1-\alpha) \cdot c + h] \cdot q_n \\
&\qquad + (p'_q + h - \alpha \cdot c) \cdot \overline{X}(p'_q) + c \cdot i_n \\
&= \alpha \cdot \overline{M}_{n-1}^{\star}(0, q_n - \overline{X}(p'_q)) - [(1-\alpha) \cdot c + h] \cdot [q_n - \overline{X}(p'_q)] \\
&\qquad + (p'_q - c) \cdot \overline{X}(p'_q) + c \cdot i_n \\
&= \alpha \cdot \overline{M}_{n-1}^{\star}(0, q'_n - \overline{X}(p')) - [(1-\alpha) \cdot c + h] \cdot [q'_n - \overline{X}(p')] \\
&\qquad + (p'_q - c) \cdot \overline{X}(p'_q) + c \cdot i_n \\
&< \alpha \cdot \overline{M}_{n-1}^{\star}(0, q'_n - \overline{X}(p')) - [(1-\alpha) \cdot c + h] \cdot [q'_n - \overline{X}(p')] \\
&\qquad + (p' - c) \cdot \overline{X}(p') + c \cdot i_n
\end{aligned}$$

$$
\begin{aligned}
&\leq \; max\{\alpha\cdot\overline{M}^{\star}_{n-1}(0, q'_n - \overline{X}(p_n)) - [(1-\alpha)\cdot c + h]\cdot q'_n \\
&\qquad +(p_n + h - \alpha\cdot c)\cdot\overline{X}(p_n) + c\cdot i_n \;:\; \bar{p}' \leq p_n \leq P_u\} \\
&= \; \overline{M}^{(21)}_n(i_n, q'_n),
\end{aligned}
$$

which implies that $\overline{M}^{(21)}_n$ is an increasing function of q_n. On the other hand, since $\overline{M}^{\star}_{n-1}$ $(0\,,\,\ddagger_{n-1})$ is independent of p_n and $\bar{P}_{hc} \leq \bar{p}$, the maximizer in (3.28) is $\bar{p}$, that is

$$
\overline{M}^{(22)}_n(i_n, q_n) = \alpha\cdot(\overline{M}^{\star}_{n-1}(0, \ddagger_{n-1}) - \mathcal{K}) + (\bar{p} - c)\cdot\overline{X}(\bar{p}) + c\cdot i_n,
$$

which is an increasing function of q_n. Hence, $\overline{M}^{(2)}_n$, given by (3.26), is an increasing function of q_n on $[\overline{X}(P_u), \overline{X}(\bar{P}_c)]$. Thus, the proof follows from (3.20).

Next we consider $\overline{M}^{\star}_n$ on the remaining q_n range of $[\overline{X}(\bar{P}_c), \infty)$ to establish the validity of condition (c) of the definition of $\mathcal{C}$ for $\overline{M}^{\star}_n$.

Lemma 5. *$\overline{M}^{\star}_n(i_n, q_n)$ is an $\alpha\mathcal{K}$-decreasing function of q_n on $[\overline{X}(\bar{P}_c), \infty)$ for $N \geq n \geq 2$.*

Proof. In the following proof we shall demonstrate that $\overline{M}^{(1)}_n$ is a decreasing function and $\overline{M}^{(2)}_n$ is an $\alpha\mathcal{K}$-decreasing function which implies that $\overline{M}^{\star}_n$, given by (3.20), is an $\alpha\mathcal{K}$-decreasing function of q_n on $[\overline{X}(\bar{P}_c), \infty)$.

If $\overline{X}(P_\ell) \leq q_n$, then $\bar{p} = P_\ell$ which implies that for the q_n range of $[\overline{X}(P_\ell), \infty)$ the function $\overline{M}^{(1)}_n$ disappears and we have $\overline{M}^{\star}_n = \overline{M}^{(2)}_n$. Therefore, it follows from (3.24) that $\overline{M}^{(1)}_n$ is a decreasing function of q_n on $[\overline{X}(\bar{P}_c), \overline{X}(P_\ell)]$.

For the analysis of $\overline{M}^{(2)}_n$ we identify two cases which are defined by q_n with respect to $\ddagger_{n-1}$ as: $q_n - \overline{X}(\bar{p}) \leq \ddagger_{n-1}$ and $q_n - \overline{X}(\bar{p}) > \ddagger_{n-1}$. We shall show that in either case $\overline{M}^{(2)}_n$ will be $\alpha\mathcal{K}$-decreasing at q_n. If $q_n - \overline{X}(\bar{p}) \leq \ddagger_{n-1}$, then it follows from (3.25) that $\overline{M}^{(2)}_n$ is given by (3.26). Therefore, considering (3.27) and (3.28) we have

$$
\begin{aligned}
\overline{M}^{(21)}_n(i_n, q_n) \;&=\; max\{\alpha\cdot\overline{M}^{\star}_{n-1}(0, q_n - \overline{X}(p_n)) - [(1-\alpha)\cdot c + h]\cdot q_n \\
&\qquad +(p_n + h - \alpha\cdot c)\cdot\overline{X}(p_n) + c\cdot i_n \;:\; \bar{p} \leq p_n \leq P_u\} \\
&\leq\; max\{\alpha\cdot\overline{M}^{\star}_{n-1}(0, \ddagger_{n-1}) - [(1-\alpha)\cdot c + h]\cdot q_n \\
&\qquad +(p_n + h - \alpha\cdot c)\cdot\overline{X}(p_n) + c\cdot i_n \;:\; \bar{p} \leq p_n \leq P_u\} \\
&=\; \overline{M}^{(22)}_n(i_n, q_n) + \alpha\cdot\mathcal{K}. \qquad (3.33)
\end{aligned}
$$

Furthermore, from (3.28) we have

$$
\begin{aligned}
\overline{M}^{(22)}_n(i_n, q_n) \;&=\; max\{\alpha\cdot(\overline{M}^{\star}_{n-1}(0, \ddagger_{n-1}) - \mathcal{K}) - [(1-\alpha)\cdot c + h]\cdot q_n \\
&\qquad +(p_n + h - \alpha\cdot c)\cdot\overline{X}(p_n) + c\cdot i_n \;:\; \bar{p} \leq p_n \leq P_u\}.
\end{aligned}
$$

If $\overline{X}(\bar{P}_c) \leq q_n \leq \overline{X}(\bar{P}_{hc})$, that is $\bar{P}_{hc} \leq \bar{p}$, then the maximizer of $(p_n + h - \alpha{\cdot}c){\cdot}\overline{X}(p_n)$ on $[\bar{p}, P_u]$ is $\bar{p}$, that is

$$\overline{M}_n^{(22)}(i_n, q_n) = \alpha{\cdot}(\overline{M}_{n-1}^{\star}(0, \ddagger_{n-1}) - \mathcal{K}) + (\bar{p} - c){\cdot}\overline{X}(\bar{p}) + c{\cdot}i_n,$$

which is a decreasing function of q_n. If, however, $\overline{X}(\bar{P}_{hc}) \leq q_n$, that is $\bar{p} \leq \bar{P}_{hc}$, then

$$\begin{aligned}\overline{M}_n^{(22)}(i_n, q_n) &= \alpha{\cdot}(\overline{M}_{n-1}^{\star}(0, \ddagger_{n-1}) - \mathcal{K}) - [(1-\alpha){\cdot}c + h]{\cdot}q_n \\ &\quad + (\bar{P}_{hc} + h - \alpha{\cdot}c){\cdot}\overline{X}(\bar{P}_{hc}) + c{\cdot}i_n,\end{aligned}$$

which is a linear decreasing function of q_n. Therefore, it follows from (3.26) and (3.33) that $\overline{M}_n^{(2)}$ is an $\alpha\mathcal{K}$-decreasing function of q_n.

On the other hand, if $q_n - \overline{X}(\bar{p}) > \ddagger_{n-1}$ (which can hold only when $\bar{p} = P_\ell$ for $\overline{X}(\bar{P}_c) \leq q_n$, because $\ddagger_{n-1} > 0$), then it follows from (3.25) and (3.9) that

$$\begin{aligned}\overline{M}_n^{(2)}(i_n, q_n) &= max\{\alpha{\cdot}\overline{M}_{n-1}^{\star}(0, q_n - \overline{X}(p_n)) - [(1-\alpha){\cdot}c + h]{\cdot}q_n \\ &\quad + (p_n + h - \alpha{\cdot}c){\cdot}\overline{X}(p_n) + c{\cdot}i_n : \ P_\ell \leq p_n \leq P_u\} \qquad (3.34)\end{aligned}$$

Since $\overline{M}_{n-1}^{\star}$ is $\mathcal{K}$-decreasing on $[\ddagger_{n-1}, \infty)$, by assumption, we have $\overline{M}_{n-1}^{\star}(0, q_n - \overline{X}(p_n)) > \overline{M}_{n-1}^{\star}(0, q_n' - \overline{X}(p_n)) - \mathcal{K}$ for all $p_n \in [P_\ell, P_u]$ and for all $q_n' > q_n$. Thus, from (3.34) we get

$$\begin{aligned}\overline{M}_n^{(2)}(i_n, q_n) &\geq max\{\alpha{\cdot}(\overline{M}_{n-1}^{\star}(0, q_n' - \overline{X}(p_n)) - \mathcal{K}) - [(1-\alpha){\cdot}c + h]{\cdot}q_n \\ &\quad + (p_n + h - \alpha{\cdot}c){\cdot}\overline{X}(p_n) + c{\cdot}i_n : \ P_\ell \leq p_n \leq P_u\} \\ &> max\{\alpha{\cdot}\overline{M}_{n-1}^{\star}(0, q_n' - \overline{X}(p_n)) - [(1-\alpha){\cdot}c + h]{\cdot}q_n' \\ &\quad + (p_n + h - \alpha{\cdot}c){\cdot}\overline{X}(p_n) + c{\cdot}i_n : \ P_\ell \leq p_n \leq P_u\} - \alpha{\cdot}\mathcal{K} \\ &= \overline{M}_n^{(2)}(i_n, q_n') - \alpha{\cdot}\mathcal{K},\end{aligned}$$

which implies that $\overline{M}_n^{(2)}$ is $\alpha\mathcal{K}$-decreasing in q_n and the proof is complete.

Finally, combining the results stated as Lemmas 3 through 5 we establish the following theorem without proof.

Theorem 3. *$\overline{M}_n^{\star}(i_n, q_n) \in \mathcal{C}(\overline{X}(\bar{P}_c), \mathcal{K})$, where $n = 1, \ldots, N$.*

According to the previous theorem, $q_n^{\star}$ is obtained from (3.9) and $\overline{\Pi}_n^{\star\star}$ is given by (3.10).

3.3.2 Special Case II

In this section we shall prove that $\overline{M}_n^{\star} \in \mathcal{C}'(\overline{X}(\bar{P}_c), \mathcal{K})$, such that the optimal procurement policy is given by (3.5). To this end, we shall follow an induction proof. First, we shall prove that $\overline{M}_1^\star \in \mathcal{C}'(\overline{X}(\bar{P}_c), \mathcal{K})$, which will be the basic step. Then, we shall assume that that $\overline{M}_{n-1}^\star \in \mathcal{C}'(\overline{X}(\bar{P}_c), \mathcal{K})$ such that $\overline{\Pi}_{n-1}^{\star\star}$ will be given by (3.12). Finally, we shall demonstrate that $\overline{M}_n^\star \in \mathcal{C}'(\overline{X}(\bar{P}_c), \mathcal{K})$, which will complete the proof.

Lemma 6. $\overline{M}_1^\star(i_1, p_2, q_1) \in \mathcal{C}'(\overline{X}(\bar{P}_c), \mathcal{K})$.

Proof. For $q_1 \leq 0$ it follows from (3.16) that

$$\overline{M}_1(i_1, p_2, p_1, q_1) = (p_2 + s - c)\cdot q_1 - s\cdot\overline{X}(p_1) + c\cdot i_1,$$

which is an increasing function of p_1. Thus,

$$\overline{M}_1^\star(i_1, p_2, q_1) = (p_2 + s - c)\cdot q_1 - s\cdot\overline{X}(P_u) + c\cdot i_1, \tag{3.35}$$

which is a linear function of q_1. Note that depending on the value of p_2, $\overline{M}_1^\star$ can be an increasing or a decreasing function. In addition, Theorem 2 indicates that for $q_1 > 0$, $\overline{M}_1^\star$ is a quasiconcave function of q_1, where $\ddagger_1 = \overline{X}(\bar{P}_c)$. Thus, combining this result with (3.35) the proof follows.

Therefore, $q_1^\star$ is obtained from (3.11) and $\overline{\Pi}_1^{\star\star}$ is given by (3.12). An immediate observation which will be referred to later is stated in the following

Corollary 1. *$\overline{M}_1^\star(i_1, p_2, \ddagger_1) = (\bar{P}_c - c)\cdot\overline{X}(\bar{P}_c) + c\cdot i_1$, where $\ddagger_1 = \overline{X}(\bar{P}_c)$.*

We now extend the result given by (3.35) to cover the other periods with the following

Lemma 7. *$\overline{M}_n^\star(i_n, p_{n+1}, q_n)$ is a linear function of q_n on $(-\infty, 0]$ for $N > n \geq 2$.*

Proof. From (3.15) we have

$$\overline{M}_2(i_2, p_3, p_2, q_2) = \alpha\cdot\overline{\Pi}_1^{\star\star}(-\overline{X}(p_2), p_2) + \alpha\cdot p_2\cdot\overline{X}(p_2) + (p_3 + s - c)\cdot q_2 + c\cdot i_2, \tag{3.36}$$

for $q_2 \leq 0$. Utilizing (3.12) to obtain $\overline{\Pi}_1^{\star\star}$ and substituting it in (3.36) we rewrite $\overline{M}_2$ as:

$$\begin{aligned}\overline{M}_2(i_2, p_3, p_2, q_2) &= \alpha\cdot max\{\overline{M}_1^\star(-\bar{X}(p_2), p_2, -\overline{X}(p_2)) \ , \ \overline{M}_1^\star(-\overline{X}(p_2), p_2, \ddagger_1) - \mathcal{K}\} \\ &\quad +\alpha\cdot p_2\cdot\overline{X}(p_2) + (p_3 + s - c)\cdot q_2 + c\cdot i_2.\end{aligned}$$

Moreover, obtaining $\overline{M}_1^\star(-\overline{X}(p_2), p_2, -\overline{X}(p_2))$ from (3.35) and $\overline{M}_1^\star(-\overline{X}(p_2), p_2, \ddagger_1)$ from Corollary 1 we write

$$\begin{aligned}\overline{M}_2(i_2, p_3, p_2, q_2) &= \alpha\cdot max\{-s\cdot\overline{X}(p_2) - s\cdot\overline{X}(P_u) \ , \ (\bar{P}_c - c)\cdot\overline{X}(\bar{P}_c) - \mathcal{K} \\ &\quad +(p_2 - c)\cdot\overline{X}(p_2)\} + (p_3 + s - c)\cdot q_2 + c\cdot i_2.\end{aligned} \tag{3.37}$$

Therefore, solving the maximization problem (3.5) for $\overline{M}_2^\star$ by utilizing (3.37) we obtain

$$\begin{aligned}\overline{M}_2^\star(i_2, p_3, q_2) &= \alpha\cdot max\{-2\cdot s\cdot\overline{X}(P_u) \ , \ 2\cdot(\bar{P}_c - c)\cdot\overline{X}(\bar{P}_c) - \mathcal{K}\} \\ &\quad +(p_3 + s - c)\cdot q_2 + c\cdot i_2 \\ &= A_{21} + (p_3 + s - c)\cdot q_2 + c\cdot i_2,\end{aligned} \tag{3.38}$$

where A_{21} is a constant which equals to either $-2\cdot s\cdot\overline{X}(P_u)$ or $2\cdot(\bar{P}_c - c)\cdot\overline{X}(\bar{P}_c) - \mathcal{K}$ depending on the magnitude of $\mathcal{K}$ with respect to the parameters s, P_u, c and the $\overline{X}(p)$ function.

To complete the proof we shall demonstrate that $\overline{M}^{\star}_3$ assumes a similar linear form to the expression given by (3.38). Repeating the analysis given by (3.36) through (3.38) for $n = 3$ we obtain the following:

$$\begin{aligned}\overline{M}_3(i_3, p_4, p_3, q_3) &= \alpha\cdot\overline{\Pi}^{\star\star}_2(-\overline{X}(p_3), p_3) + \alpha\cdot p_3\cdot\overline{X}(p_3) + (p_4 + s - c)\cdot q_3 + c\cdot i_3 \\ &= \alpha\cdot max\{\overline{M}^{\star}_2(-\overline{X}(p_3), p_3, -\overline{X}(p_3)) \;,\; \overline{M}^{\star}_2(-\overline{X}(p_3), p_3, \mathcal{Q}_2) - \mathcal{K}\} \\ &\qquad + \alpha\cdot p_3\cdot\overline{X}(p_3) + (p_4 + s - c)\cdot q_3 + c\cdot i_3. \qquad (3.39)\end{aligned}$$

Considering property (3.6) and the previous result (3.38) we get

$$\begin{aligned}\overline{M}_3(i_3, p_4, p_3, q_3) &= \alpha\cdot max\{A_{21} - s\cdot\overline{X}(p_3) \;,\; \overline{M}^{\star}_2(0, p_3, \mathcal{Q}_2) - \mathcal{K} + (p_3 - c)\cdot\overline{X}(p_3)\} \\ &\qquad + (p_4 + s - c)\cdot q_3 + c\cdot i_3,\end{aligned}$$

which implies that

$$\begin{aligned}\overline{M}^{\star}_3(i_3, p_4, q_3) &= \alpha\cdot max\{A_{21} - s\cdot\overline{X}(P_u) \;,\; \overline{M}^{\star}_2(0, p_3, \mathcal{Q}_2) - \mathcal{K} + (\bar{P}_c - c)\cdot\overline{X}(\bar{P}_c)\} \\ &\qquad + (p_4 + s - c)\cdot q_3 + c\cdot i_3 \\ &= A_{31} + (p_4 + s - c)\cdot q_3 + c\cdot i_3, \qquad (3.40)\end{aligned}$$

where A_{31} is a constant which is either $A_{21} - s\cdot\overline{X}(P_u)$ or $\overline{M}^{\star}_2(0, p_3, \mathcal{Q}_2) - \mathcal{K} + (\bar{P}_c - c)\cdot\overline{X}(\bar{P}_c)$. Hence, it is clear that repeating the above procedure n times we would obtain a series of constants, $\{A_{n1}\}$, such that

$$\overline{M}^{\star}_n(i_n, p_{n+1}, q_n) = A_{n1} + (p_{n+1} + s - c)\cdot q_n + c\cdot i_n, \qquad (3.41)$$

which is a linear function of q_n that holds for $i_n \leq q_n \leq 0$ and $N > n \geq 2$.

An immediate consequence of Lemma 7 is stated in the following

Corollary 2. *For all $q_n \leq 0$ where $N > n \geq 2$ the best price is either P_u or $\bar{P}_c$ where $P_u \geq \bar{P}_c > c$.*

According to the previous corollary, for any $q_n \leq 0$ the vendor sets a price of $\bar{P}_c$ or P_u. The former is the maximizer of the function $\alpha\cdot(p - c)\cdot\overline{X}(p)$, which represents the net pseudo-profit due to not selling $\overline{X}(p)$ in the current period but in the next assuming that all of the backlog, i.e. $\overline{X}(p)$, will be satisfied then. On the other hand, if the vendor administers the latter price, which is the highest possible, then he incurs not only the highest unit revenue but also minimizes the shortages in the current period.

Having completed the analysis of $\overline{M}_n^\star$ for $q_n \in (-\infty, 0]$, which conforms condition (a) of the definition of $C'(\overline{X}(\bar{P}_c), \mathcal{K})$, we now establish the validity of condition (b) of the same definition.

Lemma 8. *$\overline{M}_n^\star(i_n, p_{n+1}, q_n)$ is an increasing function of q_n on $[0, \overline{X}(\bar{P}_c)]$ for $N \geq n \geq 2$.*

Proof. In the following proof we shall demonstrate that $\overline{M}_n^{(1)}$ and $\overline{M}_n^{(2)}$ are increasing functions of q_n on $[0, \overline{X}(\bar{P}_c)]$, which implies that $\overline{M}_n^\star$, given by (3.20), is also an increasing function of q_n. From (3.15) and (3.21) it follows that

$$\begin{aligned}\overline{M}_n^{(1)}(i_n, p_{n+1}, q_n) &= max\{\alpha \cdot \overline{\Pi}_{n-1}^{\star\star}(q_n - \overline{X}(p_n), p_n) + (p_n - c) \cdot q_n \\ &\quad + \alpha \cdot p_n \cdot (\overline{X}(p_n) - q_n) + c \cdot i_n : \; P_\ell \leq p_n \leq \bar{p}\}.\end{aligned}$$

Substituting for $\overline{\Pi}_{n-1}^{\star\star}$ from (3.12) we have

$$\begin{aligned}\overline{M}_n^{(1)}(i_n, p_{n+1}, q_n) &= max\{\alpha \cdot max\{\overline{M}_{n-1}^\star(q_n - \overline{X}(p_n), p_n, q_n - \overline{X}(p_n)) \;, \\ &\quad \overline{M}_{n-1}^\star(q_n - \overline{X}(p_n), p_n, \updownarrow_{n-1}) - \mathcal{K}\} + (p_n - c) \cdot q_n \\ &\quad + \alpha \cdot p_n \cdot (\overline{X}(p_n) - q_n) + c \cdot i_n : \; P_\ell \leq p_n \leq \bar{p}\} \\ &= max\{\overline{M}_n^{(11)}(i_n, p_{n+1}, q_n) \;, \; \overline{M}_n^{(12)}(i_n, p_{n+1}, q_n)\}, \qquad (3.42)\end{aligned}$$

where $\overline{M}_n^{(11)}$ and $\overline{M}_n^{(12)}$ are two subproblems that are defined as:

$$\begin{aligned}\overline{M}_n^{(11)}(i_n, p_{n+1}, q_n) = max\{&\alpha \cdot \overline{M}_{n-1}^\star(q_n - \overline{X}(p_n), p_n, q_n - \overline{X}(p_n)) + (p_n - c) \cdot q_n \\ &+ \alpha \cdot p_n \cdot (\overline{X}(p_n) - q_n) + c \cdot i_n : \; P_\ell \leq p_n \leq \bar{p}\}, \qquad (3.43)\end{aligned}$$

and

$$\begin{aligned}\overline{M}_n^{(12)}(i_n, p_{n+1}, q_n) = max\{&\alpha \cdot (\overline{M}_{n-1}^\star(q_n - \overline{X}(p_n), p_n, \updownarrow_{n-1}) - \mathcal{K}) + (p_n - c) \cdot q_n \\ &+ \alpha \cdot p_n \cdot (\overline{X}(p_n) - q_n) + c \cdot i_n : \; P_\ell \leq p_n \leq \bar{p}\}. \qquad (3.44)\end{aligned}$$

Since $q_n - \overline{X}(p_n) \leq 0$ for all $p_n \in [P_\ell, \bar{p}]$, $\overline{M}_{n-1}^\star(q_n - \overline{X}(p_n), p_n, q_n - \overline{X}(p_n))$ is given by (3.41). With this substitution (3.43) can be rewritten as:

$$\begin{aligned}\overline{M}_n^{(11)}(i_n, p_{n+1}, q_n) &= max\{\alpha \cdot A_{(n-1)1} + (p_n + \alpha \cdot s - c) \cdot q_n \\ &\quad - \alpha \cdot s \cdot \overline{X}(p_n) + c \cdot i_n : \; P_\ell \leq p_n \leq \bar{p}\} \\ &= \alpha \cdot A_{(n-1)1} + (\bar{p} + \alpha \cdot s - c) \cdot q_n - \alpha \cdot s \cdot \overline{X}(\bar{p}) + c \cdot i_n. \qquad (3.45)\end{aligned}$$

Since $q_n \leq \overline{X}(\bar{P}_c)$, it follows from (3.23) that either $\bar{p} = P_u$ or $\overline{X}(\bar{p}) = q_n$. If the former holds, then it trivially follows from (3.45) that $\overline{M}_n^{(11)}$ is a linear increasing function of q_n. Otherwise, that is if $\overline{X}(\bar{p}) = q_n$, then (3.45) reduces to

$$\overline{M}_n^{(11)}(i_n, p_{n+1}, q_n) = \alpha \cdot A_{(n-1)1} + (\bar{p} - c) \cdot \overline{X}(\bar{p}) + c \cdot i_n, \qquad (3.46)$$

which is an increasing function of q_n.

For $\overline{M}_n^{(12)}$ we utilize (3.6) to represent $\overline{M}^{\star}_{n-1}(q_n - \overline{X}(p_n), p_n, \not\mathcal{L}_{n-1})$ as:

$$\overline{M}^{\star}_{n-1}(q_n - \overline{X}(p_n), p_n, \not\mathcal{L}_{n-1}) = \overline{M}^{\star}_{n-1}(0, p_n, \not\mathcal{L}_{n-1}) + c\cdot(q_n - \overline{X}(p_n)), \tag{3.47}$$

where $\overline{M}^{\star}_{n-1}(0, p_n, \not\mathcal{L}_{n-1})$ is a constant due to property (3.2). Therefore, substituting (3.47) in (3.44) we obtain

$$\begin{aligned}\overline{M}_n^{(12)}(i_n, p_{n+1}, q_n) = max\{\alpha\cdot(\overline{M}^{\star}_{n-1}(0, p_n, \not\mathcal{L}_{n-1}) - \mathcal{K}) + (1-\alpha)\cdot(p_n - c)\cdot q_n \\ + \alpha\cdot(p_n - c)\cdot\overline{X}(p_n) + c\cdot i_n :\ P_\ell \le p_n \le \bar{p}\}.\end{aligned} \tag{3.48}$$

Let p_q be the maximizer in (3.48). If $p_q \in (P_\ell, \bar{p})$, then it must satisfy

$$(1-\alpha)\cdot q_n + \alpha\cdot\frac{d}{dp}\{(p-c)\cdot\overline{X}(p)\}|_{p_q} = 0, \tag{3.49}$$

which implies that $p_q \ge \bar{P}_c > c$ and

$$\frac{dp_q}{dq_n} = \frac{(1-\alpha)}{-\alpha\cdot\frac{d^2}{dp^2}\{(p-c)\cdot\overline{X}(p)\}|_{p_q}} > 0.$$

Thus,

$$\frac{d\overline{M}_n^{(12)}(i_n, p_{n+1}, q_n)}{dq_n} = (1-\alpha)\cdot(p_q - c) > 0,$$

which states that $\overline{M}_n^{(12)}$ is an increasing function of q_n. On the other hand, if p_q is a boundary solution, then since $(1-\alpha)\cdot q_n + \alpha\cdot d\{(p-c)\cdot\overline{X}(p)\}/dp > 0$ for all $p \in [P_\ell, \bar{P}_c]$, we have $p_q = \bar{p}$, that is

$$\overline{M}_n^{(12)}(i_n, p_{n+1}, q_n) = \alpha\cdot(\overline{M}^{\star}_{n-1}(0, p_n, \not\mathcal{L}_{n-1}) - \mathcal{K}) + (\bar{p} - c)\cdot\overline{X}(\bar{p}) + c\cdot i_n,$$

which is an increasing function of q_n. Hence, it follows from (3.42) that $\overline{M}_n^{(1)}$ is an increasing function of q_n as well.

Next we consider the subproblem $\overline{M}_n^{(2)}$ given by (3.22). Writing $\overline{M}_n$ from (3.15) and substituting for $\overline{\Pi}^{\star\star}_{n-1}$ the expression given by (3.12) we have

$$\begin{aligned}\overline{M}_n^{(2)}(i_n, p_{n+1}, q_n) &= max\{\alpha\cdot\overline{\Pi}^{\star\star}_{n-1}(q_n - \overline{X}(p_n), p_n) - (c+h)\cdot q_n \\ &\qquad + (p_n + h)\cdot\overline{X}(p_n) + c\cdot i_n :\ \bar{p} \le p_n \le P_u\} \qquad (3.50) \\ &= max\{\alpha\cdot max\{\overline{M}^{\star}_{n-1}(q_n - \overline{X}(p_n), p_n, q_n - \overline{X}(p_n))\ , \\ &\qquad \overline{M}^{\star}_{n-1}(q_n - \overline{X}(p_n), p_n, \not\mathcal{L}_{n-1}) - \mathcal{K}\} - (c+h)\cdot q_n \\ &\qquad + (p_n + h)\cdot\overline{X}(p_n) + c\cdot i_n :\ \bar{p} \le p_n \le P_u\} \qquad (3.51) \\ &= max\{\overline{M}_n^{(21)}(i_n, p_{n+1}, q_n)\ ,\ \overline{M}_n^{(22)}(i_n, p_{n+1}, q_n)\}, \qquad (3.52)\end{aligned}$$

where $\overline{M}_n^{(21)}$ and $\overline{M}_n^{(22)}$ are two subproblems of $\overline{M}_n^{(2)}$ that are defined by:

$$\begin{aligned}\overline{M}_n^{(21)}(i_n, p_{n+1}, q_n) &= max\{\alpha\cdot\overline{M}_{n-1}^{\star}(q_n - \overline{X}(p_n), p_n, q_n - \overline{X}(p_n)) - (c+h)\cdot q_n \\ &\quad +(p_n + h)\cdot\overline{X}(p_n) + c\cdot i_n : \ \bar{p} \le p_n \le P_u\} \\ &= max\{\alpha\cdot\overline{M}_{n-1}^{\star}(0, p_n, q_n - \overline{X}(p_n)) - [(1-\alpha)\cdot c + h]\cdot q_n \\ &\quad +(p_n + h - \alpha\cdot c)\cdot\overline{X}(p_n) + c\cdot i_n : \ \bar{p} \le p_n \le P_u\}, \quad (3.53)\end{aligned}$$

and

$$\begin{aligned}\overline{M}_n^{(22)}(i_n, p_{n+1}, q_n) &= max\{\alpha\cdot(\overline{M}_{n-1}^{\star}(q_n - \overline{X}(p_n), p_n, \ddagger_{n-1}) - \mathcal{K}) - (c+h)\cdot q_n \\ &\quad +(p_n + h)\cdot\overline{X}(p_n) + c\cdot i_n : \ \bar{p} \le p_n \le P_u\} \\ &= max\{\alpha\cdot(\overline{M}_{n-1}^{\star}(0, p_n, \ddagger_{n-1}) - \mathcal{K}) - [(1-\alpha)\cdot c + h]\cdot q_n \\ &\quad +(p_n + h - \alpha\cdot c)\cdot\overline{X}(p_n) + c\cdot i_n : \ \bar{p} \le p_n \le P_u\}. \quad (3.54)\end{aligned}$$

It follows from (3.23) that for $q_n \in [0, \overline{X}(P_u)]$ we have $\bar{p} = P_u$, which implies that $\overline{M}_n^{(2)}$, given by (3.50), disappears and from (3.20) we obtain $\overline{M}_n^{\star} = \overline{M}_n^{(1)}$. Therefore, for the purpose of the proof, we shall restrict the analysis of $\overline{M}_n^{(2)}$ on the q_n range of $[\overline{X}(P_u), \overline{X}(\bar{P}_c)]$, which implies that $\bar{P}_c \le \bar{p}$. Moreover, considering the maximands in (3.53) and (3.54) we note that the maximizer of the function $(p + h - \alpha\cdot c)\cdot\overline{X}(p)$ on $[P_\ell, P_u]$ is $\bar{P}_{hc}$, where $\bar{P}_{hc} \le \bar{P}_c$. Hence, this function is decreasing on $[\bar{p}, P_u]$. In addition, since $q_n - \overline{X}(p_n) \ge 0$ on $[\bar{p}, P_u]$, the maximizer in (3.53), say p'_q, must satisfy

$$0 \le q_n - \overline{X}(p'_q) \le \ddagger_{n-1}. \quad (3.55)$$

Furthermore, for any q_n we choose q'_n with $\overline{X}(P_u) \le q_n < q'_n \le \overline{X}(\bar{P}_c)$ in order to demonstrate that $\overline{M}_n^{(21)}(i_n, p_{n+1}, q_n) < \overline{M}_n^{(21)}(i_n, p_{n+1}, q'_n)$. Also, we define $\bar{p}'$ by

$$\overline{X}(\bar{p}') = q'_n, \quad (3.56)$$

which implies that $\bar{P}_c \le \bar{p}' \le P_u$. Moreover, we identify p' which satisfies

$$q_n - \overline{X}(p'_q) = q'_n - \overline{X}(p'), \quad (3.57)$$

that implies $p' < p'_q$. Therefore, from (3.55), (3.56) and (3.57) it follows that

$$q'_n - \overline{X}(p') = q_n - \overline{X}(p'_q) \ge 0 = q'_n - \overline{X}(\bar{p}'),$$

which implies

$$\bar{P}_c \le \bar{p}' \le p' < p'_q \le P_u. \quad (3.58)$$

Thus, substituting (3.57) in (3.53) and considering (3.58) we get

$$\begin{aligned}
\overline{M}_n^{(21)}(i_n, p_{n+1}, q_n) &= \alpha\cdot\overline{M}_{n-1}^{\star}(0, p'_q, q_n - \overline{X}(p'_q)) - [(1-\alpha)\cdot c + h]\cdot q_n \\
&\quad +(p'_q + h - \alpha\cdot c)\cdot\overline{X}(p'_q) + c\cdot i_n \\
&= \alpha\cdot\overline{M}_{n-1}^{\star}(0, p'_q, q_n - \overline{X}(p'_q)) - [(1-\alpha)\cdot c + h]\cdot[q_n - \overline{X}(p'_q)] \\
&\quad +(p'_q - c)\cdot\overline{X}(p'_q) + c\cdot i_n \\
&= \alpha\cdot\overline{M}_{n-1}^{\star}(0, p', q'_n - \overline{X}(p')) - [(1-\alpha)\cdot c + h]\cdot[q'_n - \overline{X}(p')] \\
&\quad +(p'_q - c)\cdot\overline{X}(p'_q) + c\cdot i_n \\
&< \alpha\cdot\overline{M}_{n-1}^{\star}(0, p', q'_n - \overline{X}(p')) - [(1-\alpha)\cdot c + h]\cdot[q'_n - \overline{X}(p')] \\
&\quad +(p' - c)\cdot\overline{X}(p') + c\cdot i_n \\
&\leq max\{\alpha\cdot\overline{M}_{n-1}^{\star}(0, p_n, q'_n - \overline{X}(p_n)) - [(1-\alpha)\cdot c + h]\cdot q'_n \\
&\quad +(p_n + h - \alpha\cdot c)\cdot\overline{X}(p_n) + c\cdot i_n \;:\; \bar{p}' \leq p_n \leq P_u\} \\
&= \overline{M}_n^{(21)}(i_n, p_{n+1}, q'_n),
\end{aligned}$$

which implies that $\overline{M}_n^{(21)}$ is an increasing function of q_n. On the other hand, since $\overline{M}_{n-1}^{\star}(0,p_n, \ddagger_{n-1})$ is independent of p_n and $\bar{P}_{hc} \leq \bar{p}$, the maximizer in (3.54) is $\bar{p}$, that is

$$\overline{M}_n^{(22)}(i_n, p_{n+1}, q_n) = \alpha\cdot(\overline{M}_{n-1}^{\star}(0, p_n, \ddagger_{n-1}) - \mathcal{K}) + (\bar{p} - c)\cdot\overline{X}(\bar{p}) + c\cdot i_n,$$

which is an increasing function of q_n. Hence, $\overline{M}_n^{(2)}$, given by (3.52), is an increasing function of q_n on $[\overline{X}(P_u), \overline{X}(\bar{P}_c)]$. Thus, the proof follows from (3.20).

Next we consider $\overline{M}_n^{\star}$ on the remaining q_n range of $[\overline{X}(\bar{P}_c), \infty)$ to establish the validity of condition (c) of the definition of $\mathcal{C}'(\overline{X}(\bar{P}_c), \mathcal{K})$ for $\overline{M}_n^{\star}$.

Lemma 9. *$\overline{M}_n^{\star}(i_n, p_{n+1}, q_n)$ is an $\alpha\mathcal{K}$-decreasing function of q_n on $[\overline{X}(\bar{P}_c), \infty)$ for $N \geq n \geq 2$.*

Proof. We shall demonstrate that $\overline{M}_n^{(1)}$ is a decreasing function and $\overline{M}_n^{(2)}$ is an $\alpha\mathcal{K}$-decreasing function which implies that $\overline{M}_n^{\star}$, given by (3.20), is an $\alpha\mathcal{K}$-decreasing function of q_n on $[\overline{X}(\bar{P}_c), \infty)$.

If $\overline{X}(P_\ell) \leq q_n$, then $\bar{p} = P_\ell$ which implies that for the q_n range of $[\overline{X}(P_\ell), \infty)$ the function $\overline{M}_n^{(1)}$ disappears and we have $\overline{M}_n^{\star} = \overline{M}_n^{(2)}$. Therefore, we shall consider $\overline{M}_n^{(1)}$, under the structure given by (3.42), for the q_n range of $[\overline{X}(\bar{P}_c), \overline{X}(P_\ell)]$. It follows from (3.46) that

$$\overline{M}_n^{(11)}(i_n, p_{n+1}, q_n) = \alpha\cdot A_{(n-1)1} + (\bar{p} - c)\cdot\overline{X}(\bar{p}) + c\cdot i_n,$$

which is a decreasing function of q_n on $[\overline{X}(\bar{P}_c), \overline{X}(P_\ell)]$. Furthermore, considering $\overline{M}_n^{(12)}$ given by (3.48) we note that the function $(1-\alpha)\cdot(p_n - c)\cdot q_n + \alpha\cdot(p_n - c)\cdot\overline{X}(p_n)$ is increasing in p_n on $[P_\ell, \bar{p}]$, thus,

$$\overline{M}_n^{(12)}(i_n, p_{n+1}, q_n) = \alpha\cdot(\overline{M}_{n-1}^{\star}(0, p_n, \mathcal{S}_{n-1}) - \mathcal{K}) + (\bar{p} - c)\cdot\overline{X}(\bar{p}) + c\cdot i_n,$$

which is a decreasing function of q_n. Thus, from (3.42) we conclude that $\overline{M}_n^{(1)}$ is a decreasing function of q_n on $[\overline{X}(\bar{P}_c), \overline{X}(P_\ell)]$.

For the analysis of $\overline{M}_n^{(2)}$ we identify two cases which are defined by q_n with respect to $\mathcal{S}_{n-1}$ as: $q_n - \overline{X}(\bar{p}) \leq \mathcal{S}_{n-1}$ and $q_n - \overline{X}(\bar{p}) > \mathcal{S}_{n-1}$. We show that in either case $\overline{M}_n^{(2)}$ is $\alpha\mathcal{K}$-decreasing at q_n. If $q_n - \overline{X}(\bar{p}) \leq \mathcal{S}_{n-1}$, then it follows from (3.50) that $\overline{M}_n^{(2)}$ is given by (3.52). Considering (3.53) we have

$$\begin{aligned}
\overline{M}_n^{(21)}(i_n, p_{n+1}, q_n) &= max\{\alpha\cdot\overline{M}_{n-1}^{\star}(0, p_n, q_n - \overline{X}(p_n)) - [(1-\alpha)\cdot c + h]\cdot q_n \\
&\qquad +(p_n + h - \alpha\cdot c)\cdot\overline{X}(p_n) + c\cdot i_n : \ \bar{p} \leq p_n \leq P_u\} \\
&\leq max\{\alpha\cdot\overline{M}_{n-1}^{\star}(0, p_n, \mathcal{S}_{n-1}) - [(1-\alpha)\cdot c + h]\cdot q_n \\
&\qquad +(p_n + h - \alpha\cdot c)\cdot\overline{X}(p_n) + c\cdot i_n : \ \bar{p} \leq p_n \leq P_u\} \\
&= \overline{M}_n^{(22)}(i_n, p_{n+1}, q_n) + \alpha\cdot\mathcal{K}. \qquad (3.59)
\end{aligned}$$

Furthermore, from (3.54) we have

$$\begin{aligned}
\overline{M}_n^{(22)}(i_n, p_{n+1}, q_n) &= max\{\alpha\cdot(\overline{M}_{n-1}^{\star}(0, p_n, \mathcal{S}_{n-1}) - \mathcal{K}) - [(1-\alpha)\cdot c + h]\cdot q_n \\
&\qquad +(p_n + h - \alpha\cdot c)\cdot\overline{X}(p_n) + c\cdot i_n : \ \bar{p} \leq p_n \leq P_u\}.
\end{aligned}$$

If $\overline{X}(\bar{P}_c) \leq q_n \leq \overline{X}(\bar{P}_{hc})$, that is $\bar{P}_{hc} \leq \bar{p}$, then the maximizer of $(p_n + h - \alpha\cdot c)\cdot\overline{X}(p_n)$ on $[\bar{p}, P_u]$ is $\bar{p}$, that is

$$\overline{M}_n^{(22)}(i_n, p_{n+1}, q_n) = \alpha\cdot(\overline{M}_{n-1}^{\star}(0, p_n, \mathcal{S}_{n-1}) - \mathcal{K}) + (\bar{p} - c)\cdot\overline{X}(\bar{p}) + c\cdot i_n,$$

which is a decreasing function of q_n. If, however, $\overline{X}(\bar{P}_{hc}) \leq q_n$, that is $\bar{p} \leq \bar{P}_{hc}$, then

$$\begin{aligned}
\overline{M}_n^{(22)}(i_n, p_{n+1}, q_n) &= \alpha\cdot(\overline{M}_{n-1}^{\star}(0, p_n, \mathcal{S}_{n-1}) - \mathcal{K}) - [(1-\alpha)\cdot c + h]\cdot q_n \\
&\qquad +(\bar{P}_{hc} + h - \alpha\cdot c)\cdot\overline{X}(\bar{P}_{hc}) + c\cdot i_n,
\end{aligned}$$

which is a linear decreasing function of q_n. Therefore, it follows from (3.52) and (3.59) that $\overline{M}_n^{(2)}$ is an $\alpha\mathcal{K}$-decreasing function of q_n.

On the other hand, if $q_n - \overline{X}(\bar{p}) > \ddagger_{n-1}$ (which can hold only when $\bar{p} = P_\ell$ for $\overline{X}(\bar{P}_c) \le q_n$, because $\ddagger_{n-1} > 0$), then it follows from (3.50) and (3.12) that

$$\begin{aligned}\overline{M}_n^{(2)}(i_n, p_{n+1}, q_n) &= max\{\alpha \cdot \overline{M}_{n-1}^{\star}(0, p_n, q_n - \overline{X}(p_n)) - [(1-\alpha)\cdot c + h]\cdot q_n \\ &\quad +(p_n + h - \alpha\cdot c)\cdot \overline{X}(p_n) + c\cdot i_n \; : \; P_\ell \le p_n \le P_u\}\end{aligned} \tag{3.60}$$

Since $\overline{M}_{n-1}^{\star}$ is $\mathcal{K}$-decreasing on $[\ddagger_{n-1}, \infty)$, by assumption, we have $\overline{M}_{n-1}^{\star}(0, p_n, q_n - \overline{X}(p_n)) > \overline{M}_{n-1}^{\star}(0, p_n, q_n' - \overline{X}(p_n)) - \mathcal{K}$ for all $p_n \in [P_\ell, P_u]$ and for all $q_n' > q_n$. Thus, from (3.60) we get

$$\begin{aligned}\overline{M}_n^{(2)}(i_n, p_{n+1}, q_n) &\ge max\{\alpha\cdot(\overline{M}_{n-1}^{\star}(0, p_n, q_n' - \overline{X}(p_n)) - \mathcal{K}) - [(1-\alpha)\cdot c + h]\cdot q_n \\ &\quad +(p_n + h - \alpha\cdot c)\cdot \overline{X}(p_n) + c\cdot i_n \; : \; P_\ell \le p_n \le P_u\} \\ &> max\{\alpha\cdot \overline{M}_{n-1}^{\star}(0, p_n, q_n' - \overline{X}(p_n)) - [(1-\alpha)\cdot c + h]\cdot q_n' \\ &\quad +(p_n + h - \alpha\cdot c)\cdot \overline{X}(p_n) + c\cdot i_n \; : \; P_\ell \le p_n \le P_u\} - \alpha\cdot\mathcal{K} \\ &= \overline{M}_n^{(2)}(i_n, p_{n+1}, q_n') - \alpha\cdot\mathcal{K},\end{aligned}$$

which implies that $\overline{M}_n^{(2)}$ is $\alpha\mathcal{K}$-decreasing in q_n and the proof is complete.

Finally, combining the results stated as Lemmas 6 through 9 we establish the following theorem without proof.

Theorem 4. *For all* $p_{n+1} \in [P_\ell, P_u]$ *it follows that* $\overline{M}_n^{\star}(i_n, p_{n+1}, q_n) \in C'(\overline{X}(\bar{P}_c), \mathcal{K})$, *where* $n = 1, \ldots, N$ *and* $p_{N+1} = 0$.

According to the previous theorem, $q_n^\star$ is obtained from (3.11) and $\overline{\Pi}_n^{\star\star}$ is given by (3.12).

3.3.3 Special Case III

In this section we shall present an example which demonstrates that, under special case III, $\overline{M}_1^{\star}$ is not included in $C'(\gamma, \mathcal{K})$ in general. For the purpose of the proof consider the case where $i_1 < 0$. Then, it follows from (3.19) and (3.23) that

$$\overline{M}_1(i_1, p_1, q_1) = \begin{cases} p_1\cdot(q_1 - i_1) - s\cdot\overline{X}(p_1) + (s-c)\cdot q_1 + c\cdot i_1 & , \quad P_\ell \le p_1 \le \bar{p}, \\ (p_1 + h)\cdot\overline{X}(p_1) - p_1\cdot i_1 - (c+h)\cdot q_1 + c\cdot i_1 & , \quad \bar{p} \le p_1 \le P_u. \end{cases}$$

Since $q_1 - i_1 \geq 0$, $\overline{M}_1^\star$ is increasing in p_1 on $[P_\ell, \bar{p}]$ which implies that

$$\overline{M}_1^\star(i_1, q_1) = \begin{cases} (P_u + s - c)\cdot q_1 - (P_u - c)\cdot i_1 - s\cdot\overline{X}(P_u) & , \quad i_1 \leq q_1 \leq \overline{X}(P_u), \\ \\ max\{(p_1 + h)\cdot\overline{X}(p_1) - p_1\cdot i_1 : \ \bar{p} \leq p_1 \leq P_u\} \\ -(c+h)\cdot q_1 + c\cdot i_1 & , \quad i_1 < 0 < \overline{X}(P_u) \leq q_1 \end{cases} \tag{3.61}$$

Moreover, let us assume that the expected demand is defined by $\overline{X}(p) = a - bp$, where $a, b \in \mathcal{R}^+$, $P_\ell = 0$ and $P_u = a/b$. Under this linear demand assumption (3.61) reduces to the following (see Appendix C for the details):

$$\overline{M}_1^\star(i_1, q_1) = \begin{cases} (P_u + s - c)\cdot q_1 - (P_u - c)\cdot i_1 - s\cdot\overline{X}(P_u) & , \quad i_1 \leq q_1 \leq 0, \\ \\ (a - q_1 - c\cdot b)\cdot(q_1 - i_1)/b & , \quad \begin{cases} -b\cdot(h + P_u) < i_1 < 0 \\ 0 < q_1 < (a + b\cdot h + i_1)/2, \end{cases} \\ \\ (a - b\cdot h - i_1)^2/(4\cdot b) + h\cdot(a - q_1) - c\cdot(q_1 - i_1) & , \quad (a + b\cdot h + i_1)/2 < q_1, \end{cases} \tag{3.62}$$

It is clear that $\overline{M}_1^\star$ given by (3.62) does not necessarily satisfy the separation property (3.6). Thus, (3.9) is not optimal for the determination of $q_1^\star$, unless some restrictive assumptions about the parameter value ranges are made. Furthermore, we note that

$$argmax\{\overline{M}_1^\star(i_1, q_1) : \ i_1 \leq q_1 < \infty, \ -b\cdot(h + P_u) < i_1 < 0\} = (a - c\cdot b + i_1)/2, \tag{3.63}$$

which depends on i_1 (see Appendix D for the details). That is, the order-up-to level is a function of i_1 when $-b\cdot(h + P_u) < i_1 < 0$. This observation implies that (3.11) can not be utilized for the determination of $q_1^\star$ in general. Consequently, it is clear that under the cited backlogging rule we need a new definition for the optimal policy.

3.4 Probabilistic Demand

In the previous section it has been shown that under deterministic demand the optimal procurement quantity is determined by an $(\sigma_n, \updownarrow_n)$ policy for the lost-sales multi-period model, but this is not necessarily optimal under other backlogging rules. Since deterministic demand is a special case of probabilistic demand model, the latter argument above is also valid for probabilistic demand case. For this reason, in this section we shall concentrate on the lost-sales model, and try to characterize an $(\sigma, \updownarrow)$ type optimal policy for the determination of $q^\star$. To this end, we shall again perform an induction proof.

To start with, we need to introduce a new class of functions:

Definition 3. *$\mathcal{C}''(\gamma_1, \gamma_2, \mathcal{K})$ is a set of continuous univariate functions, say $\phi(q)$, which:*

(a) are increasing on $[0, \gamma_1]$; and

(b) have a finite maximizer on (γ_1, γ_2), say γ_3, such that $\forall q \in (\gamma_3, \gamma_2)$ we have:

$$\phi(q) > \phi(q') - \mathcal{K} \qquad \forall q' \in (q, \infty).$$

We shall refer to this definition in characterizing the shape of $\overline{M}_n^{\star}$ function which will be derived below.

Consider the pseudo-profit function under special case I (the *lost sales* model) which is given by (3.13). Evaluating the expected value of this function with respect to random demand $X(p_n)$ we obtain the following:

$$\begin{aligned}\overline{M}_n(i_n, p_n, q_n) &= (p_n + s - c)\cdot q_n - s\cdot \overline{X}(p_n) - (p_n + s + h)\cdot \Theta(p_n, q_n) + c\cdot i_n \\ &+ \alpha\cdot \overline{\Pi}_{n-1}^{\star\star}(0)\cdot[1 - F(q_n; p_n)] + \alpha\cdot \int_{X_1(p_n)}^{q_n} \overline{\Pi}_{n-1}^{\star\star}(q_n - x)\cdot f(x; p_n)\cdot dx.\end{aligned} \tag{3.64}$$

We recall that the separation property holds for $\overline{M}_n$, such that:

$$\overline{M}_n(i_n, p_n, q_n) = c\cdot i_n + \overline{M}_n(0, p_n, q_n),$$

for all n.

We initialize the induction proof by assuming that the conditions leading to unimodality of $\overline{M}_1^{\star}(0, q_1)$ holds. This implies that $\overline{M}_1^{\star}(0, q_1) \in \mathcal{C}''(\sigma_1, \infty, \mathcal{K})$ for some σ_1. Furthermore, as an induction step, we assume that $\overline{M}_{n-1}^{\star}(0, q_{n-1}) \in \mathcal{C}''(\sigma_{n-1}^1, \sigma_{n-1}^{k_{n-1}}, \mathcal{K})$, where k_{n-1} is an even integer and there exists $\sigma_{n-1}^1, \sigma_{n-1}^2, \ldots, \sigma_{n-1}^{k_{n-1}}$ which satisfy the following conditions:

- $\overline{M}_{n-1}^{\star}(0, \sigma_{n-1}^j) = \overline{M}_{n-1}^{\star}(0, \Sigma_{n-1}) - \mathcal{K}$ for $j = 1, 2, \ldots, k_{n-1}$, where Σ_{n-1} is the maximizer of $\overline{M}_{n-1}^{\star}(0, q_{n-1})$,
- $\sigma_{n-1}^1 < \sigma_{n-1}^2 < \cdots < \sigma_{n-1}^{k_{n-1}-1} < \Sigma_{n-1} < \sigma_{n-1}^{k_{n-1}}$, and
- there does not exist any $q \in (\Sigma_{n-1}, \sigma_{n-1}^{k_{n-1}})$ such that $\overline{M}_{n-1}^{\star}(0, q) = \overline{M}_{n-1}^{\star}(0, \sigma_{n-1}^{k_{n-1}})$.

Note that under these assumptions $\overline{M}_{n-1}^{\star}(0, q_{n-1})$ is an increasing function on $[0, \sigma_{n-1}^1]$ and it could have "ripples", on $(\sigma_{n-1}^1, \Sigma_{n-1}]$, about the $\overline{M}_{n-1}^{\star}(0, \Sigma_{n-1}) - \mathcal{K}$ level. Moreover, it follows that $\overline{M}_{n-1}^{\star}(0, q_{n-1})$ is $\mathcal{K}$-decreasing on $(\Sigma_{n-1}, \sigma_{n-1}^{k_{n-1}}]$, but it can exhibit a "valley" or a "peak" with a depth or a height greater than $\mathcal{K}$ on $(\sigma_{n-1}^{k_{n-1}}, \infty)$ provided that $\overline{M}_{n-1}^{\star}(0, q_{n-1}) < \overline{M}_{n-1}^{\star}(0, \Sigma_{n-1})$ for all $q_{n-1} \in (\sigma_{n-1}^{k_{n-1}}, \infty)$. Also, note that $k_1 = 2$ under the assumption that $\overline{M}_1^{\star}$ is unimodal.

It is clear that, under the above setting for $\overline{M}_{n-1}^{\star}$ function an (σ, Σ) type policy need not be optimal, because there might exist two or more "order-up-to" levels. The smallest of those is $\mathcal{S}_{n-1}$ and others, if they exist, lay on $(\sigma_{n-1}^{k_{n-1}}, \infty)$. This infavourable fact is a burden on the efforts of characterization of the optimal procurement policy as an $(\sigma, \mathcal{S})$ type. In fact, it is also encountered in the solution of the classical multi-period problem where there is no pricing decision. In that model, the general approach is to include an assumption, among the others that ensure optimality of $(\sigma_n, \mathcal{S}_n)$ policies, so as to make it impossible for the period ending inventory level, for period n say, to be in $(\sigma_{n-1}^{k_{n-1}}, \infty)$. This way, the problem is avoided by restricting the pseudo-profit functions $\overline{M}_n^\star$ within domains of $[0, \sigma_n^{k_n}]$ for $n \geq 2$. For instance, assumptions *(iv)* on page 531 in Schäl [12], *(vii)* on page 1070 in Veinott [14] or (7-28g) on page 323 in Heyman and Sobel [2] are mainly made for the cited reason. It is also customary to search for special cases under which this assumption holds. In this regard, it is sufficient to show that $\mathcal{S}_N \geq \mathcal{S}_{N-1} \geq \cdots \geq \mathcal{S}_2$ where $i_N \leq \mathcal{S}_N$, under certain conditions. Thus, it is not surprising that this infavorable issue is inherited to our model. In the following analysis we shall express this problem, which is slightly modified in comparison with the classical model, and try to establish sufficient conditions leading to optimality of $(\sigma_n, \mathcal{S}_n)$ type policies.

In (3.64) we need to replace $\overline{\Pi}_{n-1}^{\star\star}$ by an expression which is written in terms of $\overline{M}_{n-1}^\star$. For this reason, we recall the inductive assumptions and write:

$$\overline{\Pi}_{n-1}^{\star\star}(i_{n-1}) = c \cdot i_{n-1} + \begin{cases} \overline{M}_{n-1}^\star(0, \mathcal{S}_{n-1}) - \mathcal{K} & , \quad i_{n-1} \in O(n-1, k_{n-1}), \\ \overline{M}_{n-1}^\star(0, i_{n-1}) & , \quad i_{n-1} \in \overline{O}(n-1, k_{n-1}), \end{cases} \tag{3.65}$$

where

$$\begin{aligned} O(n-1, k_{n-1}) &= [0, \sigma_{n-1}^1) \cup [\sigma_{n-1}^2, \sigma_{n-1}^3) \cup [\sigma_{n-1}^4, \sigma_{n-1}^5) \cup \cdots \cup [\sigma_{n-1}^{k_{n-1}-2}, \sigma_{n-1}^{k_{n-1}-1}) \\ \overline{O}(n-1, k_{n-1}) &= [0, \sigma_{n-1}^{k_{n-1}}] \setminus O(n-1, k_{n-1}), \end{aligned}$$

provided that $i_{n-1} \in [0, \sigma_{n-1}^{k_{n-1}}]$; and if $\overline{M}_{n-1}^\star(0, \sigma_{n-1}^1) > \overline{M}_{n-1}^\star(0, \mathcal{S}_{n-1}) - \mathcal{K}$, then $\sigma_{n-1}^1 = 0$. However, we shall assume that $\mathcal{K}$ is sufficiently small so that $\sigma_{n-1}^1 > 0$. This only decreases the number of terms to carry in the analysis which in turn simplifies the mathematical representation.

It turns out that, for $k_{n-1} = 2$ the policy defined by (3.65) is an $(\sigma_{n-1}, \mathcal{S}_{n-1})$ policy. But, for $k_{n-1} > 2$ it follows that although there is one "order-up-to" level, $\mathcal{S}_{n-1}$, there are $k_{n-1}/2$ reorder intervals, union of which is denoted by $O(n-1, k_{n-1})$ where "O" stands for "order". The latter policy is called an $(\sigma_{n-1}, \mathcal{S}_{n-1})$ type policy. Here, the terminology is set in such a way that the word "type" reflects a modification on the conventional $(\sigma, \mathcal{S})$ policy with respect to the fact that there are more than one reorder intervals.

Therefore, we can replace $\overline{\Pi}_{n-1}^{\star\star}$ in (3.64) by (3.65), where $\overline{\Pi}_{n-1}^{\star\star}(0) = \overline{M}_{n-1}^\star(0, \sigma_{n-1}^1)$ which is in turn equal to $\overline{M}_{n-1}^\star(0, \mathcal{S}_{n-1}) - \mathcal{K}$ since $\sigma_{n-1}^1 > 0$. Considering,

$$\begin{aligned} 0 \leq X_1(p_n) \leq x \leq q_n - \sigma_{n-1}^1 \quad &\Leftrightarrow \quad \sigma_{n-1}^1 \leq q_n - x \leq q_n - X_1(p_n), \\ q_n - \sigma_{n-1}^1 \leq x \leq q_n \quad &\Leftrightarrow \quad 0 \leq q_n - x \leq \sigma_{n-1}^1, \end{aligned}$$

we rewrite the integral in (3.64) as:

$$\int_{X_1(p_n)}^{q_n} \overline{\Pi}^{\star\star}_{n-1}(q_n - x)\cdot f(x;p_n)\cdot dx = \left[F(q_n;p_n) - F(q_n - \sigma^1_{n-1};p_n)\right]\cdot \overline{M}^\star_{n-1}(0,\sigma^1_{n-1})$$

$$+c\cdot\Theta(p_n,q_n) + \int_{q_n - x \in \overline{O}(n-1,k_{n-1})\cap(\sigma^1_{n-1},\, q_n - X_1(p_n))} \overline{M}^\star_{n-1}(0, q_n - x)\cdot f(x;p_n)\cdot dx$$

$$+ \overline{M}^\star_{n-1}(0,\sigma^1_{n-1})\cdot \int_{q_n - x \in O(n-1,k_{n-1})\cap(\sigma^1_{n-1},\, q_n - X_1(p_n))} f(x;p_n)\cdot dx \qquad (3.66)$$

Thus, after the proposed substitution (3.64) can be written as:

$$\overline{M}_n(i_n,p_n,q_n) = \widetilde{M}(i_n,p_n,q_n) + \alpha\cdot\overline{M}^\star_{n-1}(0,\sigma^1_{n-1})$$

$$+\alpha\cdot\int_{q_n - x \in \overline{O}(n-1,k_{n-1})\cap(\sigma^1_{n-1},\, q_n - X_1(p_n))} \left[\overline{M}^\star_{n-1}(0,q_n - x) - \overline{M}^\star_{n-1}(0,\sigma^1_{n-1})\right]\cdot f(x;p_n)\cdot dx$$

$$= \widetilde{M}(i_n,p_n,q_n) + \alpha\cdot\overline{M}^\star_{n-1}(0,\sigma^1_{n-1}) + \alpha\cdot\int_{X_1(p_n)}^{q_n - \sigma^1_{n-1}} \left[\overline{M}^\star_{n-1}(0,q_n - x) - \overline{M}^\star_{n-1}(0,\sigma^1_{n-1})\right]\cdot f(x;p_n)\cdot dx, \qquad (3.67)$$

where it is understood that $q_n - X_1(p_n) \leq \sigma^{k_{n-1}}_{n-1}$, which is also implied by $q_n \leq \sigma^{k_{n-1}}_{n-1}$; and

$$\widetilde{M}(i_n,p_n,q_n) = (p_n + s - c)\cdot q_n - s\cdot\overline{X}(p_n) - (p_n + s + h - \alpha\cdot c)\cdot\Theta(p_n,q_n) + c\cdot i_n, \qquad (3.68)$$

which represents expected one-period profit with the addition of a unit salvage value of $\alpha\cdot c$. It is clear that $\widetilde{M}$ has the same functional characteristics (in p and q) as $\overline{M}_1$ since we can always view $h - \alpha\cdot c$ as an effective unit holding cost (which can be negative). Furthermore, defining

$$\begin{aligned} \widetilde{M}^\star(0,q_n) &= max\left\{\widetilde{M}(0,p,q):\ p \in [P_\ell, P_u]\right\} \\ &= \widetilde{M}(0,p_q,q), \end{aligned} \qquad (3.69)$$

we assume that the conditions ensuring unimodality of $\overline{M}^\star_1$ can be trivially extended for $\widetilde{M}^\star$ as well. In this regard, we let $\mathcal{S}$ be the maximizer of $\widetilde{M}^\star(0,q)$ on $(0,\infty)$ and σ be defined as $\widetilde{M}^\star(0,\sigma) = \widetilde{M}^\star(0,\mathcal{S}) - \mathcal{K}$ with $\sigma < \mathcal{S}$.

It follows that for $q_n < \sigma^1_{n-1}$ the integral in (3.67) vanishes, and we have:

$$\overline{M}_n(0,p_n,q_n) = \widetilde{M}(0,p_n,q_n) + \alpha\cdot\overline{M}^\star_{n-1}(0,\sigma^1_{n-1}),$$

which implies that

$$\overline{M}^\star_n(0,q_n) = \widetilde{M}^\star(0,q_n) + \alpha\cdot\overline{M}^\star_{n-1}(0,\sigma^1_{n-1}). \qquad (3.70)$$

Under the inductive assumption that $\sigma^1_{n-1} < \mathcal{S}$, we conclude that $\overline{M}^\star_n(0,q_n)$ is an increasing function on $[0,\sigma^1_{n-1})$. Later, we shall demonstrate that in fact $\sigma^1_n < \mathcal{S}$.

Next, we shall concentrate more on $\overline{M}_n^\star$: From (3.67) we obtain,

$$\overline{M}_n^\star(0,q_n) = max\{\widetilde{M}(0,p_n,q_n) + \alpha\cdot\overline{M}_{n-1}^\star(0,\sigma_{n-1}^1) + \alpha\cdot\int_{X_1(p_n)}^{q_n-\sigma_{n-1}^1}\left[\overline{M}_{n-1}^\star(0,q_n-x) - \overline{M}_{n-1}^\star(0,\sigma_{n-1}^1)\right]\cdot f(x;p_n)\cdot dx : \; p_n \in [P_\ell, P_u]\} \tag{3.71}$$

$$\geq \widetilde{M}^\star(0,q_n) + \alpha\cdot\overline{M}_{n-1}^\star(0,\sigma_{n-1}^1) + \alpha\cdot\int_{X_1(p_n)}^{q_n-\sigma_{n-1}^1}\left[\overline{M}_{n-1}^\star(0,q_n-x) - \overline{M}_{n-1}^\star(0,\sigma_{n-1}^1)\right]\cdot f(x;p_q)\cdot dx, \tag{3.72}$$

where p_q was defined in (3.69). Using (3.71) once more it follows that

$$\begin{aligned}\overline{M}_n^\star(0,q_n) &\leq max\{\widetilde{M}(0,p_n,q_n) + \alpha\cdot\overline{M}_{n-1}^\star(0,\sigma_{n-1}^1) : \; p_n \in [P_\ell, P_u]\} \\ &\quad + max\{\alpha\cdot\int_{X_1(p_n)}^{q_n-\sigma_{n-1}^1}\left[\overline{M}_{n-1}^\star(0,q_n-x) - \overline{M}_{n-1}^\star(0,\sigma_{n-1}^1)\right]\cdot f(x;p_n)\cdot dx : \; p_n \in [P_\ell, P_u]\} \\ &= \widetilde{M}^\star(0,q_n) + \alpha\cdot\overline{M}_{n-1}^\star(0,\sigma_{n-1}^1) \\ &\quad + \alpha\cdot\int_{X_1(p_n)}^{q_n-\sigma_{n-1}^1}\left[\overline{M}_{n-1}^\star(0,q_n-x) - \overline{M}_{n-1}^\star(0,\sigma_{n-1}^1)\right]\cdot f(x;p')\cdot dx,\end{aligned} \tag{3.73}$$

where p' is the appropriate maximizer. Finally, we note that $\forall q_n - x \in \overline{O}(n-1,k_{n-1}) \cap (\sigma_{n-1}^1, q_n - X_1(p_n))$ we have

$$0 \leq \overline{M}_{n-1}^\star(0,q_n-x) - \overline{M}_{n-1}^\star(0,\sigma_{n-1}^1) \leq \mathcal{K},$$

which implies that $\forall q_n \in (\sigma_{n-1}^1, \sigma_{n-1}^{k_{n-1}})$,

$$0 \leq \int_{X_1(p_n)}^{q_n-\sigma_{n-1}^1}\left[\overline{M}_{n-1}^\star(0,q_n-x) - \overline{M}_{n-1}(0,\sigma_{n-1}^1)\right]\cdot f(x;p_n)\cdot dx \;\leq \mathcal{K}\cdot F(q_n-\sigma_{n-1}^1;p_n) \;\leq\; \mathcal{K}, \tag{3.74}$$

for all p_n. Therefore, with (3.74) we conclude that $\overline{M}_n^\star(0,q_n)$ is bounded between two continuous functions, given by (3.72) and (3.73), which are at most $\alpha\cdot\mathcal{K}$ distance (vertically) apart on $q_n \in (\sigma_{n-1}^1, \sigma_{n-1}^{k_{n-1}})$.

Since,

$$\overline{M}_n^\star(0,\hat{q}_n) = max\{\overline{M}_n^\star(0,q_n) : \; 0 \leq q_n < \infty\},$$

it follows from (3.73) and (3.74) that

$$\widetilde{M}(0,\ddagger) + \alpha\cdot\overline{M}^{\star}_{n-1}(0,\sigma^{1}_{n-1}) \leq \overline{M}^{\star}_{n}(0,\ddagger)$$
$$\leq \overline{M}^{\star}_{n}(0,\ddagger_{n}) \leq \widetilde{M}^{\star}(0,\ddagger) + \alpha\cdot\overline{M}^{\star}_{n-1}(0,\sigma^{1}_{n-1}) + \alpha\cdot\mathcal{K}, \tag{3.75}$$

which implies that:

$$\sigma_{n}^{k_n - 1} < \hat{\vartheta} < \ddagger < \check{\vartheta} < \sigma_{n}^{k_n}, \tag{3.76}$$

where $\widetilde{M}^{\star}(0,\ \hat{\vartheta}\ ) = \widetilde{M}^{\star}(0,\ \check{\vartheta}\ ) = \widetilde{M}^{\star}(0,\ddagger) - (1-\alpha)\cdot\mathcal{K}$. Moreover, it can be seen that if $\sigma < \sigma^{1}_{1}$, then $\sigma < \sigma^{1}_{n}$ for $n > 1$. Also, if we define $\check{\ddagger}$ and $\hat{\ddagger}$ as:

$$\widetilde{M}^{\star}(0,\ \check{\ddagger}\ ) = \widetilde{M}^{\star}(0,\ \hat{\ddagger}\ ) = \widetilde{M}^{\star}(0,\ddagger) - \alpha\cdot\mathcal{K},$$

with $\check{\ddagger} < \ddagger < \hat{\ddagger}$, we conclude from (3.74) and (3.75) that:

$$\check{\ddagger} \leq \ddagger_{n} \leq \hat{\ddagger}\ . \tag{3.77}$$

This result indicates that if, in period n, we prefer to procure more than $\ddagger$, which is the optimal quantity to procure if we could use the leftovers in the next period, then we should not order more than $\hat{\ddagger}$ at which we breakeven with not ordering this period but in the future.

Next, we shall study $\overline{M}^{\star}_{n}$ on $(\ddagger,\infty)$. Let $q_n \in (\ddagger, \sigma^{k_{n-1}}_{n-1})$ and q'_n be an arbitrary quantity on (q_n,∞). Then, it is clear that

$$\int_{X_1(p_n)}^{q'_n - \sigma^{1}_{n-1}} \left[\overline{M}^{\star}_{n-1}(0,q'_n - x) - \overline{M}^{\star}_{n-1}(0,\sigma^{1}_{n-1})\right]\cdot f(x;p_n)\cdot dx - \mathcal{K}\ \leq\ 0, \tag{3.78}$$

for all $p_n \in [P_\ell, P_u]$. Redefining p' as the maximizing price of $\overline{M}_n$ at q'_n, i.e. $\overline{M}^{\star}_{n}(0,q'_n) = \overline{M}_{n}(0,p',q'_n)$, and recalling (3.74), (3.78) and the fact that $\widetilde{M}^{\star}$ is non-increasing on $(\ddagger,\infty)$ we can proceed as follows:

$$\begin{aligned}
\overline{M}^{\star}_{n}(0,q_n) \ &=\ max\{\widetilde{M}(0,p_n,q_n) + \alpha\cdot\overline{M}^{\star}_{n-1}(0,\sigma^{1}_{n-1}) \\
&\quad +\alpha\cdot\int_{X_1(p_n)}^{q_n - \sigma^{1}_{n-1}} \left[\overline{M}^{\star}_{n-1}(0,q_n - x) - \overline{M}^{\star}_{n-1}(0,\sigma^{1}_{n-1})\right]\cdot f(x;p_n)\cdot dx :\ p_n \in [P_\ell, P_u]\} \\
&\geq\ \widetilde{M}^{\star}(0,q_n) + \alpha\cdot\overline{M}^{\star}_{n-1}(0,\sigma^{1}_{n-1}) \\
&\quad +\alpha\cdot\int_{X_1(p_q)}^{q_n - \sigma^{1}_{n-1}} \left[\overline{M}^{\star}_{n-1}(0,q_n - x) - \overline{M}^{\star}_{n-1}(0,\sigma^{1}_{n-1})\right]\cdot f(x;p_q)\cdot dx
\end{aligned}$$

$$
\begin{aligned}
&\geq \widetilde{M}^{\star}(0,q_n) + \alpha\cdot\overline{M}^{\star}_{n-1}(0,\sigma^1_{n-1}) \\
&\geq \widetilde{M}^{\star}(0,q_n) + \alpha\cdot\overline{M}^{\star}_{n-1}(0,\sigma^1_{n-1}) \\
&\quad +\alpha\cdot\int_{X_1(p')}^{q'_n-\sigma^1_{n-1}} \left[\overline{M}^{\star}_{n-1}(0,q'_n-x) - \overline{M}^{\star}_{n-1}(0,\sigma^1_{n-1})\right]\cdot f(x;p')\cdot dx - \alpha\cdot\mathcal{K} \\
&> \widetilde{M}^{\star}(0,q'_n) + \alpha\cdot\overline{M}^{\star}_{n-1}(0,\sigma^1_{n-1}) - \alpha\cdot\mathcal{K} \\
&\quad +\alpha\cdot\int_{X_1(p')}^{q'_n-\sigma^1_{n-1}} \left[\overline{M}^{\star}_{n-1}(0,q'_n-x) - \overline{M}^{\star}_{n-1}(0,\sigma^1_{n-1})\right]\cdot f(x;p')\cdot dx \\
&\geq \widetilde{M}(0,p',q'_n) + \alpha\cdot\overline{M}^{\star}_{n-1}(0,\sigma^1_{n-1}) - \alpha\cdot\mathcal{K} \\
&\quad +\alpha\cdot\int_{X_1(p')}^{q'_n-\sigma^1_{n-1}} \left[\overline{M}^{\star}_{n-1}(0,q'_n-x) - \overline{M}^{\star}_{n-1}(0,\sigma^1_{n-1})\right]\cdot f(x;p')\cdot dx \\
&= \overline{M}^{\star}_{n}(0,q'_n) - \alpha\cdot\mathcal{K} \\
&> \overline{M}^{\star}_{n}(0,q'_n) - \mathcal{K}
\end{aligned}
\tag{3.79}
$$

Combining all of the results that we have obtained so far, we see that:

(*i*) $\overline{M}^{\star}_{n}(0,q_n) \in \mathcal{C}''(\sigma^1_n, \sigma^{k_n}_n, \mathcal{K})$ for some even k_n,

(*ii*) there exist k_n values $\sigma^1_n, \sigma^2_n, \ldots, \sigma^{k_n}_n$ such that $\sigma^1_n < \sigma^2_n < \cdots < \mathscr{S}_n < \sigma^{k_n}_n$ with $\overline{M}^{\star}_{n}(0,\mathscr{S}_n) = \overline{M}^{\star}_{n}(0,\sigma^j_n) + \mathcal{K}$ for $j = 1, 2, \cdots, k_n$. Also, there does not exist $q_n \in (\mathscr{S}_n, \sigma^{k_n}_n)$ such that $\overline{M}^{\star}_{n}(0,q_n) = \overline{M}^{\star}_{n}(0,\mathscr{S}_n) - \mathcal{K}$,

(*iii*) $\forall q_n \in (\mathscr{S}, \sigma^{k_n}_n)$ we have $\overline{M}^{\star}_{n}(0,q_n) > \overline{M}^{\star}_{n}(0,q'_n) - \mathcal{K}$ for all $q'_n \in (q_n, \infty)$,

(*iv*) $\sigma^{k_n-1}_n < \hat{\vartheta} < \mathscr{S} < \check{\gamma} < \sigma^{k_n}_n$,

(*v*) if $\sigma < \sigma^1_1$, then $\sigma < \sigma^1_n$,

(*vi*) $\check{\mathscr{S}} \leq \mathscr{S}_n \leq \hat{\mathscr{S}}$,

for $n \geq 2$ and $k_1 = 2$.

In characterizing an optimal procurement policy it is useful to establish $\overline{M}^{\star}_{n}(0,q_n)$ function as q_n approaches to 0 or as q_n tends to infinity. It follows from (3.68) and (3.70) that:

$$
\begin{aligned}
\overline{M}^{\star}_{n}(0,0) &= \widetilde{M}^{\star}(0,0) + \alpha\cdot\overline{M}^{\star}_{n-1}(0,\sigma^1_{n-1}) \\
&= -s\cdot\overline{X}(P_u) + \alpha\cdot\overline{M}^{\star}_{n-1}(0,\sigma^1_{n-1}).
\end{aligned}
$$

Therefore, it is clear that $\overline{M}^{\star}_{n}(0,q_n)$ has a finite support at $q_n = 0$

On the other extreme, $\lim_{q_1 \to \infty} \overline{M}^{\star}_1(0, q_1) = -\infty$ by Lemma 2. To show a similar result for all n we shall perform an induction proof: Suppose that

$$\lim_{q_{n-1} \to \infty} \overline{M}^{\star}_{n-1}(0, q_{n-1}) = -\infty,$$

then we have:

$$\lim_{q_n \to \infty} \overline{M}^{\star}_n(0, q_n) = max\{\lim_{q_n \to \infty} (p_n + s - c) \cdot q_n - s \cdot \overline{X}(p_n) - (p_n + s + h - \alpha \cdot c) \cdot [q_n - \overline{X}(p_n)]$$

$$+\alpha \cdot \overline{M}^{\star}_{n-1}(0, \sigma^1_{n-1}) + \alpha \cdot \int_{X_1(p_n)}^{q_n - \sigma^1_{n-1}} \left[\overline{M}^{\star}_{n-1}(0, q_n - x) - \overline{M}^{\star}_{n-1}(0, \sigma^1_{n-1})\right] \cdot f(x; p_n) \cdot dx : \ \ p_n \in [P_\ell, P_u]\}$$

$$\begin{aligned} &= \lim_{q_n \to \infty} max\{(p_n + h - \alpha \cdot c) \cdot \overline{X}(p_n) \\ &\quad +\alpha \cdot \int_{X_1(p_n)}^{\infty} \left[\overline{M}^{\star}_{n-1}(0, q_n - x) - \overline{M}^{\star}_{n-1}(0, \sigma^1_{n-1})\right] \cdot f(x; p_n) \cdot dx : \ \ p_n \in [P_\ell, P_u]\} \\ &\quad +\alpha \cdot \overline{M}^{\star}_{n-1}(0, \sigma^1_{n-1}) - [h + (1 - \alpha) \cdot c] \cdot q_n \\ &= -\infty \end{aligned}$$

Thus, $\lim_{q_n \to \infty} \overline{M}_n(0, q_n) = -\infty$ for all n.

3.4.1 Optimal Procurement Policy

It has been shown that $\overline{M}^{\star}_n(0, q_n)$, which satisfies conditions (i) through (vi), can have ripples on $q_n \in (\sigma^1_{n-1}, \Sigma_n)$, but these deviations from monotonicity are confined within a region that is defined by the functions in (3.72) and (3.73) which are at most $\alpha\mathcal{K}$ distance apart from each other. Therefore, under a general distribution, it is theoretically possible that there are more than one reorder regions for period n. In this regard, an (σ, Σ) type policy, which will be termed as $(\sigma^1_n, \sigma^2_n, \ldots, \sigma^{k_n}_n, \Sigma_n)$, is optimal for the determination of $q^{\star}_n$. That is, for $n \geq 2$ we have:

$$q^{\star}_n = \begin{cases} \Sigma_n & \text{if} \quad i_n \in O(n, k_n), \\ i_n \quad , & \quad \text{otherwise}, \end{cases}$$

provided that $i_n \leq \sigma^{k_n}_n$.

It is interesting to note that $k_n < k_{n-1}$ or $k_n > k_{n-1}$ both are possible under conditions (i) through (vi). Also, for $k_n = 2$ the above policy reduces to an (σ_n, Σ_n) policy. This can be assured when either of the following conditions hold:

(a) $\overline{M}^{\star}_n(0, q_n)$ is an increasing function of q_n on (σ^1_{n-1}, Σ),

(b) $\overline{M}^{\star}_n(0, q_n) > \overline{M}^{\star}_n(0, \sigma^1_n) = \overline{M}^{\star}_n(0, \Sigma_n) - \mathcal{K}$ for all $q_n \in (\sigma^1_n, \ \hat{\partial} \ )$.

It follows, however, that (*a*) does not hold in general (for a counter example see Figure 4.2 in the next chapter). On the other hand, condition (*b*) is not trivial to pursue due to the unknown nature of $f(x;p)$ in p and the lack of condition (*a*) to hold in general (to be used in an inductive setting for $\overline{M}^{\star}_{n-1}$). Under a given distribution, if it is possible to characterize the best price for any given q_n, one can take the analysis further. However, several best price curves that are obtained for some example problems (see Figure 4.3 in the next chapter) indicate that this is highly impossible in general. For this reason, we shall not dwell on this issue.

3.4.2 Infinite Horizon Model

In this study, we are primarily concerned with a finite period model. However, we might investigate whether there exists a limiting steady state condition as the number of periods increase. In other words, it is interesting to see whether there are restoring forces within the model such that the system approaches an equilibrium in time.

To this end, we shall drop period indices from (3.64) and rewrite $\overline{M}^{\star}$ as follows:

$$\begin{aligned}
\overline{M}^{\star}(i,q) &= max\left\{\widetilde{M}(i,p,q) + \alpha\cdot\overline{M}^{\star}(0,\sigma^1)\right. \\
&\quad \left. +\alpha\cdot\int_{X_1(p)}^{q-\sigma^1}\left[\overline{M}^{\star}(0,q-x) - \overline{M}^{\star}(0,\sigma^1)\right]\cdot f(x;p)\cdot dx :\ p\in[P_\ell,P_u]\right\}, \\
&= \widetilde{M}(i,p_q,q) + \alpha\cdot\overline{M}^{\star}(0,\sigma^1) + \alpha\cdot\int_{X_1(p_q)}^{q-\sigma^1}\left[\overline{M}^{\star}(0,q-x) - \overline{M}^{\star}(0,\sigma^1)\right]\cdot f(x;p_q)\cdot dx \\
&= \widetilde{M}(i,p_q,q) + \alpha\cdot[1 - F(q-\sigma^1;p_q)]\cdot\overline{M}^{\star}(0,\sigma^1) \\
&\quad +\alpha\cdot\int_{X_1(p_q)}^{q-\sigma^1}\overline{M}^{\star}(0,q-x)\cdot f(x;p_q)\cdot dx,
\end{aligned} \tag{3.80}$$

where p_q is the appropriate maximizer. It follows from *renewal theory* that the solution of (3.80) is:

$$\begin{aligned}
\overline{M}^{\star}(i,q) &= \widetilde{M}(i,p_q,q) + \alpha\cdot[1 - F(q-\sigma^1;p_q)]\cdot\overline{M}^{\star}(0,\sigma^1) + \int_{X_1(p_q)}^{q-\sigma^1}\widetilde{M}(0,p_q,q-x)\cdot dR_\alpha(x;p_q) \\
&\quad +\alpha\cdot\int_{X_1(p_q)}^{q-\sigma^1}\overline{M}^{\star}(0,\sigma^1)\cdot[1 - F(q-\sigma^1-x;p_q)]\cdot dR_\alpha(x;p_q),
\end{aligned} \tag{3.81}$$

where

$$R_\alpha(x;p) = \sum_{i=1}^{\infty}\alpha^i\cdot F^{(i)}(x;p),$$

and $F^{(i)}$ is the i-fold convolution of F. Furthermore, the fourth term in (3.81) can be rewritten as:

$$\alpha\cdot\overline{M}^{\star}(0,\sigma^1)\cdot R_\alpha(q-\sigma^1;p_q)-\alpha\cdot\overline{M}^{\star}(0,\sigma^1)\cdot\int_{X_1(p_q)}^{q-\sigma^1} F(q-\sigma^1-x;p_q)\cdot dR_\alpha\cdot(x;p_q),$$

which is equal to:

$$\alpha\cdot\overline{M}^{\star}(0,\sigma^1)\cdot\sum_{i=1}^{\infty}\left[\alpha^i\cdot F^{(i)}(q-\sigma^1;p_q)-\alpha^{i+1}\cdot F^{(i+1)}(q-\sigma^1;p_q)\right].$$

Using this result we can write (3.81) once more as:

$$\begin{aligned}\overline{M}^{\star}(i,q) &= \widetilde{M}(i,p_q,q)+\int_{X_1(p_q)}^{q-\sigma^1}\widetilde{M}(0,p_q,q-x)\cdot dR_\alpha(x;p_q)+\alpha\cdot\overline{M}^{\star}(0,\sigma^1)\cdot[1\\ &\quad \left.-F(q-\sigma^1;p_q)+\sum_{i=1}^{\infty}\alpha^i\cdot F^{(i)}(q-\sigma^1;p_q)-\sum_{i=1}^{\infty}\alpha^i\cdot F^{(i+1)}(q-\sigma^1;p_q)\right]\\ &= \widetilde{M}(i,p_q,q)+\int_{X_1(p_q)}^{q-\sigma^1}\widetilde{M}(0,p_q,q-x)\cdot dR_\alpha(x;p_q)\\ &\quad +\overline{M}^{\star}(0,\sigma^1)\cdot[\alpha-(1-\alpha)\cdot R_\alpha(q-\sigma^1;p_q)]\,. \end{aligned}\tag{3.82}$$

It follows from (3.80) that for $q=\sigma^1$ we have:

$$\overline{M}^{\star}(0,\sigma^1)=\widetilde{M}^{\star}(0,\sigma^1)+\alpha\cdot\overline{M}^{\star}(0,\sigma^1),$$

which, together with (3.82), implies that

$$\begin{aligned}\overline{M}^{\star}(i,q) &= \widetilde{M}(i,p_q,q)+\int_{X_1(p_q)}^{q-\sigma^1}\widetilde{M}(0,p_q,q-x)\cdot dR_\alpha(x;p_q)\\ &\quad +\widetilde{M}^{\star}(0,\sigma^1)\cdot\left[\frac{\alpha}{1-\alpha}-R_\alpha(q-\sigma^1;p_q)\right]. \end{aligned}\tag{3.83}$$

Therefore, we have:

$$\begin{aligned}\overline{M}^{\star}(i,q) &= max\left\{\widetilde{M}(i,p,q)+\int_{X_1(p)}^{q-\sigma^1}\widetilde{M}(0,p,q-x)\cdot dR_\alpha(x;p)\right.\\ &\quad \left.+\widetilde{M}^{\star}(0,\sigma^1)\cdot\left[\frac{\alpha}{1-\alpha}-R_\alpha(q-\sigma^1;p)\right]\;:\;p\in[P_\ell,P_u]\right\}. \end{aligned}\tag{3.84}$$

In (3.84) $\widetilde{M}$ function is given by (3.68) and R_α can be obtained from F; but σ^1 can not be evaluated in advance. For this reason, an iterative method is required in order to obtain the simultaneous solution of $\overline{M}^{\star}$ and σ^1. To this end, we propose the following fix-point procedure:

1. Set $\sigma^1 = \sigma$,
2. Solve for $\ddagger_\infty$, where $\overline{M}^\star(0, \ddagger_\infty) = max\{\overline{M}^\star(0, q) : \; 0 \leq q < \infty\}$,
3. Solve for σ^1, where $\overline{M}^\star(0, \sigma^1) = \overline{M}^\star(0, \ddagger_\infty) - \mathcal{K}$ with $\sigma^1 < \hat{\vartheta}$,
4. If a tolerance is not met by σ^1, then go to step 2.

We shall provide a numeric example for the infinite horizon model in the next chapter.

Chapter 4

Numerical Examples

In this chapter, we provide the results of some numerical computations. We intend to study the effect of parameter values, demand distributions, expected demand functions and demand models on the optimal solution. In addition, we display the expected pseudo-profit curves for some of the example problems in order to demonstrate different forms that these functions can assume. We also concentrate on pricing issues and plot the optimal price values versus procurement quantity to provide some evidence.

In Table 4.1 we introduce thirteen cases each of which represents a combination of parameters c, s, h, $\mathcal{K}$ and λ. We shall refer to these cases when we use them in our example problems. The first six cases are the permutations of the order of c, s and h. Next three cases are considered, in comparison with the first three cases, for the effect of changing c, s or h individually while others remain constant. Finally, the last four cases represent the combinations of $\mathcal{K}$ and λ as c, s and h are constant.

In the beginning of our numerical study, we consider a 5-period lost-sales model where the random demand is additive. That is, $X(p) = \overline{X}(p) + \varepsilon$, where ε is a random variable with $E[\varepsilon] = 0$. We assume two different distributions for ε; namely, the uniform distribution and the triangular distribution. These distributions are defined in Appendix E. In addition, we employ two different expected demand functions which are defined by two parameters a and b. They are: (1) *exponential*, $\overline{X}(p) = a \cdot e^{-b \cdot p}$ and (2) *linear*, $\overline{X}(p) = a - b \cdot p$, where $p \in [P_l, P_u]$. Furthermore, we consider a 5-period lost-sales problem with multiplicative exponential demand, and extend it to include the infinite horizon case.

Using a Pascal program that solves the N-period dynamic programming problem defined by (3.64), we have obtained the expected pseudo-profit functions $\overline{M}_1^\star(0, q_1), \ldots, \overline{M}_5^\star(0, q_5)$ and, if optimal, the control parameters (σ_1, Σ_1), ..., (σ_5, Σ_5). Tables 4.2, 4.3 and 4.4 display summaries of these results. The programs are run on a SUN Spark 460 system which took approximately 6.5 seconds of CPU time to evaluate any point on the expected pseudo-profit functions of periods 2,3,4 or 5.

It is found that the optimal values depend moderately on the distribution type. Since the triangular distribution has smaller variance than that of the uniform distribution, it is intuitive that, for the same λ value the expected pseudo-profit values under the triangular distribution are higher. Also, for the same reason the reorder and order-up-to levels under the former distribution are lower in almost all cases.

The effects of parameter values on the optimal solution can be argued by pairwise comparison of cases from the first group (1 through 6) and from the second group (7 through 9). It follows from comparing cases 1 and 7 that when c is increased three fold, expected pseudo-profit values decrease considerably in all periods and under both distributions. Also, reorder and order-up-to levels decrease almost 20 to 25 %. This change is intuitive, because when c gets larger it becomes more expensive to do business, that is the mark-up between p and c gets narrower. Increasing the price to cope with higher procurement cost declines the demand; hence, the vendor tends to decrease the stocks.

On the other hand, comparing cases 2 and 8 we note that decreasing the shortage cost three fold does not affect the optimal control parameters or the expected pseudo-profit levels considerably. Since the penalty of lost-sales decreases, we expect slightly lower stock levels and higher profits. This is exactly true for all periods under both distributions.

Finally, comparing cases 3 and 9 it follows that decreasing h three fold facilitates higher stock levels and we obtain moderately higher reorder and order-up-to levels due to lower carrying costs. In addition, due to inventory cost reduction, considerably higher expected pseudo-profits are incurred.

Other parameter combinations can also be considered. For instance, suppose that there is a financial pressure build-up against the vendor such that the supply side increases the costs and at the same time inventory costs raise. Also, assume that under these developments the shortage cost that the vendor bears declines. Then, comparing cases 2 and 7 it follows that when c raises three fold, s decreases 33 % and h increases 50 %, the expected pseudo-profit values and order-up-to levels decrease considerably. This reflects a typical behaviour that often arises in practice.

We can also discuss the effects of the fixed cost $\mathcal{K}$ and of the variance of the distribution on the optimal solutions. Table 4.3 shows the results which are obtained for various combinations of $\mathcal{K}$ and λ. Note that λ is a measure of the variance of the distributions (see Appendix E). Comparing case 10 with case 12 and case 11 with case 13 it follows that, under a larger $\mathcal{K}$ the expected pseudo-profit is considerably smaller. This is due to incresing cost of procurement. Intuitively, we would expect higher order-up-to levels under a large $\mathcal{K}$. This is exactly represented by a sudden jump in the order-up-to level of the second period in case 13 in comparison with that of case 11. It is probable that as $\mathcal{K}$ is increased gradually, the order-up-to levels will experience sudden jumps. Between these jumps they remain quite stable. In Figure 4.1 we display $\overline{M}_2^{\star}(0, q_2)$ function for cases 11 and 13 to indicate the mechanism of this sudden jump.

The effect of λ, on the other hand, is such that under a small λ value the order-up-to levels decrease and the expected pseudo-profits increase considerably. This result is intuitive, because λ is a measure of the variance. Under greater variance the vendor is subject to a greater risk of shortage. To cope with this he increases stock levels. One can argue that under greater variance the risk of leftovers is also higher, which pressures the vendor to decrease the stocks. The answer, actually, depends on the tradeoff between the cost of holding and shortage as well as the price that the vendor administers. In a lost-sales problem, however, it is true that the vendor pays more attention to shortages, because they are lost whereas the leftovers could be transferred to the next period.

In Table 4.4 we study the effect of the expected demand function on the optimal solutions of the first six cases under the uniform distribution. We consider an exponential and a linear function as defined earlier. It follows that the order-up-to levels and the expected pseudo-profit values differ considerably under different demand functions. It is clear that these functions have different price sensitivities. The linear function has less sensitivity than the exponential function. Also, for a given price value in $[P_\ell, P_u]$, the linear function yields a higher demand level than the exponential function in our setting. Intuitively, this would incur the differences between the values of the order-up-to levels and expected pseudo-profit values as mentioned earlier.

After discussing the effects of problem parameters on the optimal solution, next, we shall consider the expected pseudo-profit curves. We have established the properties of $\overline{M}_n^{\star}(0, q_n)$ function with conditions (i) through (vi) in section 3.4. In our numerical examples, we have found that in all cases these six conditions are satisfied. In Table 4.5 we show the values of the critical inventory levels which defines the regions that the values of the optimal control parameters (σ_n, Σ_n) are restricted with. It can be seen that conditions (iv), (v) and (vi) are satisfied for all cases.

In section 3.4, it had been argued that, theoretically, $\overline{M}_n^{\star}(0, q_n)$ functions can have ripples in regions $(\sigma_{n-1}^1, \Sigma_n)$ and (Σ_n, ∞). Figure 4.2 demonstrates this fact. Also regarding the order of order-up-to levels it can be seen in Figure 4.2 that $\Sigma_1 < \Sigma_2 < \Sigma_3 < \Sigma_5 < \Sigma_4$. Thus, assuming conditions under which order-up-to levels are ordered as $\Sigma_1 < \Sigma_2 < \cdots \Sigma_N$ and under this assumption declaring that (σ_n, Σ_n) policy is optimal is a restriction on the problem. Theoretically, it does not have any significance unless those conditions can be interpreted properly.

So far we have not considered pricing issues. We have mentioned in section 3.4 that pricing decision, that is the best price at an inventory level, could not be characterised analytically unless $f(x;p)$ density function yields a special structure which can be exploited. In Figure 4.3 we provide the best price curves that are obtained for case 13 under the uniform distribution and an exponential expected demand function. It is seen that the form of these curves are interesting. Intuitively, we would expect the best price to decrease at higher inventory levels. Because, to sell more the vendor must decrease the price. In fact, under some restrictions and in the absence of the fixed cost, Zabel [19] have found that for a lost-sales model with uniform demand distribution the price of the first two periods are decreasing functions of the beginning inventory level. Also, Thowsen [13] has derived the conditions under which this fact is true for his model. However, according to our findings this is not true in general. Roughly speaking, it can be said that the best price decreases in q, but there are some moderate jumps at certain levels of inventory. This behaviour is found to be characteristic for all of the problems that we have solved for with additive demand uncertainty.

To understand the reasons for having these sudden price increase points, we take a closer look at Figure 4.3 in Figure 4.4 and consider $\overline{M}_2^{\star}(0,q_2)$ in Figure 4.1. We note that the point of jump is exactly the point where $\overline{M}_2^{\star}(0,q_2)$ passes from one regime to another. Recalling that $\overline{M}_2^{\star}(0,q_2)$ is the upper envelope of all $\overline{M}_2(0,p_2,q_2)$ functions, it now is clear that upon changing a regime we also could pass from one pricing regime to another. The mechanics of this is shown in Figure 4.5.

We believe that the primary reason behind such a pricing behaviour is the presence of a fixed cost. In deciding the best price at a period, the ordering or not ordering decisions taken place in the future periods must also be considered. Referring to Figure 4.1 suppose that we are at the second period. If it is optimal to expect that in the next period we would order then we are forced to set a low price to sell everything in the current period. But, if at an inventory level we breakeven with the decision of ordering or not ordering in the next period, then it might be optimal to carry inventories for the next period bearing the fixed cost once for both periods. In this case, we are forced to increase the price so that the demand shrinks and there will be leftovers. It is clear that, the actual decision process is much more complicated than the way we describe it. However, we try to bring an insight for observing jumps, or humps, on the best price curves. In fact, the dynamic programming solves for the optimal decision by considering all possibilities within a multi-period framework and under the demand uncertainty.

In addition to the additive demand, we also consider the multiplicative demand model with exponential random term ε. That is, $X(p) = \overline{X}(p)\cdot\varepsilon$, where ε is an exponential random variable with $E[\varepsilon] = 1$. Table 4.6 shows the optimal solutions for the first six cases under an exponential expected demand function.

Comparing Table 4.6 and Table 4.2 it can be seen that under the multiplicative demand the optimal order-up-to levels in successive periods keep growing more than that of the additive demand case. Also, the optimal expected pseudo-profit levels are considerably lower, under the former model, for the same parameter set. Furthermore, it is found that best price curves under the multiplicative exponential demand are non-increasing for all periods and all cases that are considered.

For the infinite horizon problem, on the other hand, we shall demonstrate that a stationary solution $\overline{M}^{\star}(0,q)$, given by 3.84, exists and it can be obtained by the proposed fixed point procedure. Suppose that the random demand is multiplicative with an exponential random term. Under this model we have:

$$\begin{aligned} F(x;p) &= 1-e^{-x/\overline{X}(p)} \quad x \geq 0, \\ \Theta(p,q) &= q-\overline{X}(p)\cdot F(q;p) \quad q \geq 0, \\ R_{\alpha}(x;p) &= \frac{\alpha}{1-\alpha}\cdot\left[1-e^{-(1-\alpha)\cdot x/\overline{X}(p)}\right] \quad x \geq 0, \end{aligned}$$

which can be substituted in 3.84. In addition, we assume that $c = 0.5$, $s = 0.25$, $h = 0.3$, $\mathcal{K} = 8$ and $\alpha = 0.7$.

In Figure 4.6 we show the resultant $\overline{M}^{\star}(0,q)$ curve that is obtained for the example problem. In evaluating this function by a Pascal program, each iteration took approximately 40 seconds of CPU time and an accuracy of 0.1 units on σ^1 is achieved in 10 iterations. In the same figure we also provide the finite horizon solutions for various periods. It is graphically shown on the figure that, the successive finite-period solutions approach to the theoretical infinite horizon solution as n gets larger.

Case	c	s	h	$\mathcal{K}$	λ
1	0.25	0.50	0.75	8	20
2	0.25	0.75	0.50	8	20
3	0.50	0.25	0.75	8	20
4	0.75	0.25	0.50	8	20
5	0.50	0.75	0.25	8	20
6	0.75	0.50	0.25	8	20
7	0.75	0.50	0.75	8	20
8	0.25	0.25	0.50	8	20
9	0.50	0.25	0.25	8	20
10	0.50	0.25	0.30	8	20
11	0.50	0.25	0.30	8	10
12	0.50	0.25	0.30	15	20
13	0.50	0.25	0.30	15	10

Table 4.1: 13 different parameter combinations.

	Cases 1	2	3	4	5	6
$(\sigma_1, \$_1)$	(38.05, 58.73) (35.17, 54.93)	(39.96, 60.98) (36.30, 56.49)	(31.50, 50.86) (29.60, 47.91)	(27.34, 45.75) (25.56, 42.83)	(35.25, 55.25) (31.66, 50.76)	(29.19, 47.90) (26.50, 44.12)
$(\sigma_2, \$_2)$	(39.01, 60.03) (35.95, 56.09)	(41.11, 62.50) (37.42, 58.62)	(33.45, 53.49) (30.79, 50.36)	(30.92, 51.51) (27.49, 45.38)	(38.01, 59.11) (34.04, 56.45)	(33.64, 55.06) (29.29, 51.51)
$(\sigma_3, \$_3)$	(38.84, 59.83) (36.06, 56.09)	(40.92, 62.31) (37.63, 58.62)	(33.29, 53.31) (30.89, 50.36)	(30.52, 50.23) (27.56, 45.31)	(37.82, 58.77) (34.47, 56.45)	(33.01, 53.69) (29.53, 51.40)
$(\sigma_4, \$_4)$	(38.69, 59.65) (36.15, 56.09)	(40.76, 62.11) (37.82, 58.62)	(33.15, 53.15) (30.97, 50.36)	(30.38, 50.07) (27.70, 47.85)	(37.65, 58.50) (34.87, 56.45)	(32.88, 53.50) (29.82, 51.40)
$(\sigma_5, \$_5)$	(38.56, 59.49) (36.26, 55.60)	(40.61, 61.94) (38.00, 58.62)	(33.03, 53.00) (31.07, 50.36)	(30.26, 49.92) (27.83, 47.85)	(37.50, 58.38) (35.30, 61.45)	(32.74, 53.34) (30.09, 51.40)
$\bar{M}_1^*(0, \$_1)$	83.20 87.55	85.66 88.99	70.06 75.16	59.98 65.07	74.21 77.55	61.66 66.01
$\bar{M}_2^*(0, \$_2)$	152.62 159.83	157.73 163.05	129.53 137.44	113.25 120.03	139.45 143.79	118.26 123.17
$\bar{M}_3^*(0, \$_3)$	214.47 224.13	221.88 229.05	182.54 192.85	160.46 168.92	197.46 203.00	168.21 174.19
$\bar{M}_4^*(0, \$_4)$	269.58 281.33	278.98 287.87	229.79 242.14	202.52 212.49	249.06 255.94	212.66 219.79
$\bar{M}_5^*(0, \$_5)$	318.70 332.24	329.81 340.29	271.92 285.99	239.97 251.32	294.94 303.40	252.21 260.56

	7	8	9
$(\sigma_1, \$_1)$	(27.20, 45.29) (25.49, 42.50)	(38.63, 59.82) (35.51, 55.65)	(33.89, 54.08) (30.86, 49.92)
$(\sigma_2, \$_2)$	(30.04, 49.46) (27.08, 45.31)	(39.83, 61.50) (36.54, 58.62)	(36.86, 58.67) (33.12, 53.07)
$(\sigma_3, \$_3)$	(29.82, 48.79) (27.14, 45.31)	(39.62, 61.25) (36.79, 58.62)	(36.63, 57.99) (33.48, 56.45)
$(\sigma_4, \$_4)$	(29.72, 48.66) (27.20, 45.31)	(39.43, 61.02) (37.02, 58.62)	(36.43, 57.74) (33.85, 56.45)
$(\sigma_5, \$_5)$	(29.62, 48.50) (27.25, 45.31)	(9.26, 60.82) (37.22, 58.62)	(36.26, 57.50) (34.28, 61.45)
$\bar{M}_1^*(0, \$_1)$	57.34 63.32	86.21 89.53	74.77 78.08
$\bar{M}_2^*(0, \$_2)$	107.56 116.36	158.56 163.78	140.08 144.36
$\bar{M}_3^*(0, \$_3)$	152.22 163.54	222.98 229.94	198.17 203.58
$\bar{M}_4^*(0, \$_4)$	192.03 205.52	280.35 288.91	249.86 256.52
$\bar{M}_5^*(0, \$_5)$	227.50 242.86	331.45 341.46	295.84 303.92

Table 4.2: Optimal solutions for different 5-period lost-sales problems. Top values are evaluated under the additive uniform distribution and bottom ones under the additive triangular distribution both with an exponential expected demand function. Static parameters are: $P_l = 0.1$, $P_u = 4.0$, $\alpha = 0.9$, $a = 150$, $b = 0.5$.

	Cases			
	10	11	12	13
$(\sigma_1, \$_1)$	(33.61, 53.69) (30.70, 49.65)	(29.81, 48.39) (28.31, 46.35)	(26.96, 53.69) (24.55, 49.65)	(23.73, 48.39) (22.46, 46.35)
$(\sigma_2, \$_2)$	(36.44, 57.81) (32.87, 52.90)	(31.00, 50.18) (29.47, 49.72)	(30.31, 61.11) (27.39, 83.72)	(24.83, 82.61) (24.20, 81.17)
$(\sigma_3, \$_3)$	(36.23, 57.42) (33.05, 52.90)	(30.80, 49.92) (29.84, 49.72)	(29.51, 59.20) (26.46, 56.45)	(24.61, 49.97) (23.60, 49.72)
$(\sigma_4, \$_4)$	(36.05, 57.19) (33.37, 56.45)	(30.62, 49.69) (30.52, 52.22)	(29.43, 59.18) (27.58, 86.77)	(24.46, 49.76) (24.56, 82.50)
$(\sigma_5, \$_5)$	(35.88, 56.98) (33.69, 56.45)	(30.47, 49.48) (31.18, 52.23)	(29.27, 58.97) (27.04, 86.50)	(24.33, 49.50) (24.37, 52.22)
$\overline{M}_1^*(0, \$_1)$	74.22 77.74	80.09 81.85	74.22 77.74	80.09 81.85
$\overline{M}_2^*(0, \$_2)$	138.84 143.56	147.09 149.51	133.59 138.72	140.94 144.42
$\overline{M}_3^*(0, \$_3)$	196.34 202.22	206.71 210.00	185.69 191.69	195.01 199.19
$\overline{M}_4^*(0, \$_4)$	247.50 254.63	259.79 264.38	232.14 240.18	243.14 249.13
$\overline{M}_5^*(0, \$_5)$	293.02 301.49	307.03 313.27	273.45 282.58	285.98 293.26

Table 4.3: Optimal solutions of the 5-period lost-sales problem solved for cases 10 through 13. The values are evaluated under the additive uniform (top values) and the additive triangular (bottom values) distributions with an exponential expected demand function. The static parameters are: $P_\ell = 0.1$, $P_u = 4.0$, $\alpha = 0.9$, $a = 150$, $b = 0.5$.

	Cases					
	1	2	3	4	5	6
$(\sigma_1, \ddagger_1)$	(38.05, 58.73) (61.27, 80.75)	(39.96, 60.98) (63.47, 83.18)	(31.50, 50.86) (54.60, 73.93)	(27.34, 45.75) (50.03, 69.43)	(35.25, 55.25) (59.07, 78.86)	(29.19, 47.90) (52.30, 71.93)
$(\sigma_2, \ddagger_2)$	(39.01, 60.03) (62.23, 82.03)	(41.11, 62.50) (64.60, 84.65)	(33.45, 53.49) (56.77, 76.79)	(30.92, 51.51) (53.86, 74.48)	(38.01, 59.11) (62.04, 82.67)	(33.64, 55.06) (56.82, 77.82)
$(\sigma_3, \ddagger_3)$	(38.84, 59.83) (61.89, 81.65)	(40.92, 62.31) (64.25, 84.25)	(33.29, 53.31) (56.45, 76.43)	(30.52, 50.23) (53.53, 74.12)	(37.82, 58.77) (61.68, 82.26)	(33.01, 53.69) (56.47, 77.42)
$(\sigma_4, \ddagger_4)$	(38.69, 59.65) (61.60, 81.32)	(40.76, 62.11) (63.93, 83.91)	(33.15, 53.15) (56.16, 76.11)	(30.38, 50.07) (53.24, 73.79)	(37.65, 58.50) (61.36, 81.90)	(32.88, 53.50) (56.16,77.08)
$(\sigma_5, \ddagger_5)$	(38.56, 59.49) (61.33, 81.03)	(40.61, 61.94) (63.65, 83.60)	(33.03, 53.00) (55.91, 75.83)	(30.26, 49.92) (52.98, 73.50)	(37.50, 58.38) (61.07, 81.58)	(32.74, 53.34) (55.89, 76.77)
$\overline{M}_1^*(0, \ddagger_1)$	83.20 140.28	85.66 142.89	70.06 121.44	59.98 105.56	74.21 125.78	61.66 107.25
$\overline{M}_2^*(0, \ddagger_2)$	152.62 260.65	157.73 265.97	129.53 226.83	113.25 199.18	139.45 236.91	118.26 203.94
$\overline{M}_3^*(0, \ddagger_3)$	214.47 367.88	221.88 375.51	182.54 320.77	160.46 282.57	197.46 335.76	168.21 289.95
$\overline{M}_4^*(0, \ddagger_4)$	269.58 463.43	278.98 473.02	229.79 404.51	202.52 356.85	249.06 423.67	212.66 366.48
$\overline{M}_5^*(0, \ddagger_5)$	318.70 548.58	329.81 559.83	271.92 479.16	239.97 423.04	294.94 501.88	252.21 434.58

Table 4.4: Optimal solutions of the 5-period lost-sales problem that is solved under the additive uniform distribution for cases 1 through 6. The top and bottom values, respectively, are evaluated under an exponential and a linear expected demand functions. The respective functional parameters (a, b) are: $(150, 0.5)$ and $(150, 32.5)$. Other static parameters are: $P_\ell = 0.1$, $P_u = 4.0$, $\alpha = 0.9$.

Cases	σ	$\check{\Sigma}$	ϑ	Σ	γ	$\hat{\Sigma}$
1	39.20	40.20	53.26	60.25	67.54	83.03
	35.84	36.79	49.23	55.96	63.02	78.32
2	41.31	42.34	55.65	62.78	70.21	86.20
	37.21	38.18	50.99	57.96	65.33	81.85
3	33.61	34.56	47.00	53.69	60.67	75.60
	30.70	31.59	43.29	49.65	56.36	70.98
4	30.82	31.76	43.93	50.50	57.42	72.37
	27.44	28.30	39.67	45.96	52.66	67.91
5	38.20	39.20	52.22	59.24	66.59	88.41
	33.75	34.70	47.34	54.39	62.09	84.46
6	33.33	34.28	46.80	53.58	60.71	81.04
	29.11	30.00	41.96	48.69	56.06	76.94
10	36.65	37.66	50.82	57.94	65.41	84.85
11	31.22	32.13	44.02	50.47	57.25	76.20
12	29.66	30.95	48.28	57.94	68.27	102.85
13	24.96	26.11	41.73	50.47	59.83	94.20

Table 4.5: Values of the critical inventory levels which determine the feasible values that the optimal control parameters can assume. These values are evaluated under the additive uniform distribution (top values) and the additive triangular distribution (bottom values), with exponential expected demand function. The static parameters are: $P_\ell = 0.1$, $P_u = 4.0$, $\alpha = 0.9$, $a = 150$, $b = 0.5$.

	Cases					
	1	2	3	4	5	6
(σ_1, Σ_1)	(19.58, 43.09)	(24.29, 52.67)	(14.40, 32.50)	(12.94, 28.76)	(21.73, 46.48)	(16.01, 33.50)
(σ_2, Σ_2)	(28.22, 54.34)	(37.29, 68.62)	(23.16, 45.51)	(24.02, 45.50)	(40.12, 72.46)	(31.16, 57.22)
(σ_3, Σ_3)	(28.49, 58.13)	(38.94, 74.92)	(23.93, 50.25)	(26.58, 53.50)	(47.76, 86.75)	(38.02, 71.02)
(σ_4, Σ_4)	(28.03, 59.16)	(38.04, 77.10)	(23.41, 51.51)	(25.87, 56.76)	(47.51, 93.81)	(38.29, 78.10)
(σ_5, Σ_5)	(28.08, 59.51)	(38.03, 77.20)	(23.42, 51.46)	(25.75, 57.13)	(46.73, 96.01)	(37.46, 81.22)
$\overline{M}_1^*(0, \Sigma_1)$	44.24	48.94	36.50	32.21	43.19	34.45
$\overline{M}_2^*(0, \Sigma_2)$	86.12	97.57	74.09	70.12	94.92	79.09
$\overline{M}_3^*(0, \Sigma_3)$	122.71	140.24	106.93	103.80	141.45	119.87
$\overline{M}_4^*(0, \Sigma_4)$	155.25	178.12	136.02	133.45	182.60	155.90
$\overline{M}_5^*(0, \Sigma_5)$	184.43	212.05	162.07	159.94	219.26	187.93

Table 4.6: Optimal solutions of the 5-period lost-sales problem that is solved under the multiplicative exponential distribution for cases 1 through 6 with exponential expected demand function. The static parameters are: $P_\ell = 0.1$, $P_u = 4.0$, $\alpha = 0.9$, $a = 150$, $b = 0.5$.

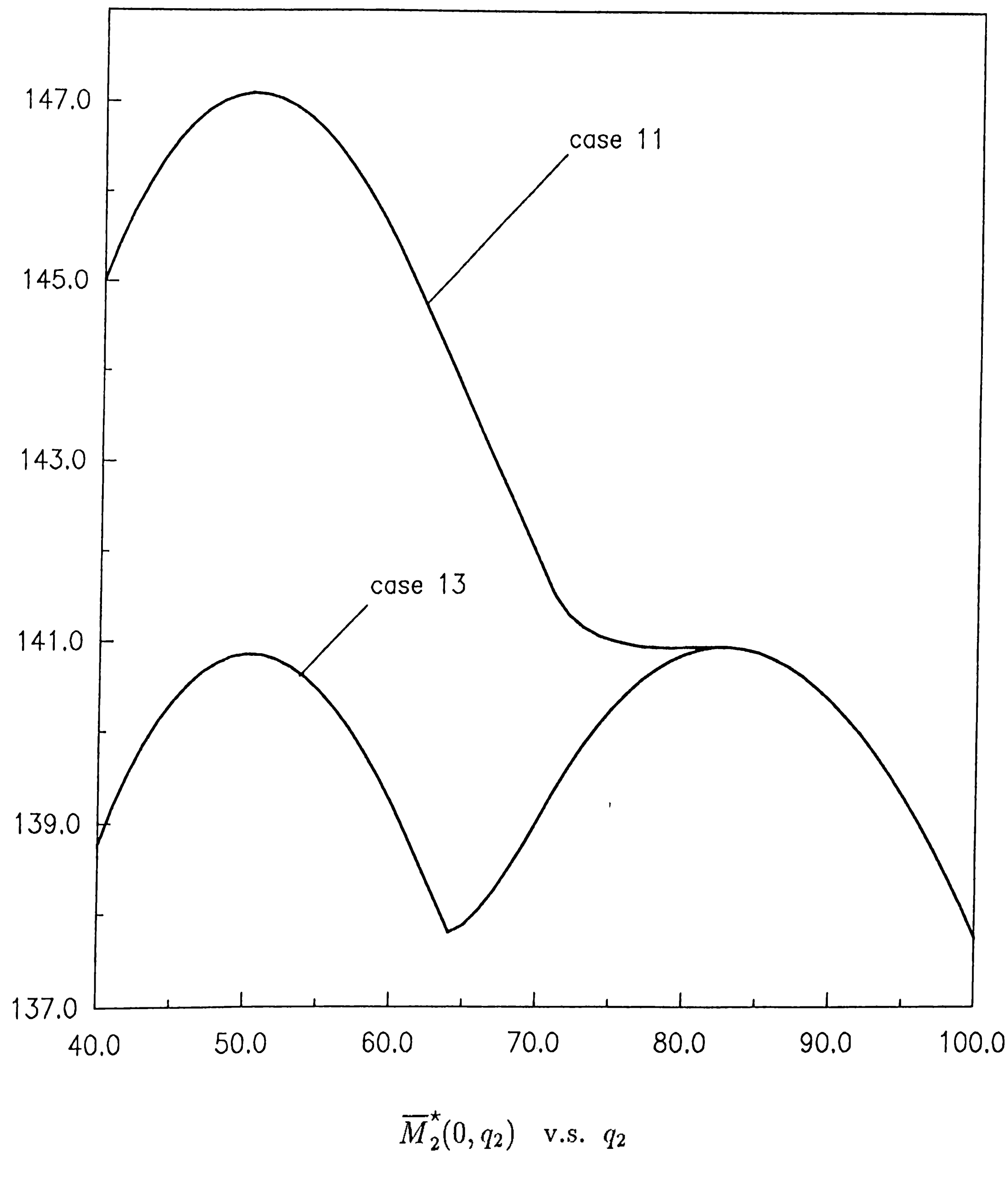
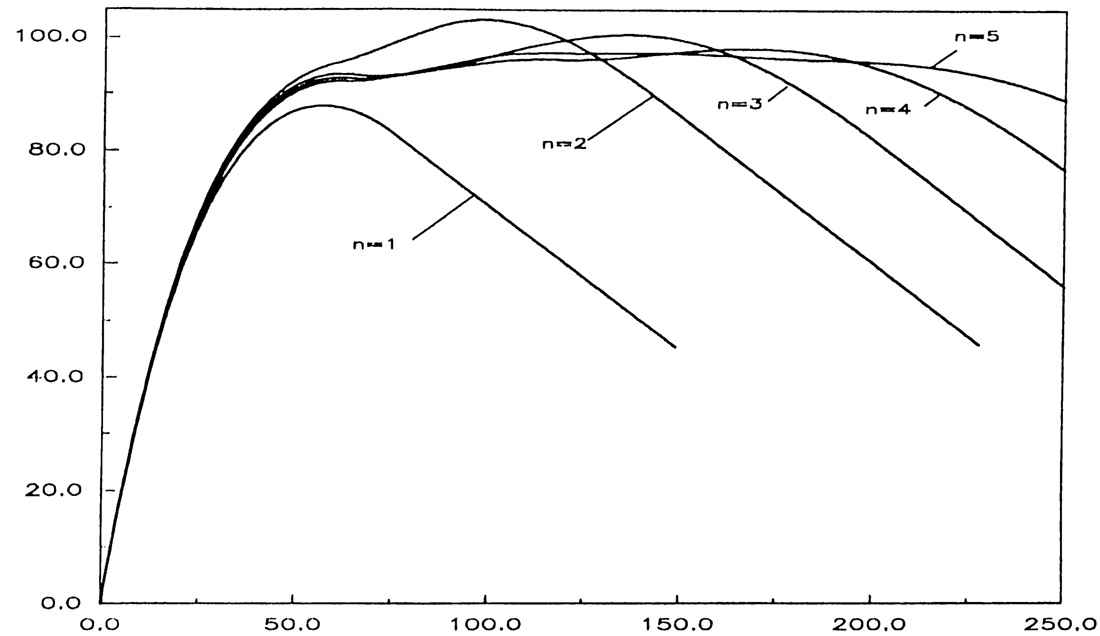
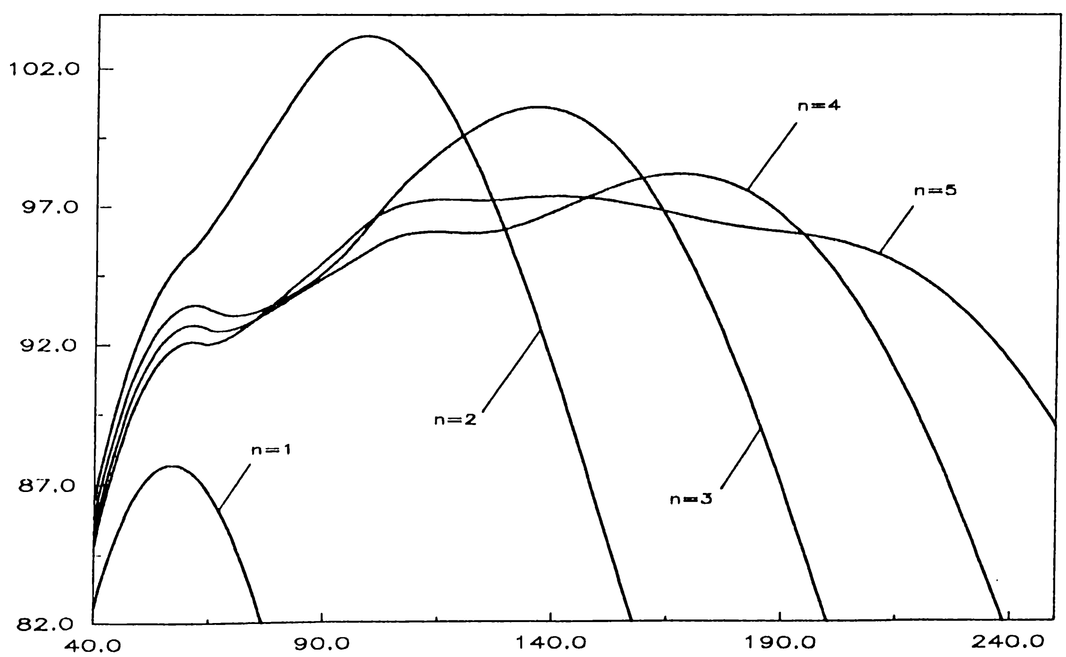


Figure 4.1: Expected pseudo-profit function of the second period which is evaluated for cases 11 and 13 under the additive uniform distribution and the exponential expected demand function with $a = 150$, $b = 0.5$ and $\alpha = 0.9$.

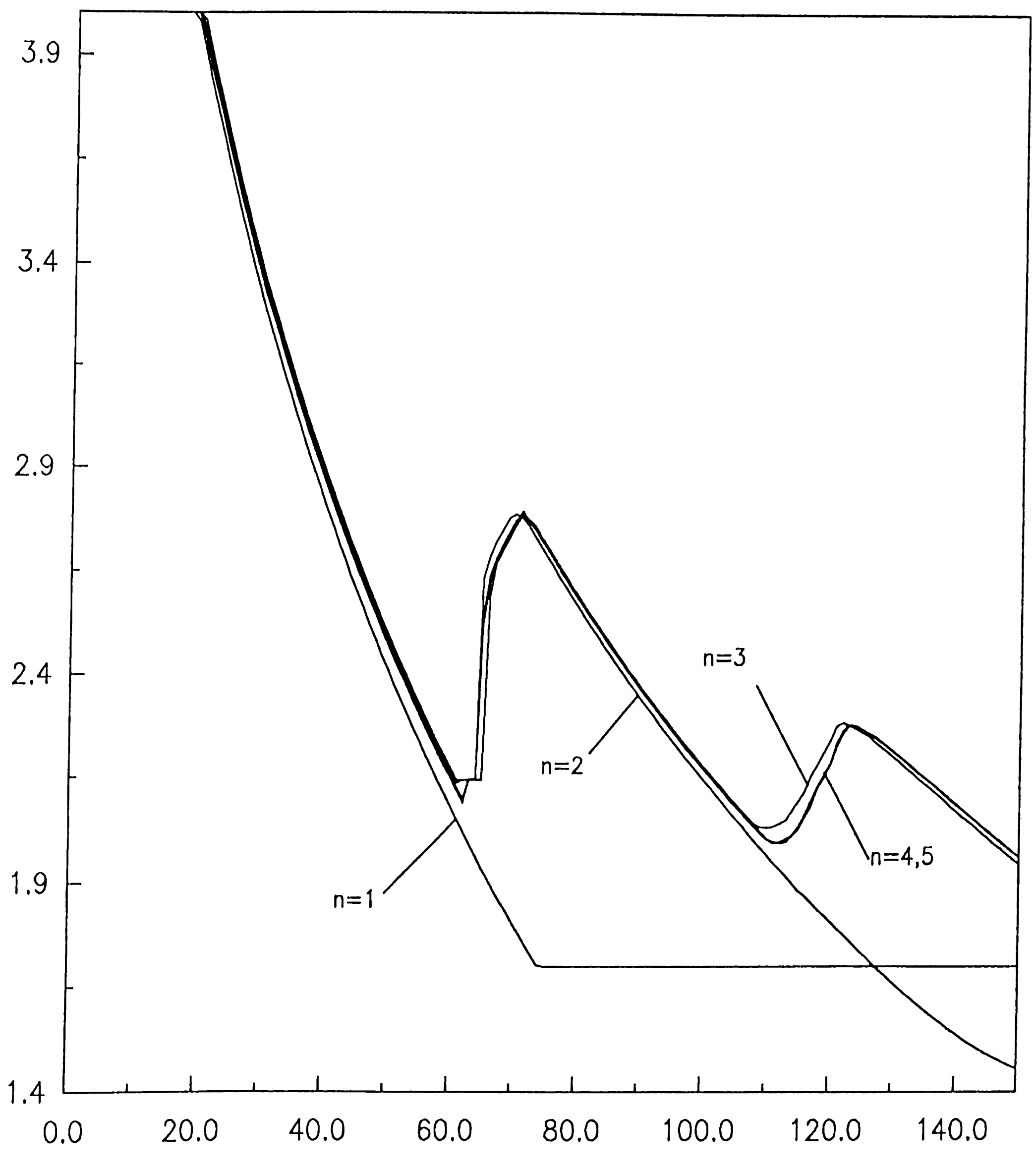


$\overline{M}_n^{\star}(0,q_n) - \overline{M}_n^{\star}(0,0)$ v.s. q_n



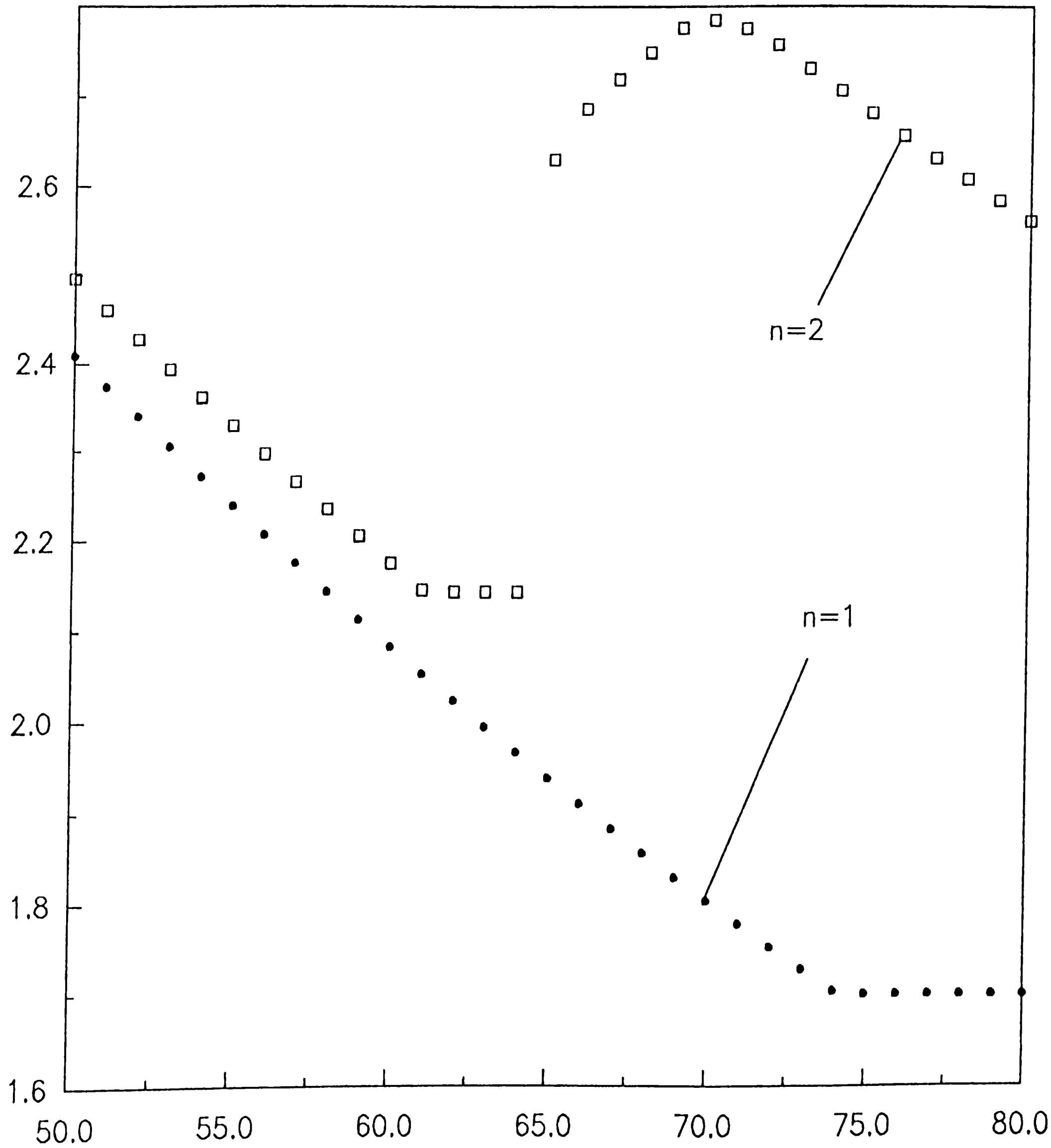
$\overline{M}_n^{\star}(0,q_n) - \overline{M}_n^{\star}(0,0)$ v.s. q_n

Figure 4.2: Expected pseudo-profit curves which are evaluated for a 5-period lost-sales model under an additive uniform demand distribution with c = 0.5, s = 0.25, h = 0.01, $\mathcal{K}$ = 15, λ = 20, a = 150, b = 0.5 and α = 0.9.



p_{q_n} v.s. q_n

Figure 4.3: Best price curves, which gives the optimal values of pricing decision, for case 13 evaluated under the additive uniform distribution and an exponential expected demand function with $a = 150$, $b = 0.5$ and $\alpha = 0.9$.



p_{q_n} v.s. q_n

Figure 4.4: Close-up of Figure 4.3.

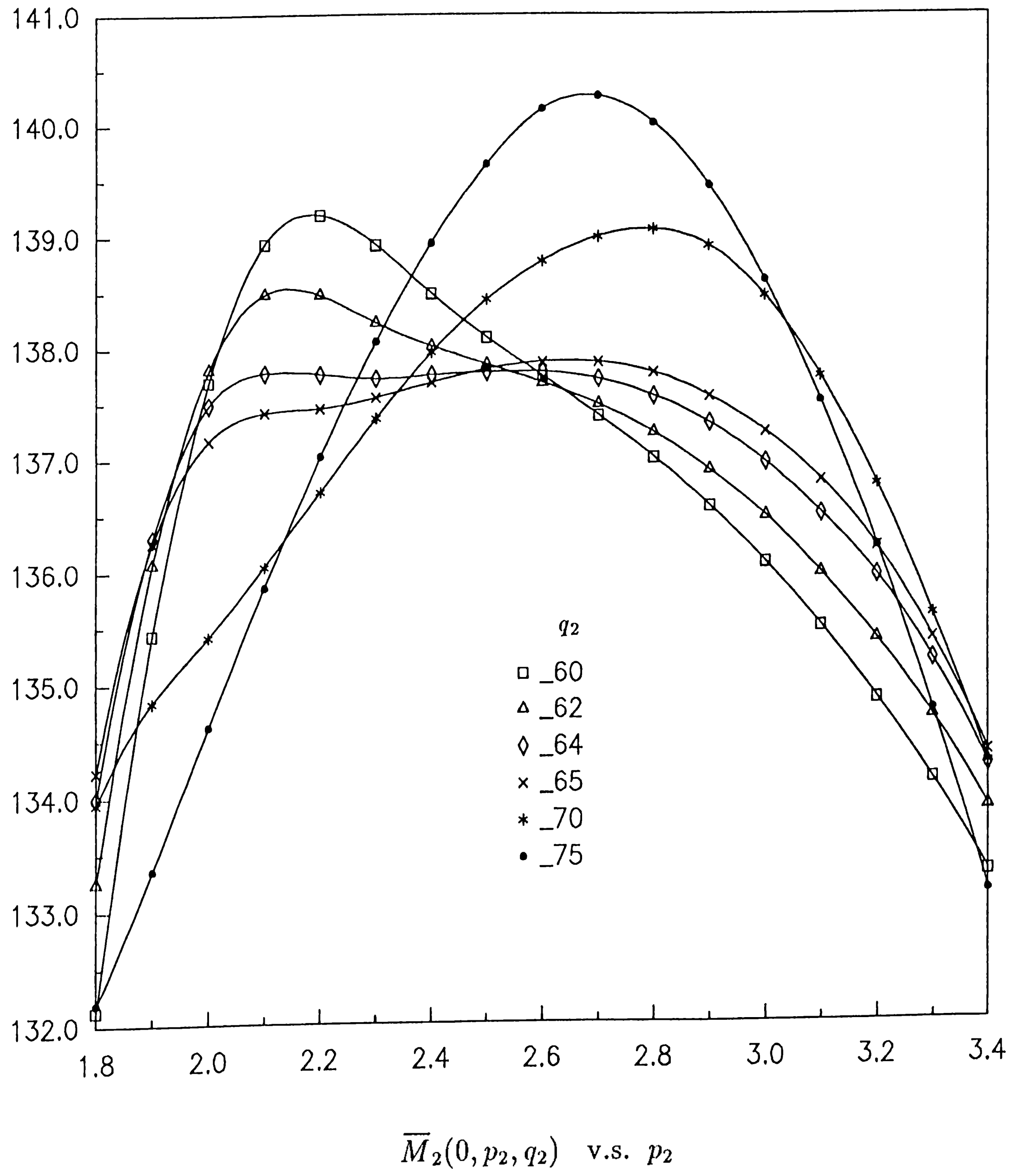


Figure 4.5: Expected pseudo-profit functions $\bar{M}_2(0, p_2, q_2)$ evaluated for case 13 under the additive uniform distribution and an exponential expected demand function with $a = 150$, $b = 0.5$ and $\alpha = 0.9$. The curves are obtained for q_2 values of 60, 62, 64, 65, 70 and 75.

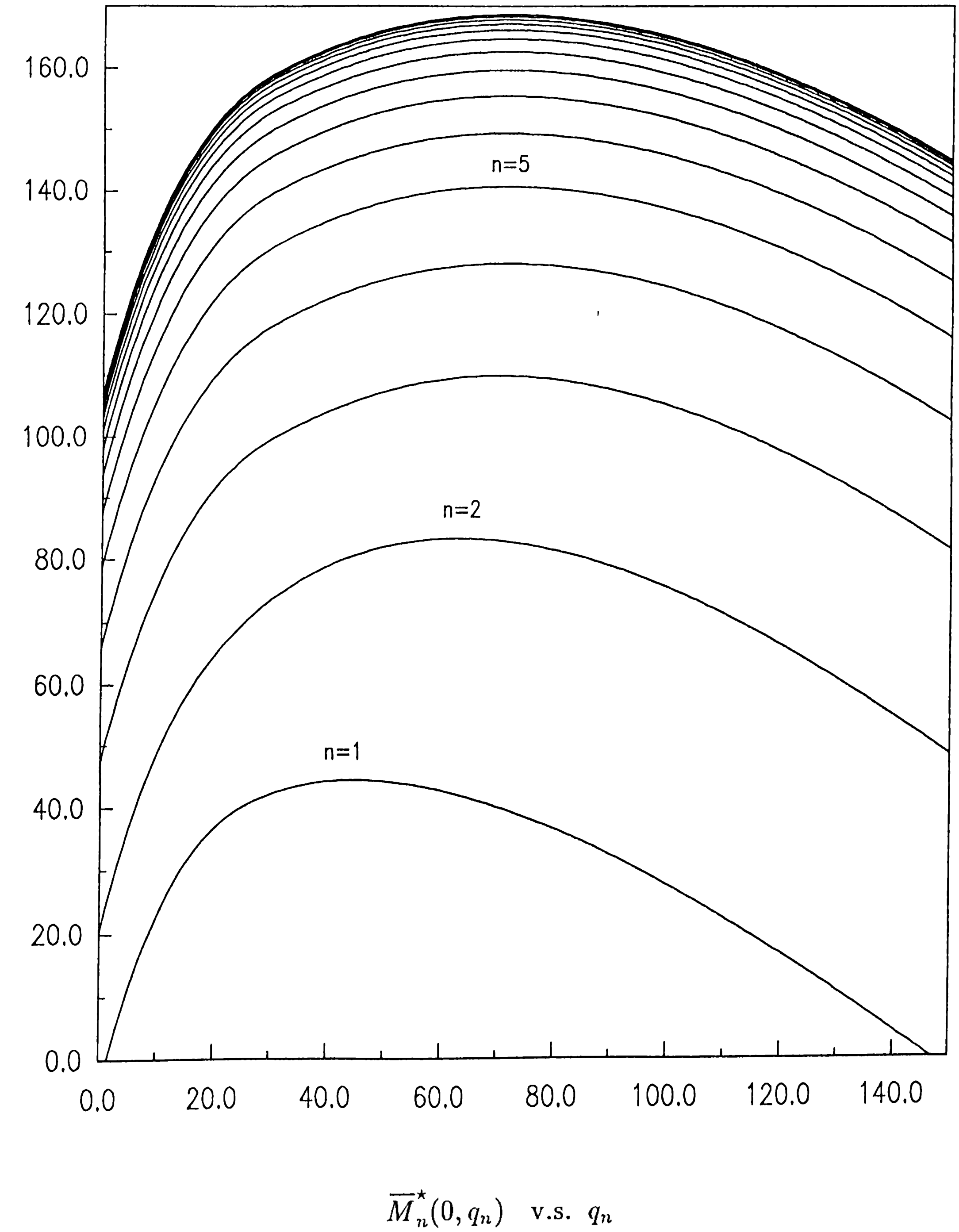


Figure 4.6: Expected pseudo-profit curves evaluated for 15 periods under the multiplicative exponential demand with exponential expected demand function. The curve at the top represents the theoretical infinite horizon expected pseudo-profit. Parameters are: $c = 0.5$, $s = 0.25$, $h = 0.3$, $K = 8$, $a = 150$, $b = 0.5$, $\alpha = 0.7$, $P_\ell = 0.1$ and $P_u = 4.0$.

Chapter 5

Conclusions

In most of the existing models the price-demand relationship has been simplified by making various assumptions about the distribution of random demand, the expected demand curve or the parameter values. Moreover, sufficient conditions have been derived in order to ensure optimality of certain inventory control policies. However, in almost all of these models these conditions could not be interpretted properly. In contrast, in this study we have approached to the problem in a pragmatic way. We have not based our analysis on a particular demand model, but assumed a general demand distribution. Our intension has been to reveal fundamental characteristics of the inventory system independent of the underlying demand model. On the other hand, to establish a link between the existing models and to provide some examples we have also studied our model under certain demand distributions.

It is observed in the literature that existing models have not emphasized possible roles that price could play when demand is backlogged. First of all, price is one of the determinants of the forgone revenue if there are shortages in any period. It is interesting that, under an optimal policy, the forgone revenue, which could arise in any period, might not ever be collected in the future periods. Therefore, a good model must take this option into account. In all of the studies referred to in here, except Thowsen's [13], the foregone revenue has not been considered adequately. It is either ignored by assuming a lost-sales model or mistakenly forgotten. In Thowsen's model, this issue is simplified by assuming a forgone revenue which is expressed as a constant ratio of backlogged revenue. The rationale behind this assumption could not be justified, because it follows from the result of this study that the optimal procurement policy is very sensitive to the role that price plays in the process of backlogging. Thus, a constant ratio of backlogged revenue could yield a policy which is considerably different from the optimal one. Also, there is a serious question about the value of that ratio. How could we determine that in any real inventory system? On the other hand, in Young's [17] infinite horizon lost-sales model it is assumed that the unsold inventory at the end of each period has an economic value that is equal to the present worth of its procurement cost. However, the economic value of unsold inventory is the present worth of the revenue that incurs in the future periods due to the sales of leftovers less the difference between procurements in the current period and the future period due to discounting. For this reason, isolating each period and thereby ignoring the binding effect of price between the periods could not be justified in a general setting.

In this perspective, we have developed an inventory system structure which embodies pricing decision in addition to procurement quantity decision. In this model, price plays several roles simultaneously. It is not only a unit income per sale, but also a factor that affects demand. Furthermore, in the case of a shortage, price is a bargaining matter between the customer and the vendor. In addition, it is an opportunity cost for the lost demand. In this study, we have considered all of these issues, to some extent, in a general setting. Under the proposed representation the existing models become special cases.

In the first part of our study we have concentrated on the single period model. We have developed the basic mathematical model which incorporates price as the second decision variable. We have generalised the price dependence of $\overline{X}(p)$, $R(p)$, $F(x;p)$ and $\Theta(p,q)$ functions. These generalizations are important to clearify the boundaries of price dependene in the most general sense. Especially, these considerations are essential in modelling for an application. In addition, we have brought another useful idea for the price variable. That is, the range of feasible price values, $[P_\ell, P_u]$. Under this setting, we have established ways of attacking the problem analytically. Thereby, we have been able to study the effects of price bounds on the optimal solutions.

In the absence of price limits, the existence issues have been studied in detail in the literature. For the existence of the best price, p_q, we have devised an alternative proof that severely relaxes the assumptions on price dependence. We have found a way of proving existence by only restricting $\overline{X}(p)$ function at its natural limits, that is as p tends to zero or infinity. Moreover, we have established the existence of a finite order-up-to level by a new proof. Under this proof, the form of the one period expected profit function could be characterised at its extremes, that is on $[0, X_1(P_u)]$ and on $[X_2(P_\ell), \infty)$. Also, we have found an upper bound on the optimal order-up-to level which could be determined by c, s, h and $\overline{X}(p)$.

We have studied the single period model under three special demand forms. In the first case, we have shown that if demand is deterministic, then the pseudo-profit function is quasiconcave on $[0, \infty)$ and, furthermore, it lays above the expected profit function for any probabilistic demand. Moreover, we have obtained the optimal order-up-to level as $\overline{X}(\bar{P}_c)$, where the best price is found to be $\bar{P}_c$ at this inventory level. It is surprising that under the optimal policy there could be shortages or leftovers although the demand is deterministic. We also have established that $\bar{P}_h$ is the lower bound of the optimal price values.

Next, we have considered the additive and the multiplicative demand models. We have verified that $c \leq \tilde{p} \leq P_c$ and $P_c \leq \tilde{p}$ under these models respectively. In addition, we have shown that, under no further assumption, the expected pseudo-profit function is unimodal for a linear expected demand function with additive uniform demand or with multiplicative exponential demand.

We have found that, under a general model, since p_q or $\tilde{q}$ could not be evaluated explicitly, the unimodality of the expected pseudo-profit function could not be justified. However, there is no major practical difficulty in verifying unimodality for a given specific problem. The reader may refer to Lau and Lau [5] for numerical examples.

In the second part of our study we have considered the multi-period problem. First, we have developed the mathematical model that takes into account the issue of bargaining on price, between the customer and the vendor, which might arise when there is shortage in any intermediate period. Then, we have introduced three special baclogging rules and derived the n-period pseudo-profit functions under each case.

We have shown that under deterministic demand, if the shortages are simply lost, then the optimal procurement quantity in each period could be determined by an (σ_n, Σ_n) policy. Under the second backlogging rule, which assumes that the customers wait only one period providing that they pay current period's price, we have identified that, though, an order-up-to level exists for every period the reorder point could depend on the previous period's price setting. Thus, an (σ_n, Σ_n) policy could not be optimal in general. In this regard, we have defined an alternative optimal procurement policy which utilizes the fact that the order-up-to levels are known. Furthermore, under the assumption that the customers wait only one period whatever the price is, we have found that, in general, not only the reorder point but also the order-up-to level could be a function of the beginning inventory level before ordering in the current period.

Therefore, we have demonstrated that depending on the type of the backlogging rule the optimal procurement strategy could be different than an (σ_n, Σ_n) policy. Hence, dwelling on the conditions which ensure optimality of such a policy, in general, is undermining the problem. It is essential that, the role of price in the process of backlogging is clearly described rather than making rough assumptions about the forgone revenue which could not be justified.

The probabilistic demand model, on the other hand, has been considered under the lost-sales assumption. It has been shown that, the expected n-period pseudo-profit function could be characterized such that an (σ_n, Σ_n) type policy is optimal if we assume that $i_n \leq \sigma_n^{k_n}$, where $\Sigma_n < \sigma_n^{k_n}$. We have also found that the optimal order-up-to level must be within $[\check{\Sigma}, \hat{\Sigma}]$ where $\check{\Sigma}$ and $\hat{\Sigma}$ could be determined from the problem parameters. It has been demonstrated that unimodality of the single period expected pseudo-profit function is essential in proving the above results.

As a special case, we have also considered the infinite horizon lost-sales probabilistic demand problem. We have shown that the expected pseudo-profit function could be obtained through a renewal theoretic approach. To demonstrate this, we have provided an example problem.

The effects of the parameter values on the optimal solution have been discussed on some example problems. It has been shown that problem parameter values, under the lost-sales probabilistic demand model, could effect the values of the optimal control parameters considerably. Also, type of the demand distribution and form of the expected demand function could affect the optimal solution.

The pricing issues are also discussed in the last chapter. The striking result is the fact that, due to presence of the fixed cost, the best price is not always decreasing at higher inventory levels. We have identified some example problems in which the best price, for $n > 1$, temporarily increases as q_n gets higher, but starts decreasing later again.

There are possible extensions to our model which could be considered as future research opportunities. For instance, the model can be further generalized by assuming that the cost parameters and/or the demand distribution are different in each period. In fact, our theoretical work will be exactly valid under this extension, but we have to rewrite the mathematical model and modify the results accordingly. This generalization would enable us to identify, for example, the effects of a rise or a fall in demand (with a certain pattern in time) on the optimal solution. Similarly, effects of the inflation rate on the optimal procurement and pricing decisions of the vendor could be investigated by including an inflation rate factor.

Another potential issue is the fact that there are several other factors which affect demand besides the retail price. For instance, income level of the customers, sales effort, competitors' price or substitute's price are possible ones. The last two factors link the model with the game-theoretic applications under which the analysis is severely limited due to mathematical intractabilities. The reader may refer to Kirman and Sobel [4] or to Nti [10]. The sales effort (advertisement and etc.) issue is studied by Gerchak and Parlar [1] for the single period model, which does not include the pricing decision. Thus, to start with, their model could be extended in the proposed direction, and the effect of price and sales effort could be simultaneously studied. Since this analysis is subject to analytical difficulties, a simulation study similar to the one performed in [1] would be appropriate.

In this study, we have assumed that the vendor is maximizing the expected profit. This intrinsically means that the vendor is risk neutral. As an extension, therefore, we can incorporate risk attitude of the vendor by defining a utility function, and maximize the utility of the expected profit. The general setting for this model is explained by Leland in [6].

Apart from structural changes, some technical points could also be emphasized. For instance, the proof of unimodality of the single period expected pseudo-profit function could be further studied. A realistic price-demand relationship could be formulated under which the unimodality is demonstrated. Moreover, the connection of unimodality with the monopoly power of the vendor could be investigated. Under the multi-period model, on the other hand, the characteristics of the $\overline{M}_n^\star(0, q_n)$ function could be further revealed by improving the six conditions which are formulated earlier. In this regard, for instance, we might try to show that the ripples in the range of $(\sigma_n^1, \ \hat{\vartheta}\ )$ would always remain above the $\overline{M}_n^\star(0, \hat{q}_n) - \mathcal{K}$ level so that $k_n = 2$ for all n. This would enable us to prove the optimality of the $(\sigma_n, \hat{q}_n)$ policy under the rule that $i_n \leq \sigma_n^{k_n}$ for $n > 1$.

Appendix A

For $R(p) = p\cdot\overline{X}(p)$ we have

$$R'(p) = \overline{X}(p) + p\cdot\overline{X}'(p), \tag{A.1}$$
$$R''(p) = 2\cdot\overline{X}'(p) + p\cdot\overline{X}''(p). \tag{A.2}$$

Lemma A1. *$R(p)$ is not pseudoconcave for all monotone decreasing $\overline{X}(p)$ functions.*

Proof. If we let $\overline{X}(p) = 600\cdot e^{-0.15\cdot p} + 1.5\cdot Sin(2\cdot\pi\cdot p)$, which is a monotone decreasing function of p on $(0, 8)$, then $R(p)$ is not a pseudoconcave function on $(0, 8)$.

Lemma A2. *If $\overline{X}(p)$ is a convex decreasing function, then $R(p)$ is pseudoconcave on $(0,\infty)$.*

Proof. Since $\overline{X}(p)$ is a convex decreasing function, $\forall p, p_1 \in (0,\infty)$ we have

$$\overline{X}(p_1) - \overline{X}(p) \geq (p_1 - p)\cdot\overline{X}'(p_1), \tag{A.3}$$

$$p_1 < (>) p \;\Leftrightarrow\; \overline{X}(p_1) > (<) \overline{X}(p). \tag{A.4}$$

By definition, $R(p)$ will be pseudoconcave at $p_1 \in (0,\infty)$ if it is differentiable at p_1 and

$$R'(p_1)\cdot(p - p_1) \leq 0 \;\Rightarrow\; R(p) \leq R(p_1), \;\; \forall p \in (0,\infty). \tag{A.5}$$

From (A.1) and (A.5) we get

$$R'(p_1)\cdot(p - p_1) = \overline{X}(p_1)\cdot(p - p_1) + p_1\cdot(p - p_1)\cdot\overline{X}'(p_1) \leq 0. \tag{A.6}$$

By (A.3) we have

$$p_1\cdot(p - p_1)\cdot\overline{X}'(p_1) \geq p_1\cdot[\overline{X}(p) - \overline{X}(p_1)]. \tag{A.7}$$

It follows from (A.6) and (A.7) that

$$(A.6) \;\Rightarrow\; \overline{X}(p_1)\cdot(p - p_1) + p_1\cdot[\overline{X}(p) - \overline{X}(p_1)] \leq 0, \tag{A.8}$$

and

$$(A.8) \;\Leftrightarrow\; p\cdot\overline{X}(p_1) + p_1\cdot[\overline{X}(p) - \overline{X}(p_1)] \leq p_1\cdot\overline{X}(p_1) = R(p_1). \tag{A.9}$$

Adding and subtracting $p\cdot\overline{X}(p)$ on the L.H.S. of (A.9) and collecting terms we obtain

$$(A.9) \;\Leftrightarrow\; R(p) + (p - p_1)\cdot[\overline{X}(p_1) - \overline{X}(p)] \leq R(p_1). \tag{A.10}$$

From (A.4) we have

$$(p - p_1)\cdot[\overline{X}(p_1) - \overline{X}(p)] \geq 0. \tag{A.11}$$

Therefore, by (A.10) and (A.11) we get

$$(A.9) \Rightarrow R(p) \leq R(p_1).$$

Since p_1 was arbitrary the proof is valid for all $p_1 \in (0, \infty)$.

Theorem A1. *If $\overline{X}(p)$ is a convex or concave decreasing function, then $R(p)$ is pseudoconcave on $(0, \infty)$.*

Proof. If $\overline{X}(p)$ is concave, then from (A.2) it follows that $R(p)$ is concave on $(0, \infty)$. Also by Lemma A2, $R(p)$ is pseudoconcave on $(0, \infty)$ for a convex decreasing function.

Corollary A1. *The function $T(p) = (p + a)\cdot\overline{X}(p)$ is pseudoconcave on $(0, \infty)$, where $a \in \mathcal{R}$.*

Proof. Making a coordinate change by $p_2 \leftarrow p + a$ and introducing the function $Y(p_2) = \overline{X}(p_2 - a)$ we obtain

$$T(p) = (p + a)\cdot\overline{X}(p) = p_2\cdot\overline{X}(p_2 - a) = p_2\cdot Y(p_2).$$

If $a < 0$, then $p_2 \cdot Y(p_2)$ is monotone increasing on $(a, 0)$. Since $Y(p_2)$ is monotone decreasing on $(0, \infty)$, it follows by Theorem A1 that $p_2\cdot Y(p_2)$ is a pseudoconcave function on $(0, \infty)$. Therefore, $p_2\cdot Y(p_2)$ is a pseudoconcave function on (a, ∞) which implies that $T(p)$ is a pseudoconcave function on $(0, \infty)$.

If $a > 0$, then we can extend $Y(p_2)$ on $(0, a)$ by a straight line (or by another appropriate function) which complies with the assumptions of Theorem A1. From the same theorem it follows that $p_2\cdot Y(p_2)$ is pseudoconcave on $(0, \infty)$. Hence, it is also pseudoconcave in its open subset (a, ∞). Thus, $T(p)$ is a pseudoconcave function on $(0, \infty)$.

Corollary A2. *$\forall a, b \in \mathcal{R}$ such that $a \geq b$ we have $p_a \leq p_b$ where*

$$p_a = argsup\{(p + a)\cdot\overline{X}(p) : \; p \in (0, \infty)\},$$

$$p_b = argsup\{(p + b)\cdot\overline{X}(p) : \; p \in (0, \infty)\}.$$

Proof. We define two functions: $A(p) = (p+a)\cdot\overline{X}(p)$ and $B(p) = (p+b)\cdot\overline{X}(p)$. From Corollary A1 it follows that $A(p)$ and $B(p)$ are pseudoconcave functions on $(0, \infty)$. Moreover, we note that

$$A(p) \geq B(p) \quad \forall p \in (0, \infty) \tag{A.12}$$

If $p_a, p_b \in (0, \infty)$, then they should satisfy the first order conditions

$$A'(p_a) = 0 \quad \text{and} \quad B'(p_b) = 0.$$

We rewrite $A(p)$ as $A(p) = B(p) + (a - b)\cdot\overline{X}(p)$, which leads to:

$$A'(p) = B'(p) + (a - b)\cdot\overline{X}'(p).$$

Evaluating the above equation for $p = p_a$ we get

$$A'(p_a) = B'(p_a) + (a - b)\cdot \overline{X}'(p_a),$$

which implies

$$B'(p_a) = -(a - b)\cdot \overline{X}'(p_a) \geq 0. \tag{A.13}$$

That is, $B(p)$ is non-decreasing at $p = p_a$. Since $B(p)$ is a pseudoconcave function we deduce that $p_a \leq p_b$.

If p_a and p_b are both non-interior point solutions, then (A.12) implies that $p_a = p_b = 0$.

If $p_a = 0$ and $p_b \in (0, \infty)$, then $p_a \leq p_b$.

If $p_a \in (0, \infty)$, then from (A.13) we conclude that $p_a \leq p_b$.

Corollary A3. *$(p - c)\cdot q$ is a pseudoconcave function of q on $(\overline{X}(P_u), \overline{X}(P_\ell))$, where $q = \overline{X}(p)$.*

Proof. $\overline{X}(p)$ is a decreasing function of p. Therefore, its inverse, $\overline{X}^{-1}(q)$, is decreasing on $(\overline{X}(P_u), \overline{X}(P_\ell))$. By Theorem A1, $q\cdot \overline{X}^{-1}(q)$ is pseudoconcave on $(\overline{X}(P_u), \overline{X}(P_\ell))$. Thus, $q\cdot \overline{X}^{-1}(q) - c\cdot q = (p - c)\cdot q$ is also pseudoconcave on $(\overline{X}(P_u), \overline{X}(P_\ell))$.

Appendix B

The discontinuity in $\overline{M}^{\star}(q)$, if it exists, can not be of second kind. Because, $\overline{M}(p,q)$ is everywhere defined on $p \in [P_{\ell}, P_u]$ and $q \in [i, \infty)$. That is, $\forall q \in [i,\infty)$ $\exists p_q \in [P_{\ell}, P_u]$ such that $\overline{M}^{\star}(q) = \overline{M}(p_q, q)$.

If at $q = \hat{q}$, $\overline{M}(q)$ has a first kind discontinuity, then

$$\lim_{q \to \hat{q}^-} \overline{M}^{\star}(q) = \lim_{q \to \hat{q}^-} max\{\overline{M}(p,q) : \; p \in [P_{\ell}, P_u]\}$$

$$\neq \lim_{q \to \hat{q}^+} max\{\overline{M}(p,q) : \; p \in [P_{\ell}, P_u]\} = \lim_{q \to \hat{q}^+} \overline{M}^{\star}(q).$$

We define

$$\hat{p}^+ = argmax\{\overline{M}(p, \hat{q}^+) : \; p \in [P_{\ell}, P_u]\},$$

and

$$\hat{p}^- = argmax\{\overline{M}(p, \hat{q}^-) : \; p \in [P_{\ell}, P_u]\},$$

for any given $\hat{q}^+$ and $\hat{q}^-$, respectively. It follows from the q-continuity of $\overline{M}(p,q)$ that

$$\lim_{q \to \hat{q}^-} \overline{M}^{\star}(q) \neq \lim_{q \to \hat{q}^+} \overline{M}^{\star}(q) \;\Rightarrow\; \hat{p}^- \neq \hat{p}^+.$$

Also, it can be seen that

$$\lim_{q \to \hat{q}^+} \overline{M}^{\star}(q) = \lim_{q \to \hat{q}^+} \overline{M}(\hat{p}^+, q),$$

and

$$\lim_{q \to \hat{q}^-} \overline{M}^{\star}(q) = \lim_{q \to \hat{q}^-} \overline{M}(\hat{p}^-, q).$$

Moreover, $\forall p \in [P_{\ell}, P_u]$ we have

$$\overline{M}(p, \hat{q}^+) \leq \overline{M}(\hat{p}^+, \hat{q}^+),$$

and

$$\overline{M}(p, \hat{q}^-) \leq \overline{M}(\hat{p}^-, \hat{q}^-).$$

Since $\overline{M}(\hat{p}^-, q)$ and $\overline{M}(\hat{p}^+, q)$ are concave functions of q, it follows from above that for some $\epsilon > 0$ with $\hat{q}^- = \hat{q} - \epsilon$ and $\hat{q}^+ = \hat{q} + \epsilon$ we have

$$\overline{M}(\hat{p}^+, \hat{q}) = \overline{M}(\hat{p}^-, \hat{q}),$$

in the limit as $\epsilon \to 0$. Therefore,

$$\lim_{\epsilon \to 0} \overline{M}^\star(\hat{q} - \epsilon) \neq \lim_{\epsilon \to 0} \overline{M}^\star(\hat{q} + \epsilon) \;\Rightarrow\; \hat{p}^- \neq \hat{p}^+$$

$$\Rightarrow\; \overline{M}(\hat{p}^+, \hat{q}) = \overline{M}(\hat{p}^-, \hat{q}) = \overline{M}^\star(\hat{q}),$$

which is a contradiction. Hence, $\overline{M}^\star(q)$ can not have any first kind discontinuity. It is also clear that $\overline{M}^\star(q)$ does not have any removable discontinuity. Consequently, $\overline{M}^\star(q)$ is continuous in q on $[i, \infty)$.

Appendix C

We drop the subscript "1" from the variables. Thus, from (3.61) it follows that for $\overline{X}(P_u) \leq q$ we have:

$$\begin{aligned} \overline{M}_1^\star(i,q) &= max\{-b\cdot p^2 + (a - b\cdot h - i)\cdot p + a\cdot h : \ \bar{p} \leq p \leq P_u\} \\ &\quad -(c+h)\cdot q + c\cdot i, \end{aligned} \tag{C.14}$$

where $\bar{p} = (a-q)/b$. Since the maximand in (C.14) is a quadratic function, we have

$$max\{-b\cdot p^2 + (a - b\cdot h - i)\cdot p + a\cdot h : \ \bar{p} \leq p \leq P_u\}$$

$$= \begin{cases} P_u & , \quad P_u \leq (a - b\cdot h - i)/(2\cdot b), \\ (a - b\cdot h - i)/(2\cdot b) & , \quad \bar{p} < (a - b\cdot h - i)/(2\cdot b) < P_u, \\ \bar{p} & , \quad (a - b\cdot h - i)/(2\cdot b) \leq \bar{p} \end{cases}$$

$$= \begin{cases} (a - b\cdot h - i)/(2\cdot b) & , \quad -b\cdot(h + P_u) < i < 0 < (a + b\cdot h + i)/2 < q, \\ (a - b\cdot h - i)/(2\cdot b) & , \quad i < -b\cdot(h + P_u) < (a + b\cdot h + i)/2 < 0 < q, \\ \bar{p} & , \quad -b\cdot(h + P_u) < i < 0 < q < (a + b\cdot h + i)/2. \end{cases}$$

Note that $i < (a + b\cdot h + i)/2 \Leftrightarrow i < a + b\cdot h = b\cdot(h + P_u)$.

Appendix D

We drop the subscript "1" from the variables. It follows from (3.62) that

$$argmax\{\overline{M}^{\star}(i,q):\ \ i \le q < \infty,\ \ i < 0\}$$

$$= \begin{cases} 0 & , \quad i \le -b\cdot(h+P_u), \\ argmax\{(a-q-c\cdot b)\cdot(q-i)/b:\ \ 0 \le q \le (a+b\cdot h+i)/2\} & , \quad -b\cdot(h+P_u) < i < 0, \end{cases}$$

$$= \begin{cases} 0 & , \quad i \le -b\cdot(P_u - c), \\ (a-c\cdot b+i)/2 & , \quad -b\cdot(P_u - c) < i < 0. \end{cases}$$

Note that $(a - c\cdot b + i)/2 < 0 \Leftrightarrow i < -b\cdot(P_u - c) \Rightarrow i < -b\cdot(P_u + h)$ and also $(a + b\cdot h + i) > (a - c\cdot b + i)$.

Appendix E

Uniform distribution:

$$\begin{aligned} g(\varepsilon) &= \frac{1}{2\cdot\lambda}, \quad \varepsilon \in [-\lambda, \lambda] \\ G(\varepsilon) &= \frac{\varepsilon+\lambda}{2\cdot\lambda} \quad \varepsilon \in [-\lambda, \lambda]. \\ E[\varepsilon] &= 0 \\ \mathrm{Var}(\varepsilon) &= \lambda^2/3 \end{aligned}$$

Under the uniform density function the expected leftovers function will be:

$$\Theta(p,q) = \begin{cases} 0 & , \quad q-\overline{X}(p) \leq -\lambda \\ \frac{(q-\overline{X}(p)+\lambda)^2}{4\cdot\lambda} & , \quad -\lambda < q-\overline{X}(p) \leq \lambda \\ q-\overline{X}(p) & , \quad \lambda < q-\overline{X}(p). \end{cases}$$

Triangular distribution:

$$g(\varepsilon) = \begin{cases} \frac{\varepsilon+\lambda}{\lambda^2} & , \quad -\lambda \leq \varepsilon < 0, \\ \frac{\lambda-\varepsilon}{\lambda^2} & , \quad 0 \leq \varepsilon \leq \lambda, \\ 0 & , \quad \text{otherwise} \end{cases}$$

$$G(\varepsilon) = \begin{cases} 0 & , \quad \varepsilon \leq -\lambda \\ \frac{(\varepsilon+\lambda)^2}{2\cdot\lambda^2} & , \quad -\lambda < \varepsilon \leq 0 \\ 1-\frac{(\lambda-\varepsilon)^2}{2\cdot\lambda^2} & , \quad 0 < \varepsilon \leq \lambda \\ 1 & , \quad \lambda < \varepsilon \end{cases}$$

$$\begin{aligned} E[\varepsilon] &= 0 \\ \mathrm{Var}(\varepsilon) &= \lambda^2/6 \end{aligned}$$

Under the triangular density function the expected leftovers function will be:

$$\Theta(p,q) = \begin{cases} 0 & , \quad q - \overline{X}(p) \leq -\lambda \\ \frac{(q-\overline{X}(p)+\lambda)^3}{6\cdot\lambda^2} & , \quad -\lambda < q - \overline{X}(p) \leq 0 \\ q - \overline{X}(p) - \frac{(q-\overline{X}(p)-\lambda)^3}{6\cdot\lambda^2} & , \quad 0 < q - \overline{X}(p) \leq \lambda \\ q - \overline{X}(p) & , \quad \lambda < q - \overline{X}(p). \end{cases}$$

Bibliography

[1] Gerchak Y. and Parlar M., "A Single Period Inventory Problem With Partially Controllable Demand", *Computers and Operations Research* 14 (1987) 1-9.

[2] Heyman D.P. and Sobel M.J., *Stochastic Models in Operations Research*, Vol 2, McGraw-Hill, New York, (1984).

[3] Karlin S. and Carr R.C., "Prices And Optimal Inventory Policy", Chapter 10 in Arrow K.J., Karlin S. and Scarf H., *Studies In Applied Probability And Management Science*, Stanford University Press, Stanford, California, 1962.

[4] Kirman A.P. and Sobel M.J., "Dynamic Oligopoly With Inventories", *Econometrica* 42 (1974).

[5] Lau A.Hing-Ling and Lau Hon-Shiang, "The Newsboy Problem With Price-Dependent Demand Distribution", *IIE Transactions 20 (1988) 168-175.*

[6] Leland H.E., "Theory Of The Firm Facing Uncertain Demand", *The American Economic Review* 62 (1972) 278-91.

[7] Mills E.S., "Uncertainty and Price Theory", *Quarterly Journal of Economics* 73 (1959) 116-130.

[8] Mills E.S., *Price, Output, And Inventory Policy*, John Wiley & Sons, 1962.

[9] Nevins A.J., "Some Effects of Uncertainty: Simulation of a Model of Price", *The Quarterly Journal of Economics* 80 (1966) 73-87.

[10] Nti K.O., "Competitive Procurement Under Demand Uncertainty", *Management Science* 33 (1987).

[11] Porteus E.L. "On the Optimality of Generalized (s,S) Policies", *Management Science* 17 (1971).

[12] Schäl M., "On the Optimality od (s,S)-Policies in Dynamic Inventory Models With Finite Horizon", *SIAM Journal of Applied Mathematics* 30 (1976) 528-537.

[13] Thowsen G.T., "A Dynamic Nonstationary Inventory Problem For A Price/Quantity Setting Firm", *Naval Research Logistics Quarterly* 22 (1975) 461-76.

[14] Veinott A.F.Jr., "On the Optimality of (s,S) Inventory Policies: New Conditions and a New Proof", *SIAM Journal of Applied Mathematics* 14 (1966) 1067-83.

[15] Whitin T.M., "Inventory Control And Price Theory", *Management Science* 2 (1955) 61-68.

[16] Young L., "Price, Inventory and the Structure of Uncertain Demand", *New Zealand Journal of Operational Research* 6 (1978) 157-177.

[17] Young L., "Uncertainty, Market Structure, and Resource Allocation", *Oxford Economical Papers* 46 (1979) 47-59.

[18] Zabel E., "Monopoly and Uncertainty", *Review of Economic Studies* 37 (1970) 205-219.

[19] Zabel E., "Multiperiod Monopoly Under Uncertainty", *Journal of Economic Studies* 5 (1972) 524-536.