PAINLEVE TEST AND THE PAINLEVE EQUATIONS HIERARCHIES

A THESIS
SUBMITTED TO THE DEPARTMENT OF MATHEMATICS
AND THE INSTITUTE OF ENGINEERING AND SCIENCES
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

By
Fahd Jrad
January, 2001

PAINLEVÉ TEST AND THE PAINLEVÉ EQUATIONS HIERARCHIES

A THESIS

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS

AND THE INSTITUTE OF ENGINEERING AND SCIENCES

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

By

Fahd Jrad

January, 2001

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

Assoc. Prof. Dr. Uğurhan Muğan(Principal Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

Prof. Dr. Mefharet Kocatepe

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

Prof. Dr. Bilal Tanatar

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

Assoc. Prof. Dr. Atalay Karasu

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

Asst. Prof. Dr. Azer Kerimov

Approved for the Institute of Engineering and Sciences:

Prof. Dr. Mehmet Baray
Director of Institute of Engineering and Sciences

ABSTRACT

PAINLEVÉ TEST AND THE PAINLEVÉ EQUATIONS HIERARCHIES

Fahd Jrad
Ph. D. in Mathematics
Supervisor: Assoc. Prof. Dr. Uĝurhan Muĝan
January, 2001

Recently there has been a considerable interest in obtaining higher order ordinary differential equations having the Painlevé property. In this thesis, starting from the first, the second and the third Painlevé transcendents polynomial and non-polynomial type higher order ordinary differential equations having the Painlevé property have been obtained by using the singular point analysis.

Keywords : Painlevé property, movable singularity, resonances, compatibility conditions.

ÖZET

PAINLEVÉ TESTİ VE PAINLEVÉ DENKLEMLERİNİN HİYERARŞİLERİ

Fahd Jrad
Matematik Bölümü Doktora
Tez Yöneticisi: Assoc. Prof. Dr. Uĝurhan Muĝan
Ocak, 2001

Son zamanlarda Painlevé özelliĝine sahip, yüksek dereceli adi diferansiyel denklemleri bulmaya ilgi oluşmuştur. Bu tezde, birinci, ikinci ve üçüncü Painlevé denklemlerinden başlayarak, Painlevé özelliĝine sahip yüksek dereceli polinom ve polinom olmayan adi diferansiyel denklemler tekil nokta analizi kullanılarak bulunmuştur.

Anahter Kelimeler: Painlevé özelliĝi, Hareketli tekil nokta, Rezonans, Uyumluluk şartları.

ACKNOWLEDGMENT

I am really indebted to my supervisor Assoc. Prof. Dr. Uĝurhan Muĝan for his supervision, guidance, encouragement, help and critical comments while developing this thesis.

I would like to thank the chairperson of the mathematics department at Bilkent university Prof. Dr. Mefharet Kocatepe for her help whenever I needed.

I would like to thank Assoc. Prof. Dr. Atalay Karasu for his useful suggestions and support during my research in this thesis.

Words can never express how I am grateful to my family for their endless love and support in good and bad times.

I would like to thank all of my friends for their help and cooperation.

TABLE OF CONTENTS

Chapter 1

Introduction

An ordinary differential equation (ODE) is said to be of Painlevé type, or have the Painlevé property, if the only movable singularities of its solutions are poles. Movable singularity means that its location depends on the constant of integration of the differential equation.

The Ricatti equation

$$y' = a(z)y^2 + b(z)y + c(z), \tag{1.1}$$

where a, b and c are locally analytic functions in z is the only example of the first-order first-degree differential equation which has the Painlevé property. Fuchs [3, 4] considered the equation of the form

$$F(z, y, y') = 0, \tag{1.2}$$

where F is polynomial in y and y' and locally analytic in z, such that the movable branch points are absent, that is, the generalization of Riccati equation. The irreducible form of the first order algebraic differential equation of the second-degree is

$$a_0(z)(y')^2 + \sum_{i=0}^{2} b_i(z)y^i y' + \sum_{j=0}^{4} c_j(z)y^j = 0, \tag{1.3}$$

where b_i, c_j are analytic functions of z and $a_0(z) \neq 0$. Briot and Bouquet [3] considered the subcase of (1.2). That is, first order binomial equations of degree m:

$$(y')^m + F(z, y) = 0, \tag{1.4}$$

where $F(z, y)$ is a polynomial of degree at most $2m$ in y and m is a positive integer. It was found that there are six types of equation of the form (1.4).

But, all these equations are either reducible to a linear equation or solvable by means of elliptic functions [3].

The most well known second-order first-degree Painlevé type equations are $\mathrm{P}_I, P_{II}, ..., P_{VI}$ discovered by Painlevé and his school [1, 2, 3] around the turn of the last century. They classified all equations of the form

$$y'' = F(z, y, y'), \tag{1.5}$$

where F is rational in y', algebraic in y and locally analytic in z. They found fifty such equations, but six of them

$$\begin{aligned}
\mathrm{P}_I \quad &: y'' = 6y^2 + z,\\
\mathrm{P}_{II} \quad &: y'' = 2y^3 + zy + \alpha,\\
\mathrm{P}_{III} \quad &: y'' = \tfrac{(y')^2}{y} - \tfrac{y'}{z} + \gamma y^3 + \tfrac{\alpha}{z}y^2 + \tfrac{\beta}{z} + \tfrac{\delta}{y},\\
\mathrm{P}_{IV} \quad &: y'' = \tfrac{(y')^2}{2y} + \tfrac{3}{2}y^3 + 4zy^2 + 2(z^2 - \alpha)y + \tfrac{\beta}{y},\\
\mathrm{P}_V \quad &: y'' = \tfrac{3y-1}{2y(y-1)}(y')^2 - \tfrac{1}{z}y'\\
&\quad + \tfrac{\alpha}{z^2}y(y-1)^2 + \tfrac{\beta(y-1)^2}{z^2 y} + \tfrac{\gamma}{z}y + \tfrac{\delta y(y+1)}{y-1},\\
\mathrm{P}_{VI} &: y'' = \tfrac{1}{2}(\tfrac{1}{y} + \tfrac{1}{y-1} + \tfrac{1}{y-z})(y')^2 - (\tfrac{1}{z} + \tfrac{1}{z-1} + \tfrac{1}{y-z})y'\\
&\quad + \tfrac{y(y-1)(y-z)}{z^2(z-1)^2)}(\alpha + \tfrac{\beta z}{y^2} + \tfrac{\gamma(z-1)}{(y-1)^2} + \tfrac{\delta z(z-1)}{(y-z)^2}),
\end{aligned} \tag{1.6}$$

are the only irreducible ones and define new transcendents. Any of the other forty four equations either can be integrated in terms of the known functions or can be reduced to one of the six equations by using the Möbius transformation. Although the Painlevé equations were discovered from strictly mathematical considerations, they have appeared in many physical problems, and possess rich internal structure.

Second-order second-degree Painlevé type equations of the following form

$$(y'')^2 = E(z, y, y')y'' + F(z, y, y'), \tag{1.7}$$

where E and F are assumed to be rational in y, y' and locally analytic in z were subject of the articles [8, 13, 18]. In [8, 13], the special form, $E = 0$, and hence F is polynomial in y and y' of (1.7) was considered. Also, in this case no new Painlevé type equation was discovered, since all of them can be solved either in terms of the known functions or one of the six Painlevé transcendents. In [18], it was shown that all the second-degree equations obtained in [8, 13], $E = 0$ case, and second-degree equations such that $E \neq 0$ can be obtained from $\mathrm{P}_I, ..., P_{VI}$ by using the following transformations which preserve the Painlevé property

$$u(z, \hat{\alpha}) = \frac{y' + \sum_{i=0}^{2} a_i(z)y^i}{\sum_{i=0}^{2} b_i(z)y^i}, \tag{1.8}$$

and

$$u(z,\hat{\alpha}) = \frac{(y')^2 + \sum_{i=0}^{2} a_i(z) y^i y' + \sum_{j=0}^{4} b_j(z) y^j}{\sum_{i=0}^{2} c_i(z) y^i y' + \sum_{j=0}^{4} d_j(z) y^j} = 0, \tag{1.9}$$

where a_i, b_j, c_i, d_j are analytic functions of z. That is, if y solves one of the Painlevé equation with parameter set α then u solves a second-order second-degree Painlevé type equation of the form (1.7) with the parameter set $\hat{\alpha}$.

The special form, polynomial-type, of the third order Painlevé type equations

$$y''' = F(z, y, y', y''), \tag{1.10}$$

where F is polynomial in y, y' and y'' and locally analytic in z was considered in [5, 7]. The most well known third order equation is Chazy's "natural-barrier" equation

$$y''' = 2yy'' - 3y'^2 + \frac{4}{36 - n^2}(6y' - y^2)^2. \tag{1.11}$$

The case $n = \infty$ appears in several physical problems. The equation (1.11) is integrable for all real and complex n and $n = \infty$. Its solutions are rational for $2 \leq n \leq 5$, and have a circular natural barrier for $n \geq 7$ and $n = \infty$. Bureau [7] considered the third order equation of Painlevé type of the following form

$$y''' = P_1(y)y'' + P_2(y)y'^2 + P_3(y)y' + P_4(y), \tag{1.12}$$

where $P_n(y)$ is a polynomial in y of degree n with analytic coefficients in z. In [12] Martynov investigated Painlevé type equations of the form

$$y''' = (1 - \frac{1}{\nu})\frac{(y'' - 2yy')^2}{y' - y^2} + ayy'' + b(y')^2 + cy^2y' + dy^4 + a_1\frac{y'y''}{y} + b_1\frac{(y')^3}{y^2} \tag{1.13}$$

where a, b, c, d, a_1, b_1 are constants and $d \neq 0$. In [10], Exton attempted to classify equations of the form

$$\begin{aligned} y''' = {} & b\tfrac{y'y''}{y} + c\tfrac{(y')^3}{y^2} + (e_2y^2 + e_1y + e_0)\tfrac{y''}{y} + (f_1y^2 + f_2y + f_0)\tfrac{(y')^2}{y^2} \\ & + (g_4y^4 + g_3y^3 + g_2y^2 + g_1y + g_0)\tfrac{y'}{y^2} \\ & + (h_6y^6 + h_5y^5 + h_4y^4 + h_3y^3 + h_2y^2 + h_1y + h_0)\tfrac{1}{y^2} \end{aligned} \tag{1.14}$$

where b, c are constant and the other coefficients are locally analytic in z.

In [7, 14] fourth order polynomial-type equations of the form

$$y^{(4)} = ayy''' + by'y'' + cy^2y'' + dyy'^2 + ey^3y' + fy^5 + F(z, y), \tag{1.15}$$

where

$$\begin{aligned} F(z, y) = {} & a_0y''' + (c_1y + c_0)y' + d_0y'^2 + (e_2y^2 + e_1y + e_0)y' \\ & + f_4y^4 + f_3y^3 + f_2y^2 + f_1y + f_0, \end{aligned} \tag{1.16}$$

and all the coefficients a, b, c, d, e, f with or without subscripts are assumed to be analytic functions of z were investigated.

Besides their mathematically rich internal structure and appearance in many physical problems, Painlevé equations play an important role for the completely integrable partial differential equations (PDE). Ablowitz, Ramani and Segur [20] demonstrated a close connection between completely integrable PDE solvable by inverse scattering transform and the Painlevé equations. They conjectured that every non-linear ODE obtained by an exact reduction of a non-linear PDE solvable by inverse scattering transform has the Painlevé property. They gave an algorithmic method to test the given equation. The test provides the necessary conditions a given PDE is completely integrable . Weiss, Tabor and Carnavale [23] introduced the Painlevé property for PDE's or Painlevé PDE test as a method of applying the Painlevé ODE test directly to a given PDE without having to reduce it to an ODE.

Recently, Kudryashov [16], Clarkson, Joshi and Pickering [17] obtained the higher order Painlevé type equations, the first and second Painlevé hierarchy, by similarity reduction from the Korteweg-de-Vries (KdV) and the modified Korteweg-de Vries (mKdV) hierarchies respectively.

In this work hierarchies of the first, second and third Painlevé equations are investigated by using the Painlevé ODE test, singular point analysis. It is possible to obtain the Painlevé type equation of any order, as well as the known ones, starting from a Painlevé equation. Singular point analysis, an algorithm introduced by Ablowitz, Ramani, Segur [20] to test whether a given ODE satisfies the necessary conditions to be of Painlevé type. It consists of seeking a Laurent series expansion solution of the given ODE in the neighborhood of a movable singularity and requires this series solution to be single-valued and self-consistent.

The singular point analysis can be summarized as follows : Let

$$y^{(n)} = F(z, y, y', ..., y^{(n-1)}), \tag{1.17}$$

be an nth order ODE where F is analytic in z and rational in the other arguments. Then $y(z)$ is expanded as

$$y(z) = \sum_{j=0}^{\infty} y_j (z - z_0)^{j+\alpha}, \tag{1.18}$$

where z_0 is an arbitrary singularity and $\mathcal{R}(\alpha) < 0$. The singular point analysis consists of three basic steps:

1- The leading order analysis: substitute $y = y_0(z-z_0)^\alpha$ in equation (1.17). For certain values of integer α, two or more terms balance. These balancing terms are called leading or dominant terms. After finding α, one can determine y_0.
2- The resonances: For each choice (α, y_0) from step 1, substitute

$$y = y_0(z-z_0)^\alpha + \delta(z-z_0)^{r+\alpha}, \tag{1.19}$$

where δ is an arbitrary constant, in the part of (1.17) that contains the dominant or the leading terms only. This equation reduces to $Q(r)\delta(z-z_0)^{r+n+\alpha} = 0$. The roots of the polynomial $Q(r)$ are called the resonances. It should noted that -1 must be a resonance that corresponds to the arbitrariness of z_0 and the other $n-1$ resonances must be distinct integers $\neq -1$.
3- The compatibility conditions: For each choice (α, y_0) substitute the series (1.18) in (1.17) to get the relation relation for the coefficients y_j :

$$(j+1)(j-r_1)...(j-r_{n-1})y_j = F_j(y_0, y_1, ..., y_{j-1}) \tag{1.20}$$

where r_i, $i = 1, 2, ..., n-1$, are the roots of $Q(r)$. If at each nonnegative r_i, the compatibility condition $F_{r_i} = 0$ is satisfied, then equation (1.17) meets the necessary conditions to have the Painlevé property.

Painlevé test was improved in such a way that negative resonances can be treated [24]. In this work , we will consider only the "principal branch" that is, all the resonances r_i (except $r_0 = -1$) are positive real distinct integers and the number of resonances is equal to order of the differential equation for a possible choice of (α, y_0). Then, the compatibility conditions give full set of arbitrary integration constants. The other possible choices of (α, y_0) may give "secondary branch" which possess several distinct negative integer resonances. Negative but distinct integer resonances give no conditions which contradict integrability [21].

The procedure to obtain higher order Painlevé type equations starting any Painlevé equation may be summarized as follows:
I. Take an nth order Painlevé type differential equation of the form (1.17). If $y \sim y_0(z-z_0)^\alpha$ as $z \to z_0$, then α is a negative integer for certain values of y_0. Moreover, the highest derivative term is one of the dominant terms. Then the dominant terms are of order $\alpha - n$. There are n resonances $r_0 = -1, r_1, r_2, ...r_{n-1}$, with all r_i, $i = 1, 2, ..., (n-1)$ being nonnegative distinct integers such that $Q(r_j) = 0$, $j = 0, 1, 2, ..., (n-1)$. The compatibility conditions, for the simplified equation that retains only dominant terms of (1.17) are identically satisfied. Differentiating the simplified equation with respect to

z yields

$$y^{(n+1)} = G(z, y, y', ..., y^{(n)}). \tag{1.21}$$

where G contains the terms of order $\alpha - n - 1$, and the resonances of (1.21) are the roots of $Q(r_j)(\alpha + r - n) = 0$. Hence, equation (1.21) has a resonance $r_n = n - \alpha$ additional to the resonances of (1.17). Equation (1.21) passes the Painlevé test provided that $r_n \neq r_i, \quad i = 1, 2, ..., (n-1)$ and positive integer. Moreover the compatibility conditions are identically satisfied, that is $z_0, y_{r_1}, ..., y_{r_n}$ are arbitrary.

II. Add the dominant terms which are not contained in G. Then the resonances of the new equation are the zeros of a polynomial $\tilde{Q}(r)$ of order $n+1$. Find the coefficients of $\tilde{Q}(r)$ such that there is at least one principal Painlevé branch. That is, all $n+1$ resonances (except $r_0 = -1$) are positive distinct integers for at least one possible choice of (α, y_0). The other possible choices of (α, y_0) may give the secondary Painlevé branch, that is all the resonances are distinct integers.

III. Add the non-dominant terms which are the terms of weight less than $\alpha - n - 1$, with (locally) analytic coefficients of z. Find the coefficients of the non-dominant terms by using the compatibility conditions.

In this work we apply the procedure to the first, the second and the third Painlevé equations. In Chapter 2, we start with the first Painlevé equation P_I and obtain the third, fourth, fifth and sixth order equations of Painlevé type. In Chapter 3, we start with the second Painlevé equation P_{II} and obtain the third, fourth and some of the fifth and sixth order equations with the Painlevé property . In Chapter 4, we start with the third Painlevé equation P_{III} and obtain third order equations of Painlevé type.

Chapter 2

The first Painlevé hierarchy

In this chapter, we apply the procedure to the first Painlevé equations and give Painlevé type equations , of order three, four, five and six.

2.1 Third order equations: $\mathrm{P}_I^{(3)}$

The first Painlevé equation, P_I is

$$y'' = 6y^2 + z \tag{2.1}$$

Painlevé test gives that there is only one branch and

$$(\alpha, y_0) = (-2, 1) \qquad Q(r) = r^2 - 5r - 6, \tag{2.2}$$

The dominant terms are y'' and y^2 which are of order -4 as $z \to z_0$. Taking the derivative of the simplified equation gives

$$y''' = ayy' \tag{2.3}$$

where a is a constant which can be introduced by replacing y with λy, such that $12\lambda = a$. For the equation (2.3), $(\alpha, y_0) = (-2, 12/a)$. No more polynomial type term of weight -5 with constant coefficients can be added to (2.3). The resonances of (2.3) are the zeros of

$$\tilde{Q}(r) = Q(r)(r-4). \tag{2.4}$$

Hence, the resonances are $(r_0, r_1, r_2) = (-1, 4, 6)$. Next step is to add the terms of weight greater than -5 of z. That is,

$$y''' = ayy' + A_1(z)y'' + A_2(z)y^2 + A_3(z)y' + A_4(z)y + A_5(z). \tag{2.5}$$

where A_i $i = 1, ..., 5$ are (locally) analytic functions in z. The linear transformation

$$y(z) = \mu(z)u(t) + \nu(z), \qquad t = \rho(z), \tag{2.6}$$

where μ, ν and ρ are analytic functions of z preserves the Painlevé property. By using the transformation (2.6), one can set

$$6A_1 + A_2 = 0, \qquad A_3 = 0, \qquad a = 12. \tag{2.7}$$

Then, substituting

$$y = y_0(z - z_0)^{-2} + \sum_{j=1}^{6} y_j(z - z_0)^{j-2}, \tag{2.8}$$

into equation (2.5) gives that

$$y_0 = 1, \quad y_1 = 0, \quad y_2 = 0, \quad y_3 = A_4(z_0)/12. \tag{2.9}$$

The recursion relation for $j = 4$ implies that, if $y_4 =$ arbitrary, then

$$A_4' - A_1 A_4 = 0, \tag{2.10}$$

and for $j = 5$

$$y_5 = -\frac{1}{72}[12A_5^{(0)} + 20A_2^{(0)}y_4 + 12A_4^{(2)} + 2A_2^{(1)}A_4^{(0)}] \tag{2.11}$$

where $A_i^{(k)}$, $k = 0, 1, 2, ...$ denote the coefficient of the k^{th} order term of Taylor series expansion of the function $A_i(z)$ about $z = z_0$. The compatibility condition at the resonance $r_2 = 6$ implies that

$$\begin{gathered} A_1' + A_1^2 = 0, \\ -6(A_1A_5 + A_5') - A_4(A_4 - A_1A_1') + 3A_4A_1'' - 3A_1A_4'' - A_4''' = 0, \end{gathered} \tag{2.12}$$

if y_6 is arbitrary. According to (2.12.a), there are two cases should be considered separately:

I. $A_1(z) = 0$: Equations (2.7),(2.10) and (2.12.b) imply that $A_2 = 0$, $A_4 = c_1 =$ constant,

$A_5(z) = -(c_1^2/6)z + c_2$, $c_2 =$ constant. Then the canonical form of the third order Painlevé type equation is

$$y''' = 12yy' + c_1y - \frac{1}{6}c_1^2 z + c_2. \tag{2.13}$$

If $c_1 = c_2 = 0$, then (2.13) has the first integral

$$y'' = 6y^2 + k, \qquad k = \text{constant}, \tag{2.14}$$

which has the solution in terms of the elliptic functions. If $c_1 \neq 0$, then replacing $z + c_2/k^2$ by z where $k = -c_1/6$, and then replacing y by βy and z by γz such that $\gamma^2\beta = 1$ and $k\gamma^3 = -1$ in (2.13). Then it takes the form of

$$y''' = 12yy' + 6y - 6z. \tag{2.15}$$

If one lets $y = u'$, integrates with respect to z once and replaces u by $u - c/6$ to eliminate the integration constant c, then (2.15) gives

$$u''' = 6u'^2 + 6u - 3z^2. \tag{2.16}$$

Equation (2.16) was also given by Chazy and Bureau [5, 7].

II. $A_1(z) = 1/(z - c_1)$: Equations (2.7),(2.10) and (2.12.b) give

$$A_2 = -\frac{6}{z - c_1}, \qquad A_4 = c_2(z - c_1), \qquad A_5 = -\frac{1}{24}c_2^2(z - c_1)^3 + \frac{c_3}{z - c_1}. \tag{2.17}$$

where c_i, $i = 1, 2, 3$, are constants. Then the canonical form after replacing $z - c_1$ by z is

$$y''' = 12yy' + \frac{1}{z}(y'' - 6y^2) + c_2 zy + \frac{c_3}{z} - \frac{c_2^2}{24}z^3. \tag{2.18}$$

Equation (2.18) was also considered in [7]. Replacing z by γz and y by βy, such that $\gamma^2\beta = 1$ and $c_2\gamma^4 = 12$ reduces the equation (2.18) to

$$y''' = 12yy' + \frac{1}{z}(y'' - 6y^2 - k) + 12zy - 6z^3, \tag{2.19}$$

where k is an arbitrary constant. Integrating (2.19) once yields

$$(u'' - 6u^2 - \frac{k_1}{4})^2 = z^2(u'^2 - 4u^3 - \frac{k_1}{2}u), \tag{2.20}$$

where $k_1 = -(k + 72)/3$ and $u = y - z^2/12$. There exists one-to-one correspondence between $u(z)$ and solution of the fourth Painlevé equation [18].

2.2 Fourth order equations: $\mathrm{P}_I^{(4)}$

Differentiating (2.3) with respect to z gives the terms $y^{(4)}, y'^2, yy''$, all of which are of order -6 for $\alpha = -2$ and as $z \to z_0$.Adding the term y^3 which is also of order -6, gives the following simplified equation

$$y^{(4)} = a_1 y'^2 + a_2 yy'' + a_3 y^3, \tag{2.21}$$

where a_i, $i = 1, 2, 3$ are constants. Substituting

$$y = y_0(z - z_0)^{-2} + \delta(z - z_0)^{r-2}, \tag{2.22}$$

into above equation gives the following equations for resonance r and for y_0 respectively,

$$Q(r) = (r+1)[r^3 - 15r^2 + (86 - a_2y_0)r + 2(2a_1y_0 + 3a_2y_0 - 120)] = 0,$$
$$a_3y_0^2 + 2(2a_1 + 3a_2)y_0 - 120 = 0. \tag{2.23}$$

Equation (2.23.b) implies that in general, there are two branches of Painlevé expansion, if $a_3 \neq 0$. Now, one should determine y_{0j}, $j = 1,2$ and a_i such that at least one of the branches is the principal branch. That is, all the resonances (except $r_0 = -1$ which is common for both branches) are distinct positive integers for one of $(-2, y_{0j})$, $j = 1,2$. Negative but distinct resonances for the secondary branch may be allowed, since they give no conditions which contradict the Painlevé property. If y_{01}, y_{02} are the roots of (2.23.b), by setting

$$P(y_{0j}) = -2[(2a_1 + 3a_2)y_{0j} - 120], \qquad j = 1,2 \tag{2.24}$$

and if (r_{11}, r_{12}, r_{13}), (r_{21}, r_{22}, r_{23}) are the resonances corresponding to the branches $(-2, y_{01})$ and $(-2, y_{02})$ respectively, then one can have

$$\prod_{i=1}^{3} r_{1i} = P(y_{01}) = p_1, \qquad \prod_{i=1}^{3} r_{2i} = P(y_{02}) = p_2, \tag{2.25}$$

where p_1, p_2 are integers and such that, at least one of them is positive. Equation (2.23.b) gives

$$y_{01} + y_{02} = -\frac{2}{a_3}(2a_1 + 3a_2), \qquad y_{01}y_{02} = -\frac{120}{a_3}. \tag{2.26}$$

Then equation (2.24) can be written as

$$P(y_{01}) = 120(1 - \frac{y_{01}}{y_{02}}), \qquad P(y_{02}) = 120(1 - \frac{y_{02}}{y_{01}}). \tag{2.27}$$

Then, for $p_1p_2 \neq 0$, p, p_2 satisfy the following Diophantine equation

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{120}. \tag{2.28}$$

Now, one should determine all integer solutions of Diophantine equation under certain conditions. Equation (2.23.a) implies that $\sum_{i=1}^{3} r_{1i} = \sum_{i=1}^{3} r_{2i} = 15$. Let (r_{11}, r_{12}, r_{13}) be the distinct positive integers, then $r_{11} + r_{12} + r_{13} = 15$ implies that there are 12 possible choices of (r_1, r_2, r_3). Then (2.28) has negative integer solutions p_2 for each of the possible values of p_1 except $p_1 = 120$. $p_1 = 120$ case which corresponds to $(r_1, r_2, r_3) = (4,5,6)$ will be considered later. The equations (2.26), (2.27.a) and $\sum_{i \neq j} r_{1i}r_{1j} = 86 - a_2y_{01}$ determine y_{01}, y_{02}, a_1, a_3 in terms of a_2. Hence, all the coefficients of (2.23.a) are determined such that its roots (r_{11}, r_{12}, r_{13}) corresponding to y_{01} are positive distinct integers, and $\prod_{i=1}^{3} r_{2i} = p_2 < 0$ and integer for y_{02}. Then, it should be checked that whether the resonances (r_{21}, r_{22}, r_{23}) are distinct integers (i.e the existence of the secondary branch). There are 4 cases out of 11 cases such that (r_{11}, r_{12}, r_{13}) corresponding to y_{01} being positive distinct integers and (r_{21}, r_{22}, r_{23}) corresponding to y_{02} being distinct integers. These cases are as follows:

Case 1:

$$y_{01} = \frac{30}{a_2}: \quad (r_{11}, r_{12}, r_{13}) = (2, 3, 10)$$
$$y_{02} = \frac{60}{a_2}: \quad (r_{21}, r_{22}, r_{23}) = (-2, 5, 12)$$
$$a_1 = 0, \quad a_3 = -\frac{1}{15}a_2^2$$
$$y^{(4)} = a_2(yy'' - \frac{1}{15}a_2 y^3) \tag{2.29}$$

Case 2:

$$y_{01} = \frac{20}{a_2}: \quad (r_{11}, r_{12}, r_{13}) = (2, 5, 8)$$
$$y_{02} = \frac{60}{a_2}: \quad (r_{21}, r_{22}, r_{23}) = (-3, 8, 10)$$
$$a_1 = \frac{1}{2}a_2, \quad a_3 = -\frac{1}{10}a_2^2$$
$$y^{(4)} = a_2(yy'' + \frac{1}{2}y'^2 - \frac{1}{10}a_2 y^3) \tag{2.30}$$

Case 3:

$$y_{01} = \frac{18}{a_2}: \quad (r_{11}, r_{12}, r_{13}) = (3, 4, 8)$$
$$y_{02} = \frac{90}{a_2}: \quad (r_{21}, r_{22}, r_{23}) = (-5, 8, 12)$$
$$a_1 = \frac{1}{2}a_2, \quad a_3 = -\frac{2}{27}a_2^2$$
$$y^{(4)} = a_2(yy'' + \frac{1}{2}y'^2 - \frac{2}{27}a_2 y^3) \tag{2.31}$$

Case 4:

$$y_{01} = \frac{15}{a_2}: \quad (r_{11}, r_{12}, r_{13}) = (3, 5, 7)$$
$$y_{02} = \frac{120}{a_2}: \quad (r_{21}, r_{22}, r_{23}) = (-7, 10, 12)$$
$$a_1 = \frac{3}{4}a_2, \quad a_3 = -\frac{1}{15}a_2^2$$
$$y^{(4)} = a_2(yy'' + \frac{3}{4}y'^2 - \frac{1}{15}a_2 y^3) \tag{2.32}$$

For each case the compatibility conditions are identically satisfied. To find the canonical form of the fourth order equations of Painlevé type, one should add non-dominant terms with the coefficients which are analytic functions of z. That is, one should consider the following equation

$$y^{(4)} = a_1 y'^2 + a_2 yy'' + a_3 y^3 + A_1(z)y''' + A_2(z)yy' + A_3(z)y'' + A_4(z)y^2 + A_5(z)y' + A_6(z)y + A_7(z). \tag{2.33}$$

The coefficients A_i, $i = 1, ..., 7$ are (locally) analytic functions in z and can be determined by using the compatibility conditions.

Case 1. By using the transformation (2.6), one can set

$$12A_1 + A_2 = 0, \qquad A_3 = 0, \qquad a_2 = 30. \tag{2.34}$$

Substituting

$$y = y_{01}(z - z_0)^{-2} + \sum_{j=1}^{r_3} y_j (z - z_0)^{j-2} \tag{2.35}$$

into equation(2.33) gives the recursion relation for y_j. The recursion relation yields $y_1 = 0$ for $j = 1$ and for $j = r_{11} = 2$, $A_4 = 0$ if y_2 is arbitrary. If y_3 is arbitrary, then $A_2 = A_5 = 0$ and then (2.34.a) implies that $A_1 = 0$. Recursion relation for $j = r_{13} = 10$ implies that $A_6 = c_1$ =constant and $A_7 = c_2$ =constant if y_{10} is arbitrary. Therefore, the canonical form is,

$$y^{(4)} = 30yy'' - 60y^3 + c_1 y + c_2. \tag{2.36}$$

Equation (2.36) was also obtained by Cosgrove [15]. For $c_1 = 0$, replacing y by $-y$ yields

$$y^{(4)} = -30yy'' - 60y^3 + c_2, \tag{2.37}$$

$y(z)$ is the stationary solution of Caudrey-Dodd-Gibbon equation [25].

Case 2: Linear transformation (2.6) allows one to set

$$12A_1 + A_2 = 0, \qquad A_3 = 0, \qquad a_2 = 20. \tag{2.38}$$

Then, the compatibility conditions imply that $A_4 = 0$ for $j = 2$, $A_2 = A_5 = 0$, $A_6(z) = c_1$ = constant for $j = 5$ and $A_7 = c_2 z + c_3$, c_2 and c_3 are constant, for $j = 8$. Then the canonical form for this case is,

$$y^{(4)} = 10(2yy'' + y'^2 - 4y^3) + c_1 y + c_2 z + c_3. \tag{2.39}$$

One can always choose $c_3 = 0$ by replacing $z + c_3/c_2$ by z. Replacing y by $-y/4$ in (2.39) gives

$$y^{(4)} + 5yy'' + \frac{5}{2}y'^2 + \frac{5}{2}y^3 + k_1 y + k_2 z = 0. \tag{2.40}$$

where k_i =constant. Equation (2.40) was also introduced by Kudryashov and Cosgrove [16], [15].

Case 3: By using the linear transformation (2.6), one can set

$$12A_1 + A_2 = 0, \qquad 6A_3 + A_4 = 0, \qquad a_2 = 18. \tag{2.41}$$

Then, the compatibility conditions imply that $A_2 = A_5 = A_6 = 0$ and $A_3 = c_1$, $A_4 = -6c_1$, $A_7 = c_2 z + c_3$, where c_i, $i = 1, 2, 3$ are constants. Therefore, the canonical form of the fourth order Painlevé type equation for this case is

$$y^{(4)} = 18yy'' + 9y'^2 - 24y^3 + c_1 y'' - 6c_1 y^2 + c_2 z + c_3. \tag{2.42}$$

Equation (2.42) was also obtained in [15]. For $c_2 \neq 0$, replacing $z + c_3/c_2$ by z and then replacing z by γz and y by βy such that
$\beta\gamma^2 = 1$, $c_2\gamma^7 = 1$ reduces the (2.42) into the following form

$$y^{(4)} = 18yy'' + 9y'^2 - 24y^3 + k_1 y'' - 6k_1 y^2 + z, \tag{2.43}$$

where $k_1 = c_1\gamma^2$.

Case 4: Linear transformation (2.6) allows one to set

$$12A_1 + A_2 = 0, \qquad A_4 = 0, \qquad a_2 = 15. \tag{2.44}$$

Then the compatibility conditions at the resonances $j = 3, 5, 7$ imply that, if y_3, y_5, y_7 are arbitrary then $A_2 = A_3 = A_5 = 0$ and $A_6 = c_1 =$ constant, $A_7 = c_2 =$ constant. Therefore the canonical form is

$$y^{(4)} = 15yy'' + \frac{45}{4}y'^2 - 15y^3 + c_1 y + c_2. \tag{2.45}$$

If one sets $y = -2u$ then (2.45) takes the form of

$$u^{(4)} + 30uu'' + \frac{45}{2}u'^2 + 60u^3 + k_1 u + k_2 = 0, \tag{2.46}$$

where $k_1 = -c_1$, $k_2 = c_2/2$. $u(z)$ is the stationary solution of Kuperschmidt equation [25] for $k_1 = 0$ and it was also given in [15].

If $a_3 = 0$, equation (2.23) reduces to

$$\begin{gathered} Q(r) = (r+1)[r^3 - 15r^2 + (86 - a_2 y_0)r - 120] = 0, \\ (2a_1 + 3a_2)y_0 - 60 = 0, \end{gathered} \tag{2.47}$$

and hence, there is only one Painlevé branch which has to be the principal branch. (2.47.a) implies that $r_0 = -1$ and $\sum_{i=1}^{3} r_i = 15$ which gives 12 possible positive distinct integers (r_1, r_2, r_3). But, $\prod_{i=1}^{3} r_i = 120$ implies that $(r_1, r_2, r_3) = (4, 5, 6)$ is the only possible choice of the resonances. Equation

(2.47.b) and $\sum_{i\neq j} r_i r_j = 86 - a_2 y_0$ imply that $a_1 = a_2$. Then, the simplified equation is

$$y^{(4)} = a_1(yy'' + y'^2). \tag{2.48}$$

Adding the non-dominant terms with the analytic coefficients of z gives

$$\begin{aligned} y^{(4)} = a_1(yy'' + y'^2) + A_1(z)y''' + A_2(z)yy' + A_3(z)y'' + \\ A_4(z)y^2 + A_5(z)y' + A_6(z)y + A_7(z) \end{aligned} \tag{2.49}$$

One can always set

$$12A_1 + A_2 = 0, \qquad A_3 = 0, \qquad a_2 = 12, \tag{2.50}$$

by using the linear transformation (2.6). The compatibility conditions at the resonances $r = 4, 5, 6$ imply that y_4, y_5, y_6 are arbitrary and $A_2 = A_4 = 0$ and

$$A_5 = \frac{c_1}{2}z + c_2, \qquad A_6 = c_1, \qquad A_7 = -\frac{1}{6}(\frac{c_1}{2}z + c_2)^2, \tag{2.51}$$

where c_1, c_2 are constants. Hence , the canonical form is

$$y^{(4)} = 12(yy'' + y'^2) + (\frac{c_1}{2}z + c_2)y' + c_1 y - \frac{1}{6}(\frac{c_1}{2}z + c_2)^2. \tag{2.52}$$

If $c_1 = 0$, then integrating (2.52) once gives the equation(2.15). If $c_1 \neq 0$, letting $c_1 = -12k_1$, $c_2 = -6k_2$ first, and replacing $z + k_2/k_1$ by z, and then replacing z by γz, y by βy, such that $\beta\gamma^2 = 1$, $k_1\gamma^4 = 1$ then the equation (2.52) takes the form of

$$y^{(4)} = 12(yy')' - 6zy' - 12y - 6z^2. \tag{2.53}$$

If one lets $y = -u'$ and integrates the resulting equation once then (2.53) yields

$$u^{(4)} + 12u'u'' = 6zu' + 6u + 2z^3 - k, \tag{2.54}$$

after replacing u by βu, z by γz such that $\beta\gamma = -1$, $\gamma^4 = -1$. Equation (2.54) was also obtained by Bureau [7] and which belongs to hierarchy of the second Painlevé equation.

2.3 Fifth order equations: $\mathrm{P}_I^{(5)}$

Differentiating (2.21) with respect to z gives the terms $y^{(5)}$, yy''', $y'y''$, y^2y' which are all the dominant terms for $\alpha = -2$ and $z \to z_0$. Therefore, the simplified equation is

$$y^{(5)} = a_1 yy''' + a_2 y'y'' + a_3 y^2 y', \tag{2.55}$$

where a_i, $i = 1,2,3$ are constants. Substituting (2.22) into (2.55) gives the following equations for the resonance r and y_0,

$$\begin{aligned}(r+1)\{r^4 - 21r^3 + (176 - a_1 y_0)r^2 + [2(5a_1 + a_2)y_0 - 378]r \\ +[1800 - 18(2a_1 + a_2)y_0 - a_3 y_0^2]\} = 0, \\ a_3 y_0^2 + 6(2a_1 + a_2)y_0 - 360 = 0.\end{aligned} \tag{2.56}$$

Equation (2.56.a) implies that one of the resonance $r_0 = -1$ which corresponds to arbitrariness of z_0. (2.56.b) implies the existence of two Painlevé branches corresponding to $(-2, y_{0i})$, $i = 1,2$. Let $(r_{11}, r_{12}, r_{13}, r_{14})$ and $(r_{21}, r_{22}, r_{23}, r_{24})$ be the resonances corresponding toy_{01} and y_{02} respectively. Setting,

$$P(y_{0j}) = 1800 - 18(2a_1 + a_2)y_{0j} - a_3 y_{0j}^2, \qquad j = 1,2 \tag{2.57}$$

then, (2.56.a) implies that

$$\prod_{i=1}^{4} r_{1i} = P(y_{01}) = p_1, \qquad \prod_{i=1}^{4} r_{2i} = P(y_{02}) = p_2, \tag{2.58}$$

where p_1, p_2 are integers such that at least one of them is positive, to have the principal branch. From equation (2.56.b), one can have

$$a_3 = -\frac{360}{y_{01}y_{02}}, \qquad 2a_1 + a_2 = \frac{60}{y_{01}y_{02}}(y_{01} + y_{02}). \tag{2.59}$$

By using the above equation, (2.57) yields the following Diophantine equation, if $p_1 p_2 \neq 0$

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{720}, \tag{2.60}$$

Now, one should determine all possible integer solutions (p_1, p_2) of (2.60). (2.56.a) implies that $\sum_{i=1}^{4} r_{ji} = 21$ $j = 1,2$. Then, there are 27 possible cases for $(r_{11}, r_{12}, r_{13}, r_{14})$ (i.e. 27 possible values of p_1) such that r_{1i}'s are positive distinct integers. Diophantine equation implies that there are 12 cases out of 27 cases such that both $p_1 > 0, p_2 < 0$ are integers. By using the equations

$$\sum_{i\neq j} r_{1i}r_{1j} = 176 - a_1 y_{01}, \qquad \sum_{i\neq j\neq k} r_{1i}r_{1j}r_{1k} = -2[(5a_1 + a_2)y_{01} - 378] \tag{2.61}$$

and (2.59), y_{01}, y_{02}, a_2, a_3 can be obtained in terms of a_1 for each 12 possible integer values of (p_1, p_2). But, there are only 4 cases out of 12 cases such that the resonances $(r_{21}, r_{22}, r_{23}, r_{24})$ corresponding to y_{02} are distinct integers. These cases and the corresponding simplified equations are as follows:

Case 1:

$$\begin{aligned} y_{01} = \tfrac{30}{a_1}: &\quad (r_{11}, r_{12}, r_{13}, r_{14}) = (2,3,6,10) \\ y_{02} = \tfrac{60}{a_1}: &\quad (r_{21}, r_{22}, r_{23}, r_{24}) = (-2,5,6,12) \\ a_2 = a_1, &\quad a_3 = -\tfrac{1}{5}a_1^2 \end{aligned}$$

$$y^{(5)} = a_1(yy''' + y'y'' - \tfrac{1}{5}a_1 y^2 y') \tag{2.62}$$

Case 2:

$$y_{01} = \frac{15}{a_1}: \qquad (r_{11}, r_{12}, r_{13}, r_{14}) = (3, 5, 6, 7)$$
$$y_{02} = \frac{120}{a_1}: \qquad (r_{21}, r_{22}, r_{23}, r_{24}) = (-7, 6, 10, 12)$$
$$a_2 = \tfrac{5}{2}a_1, \quad a_3 = -\tfrac{1}{5}a_1^2$$
$$y^{(5)} = a_1(yy''' + \tfrac{5}{2}y'y'' - \tfrac{1}{5}a_1y^2y') \tag{2.63}$$

Case 3:

$$y_{01} = \frac{18}{a_1}: \qquad (r_{11}, r_{12}, r_{13}, r_{14}) = (3, 4, 6, 8)$$
$$y_{02} = \frac{90}{a_1}: \qquad (r_{21}, r_{22}, r_{23}, r_{24}) = (-5, 6, 8, 12)$$
$$a_2 = 2a_1, \qquad a_3 = -\tfrac{2}{9}a_1^2$$
$$y^{(5)} = a_1(yy''' + 2y'y'' - \tfrac{2}{9}a_1y^2y') \tag{2.64}$$

Case 4:

$$y_{01} = \frac{20}{a_1}: \qquad (r_{11}, r_{12}, r_{13}, r_{14}) = (2, 5, 6, 8)$$
$$y_{02} = \frac{60}{a_1}: \qquad (r_{21}, r_{22}, r_{23}, r_{24}) = (-3, 6, 8, 10) \tag{2.65}$$
$$a_2 = 2a_1, \qquad a_3 = -\tfrac{3}{10}a_1^2$$
$$y^{(5)} = a_1(yy''' + 2y'y'' - \tfrac{3}{10}a_1y^2y') \tag{2.66}$$

The compatibility conditions for all 4 cases are identically satisfied. To obtain the canonical form of the fifth order equation of Painlevé type, one should add the non-dominant terms of weight < 7 for $\alpha = -2$ with analytic coefficients of z. Therefore, the general form is

$$\begin{aligned} y^{(5)} &= a_1yy''' + a_2y'y'' + a_3y^2y' + A_1(z)y^{(4)} + \\ A_2(z)y''' &+ A_3(z)yy'' + A_4(z)y'' + A_5(z)y'^2 + A_6(z)yy' + \\ A_7(z)y' &+ A_8(z)y^3 + A_9(z)y^2 + A_{10}(z)y + A_{11}(z). \end{aligned} \tag{2.67}$$

The coefficients $A_1(z), ..., A_{11}(z)$ can be determined by using the compatibility conditions. Substituting

$$y = y_{01}(z - z_0)^{-2} + \sum_{j=1}^{r_4} y_j(z - z_0)^{j-2}, \tag{2.68}$$

into (2.67) gives the recursion relation for y_j. The recursion relations for $j = r_{11}, r_{12}, r_{13}, r_{14}$ give the compatibility conditions if $y_{r_{11}}, y_{r_{12}}, y_{r_{13}}, y_{r_{14}}$ are arbitrary.

Case 1: By using the linear transformation (2.6), one can set

$$120A_1 + 6A_3 + 4A_5 + A_8 = 0, \qquad A_6 = 0, \qquad a_1 = 30, \tag{2.69}$$

then, $y_{01} = 1$ and $y_1 = 0$. The compatibility conditions at $j = 2, 3, 6, 10$ imply that all the coefficients are zero except

$$A_7 = c_1 z + c_2, \qquad A_{10} = 2c_1, \tag{2.70}$$

where c_1, c_2 are constants. Then the canonical form for this case is

$$y^{(5)} = 30(yy''' + y'y'' - 6y^2y') + (c_1 z + c_2)y' + 2c_1 y. \tag{2.71}$$

Equation (2.71) was also obtained in [15]. If $c_1 \neq 0$, replacing $z + c_2/c_1$ by z and then replacing z by γz and y by βy such that $\gamma^2\beta = 1,\ c_1\gamma^5 = 1$ in (2.71) gives

$$y^{(5)} = 30(yy''' + y'y'' - 6y^2y') + zy' + 2y. \tag{2.72}$$

Case 2: One can always choose

$$120A_1 + 6A_3 + 4A_5 + A_8 = 0, \qquad 12A_2 + A_6 = 0, \qquad a_1 = 15, \tag{2.73}$$

by using the linear transformation (2.6). Then $y_{01} = 1,\ y_1 = y_2 = 0$. The compatibility conditions at $j = 3, 5, 6, 7$ imply that all the coefficients are zero except

$$A_7 = c_1 z + c_2, \qquad A_{10} = 2c_1, \tag{2.74}$$

where c_1, c_2 are constants. Then the canonical form for this case is

$$y^{(5)} = 15(yy''' + \frac{5}{2}y'y'' - 3y^2y') + (c_1 z + c_2)y' + 2c_1 y. \tag{2.75}$$

Equation (2.75) was also given in [15]. If $c_1 \neq 0$, replacing $z + c_2/c_1$ by z and then replacing z by γz and y by βy such that $\gamma^2\beta = 1,\ c_1\gamma^5 = 1$ in (2.75) gives

$$y^{(5)} = 15(yy''' + \frac{5}{2}y'y'' - 3y^2y') + zy' + 2y. \tag{2.76}$$

Case 3: By using the transformation (2.6) one can set $y_{01} = 1,\ y_1 = y_2 = 0$. That is,

$$120A_1 + 6A_3 + 4A_5 + A_8 = 0, \qquad 12A_2 + 9A_6 = 0, \qquad a_1 = 18. \tag{2.77}$$

The compatibility conditions at $j = 3, 4, 6, 8$ give

$$6A_4 + A_7 = 0, \tag{2.78}$$

$$-6A_3 + 4A_5 - 3A_8 = 0, \qquad A_7 = 0, \tag{2.79}$$

$$24A_6' - 48A_9 - A_6A_8 = 0, \qquad -24A_{10}' + A_8A_{10} = 0, \tag{2.80}$$

and

$$8A_5 + 3A_8 = 0, \qquad 24A_8' + A_8^2 = 0, \qquad 24A_9' + A_8A_9 = 0, \tag{2.81}$$

respectively. The equation (2.81.b) implies that there are two cases that should be considered separately.

a. $A_8(z) = 0$: The equations (2.77)-(2.81) and the compatibility condition at $j = 8$ implies that all the coefficients are zero except

$$A_6 = c_1, \qquad A_2 = -\frac{1}{6}c_1, \qquad A_{11} = c_2, \tag{2.82}$$

where c_1, c_2 are constants. Then, the canonical form of the equation for this case is

$$y^{(5)} = 18(yy''' + 2y'y'' - 4y^2y') - \frac{1}{6}c_1y''' + c_1yy' + c_2. \tag{2.83}$$

Equation (2.83) was given in [15].

b. $A_8(z) = 24/(z-c)$: For simplicity, let the constant $c = 0$. Then the equations (2.77)-(2.81) and the compatibility condition at $j = 8$ implies that there are two following distinct cases:

i.

$$\begin{aligned} A_1 = \frac{1}{z},\ \ A_2 = \frac{c_2}{6},\ \ A_3 = -\frac{18}{z},\ \ A_4 = -\frac{c_2}{6z},\ \ A_5 = -\frac{9}{z}, \\ A_6 = -2c_2,\ \ A_7 = \frac{c_2}{z},\ \ A_{10} = 0,\ \ A_{11} = \frac{c_1}{z}, \end{aligned} \tag{2.84}$$

where c_1, c_2 are constants. Then, the canonical form is

$$\begin{aligned} y^{(5)} = 18(yy''' + 2y'y'' - 4y^2y') + \tfrac{1}{z}y^{(4)} + \tfrac{c_2}{6}y''' - \tfrac{18}{z}yy'' \\ - \tfrac{c_2}{6z}y'' - \tfrac{9}{z}y'^2 - 2c_2yy' + \tfrac{24}{z}y^3 + \tfrac{c_2}{z}y^2 + \tfrac{c_1}{z}. \end{aligned} \tag{2.85}$$

Equaton (2.85) was also given in [15]. When $c_2 = 0$; if one lets

$$u = y^{(4)} - 3(6yy'' + 3y'^2 - 8y^3), \tag{2.86}$$

Then equation (2.85) can be written as

$$u' = \frac{1}{z}u + \frac{c_1}{z}. \tag{2.87}$$

Hence, (2.85) has the first integral

$$y^{(4)} = 3(6yy'' + 3y'^2 - 8y^3) + kz - c_1, \tag{2.88}$$

where k is an arbitrary constant. Equation (2.88) is nothing but the equation (2.43) with $k_1 = 0$.

ii. $A_4 = A_7 = A_9 = 0$ and,

$$\begin{aligned} A_1 = \frac{1}{z},\ \ A_2 = -\frac{c_3}{2}z,\ \ A_3 = -\frac{18}{z},\ \ A_5 = -\frac{9}{z}, \\ A_6 = 6c_3z,\ \ A_{10} = \frac{c_3^2}{2}z,\ \ A_{11} = -\frac{c_3^3}{36}z^2 + \frac{c_4}{z}, \end{aligned} \tag{2.89}$$

where c_3, c_4 are constants. Then, the canonical form is

$$\begin{aligned} y^{(5)} = {} & 18(yy''' + 2y'y'' - 4y^2y') + \tfrac{1}{z}y^{(4)} - \tfrac{c_3}{2}zy''' - \tfrac{18}{z}yy'' \\ & - \tfrac{9}{z}y'^2 + 6c_3zyy' + \tfrac{24}{z}y^3 + \tfrac{c_3^2}{2}zy - \tfrac{c_3^3}{36}z^2 + \tfrac{c_4}{z}. \end{aligned} \tag{2.90}$$

When $c_3 = 0$, (2.90) has the first integral same as (2.85).

Case 4: By using the transformation one can set

$$120A_1 + 6A_3 + 4A_5 + A_8 = 0, \qquad A_6 = 0, \qquad a_1 = 20. \tag{2.91}$$

The compatibility conditions at $j = 2$ and $j = 5$ implies that $A_2 = 0$ and $A_4 = 0$ respectively. The compatibility conditions at $j = 6, 8$ implies

$$4A_5 + A_8 = 0, \tag{2.92}$$

and

$$A_7 = 0, \qquad -7A_3 + 6A_5 - 2A_8 = 0, \qquad 40A_8' + A_8^2 = 0, \qquad 40A_{10}' + A_8A_{10} = 0, \tag{2.93}$$

respectively. Therefore there are two cases should be considered separately: a) $A_8(z) = 0$ and b) $A_8(z) = 40/z$ (for simplicity the integration constant is set to zero).

a) $A_8(z) = 0$: The equations (2.91)-(2.93) implies that all the coefficients are zero except $A_7 = c_1z + c_2$, $A_{10} = 2c_1$ and $A_{11} = c_3$ where c_i are constants. Then, the canonical form is

$$y^{(5)} = 20(yy''' + 2y'y'' - 6y^2y') + (c_1z + c_2)y' + 2c_1y + c_3. \tag{2.94}$$

b) $A_8(z) = 40/z$: The equations (2.91)-(2.93) and the compatibility conditions at $j = 5, 8$ imply that

$$\begin{aligned} &A_1 = \tfrac{1}{z}, \quad A_2 = 0, \quad A_3 = -\tfrac{20}{z}, \quad A_4 = 0, \quad A_5 = -\tfrac{10}{z}, \\ &A_6 = 0, \quad A_7 = -k_1, \quad A_9 = 0, \quad A_{10} = \tfrac{k_1}{z}, \quad A_{11} = \tfrac{k_2}{z}, \end{aligned} \tag{2.95}$$

where k_1, k_2 are constants. Then, the canonical form is

$$\begin{aligned} y^{(5)} = {} & 20(yy''' + 2y'y'' - 6y^2y') + \frac{1}{z}y^{(4)} - \frac{20}{z}yy'' \\ & -\frac{10}{z}y'^2 - k_1y' + \frac{40}{z}y^3 + \frac{k_1}{z}y + \frac{k_2}{z}. \end{aligned} \tag{2.96}$$

When $k_1 = 0$: if one lets

$$u = y^{(4)} - 10(2yy'' + y'^2 - 4y^3). \tag{2.97}$$

Then equation (2.96) can be written as

$$u' = \frac{1}{z}u + \frac{k_2}{z}. \tag{2.98}$$

Hence, the first integral of (2.96) is

$$y^{(4)} = 10(2yy'' + y'^2 - 4y^3) + k_3 z - k_2, \qquad (2.99)$$

where k_3 is an arbitrary constant. Replacing y by $-y/4$ in (2.99) gives (2.40) with $k_1 = 0$.

2.4 Sixth order equations: $\mathrm{P}_I^{(6)}$

Differentiating (2.55) with respect to z gives the terms $y^{(6)}$, $yy^{(4)}$, $y'y'''$, y''^2, $y^2y''$and yy'^2 all of which are of order -8 for $\alpha = -2$ as $z \to z_0$. Adding the term y^4 which is also of order -8 gives the following simplified equation

$$y^{(6)} = a_1 yy^{(4)} + a_2 y'y''' + a_3 y''^2 + a_4 y^2 y'' + a_5 yy'^2 + a_6 y^4, \qquad (2.100)$$

where a_i, $i = 1, 2, ..., 6$ are constants. Substituting (2.22) into (2.100) gives the following equations for the resonance r and y_0,

$$\begin{aligned}(r+1)\{r^5 - 28r^4 + (323 - a_1 y_0)r^3 + [(15a_1 + 2a_2)y_0 - 1988]r^2 \\ -[a_4 y_0^2 + 2(43a_1 + 10a_2 + 6a_3)y_0 - 7092]r \\ +2[(2a_5 + 3a_4)y_0^2 + 12(10a_1 + 4a_2 + 3a_3)y_0 - 7560]\} = 0, \\ a_6 y_0^3 + 2(3a_4 + 2a_5)y_0^2 + 12(10a_1 + 4a_2 + 3a_3)y_0 - 5040 = 0 \qquad (2.101)\end{aligned}$$

Equation (2.101.a) implies that one of the resonance $r_0 = -1$ which corresponds to arbitrariness of z_0. Two cases should be considered separately a) $a_6 = 0$ and b) $a_6 \neq 0$.

a) $a_6 = 0$: There are two Painlevé branches corresponding to $(-2, y_{0j})$, $j = 1, 2$, where y_{0j}'s are the roots of

$$(3a_4 + 2a_5)y_0^2 + 6(10a_1 + 4a_2 + 3a_3)y_0 - 2520 = 0. \qquad (2.102)$$

Then, one has

$$y_{01} + y_{02} = -\frac{6(10a_1 + 4a_2 + 3a_3)}{3a_4 + 2a_5}, \qquad y_{01}y_{02} = -\frac{2520}{3a_4 + 2a_5}. \qquad (2.103)$$

Let $r_{11}, r_{12}, ..., r_{15}$ and $r_{21}, r_{22}, ..., r_{25}$ be the roots (additional to $r_0 = -1$) of (2.101.a) corresponding to y_{01} and y_{02} respectively. Setting

$$P(y_{0j}) = -2[(2a_5 + 3a_4)y_{0j}^2 + 12(10a_1 + 4a_2 + 3a_3)y_{0j} - 7560], \quad j = 1, 2. \quad (2.104)$$

then, (2.101.a) implies that

$$\prod_{i=1}^{5} r_{1i} = P(y_{01}) = p_1, \qquad \prod_{i=1}^{5} r_{2i} = P(y_{02}) = p_2 \tag{2.105}$$

and

$$\sum_{i=1}^{5} r_{1i} = \sum_{i=1}^{5} r_{2i} = 28, \tag{2.106}$$

where p_1, p_2 are integers, and at least one of them is positive. Now, one should determine y_{0j}, $j = 1, 2$, and a_i, $i = 1, 2, ..., 5$ such that there is at least one principal branch. Let the branch corresponding to y_{01} be the principal branch, then $p_1 > 0$. Equation (2.104) gives

$$P(y_{01}) = 5040(1 - \frac{y_{01}}{y_{02}}) = p_1, \qquad P(y_{02}) = 5040(1 - \frac{y_{02}}{y_{01}}) = p_2, \tag{2.107}$$

by using the (2.103). Therefore, p_1, p_2 satisfy the following Diophantine equation, if $p_1 p_2 \neq 0$

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{5040}. \tag{2.108}$$

Equation(2.106) implies that there are 57 possible cases of $(r_{11}, r_{12}, ..., r_{15})$ such that r_{1i}'s are positive distinct integers. Diophantine equation has 27 integer solutions (p_1, p_2) such that $p_2 < 0$. For each 27 cases of (p_1, p_2), y_{0j}, $j = 1, 2$, and a_i, $i = 2, ..., 5$ can be obtained from (2.103), (2.107) and

$$\begin{gathered} \textstyle\sum_{i \neq j} r_{1i} r_{1j} = 323 - a_1 y_{01}, \quad \sum_{i \neq j \neq k} r_{1i} r_{1j} r_{1k} = -[(15a_1 + 2a_2) y_{01} + 1988], \\ \textstyle\sum_{i \neq j \neq k \neq l} r_{1i} r_{1j} r_k r_{1l} = -a_4 y_{01}^2 - 2(43a_1 + 10a_2 + 6a_3) y_{01} + 7092. \end{gathered} \tag{2.109}$$

in terms of a_1. But, there are only 3 cases out of 27 cases such that the resonances $(r_{21}, r_{22}, ..., r_{25})$ corresponding to y_{02} are distinct integers. These cases and the corresponding simplified equations are as follows:

Case 1:

$$\begin{gathered} y_{01} = \tfrac{20}{a_1}: \quad (r_{11}, r_{12}, r_{13}, r_{14}, r_{15}) = (2, 5, 6, 7, 8) \\ y_{02} = \tfrac{60}{a_1}: \quad (r_{21}, r_{22}, r_{23}, r_{24}, r_{25}) = (-3, 6, 7, 8, 10) \\ a_2 = 3a_1, \quad a_3 = 2a_1, \quad a_4 = -\tfrac{3}{10} a_1^2, \quad a_5 = -\tfrac{3}{5} a_1^2, \\ y^{(6)} = a_1 (y y^{(4)} + 3y' y''' + 2y''^2 - \tfrac{3}{10} a_1 y^2 y'' - \tfrac{3}{5} a_1 y y'^2) \end{gathered} \tag{2.110}$$

Case 2:

$$\begin{gathered} y_{01} = \tfrac{18}{a_1}: \quad (r_{11}, r_{12}, r_{13}, r_{14}, r_{15}) = (3, 4, 6, 7, 8), \\ y_{02} = \tfrac{90}{a_1}: \quad (r_{21}, r_{22}, r_{23}, r_{24}, r_{25}) = (-5, 6, 7, 8, 12) \\ a_2 = 3a_1, \quad a_3 = 2a_1, \quad a_4 = -\tfrac{2}{9} a_1^2, \quad a_5 = -\tfrac{4}{9} a_1^2, \\ y^{(6)} = a_1 (y y^{(4)} + 3y' y''' + 2y''^2 - \tfrac{2}{9} a_1 y^2 y'' - \tfrac{4}{9} a_1 y y'^2) \end{gathered} \tag{2.111}$$

Case 3:

$$y_{01} = \tfrac{30}{a_1}: \quad (r_{11}, r_{12}, r_{13}, r_{14}, r_{15}) = (2, 3, 6, 7, 10),$$
$$y_{02} = \tfrac{60}{a_1}: \quad (r_{21}, r_{22}, r_{23}, r_{24}, r_{25}) = (-2, 5, 6, 7, 12)$$
$$a_2 = 2a_1, \quad a_3 = a_1, \quad a_4 = -\tfrac{1}{5}a_1^2, \quad a_5 = -\tfrac{2}{5}a_1^2,$$
$$y^{(6)} = a_1(yy^{(4)} + 2y'y''' + y''^2 - \tfrac{1}{5}a_1y^2y'' - \tfrac{2}{5}a_1yy'^2) \qquad (2.112)$$

The compatibility conditions are identically satisfied for the first two cases but not for the third case. Therefore, the third case will not be considered.

To obtain the canonical form of the sixth order Painlevé type equation when $a_6 = 0$, one should add the non-dominant terms with analytic coefficients of z. That is,

$$\begin{aligned} y^{(6)} &= a_1yy^{(4)} + a_2y'y''' + a_3y''^2 + a_4y^2y'' + a_5yy'^2 \\ &+A_1(z)y^{(5)} + A_2(z)y^{(4)} + A_3(z)yy''' + A_4(z)y''' + A_5(z)y'y'' \\ &+A_6(z)yy'' + A_7(z)y'' + A_8(z)y^2y' + A_9(z)yy' + A_{10}(z)y'^2 \\ &+A_{11}(z)y' + A_{12}(z)y^3 + A_{13}(z)y^2 + A_{14}(z)y + A_{15}(z) \end{aligned} \qquad (2.113)$$

The coefficients $A_1(z), ..., A_{15}(z)$ can be determined by using the compatibility conditions at the resonances. Substituting

$$y = y_{01}(z - z_0)^{-2} + \sum_{j=1}^{r_5} y_j(z - z_0)^{j-2}, \qquad (2.114)$$

into (2.113) gives the recursion relation for y_j. Then, one can find $A_1, ..., A_{15}$ such that the recursion relations for $j = r_{11}, r_{12}, r_{13}, r_{14}, r_{15}$ are identically satisfied, and hence $y_{r_{11}}, y_{r_{12}}, y_{r_{13}}, y_{r_{14}}, y_{r_{15}}$ are arbitrary.

Case 1: By using the linear transformation (2.6), one can set

$$360A_1 + 12A_3 + 6A_5 + A_8 = 0, \quad A_6 = 0, \quad a_1 = 20, \qquad (2.115)$$

then, $y_{01} = 1$ and $y_1 = 0$. The compatibility conditions at $j = 2, 5, 6, 7, 8$ imply that all the coefficients are zero except

$$A_7 = c_1z + c_2, \quad A_{11} = 3c_1, \qquad (2.116)$$

where c_1, c_2 are constants. Then the canonical form for this case is

$$y^{(6)} = 20(yy^{(4)} + 3y'y''' + 2y''^2 - 6y^2y'' - 12yy'^2) + (c_1z + c_2)y'' + 3c_1y' \quad (2.117)$$

If $c_1 \neq 0$, replacing $z + c_2/c_1$ by z and then replacing z by γz and y by βy such that $\gamma^2\beta = 1$, $c_1\gamma^5 = 1$ in (2.117) gives

$$y^{(6)} = 20(yy^{(4)} + 3y'y''' + 2y''^2 - 6y^2y'' - 12yy'^2) + zy'' + 3y' \qquad (2.118)$$

Case 2: One can always choose $y_{01}=1$, and $y_1=y_2=0$ by choosing

$$360A_1+12A_3+6A_5+A_8=0,\quad 120A_2+6A_6+4A_{10}+A_{12}=0,\quad a_1=18, \tag{2.119}$$

Then, the recursion relation imply that if, $y_3, y_4, y_6, y_7,$ and y_8 are arbitrary then $A_1=A_3=A_5=A_7=A_8=A_{12}=A_{13}=0$ and

$$A_2=-\frac{1}{12}(c_1z+c_2),\ A_4=-\frac{1}{6}c_1,\ A_6=A_{10}=c_1z+c_2,\quad A_9=2c_1,$$
$$A_{11}=\frac{c_1}{72}(c_1z+c_2),\ A_{14}=\frac{1}{36}c_1^2,\ A_{15}=-\frac{c_1^2}{2592}(c_1z+c_2) \tag{2.120}$$

where c_1, c_2 are arbitrary constants. Then the canonical form for this case is

$$\begin{aligned} y^{(6)} &= 18(yy^{(4)}+3y'y'''+2y''^2-4y^2y''-8yy'^2)-\tfrac{1}{12}(c_1z+c_2)y^{(4)} \\ &\quad -\tfrac{c_1}{6}y'''+(c_1z+c_2)yy''+2c_1yy'+(c_1z+c_2)y'^2 \\ &\quad +\tfrac{c_1}{72}(c_1z+c_2)y'+\tfrac{c_1^2}{36}y-\tfrac{c_1^2}{2592}(c_1z+c_2) \end{aligned} \tag{2.121}$$

If $c_1\neq 0$, replacing $z+c_2/c_1$ by z and then replacing z by γz and y by βy such that $\gamma^2\beta=1,\ c_1\gamma^3=36$ in (2.121) gives

$$\begin{aligned} y^{(6)} &= 18(yy^{(4)}+3y'y'''+2y''^2-4y^2y''-8yy'^2)-3zy^{(4)} \\ &\quad -6y'''+36z(yy''+y'^2)+6(12yy'+3zy'+6y-3z). \end{aligned} \tag{2.122}$$

b) $a_6\neq 0$: Equation (2.101.b) implies that there are three Painlevé branches corresponding to $(-2, y_{0j})$, $j=1,2,3$ where y_{0j} are the roots of (2.101.b). (2.101.b) implies that

$$\prod_{j=1}^{3} y_{0j}=\frac{5040}{a_6},\qquad \sum_{j=1}^{3} y_{0j}=-\frac{2(3a_4+2a_5)}{a_6},$$
$$\sum_{i\neq j} y_{0i}y_{0j}=\frac{12}{a_6}(10a_1+4a_2+3a_3). \tag{2.123}$$

If the resonances (except $r_0=-1$) are r_{1i}, r_{2i}, r_{3i} $i=1,2,...,5$ corresponding to y_{01}, y_{02}, y_{03} respectively. If one sets,

$$P(y_{0j})=-2[(2a_5+3a_4)y_{0j}^2+12(10a_1+4a_2+3a_3)y_{0j}-7560], \tag{2.124}$$

then, (2.101.a) implies that

$$\prod_{i=1}^{5} r_{1i}=P(y_{01}),\qquad \prod_{i=1}^{5} r_{2i}=P(y_{02}),\qquad \prod_{i=1}^{5} r_{3i}=P(y_{03}) \tag{2.125}$$

and

$$\sum_{i=1}^{5} r_{1i}=\sum_{i=1}^{5} r_{2i}=\sum_{i=1}^{5} r_{3i}=28 \tag{2.126}$$

The condition of $r_{1i}, r_{2i},\ r_{3i}$ being integers and (2.124), (2.125) give

$$P(y_{01}) = p_1, \qquad P(y_{02}) = p_2, \qquad P(y_{03}) = p_3 \tag{2.127}$$

where p_1, p_2, p_3 are integers, and at least one is positive. Then the equations (2.123) and (2.124) give

$$\begin{aligned} p_1 &= 5040(1 - \frac{y_{01}}{y_{02}})(1 - \frac{y_{01}}{y_{03}}) \\ p_2 &= 5040(1 - \frac{y_{02}}{y_{01}})(1 - \frac{y_{02}}{y_{03}}) \\ p_3 &= 5040(1 - \frac{y_{03}}{y_{01}})(1 - \frac{y_{03}}{y_{02}}). \end{aligned} \tag{2.128}$$

By setting, $\kappa = y_{02} - y_{03},\ \mu = y_{03} - y_{01}$, and $\nu = y_{01} - y_{02}$, then (2.128) yields

$$p_1 = -5040\frac{\mu\nu}{y_{02}y_{03}}, \quad p_2 = -5040\frac{\kappa\nu}{y_{01}y_{03}}, \quad p_3 = -5040\frac{\kappa\mu}{y_{01}y_{02}}. \tag{2.129}$$

Thus,

$$\sum_{i\neq j} p_i p_j = (5040)^2\kappa\mu\nu(\frac{\kappa}{y_{01}} + \frac{\mu}{y_{02}} + \frac{\nu}{y_{03}}). \tag{2.130}$$

But,

$$\frac{\kappa}{y_{01}} + \frac{\mu}{y_{02}} + \frac{\nu}{y_{03}} = -\frac{\kappa\mu\nu}{y_{01}y_{02}y_{03}}. \tag{2.131}$$

Therefore,

$$\sum_{i\neq j} p_i p_j = -(5040)^2\frac{\kappa^2\mu^2\nu^2}{y_{01}^2 y_{02}^2 y_{03}^2} = \frac{1}{5040}p_1p_2p_3. \tag{2.132}$$

So that, $p_i,\ i = 1, 2, 3$, satisfy the following Diophatine equation

$$\sum_{i=1}^{3}\frac{1}{p_i} = \frac{1}{5040}. \tag{2.133}$$

If the principal branch corresponds to $(-2, y_{01})$, then the resonances $r_{1i},\ i = 1, 2, ..., 5$ are positive distinct integers and thus p_1 is a positive integer. Equation (2.129) yields

$$p_1p_2p_3 = -(5040)^3\frac{\kappa^2\mu^2\nu^2}{y_{01}^2 y_{02}^2 y_{03}^2}. \tag{2.134}$$

Therefore, either p_2 or p_3 is a negative integer. $\sum r_{1i} = 28$ and r_{1i} being distinct positive integers imply that there are 57 possible values of p_1. Then, one should find all integer solutions (p_2, p_3) of (2.133) for each possible values of p_1. There are 3740 possible integer values of (p_1, p_2, p_3) such that $p_1, p_2 > 0$ and $p_3 < 0$. Equations (2.123), (2.128) and

$$\begin{aligned} \textstyle\sum_{i\neq j} r_{1i}r_{1j} &= 323 - a_1y_{01} \\ \textstyle\sum_{i\neq j\neq k} r_{1i}r_{1j}r_{1k} &= -[(15a_1 + 2a_2)y_{01} - 1988], \\ \textstyle\sum_{i\neq j\neq k\neq l} r_{1i}r_{1j}r_{1k}r_{1l} &= -a_4y_{01}^2 - 2(43a_1 + 10a_2 + 6a_3)y_{01} + 7092 \end{aligned} \tag{2.135}$$

determine all the coefficients of (2.101.a) in terms of a_1 for all possible values of (p_1, p_2, p_3). Now one should find the roots r_{2i}, r_{3i} of (2.101.a). There are only 3 cases such that r_{2i}, r_{3i} are being distinct integers. The cases and the corresponding simplified equations are as follows:

Case 1:

$$y_{01} = \tfrac{36}{a_1}: \quad (r_{11}, r_{12}, r_{13}, r_{14}, r_{15}) = (2, 3, 4, 9, 10)$$
$$y_{02} = \tfrac{252}{a_1}: \quad (r_{21}, r_{22}, r_{23}, r_{24}, r_{25}) = (-5, -7, 10, 12, 18)$$
$$y_{03} = \tfrac{72}{a_1}: \quad (r_{31}, r_{32}, r_{33}, r_{34}, r_{35}) = (-2, 3, 5, 10, 12)$$
$$a_2 = \tfrac{5}{3}a_1, \quad a_3 = \tfrac{5}{6}a_1, \quad a_4 = a_5 = -\tfrac{5}{18}a_1^2, \quad a_6 = \tfrac{5}{648}a_1^3$$
$$y^{(6)} = a_1(yy^{(4)} + \tfrac{5}{3}y'y''' + \tfrac{5}{6}y''^2 - \tfrac{5}{18}a_1y^2y'' - \tfrac{5}{18}a_1yy'^2 + \tfrac{5}{648}a_1^2y^4) \quad (2.136)$$

Case 2:

$$y_{01} = \tfrac{28}{a_1}: \quad (r_{11}, r_{12}, r_{13}, r_{14}, r_{15}) = (2, 4, 5, 7, 10)$$
$$y_{02} = \tfrac{168}{a_1}: \quad (r_{21}, r_{22}, r_{23}, r_{24}, r_{25}) = (-3, -5, 10, 12, 14)$$
$$y_{03} = \tfrac{84}{a_1}: \quad (r_{31}, r_{32}, r_{33}, r_{34}, r_{35}) = (-3, 2, 7, 10, 12)$$
$$a_2 = 2a_1, \quad a_3 = \tfrac{3}{2}a_1, \quad a_4 = a_5 = -\tfrac{5}{14}a_1^2, \quad a_6 = \tfrac{5}{392}a_1^3,$$
$$y^{(6)} = a_1(yy^{(4)} + 2y'y''' + \tfrac{3}{2}y''^2 - \tfrac{5}{14}a_1y^2y'' - \tfrac{5}{14}a_1yy'^2 + \tfrac{5}{392}a_1^2y^4) \quad (2.137)$$

Case 3:

$$y_{01} = \tfrac{21}{a_1}: \quad (r_{11}, r_{12}, r_{13}, r_{14}, r_{15}) = (3, 4, 5, 7, 9)$$
$$y_{02} = \tfrac{336}{a_1}: \quad (r_{21}, r_{22}, r_{23}, r_{24}, r_{25}) = (-5, -11, 12, 14, 18)$$
$$y_{03} = \tfrac{105}{a_1}: \quad (r_{31}, r_{32}, r_{33}, r_{34}, r_{35}) = (-5, 3, 7, 11, 12)$$
$$a_2 = \tfrac{5}{2}a_1, \quad a_3 = \tfrac{7}{4}a_1, \quad a_4 = -\tfrac{2}{7}a_1^2, \quad a_5 = -\tfrac{5}{14}a_1^2, \quad a_6 = \tfrac{1}{147}a_1^3,$$
$$y^{(6)} = a_1(yy^{(4)} + \tfrac{5}{2}y'y''' + \tfrac{7}{4}y''^2 - \tfrac{2}{7}a_1y^2y'' - \tfrac{5}{14}a_1yy'^2 + \tfrac{1}{147}a_1^2y^4) \quad (2.138)$$

For all three cases, the compatibility conditions are identically satisfied. To obtain the canonical form of the sixth order Painlevé type equation, one should add the non-dominant terms with analytic coefficients of z. That is,

$$\begin{aligned} y^{(6)} &= a_1yy^{(4)} + a_2y'y''' + a_3y''^2 + a_4y^2y'' + a_5yy'^2 + a_6y^4 \\ &+A_1(z)y^{(5)} + A_2(z)y^{(4)} + A_3(z)yy''' + A_4(z)y''' + A_5(z)y'y'' \\ &+A_6(z)yy'' + A_7(z)y'' + A_8(z)y^2y' + A_9(z)yy' + A_{10}(z)y'^2 \\ &+A_{11}(z)y' + A_{12}(z)y^3 + A_{13}(z)y^2 + A_{14}(z)y + A_{15}(z) \end{aligned} \quad (2.139)$$

The coefficients $A_1(z), ..., A_{15}(z)$ can be determined by using the compatibility conditions at the resonances. Substituting (2.114) into (2.139) gives the recursion relation for y_j. Then, one can find $A_1, ..., A_{15}$ such that the recursion relations for $j = r_{11}, r_{12}, r_{13}, r_{14}, r_{15}$ are identically satisfied, and hence $y_{r_{11}}, y_{r_{12}}, y_{r_{13}}, y_{r_{14}}, y_{r_{15}}$ are arbitrary.

Case 1. By using the linear transformation (2.6), one can set

$$360A_1 + 12A_3 + 6A_5 + A_8 = 0, \qquad A_{12} = 0, \qquad a_1 = 36, \tag{2.140}$$

then, $y_{01} = 1$ and $y_1 = 0$. The compatibility conditions at $j = 2, 3, 4, 9, 10$ imply that all the coefficients are zero except

$$A_7 = -\frac{c_1}{6}, \qquad A_{13} = c_1, \qquad A_{14} = c_2, \qquad A_{15} = c_3, \tag{2.141}$$

where c_i's are arbitrary constants. Therefore, the canonical form for this case is

$$\begin{aligned} y^{(6)} = 36(yy^{(4)} + \frac{5}{3}y'y''' + \frac{5}{6}y''^2 - 10y^2y'' - 10yy'^2 + 10y^4) \\ -\frac{c_1}{6}y'' + c_1y^2 + c_2y + c_3 \end{aligned} \tag{2.142}$$

Case 2. One can always choose $y_{01} = 1$, and $y_1 = 0$ by setting

$$360A_1 + 12A_3 + 6A_5 + A_8 = 0, \qquad A_{12} = 0, \qquad a_1 = 28, \tag{2.143}$$

Then, the recursion relation imply that if, y_2, y_4, y_5, y_7, and y_{10} are arbitrary then all the coefficients are zero except

$$A_7 = -\frac{c_1}{6}, \qquad A_{13} = c_1, \qquad A_{14} = c_2, \qquad A_{15} = c_3z + c_4, \tag{2.144}$$

where c_i's are arbitrary constants. Then the canonical form is

$$\begin{aligned} y^{(6)} = 28(yy^{(4)} + 2y'y''' + \frac{3}{2}y''^2 - 10y^2y'' - 10yy'^2 + 10y^4) \\ -\frac{c_1}{6}y'' + c_1y^2 + c_2y + c_3z + c_4 \end{aligned} \tag{2.145}$$

(2.145) can also be obtained by the similarity reduction of the hierarchy of the (KdV) equation [16].

Case 3: One can always set $y_{01} = 1$, and $y_1 = y_2 = 0$ by choosing

$$360A_1+12A_3+6A_5+A_8 = 0, \quad 120A_2+6A_6+4A_{10}+A_{12} = 0, \quad a_1 = 21, \tag{2.146}$$

Then, the recursion relation imply that if, y_3, y_4, y_5, y_7, and y_9 are arbitrary then all the coefficients are zero except

$$A_2 = \frac{c_1}{15}, \quad A_6 = -c_1, \quad A_{10} = -\frac{3}{4}c_1, \quad A_{14} = c_2, \quad A_{15} = c_3, \tag{2.147}$$

where c_i's are arbitrary constants. Then the canonical form is

$$\begin{aligned} y^{(6)} = 21(yy^{(4)} + \frac{5}{2}y'y''' + \frac{7}{4}y''^2 - 6y^2y'' - \frac{15}{2}yy'^2 + 3y^4) \\ -\frac{c_1}{15}y^{(4)} - c_1yy'' - \frac{3}{4}c_1y'^2 + c_2y + c_3. \end{aligned} \tag{2.148}$$

Chapter 3

The second Painlevé hierarchy

In this chapter we apply the procedure to the second Painlevé equation and present Painlevé type equations of order three, four, five and six.

3.1 Third order equations: $\mathrm{P}_{II}^{(3)}$

The second Painlevé equation, P_{II} is

$$y'' = 2y^3 + zy + \nu. \tag{3.1}$$

Painlevé test gives that there are two branches with common resonances are $(-1, 4)$. The dominant terms of (3.1) are y'' and $2y^3$ which are of order -3 as $z \to z_0$. Taking the derivative of the simplified equation gives

$$y''' = ay^2y' \tag{3.2}$$

where a is a constant which can be introduced by replacing y with λy, such that $6\lambda^2 = a$. Adding the polynomial type terms of order -4 gives the following simplified equation

$$y''' = a_1yy'' + a_2y'^2 + a_3y^2y' + a_4y^4. \tag{3.3}$$

where a_i, $i = 1, ..., 4$ are constants. Substituting

$$y = y_0(z - z_0)^{-1} + \delta(z - z_0)^{r-1}, \tag{3.4}$$

into the simplified equation, to leading order in δ, gives the equation $Q(r) = 0$ for the resonance r, and for y_0 respectively

$$\begin{gathered} Q(r) = (r+1)\{r^2 - (a_1y_0 + 7)r - [a_3y_0^2 - 2(2a_1 + a_2)y_0 - 18]\} = 0, \\ a_4y_0^3 - a_3y_0^2 + (2a_1 + a_2)y_0 + 6 = 0. \end{gathered} \tag{3.5}$$

Equation (3.5.b) implies that, in general, there are three branches of Painlevé expansion if $a_4 \neq 0$. Now, one should determine y_{0j}, $j = 1, 2, 3$ and a_i such that at least one of the branch is the principal branch. There are three cases which should be considered separately.

Case I: $a_3 = a_4 = 0$: In this case there is only one branch. The resonance equation (3.5.a) implies that $r_1 r_2 = 6$. Therefore, there are following four cases:

$$\begin{aligned} &\mathrm{a}: \ y_{01} = -\tfrac{6}{a_2}: \ (r_1, r_2) = (1, 6), \ a_1 = 0, \\ &\mathrm{b}: \ y_{01} = -\tfrac{2}{a_1}: \ (r_1, r_2) = (2, 3), \ a_1 = a_2 \\ &\mathrm{c}: \ y_{01} = -\tfrac{12}{a_1}: \ (r_1, r_2) = (-2, -3), \ a_1 = -\tfrac{2}{3} a_2, \\ &\mathrm{d}: \ y_{01} = -\tfrac{14}{a_1}: \ (r_1, r_2) = (-1, -6). \end{aligned} \tag{3.6}$$

The case d will not be considered since $r = -1$ is a double resonance. The compatibility conditions are identically satisfied for the first two cases. To find the canonical form of the third-order equations of Painlevé type, one should add non-dominant terms with the coefficients which are analytic functions of z. That is, one should consider the following equation for each case

$$y''' = a_1 y y'' + a_2 y'^2 + A_1 y'' + A_2 y y' + A_3 y^3 + A_4 y' + A_5 y^2 + A_6 y + A_7. \tag{3.7}$$

where $A_k(z)$, $k = 1, ..., 7$ are analytic functions of z. Substituting

$$y = y_0 (z - z_0)^{-1} + \sum_{j=1}^{6} y_j (z - z_0)^{j-1}, \tag{3.8}$$

into equation (3.7) gives the recursion relation for y_j. Then one can find A_k such that the recursion relation, i.e. the compatibility conditions for $j = r_1, r_2$ are identically satisfied, and hence y_{r_1}, y_{r_2} are arbitrary.

I.a: By using the transformation(2.6), one can set $A_4 - A_5 = 0$, $A_1 = 0$, and $a_2 = -6$. The compatibility condition at the resonance $r_1 = 1$ gives $A_2 = A_3$. The arbitrariness of y_1 in the recursion relation for $j = 6$ and the recursion relation yield that

$$A_5'' - A_5^2 = 0, \qquad A_6'' - A_5 A_6 = 0, \qquad A_7'' - \frac{1}{3} A_5 A_7 = \frac{1}{6} A_6^2, \qquad A_3 = 0. \tag{3.9}$$

According the equation (3.9.a), there are three cases should be considered separately.

I.a.i: $A_5 = 0$: From the equation (3.9), all the coefficients A_k can be determined uniquely. The canonical form of the third order equation for this case is

$$y''' = -6y'^2 + (c_1 z + c_2) y + \frac{1}{72} c_1^2 z^4 + \frac{1}{18} c_1 c_2 z^3 + \frac{1}{12} c_2^2 z^2 + c_3 z + c_4, \tag{3.10}$$

where $c_i,\ i = 1, ..., 4$ are constants. If $c_1 = c_2 = 0$, then (3.10) can be written as

$$u'' = 6u^2 - c_3 z - c_4 \tag{3.11}$$

where $u = -y'$. If $c_3 = 0$ then the solution of (3.11) can be written in terms of the elliptic function. If $c_3 \neq 0$, (3.11) can be transformed into the first Painlevé equation. If $c_1 = 0,\ c_2 \neq 0$, (3.10) takes the following form by replacing y by γy and z by δz such that $\gamma\delta = 1,\ c_2\delta^3 = 6$

$$y''' = -6y'^2 + 6y + 3z^2 + \tilde{c_3} z + \tilde{c_4}, \tag{3.12}$$

where $\tilde{c_3} = c_3\delta^5,\ \tilde{c_4} = c_4\delta^4$. Equation (3.12) was also given in [5] and [7]. If $c_1 \neq 0,\ c_2 = 0$, replacing y by γy and z by δz in (3.10) such that $\gamma\delta = 1,\ c_1\delta^4 = 12$ yields

$$y''' = -6y'^2 + 12zy + 2z^4 + \tilde{c_3} z + \tilde{c_4}, \tag{3.13}$$

where $\tilde{c_3} = c_3\delta^5,\ \tilde{c_4} = c_4\delta^4$. Equation (3.13) was also given by Chazy [5] and Bureau [7]. It should be noted that (3.10) can be reduced to (3.13) by replacing z by $z - (c_2/c_1)$ and then replacing y by γy and z by δz such that $\gamma\delta = 1,\ c_1\delta^4 = 12$.

I.a.ii: $A_5 = \frac{6}{(z+c)^2}$: Without loss of generality the integration constant c can be set to zero. From (3.9), the coefficients A_k can be determined and the canonical form of the equation is

$$\begin{aligned} y''' = -6y'^2 + 6z^{-2}(y' + y^2) + (c_1 z^3 + c_2 z^{-2})y + c_3 z^2 + c_4 z^{-1} \\ + \frac{1}{18}\left(\frac{1}{18}c_1^2 z^8 + \frac{3}{2}c_1 c_2 z^3 + \frac{3}{4}c_2^2 z^{-2}\right), \end{aligned} \tag{3.14}$$

where $c_i,\ i = 1, ..., 4$ are constants. If $c_1 = c_2 = 0$, (3.14) is a special case of the equation given by Chazy [5]. If $c_1 = 0,\ c_2 \neq 0$, (3.14) takes the following form by replacing y by γy and z by δz such that $\gamma\delta = 1,\ c_2\delta = 24$

$$y''' = -6y'^2 + 6z^{-2}(y' + y^2 + 4y) + \tilde{c_3} z^2 + \tilde{c_4} z^{-1} + 24z^{-2} \tag{3.15}$$

where $\tilde{c_3} = c_3\delta^6$ and $\tilde{c_4} = c_4\delta^3$. The equation (3.15) is given in [7]. If $c_1 \neq 0,\ c_2 = 0$, then equation (3.14) takes the form of

$$y''' = -6y'^2 + \frac{6}{z^2}(y' + y^2) + 18z^3 y + z^8 + \tilde{c_3} z^2 + \tilde{c_4}\frac{1}{z} \tag{3.16}$$

where $\tilde{c_3},\ \tilde{c_4}$ are constants and equation (3.16) was also given in [7].

I.a.iii: If one replaces A_5 with $6\hat{A}_5$, A_6 with $6\hat{A}_6$ and A_7 with $6\hat{A}_7$, then the equations (3.9) yields

$$\hat{A}_5'' - 6\hat{A}_5^2 = 0, \qquad \hat{A}_6'' - 6\hat{A}_5\hat{A}_6 = 0, \qquad \hat{A}_7'' - 2\hat{A}_5\hat{A}_7 = \hat{A}_6^2. \tag{3.17}$$

Integrating (3.17.a) once gives

$$\hat{A}_5'^2 = 4\hat{A}_5^3 - \alpha_1, \tag{3.18}$$

where α_1 is an integration constant. Then

$$\hat{A}_5 = \mathcal{P}(z, 0, \alpha_1) \tag{3.19}$$

where $\mathcal{P}$ is Weierstrass elliptic function. If $\hat{A}_6 = 0$, (3.17.c) implies that $\hat{A}_7$ satisfies the Lamé's equation. Hence,

$$\hat{A}_7 = c_1 e^{-z\zeta(a)} \frac{\sigma(z+a)}{\sigma(z)} + c_2 e^{z\zeta(a)} \frac{\sigma(z-a)}{\sigma(z)} \tag{3.20}$$

where ζ is ζ-Weierstrass function such that $\zeta' = -\mathcal{P}(z)$, σ is σ-Weierstrass function such that $\frac{\sigma'(z)}{\sigma(z)} = \zeta(z)$ and a is a parameter such that $\mathcal{P}(a, 0, \alpha_1) = 0$. Then the equation

$$y''' = -6y'^2 + 6\mathcal{P}(z, 0, \alpha_1)(y' + y^2) + \tilde{c_1} e^{-z\zeta(a)} \frac{\sigma(z+a)}{\sigma(z)} + \tilde{c}_2 e^{z\zeta(a)} \frac{\sigma(z-a)}{\sigma(z)} \tag{3.21}$$

where $\tilde{c_1} = 6c_1$ and $\tilde{c_2} = 6c_2$. Equation (3.21) was also considered in [5].

I.b: The coefficients $A_k(z)$, $k = 1, ..., 7$ of the non-dominant terms can be found by using the linear transformation (2.6) and the compatibility conditions. The linear transformation (2.6) allows one to set $a_2 = -2$, $A_1(z) = 0$, $A_2(z) - A_3(z) = 0$ and the compatibility conditions give that $A_2(z) = A_6(z) = 0$ and $A_4(z) = A_5(z)$. So, the canonical form of the equation is

$$y''' = -2(yy'' + y'^2) + A_4(y' + y^2) + A_7, \tag{3.22}$$

where A_4 and A_7 are arbitrary analytic functions of z. If one lets $u = y' + y^2$, then (3.22) can be reduced to a linear equation for u. Equation (3.22) was also given in [7].

I.c: Without loss of generality one can choose $a_1 = 2$, then the simplified equation is

$$y''' = 2yy'' - 3y'^2, \tag{3.23}$$

which was also considered in [5, 7]. Since all the resonances are negative distinct integers then there are no compatibility conditions and hence no non-dominant terms can be introduced.

Case II. $a_4 = 0$: In this case y_0 satisfies the following quadratic equation

$$a_3 y_0^2 - (2a_1 + a_2)y_0 - 6 = 0, \tag{3.24}$$

Therefore, there are two branches corresponding to $(-1, y_{0j})$, $j = 1, 2$. The resonances satisfy the equation (3.5.a). Now, one should determine y_{0j} and a_i, $i = 1, 2, 3$ such that one of the branches is the principal branch. If y_{0j} are the roots of (3.24), by setting

$$P(y_{0j}) = -[a_3 y_{0j}^2 - 2(2a_1 + a_2)y_{0j} - 18], \quad j = 1, 2 \tag{3.25}$$

and if (r_{j1}, r_{j2}) are the resonances corresponding to y_{0j}, then one has

$$r_{j1} r_{j2} = P(y_{0j}) = p_j, \quad j = 1, 2 \tag{3.26}$$

where p_j are integers and such that at least one is positive. Equation (3.24) gives that

$$a_3 = -\frac{6}{y_{01} y_{02}}, \qquad 2a_1 + a_2 = a_3(y_{01} + y_{02}). \tag{3.27}$$

Then (3.25) can be written as

$$P(y_{01}) = 6(1 - \frac{y_{01}}{y_{02}}), \qquad P(y_{02}) = 6(1 - \frac{y_{02}}{y_{01}}). \tag{3.28}$$

For $p_1 p_2 \neq 0$, p_j satisfy the following Diophantine equation

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{6}. \tag{3.29}$$

Now, one should determine all finite integer solutions of Diophantine equation. One solution of (3.29) is $(p_1, p_2) = (12, 12)$. The following cases should be considered: **i)** If $p_1 > 0$, $p_2 > 0$ and $p_1 < p_2$, then $p_1 > 6$ and $p_2 > 12$. **ii)** If $p_1 > 0$, $p_2 < 0$, then $p_1 < 6$. Based on these observations there are following nine integer solutions of Diophantine equation.

$$\begin{aligned}(p_1, p_2) = &(12, 12), (7, 42), (8, 24), (9, 18), (10, 15),\\ &(2, -3), (3, -6), (4, -12), (5, -30).\end{aligned} \tag{3.30}$$

For each (p_1, p_2), one should write (r_{j1}, r_{j2}) such that r_{ji} are distinct integers and $r_{j1} r_{j2} = p_j$, $j = 1, 2$. Then y_{0j} and a_i can be obtained from (3.27), (3.28) and

$$r_{j1} + r_{j2} = a_1 y_{0j} + 7, \quad j = 1, 2 \tag{3.31}$$

in terms one of the a_i. There are following five cases such that all the resonances are distinct integers for both branches. The resonances and the simplified equations for these cases are

$$\text{II.a:} \quad y_{01}^2 = \frac{6}{a_3} : \ (r_{11}, r_{12}) = (3, 4), \qquad y_{02} = -y_{01} : \ (r_{21}, r_{22}) = (3, 4),$$
$$y''' = a_3 y^2 y' \tag{3.32}$$

II.b: $y_{01} = -\frac{1}{a_1}: \ (r_{11}, r_{12}) = (2,4), \qquad y_{02} = \frac{3}{a_1}: \ (r_{21}, r_{22}) = (4,6),$

$$y''' = a_1(yy'' + 2y'^2 + 2a_1y^2y') \tag{3.33}$$

II.c: $y_{01} = -\frac{3}{a_1}: \ (r_{11}, r_{12}) = (1,3), \qquad y_{02} = -\frac{6}{a_1}: \ (r_{21}, r_{22}) = (-2,3),$

$$y''' = a_1(yy'' + y'^2 - \frac{1}{3}a_1y^2y') \tag{3.34}$$

II.d: $y_{01} = -\frac{2}{a_1}: \ (r_{11}, r_{12}) = (1,4), \qquad y_{02} = -\frac{6}{a_1}: \ (r_{21}, r_{22}) = (-3,4),$

$$y''' = a_1(yy'' + 2y'^2 - \frac{1}{2}a_1y^2y') \tag{3.35}$$

II.e: $y_{01} = -\frac{1}{a_1}: \ (r_{11}, r_{12}) = (1,5), \qquad y_{02} = -\frac{6}{a_1}: \ (r_{21}, r_{22}) = (-5,6),$

$$y''' = a_1(yy'' + 5y'^2 - a_1y^2y'). \tag{3.36}$$

For each case the compatibility conditions for the simplified equations are identically satisfied. To find the canonical form of the third order equations of Painlevé type, one should add non-dominant terms with the coefficients which are analytic functions of z.

II.a: Using the linear transformation (2.6), one can set $2A_1 + A_3 = 0, \ A_2 = 0$ and $a_3 = 6$. The compatibility conditions at $j = 3, 4$ for the both branches allows one to determine the coefficients A_k. The canonical form of the equation for this case is

$$y''' = 6y^2y' - (\frac{1}{2}c_1^2z^2 - c_2z - c_3)y' + c_1y^2 - (c_1^2z - c_2)y - \frac{1}{4}c_1^3z^2 + \frac{1}{2}c_1c_2z + \frac{1}{2}c_1c_3. \tag{3.37}$$

where c_1, c_2, c_3 are constants. If one replaces $z - \frac{c_2}{c_1^2}$ by z, then y by γy and z by δz such that $\gamma\delta = 1, \ c_1\delta^2 = -2$, then (3.37) yields

$$u''' = 6u^2u' + 12zuu' + 4(z^2 + k)u' + 4zu + 4u^2, \tag{3.38}$$

where $u = y - z$ and k is a constant. Equation (3.38) was also considered in [5, 7], and its first integral is the forth Painlevé equation. If $c_1 = c_2 = 0$, (3.37) can be solved in terms of the elliptic functions. If $c_1 = 0, \ c_2 \neq 0$, (3.37) gives

$$y''' = 6y^2y' + c_2(z + \frac{c_3}{c_2})y' + c_2y. \tag{3.39}$$

If one introduces $t = z + \frac{c_3}{c_2}$ then the first integral of (3.39) is the second Painlevé equation.

II.b: On using the linear transformation (2.6) one can always choose $2A_1 + A_3 = 0, \ A_2 = 0$, and $a_1 = -1$. Then the compatibility conditions for the both branches, that is the arbitrariness of y_{21} and y_{41} for the first branch and y_{42} and

y_{62} for the second branch imply that all the coefficients A_k of non-dominant terms, are zero. So the canonical form for this case is

$$y''' = -yy'' - 2y'^2 + 2y^2y'. \tag{3.40}$$

Equation (3.40) was also given in [5, 7].

II.c: By using the linear transformation (2.6), one can always set $A_3 = A_5 = 0$, and $a_1 = -3$. Then, the compatibility conditions at $j = 1, 3$ give that $A_1 = c_1/2$, $A_2 = c_1, c_1 =$ constant, and $A_4' = A_6$. Then the canonical form of the equation is

$$y''' = -3yy'' - 3y'^2 - 3y^2y' + \frac{1}{2}c_1y'' + c_1yy' + A_4y' + A_4'y + A_7. \tag{3.41}$$

The first integral of (3.41) gives that

$$u'' = -3uu' - u^3 + B_1u + B_2 \tag{3.42}$$

where $u = y-(c_1/6)$, and B_1, B_2 are arbitrary analytic functions of z. Equation (3.41) was also considered in [5].

II.d: One can always choose $A_3 = A_5 = 0$, and $a_1 = -2$ by using linear transformation (2.6). The arbitrariness of y_{11} and y_{41} for the first branch and y_{42} for the second branch imply that $A_1 = A_2 = A_7 = 0$,and $A_4' = 2A_6$. Then the canonical form is

$$y''' = -2yy'' - 4y'^2 - 2y^2y' + A_4y' + \frac{1}{2}A_4'y. \tag{3.43}$$

The first integral of (3.43) is

$$y'' = \frac{y'^2}{2y} - 2yy' - \frac{y^3}{2} + A_4y + \frac{c}{y}, \qquad c = \text{constant} \tag{3.44}$$

The equation (3.43) was also considered in [5, 7].

II.e: By the linear transformation (2.6), one can choose $A_1 = A_3 = 0$, and $a_1 = -1$. Then the compatibility conditions give that $A_2 = A_5 = 0$, $A_6 = A'_4/3$, $A_7 = -A_4''/3$. After replacing y by $-y$ and A_4 by $3A_4$ the canonical form of the equation for this case is

$$y''' = yy'' + 5y'^2 - y^2y' + 3A_4y' + A_4'y + A_4''. \tag{3.45}$$

Equation (3.45) has the first integral

$$\begin{aligned}(y'' - yy' - y^3 + A_4y + A'_4)^2 &= \frac{8}{3}(y' - y^2)(y' + \frac{y^2}{2} + \frac{3}{2}A_4)\\ &+4(y' - y^2)(2A_4y^3 + A'_4y + A''_4) + 4A^2_4y^2 + 4A_4A'_4y + 4A_4'^2 + c,\end{aligned} \tag{3.46}$$

where A_4 is an arbitrary function of z and c is an arbitrary integration constant. Equation (3.45) was also considered in [5, 7].

Case III: $a_4 \neq 0$: In this case there are three branches corresponding to $(-1, y_{0j})$, $j = 1, 2, 3$ where y_{0j} are the roots of (3.5.b). (3.5.b) implies that

$$\sum_{j=1}^{3} y_{0j} = \frac{a_3}{a_4}, \qquad \sum_{i \neq j} y_{0i} y_{0j} = \frac{1}{a_4}(2a_1 + a_2), \qquad \prod_{j=1}^{3} y_{0j} = -\frac{6}{a_4}. \tag{3.47}$$

If the resonances (except $r_0 = -1$, which is common for all branches) are r_{ji}, $i = 1, 2$ corresponding to y_{0j}, and if one sets

$$P(y_{0j}) = -[a_3 y_{0j}^2 - 2(2a_1 + a_2) y_{0j} - 18], \qquad j = 1, 2, 3 \tag{3.48}$$

then (3.5.a) implies that

$$\prod_{i=1}^{2} r_{ji} = P(y_{0j}) = p_j, \tag{3.49}$$

where p_j are integers and in order to have a principal branch, at least one of them is positive. The equations (3.47) and (3.48) give

$$\begin{aligned} p_1 &= 6\left(1 - \frac{y_{01}}{y_{02}}\right)\left(1 - \frac{y_{01}}{y_{03}}\right), \\ p_2 &= 6\left(1 - \frac{y_{02}}{y_{01}}\right)\left(1 - \frac{y_{02}}{y_{03}}\right), \\ p_3 &= 6\left(1 - \frac{y_{03}}{y_{01}}\right)\left(1 - \frac{y_{03}}{y_{02}}\right) \end{aligned} \tag{3.50}$$

and hence, p_j satisfy the following Diophantine equation

$$\sum_{j=1}^{3} \frac{1}{p_j} = \frac{1}{6}. \tag{3.51}$$

Moreover the equation (3.50) gives that

$$\prod_{j=1}^{3} p_j = -\frac{6^3}{(y_{01} y_{02} y_{03})^2} (y_{01} - y_{02})^2 (y_{01} - y_{03})^2 (y_{02} - y_{03})^2, \tag{3.52}$$

if $a_1 \neq 0$. That is, if $p_1 > 0$ then either p_2 or p_3 is a negative integer. One should consider the case $a_1 = 0$ separately.

III.a: $a_1 = 0$: In this case the sum of the resonances for all three branches are fixed and

$$\sum_{i=1}^{2} r_{ji} = 7, \qquad j = 1, 2, 3. \tag{3.53}$$

Under this condition, the solutions of the Diophantine equation (3.51) are $(p_1, p_2, p_3) = (10, 10, -30), (10, 12, -60)$.

III.a.i: $(p_1, p_2, p_3) = (10, 10, -30)$: The equation (3.50) can be written as

$$p_1(y_{02} - y_{03}) = ky_{01}, \quad p_2(y_{03} - y_{01}) = ky_{02}, \quad p_3(y_{01} - y_{02}) = ky_{03}, \tag{3.54}$$

where

$$k = \frac{6}{y_{01}y_{02}y_{03}}(y_{01} - y_{02})(y_{02} - y_{03})(y_{01} - y_{03}). \tag{3.55}$$

For $k = \pm 10\sqrt{5}$, the system (3.54) has nontrivial solutions y_{0j}, $j = 1, 2, 3$. For these values of y_{0j} the resonances and the coefficients a_i, $i = 2, 3, 4$ are as follows

$$\begin{gathered} y_{01} = \nu(1 - \sqrt{5}): \ (r_{11}, r_{12}) = (2, 5), \ y_{02} = \nu(1 + \sqrt{5}): \ (r_{21}, r_{22}) = (2, 5), \\ y_{03} = 6\nu: \quad (r_{31}, r_{32}) = (-3, 10), \\ a_2 = \tfrac{2}{\nu}, \quad a_3 = \tfrac{2}{\nu^2}, \quad a_4 = \tfrac{1}{4\nu^3}, \quad \nu = \text{constant}. \end{gathered} \tag{3.56}$$

for both values of k. For these values of y_{0j} and a_i the simplified equation passes the Painlevé test for all branches. The linear transformation (2.6) and the compatibility conditions at the resonances of the first and second branches are enough to determine all the coefficients $A_k(z)$ of the non-dominant terms. The canonical form of the equation for this case is,

$$y''' = 12y'^2 + 72y^2y' + 54y^4 + c_1. \tag{3.57}$$

where c_1 is an arbitrary constant. (3.57) can be obtained with the choice of $\nu = 1/(1 - \sqrt{5})$ and replacing y with $6y/(1 - \sqrt{5})$. (3.57) was also given in [5, 14].

III.a.ii: $(p_1, p_2, p_3) = (10, 12, -60)$: For this case the equation (3.54) has non-trivial solution y_{0j} for $k = \pm 20\sqrt{3}$. Then y_{0j}, a_i and the corresponding resonances are as follows:

$$\begin{gathered} y_{01} = -\tfrac{1}{\delta}(-1 \pm \sqrt{3}): \ (r_{11}, r_{12}) = (2, 5), \ y_{02} = \pm\tfrac{\sqrt{3}}{\delta}: \ (r_{21}, r_{22}) = (3, 4), \\ y_{03} = -\tfrac{1}{\delta}(-6 \pm \sqrt{3}): \ (r_{31}, r_{32}) = (-5, 12), \\ a_2 = 3\tfrac{7\pm3\sqrt{3}}{11}\delta, \quad a_3 = \tfrac{40\pm14\sqrt{3}}{11}\delta^2, \quad a_4 = \tfrac{7\pm3\sqrt{3}}{11}\delta^3, \quad \delta = \text{constant}. \end{gathered} \tag{3.58}$$

By using the linear transformation (2.6), one can choose $\delta = \pm\sqrt{3}$ and $A_1 = A_2 = 0$. All the other coefficients A_k of the non-dominant terms can be determined from the compatibility conditions at the resonances of the first and second branches. The canonical form for this case is as follows:

$$\begin{aligned} y''' = {} & \tfrac{27\pm21\sqrt{3}}{11}(y'^2 + y^4) + \tfrac{120\pm42\sqrt{3}}{11}y^2y' + c(\pm\tfrac{1\mp\sqrt{3}}{\sqrt{3}}y' + y^2) \\ & - \tfrac{231\pm143\sqrt{3}}{132}c^2. \end{aligned} \tag{3.59}$$

or

$$y''' = 6y^2y' + \tfrac{3}{11}(9 \pm 7\sqrt{3})(y' + y^2)^2 - \tfrac{1}{22}(4 \mp 3\sqrt{3})c_1 y' + \tfrac{1}{44}(3 \mp 5\sqrt{3})c_1 y^2 - \tfrac{1}{352}(9 \pm 7\sqrt{3})c_1^2. \tag{3.60}$$

where $c_1 = 44c/(3 \mp 5\sqrt{3})$. Equation (3.60) was also considered in [14].

III.b: $a_1 \neq 0$: Since $p_1, p_2 > 0$, $p_3 < 0$, equation (3.52) can be written as

$$p_1 p_2 \hat{p}_3 = 6n^2 \tag{3.61}$$

where n is a constant and $\hat{p}_3 = -p_3$. Then the Diophantine equation (3.51) yields

$$p_1 p_2 = \hat{p}_3(p_1 + p_2) - n^2 \tag{3.62}$$

and since $(p_1 - p_2)^2 \geq 0$ then

$$(p_1 + p_2)^2 - 4\hat{p}_3(p_1 + p_2) + 4n^2 \geq 0 \tag{3.63}$$

Therefore $-n \leq \hat{p}_3 \leq n$ so $0 < \hat{p}_3 \leq n$. Hence, one may assume that n is a positive integer. When $\hat{p}_3 = n$ the equations (3.61) and (3.62) give that $(p_1, p_2, p_3) = (6, n, -n)$ as the solution of the Diophantine equation. For the case of $\hat{p}_3 < n$, if one assumes that $p_1 < p_2$ (if $p_1 = p_2$, (3.62) implies that p_1 and p_2 are complex numbers) then the Diophantine equation (3.51) implies that $p_1 < 12$. The equations (3.51) and (3.62) give that

$$(p_1\hat{p}_3)^2 = n^2[6p_1 - (6 - p_1)\hat{p}_3], \qquad (p_1 p_2)^2 = n^2[6p_1 + (6 - p_1)p_2] \tag{3.64}$$

for $p_1 < 6$ and for $6 < p_1 < 12$ respectively. Equation (3.64) imply that $[6p_1 - (6 - p_1)\hat{p}_3]$ and $[6p_1 + (6 - p_1)p_2]$ must be square of integers. By using these results, p_j, the multiplication of the resonances for the branches corresponding the y_{0j}, $j = 1, 2, 3$, are

$$\begin{aligned}(p_1, p_2, p_3) = {} & (4, 6, -10), (5, 870, -26), (5, 195, -21), (7, 41, -1722) \\ & (7, 38, -399), (7, 33, -154), (8, 22, -264), (8, 16, -48), \\ & (9, 15, -90), (10, 14, -210), (11, 13, -858)\end{aligned} \tag{3.65}$$

For each values of (p_1, p_2, p_3) given in (3.65) one should follow the given steps below for $(4, 6, -10)$. When $(p_1, p_2, p_3) = (4, 6, -10)$: $p_1 = 4$ implies that integer possible values of $r_{1i}, i = 1, 2$ are $(r_{11}, r_{12}) = (1, 4), (-1, -4)$. Then

$$r_{j1} + r_{j2} = a_1 y_{0j} + 7, \qquad j = 1, 2, 3 \tag{3.66}$$

implies that $y_{01} = -2/a_1$ and $y_{01} = -12/a_1$ for $(r_{11}, r_{12}) = (1, 4), (-1, -4)$ respectively. On the other hand y_{0j} satisfies the equation (3.54) for $k = \pm 20$. For $k = 20$, $y_{02} = -9y_{01}/14$. But the resonance equation for the second branch

$$r_{2i}^2 - (7 + a_1 y_{02})r_{2i} + p_2 = 0, \qquad i = 1, 2 \tag{3.67}$$

implies that $7 + a_1 y_{02}$ has to be an integer. So, in order to have integer resonances (r_{21}, r_{22}) for the second branch, $a_1 y_{02}$ has to be integer. Similar argument holds for the third branch. But for $k = 20$ both y_{01} and y_{02} are not integers. Also, for $k = -20$ the resonances for the second and third branches are not integers. Following the same steps one can not find the integer resonances for the second and third branches for all the other cases of (p_1, p_2, p_3) given in (3.65). When $(p_1, p_2, p_3) = (6, n, -n)$, the equation (3.54) has non-trivial solution y_{0j} for $k = \pm n$. $y_{01} = 0$, $y_{02} = y_{03}$ for $k = n$ and $y_{01} = 12\nu$, $y_{02} = \nu(6 - n)$, $y_{03} = \nu(6 + n)$ for $k = -n$ where ν is an arbitrary constant. Since $y_{01} = 0$ for $k = n$, this case will not be considered. For $k = -n$, $2a_1 + a_2$, a_3 and a_4 can be determined from the equation (3.47) as follows:

$$2a_1 + a_2 = -\frac{180 - n^2}{2\nu(36 - n^2)}, \quad a_3 = -\frac{12}{\nu^2(36 - n^2)}, \quad a_4 = -\frac{1}{2\nu^3(36 - n^2)}. \tag{3.68}$$

Since, $p_1 = 6$, then all possible distinct integer resonances for the first branch are $(r_{11}, r_{12}) = (-1, -6), (-2, -3), (1, 6), (2, 3)$. The case $(-1, -6)$, because of the double resonance at $r_0 = r_{11} = -1$, will not be considered. For $(r_{11}, r_{12}) = (1, 6)$, (3.66) implies that $a_1 = 0$. This case was considered in case III.a. For the other possible resonances, one can obtain the a_i, $i = 1, 2, 3, 4$, and y_{0j}, $j = 1, 2, 3$. Once the coefficients of the resonance equation (3.5) are known one should look at the distinct integer resonances for the second and third branches. We have only two cases, such that all the resonances are distinct integers for all branches. The resonances and the corresponding simplified equations are as follows:

$$\begin{aligned}
\text{III.b.i: } y_{01} &= -\tfrac{12}{a_1}: \ (r_{11}, r_{12}) = (-2, -3), \\
y_{02} &= -\tfrac{1}{a_1}(6 - n): \ (r_{21}, r_{22}) = (1, n), \\
y_{03} &= -\tfrac{1}{a_1}(6 + n): \ (r_{31}, r_{32}) = (1, -n), \\
y''' &= a_1[yy'' + \tfrac{3(12+n^2)}{2(36-n^2)}y'^2 - \tfrac{12}{36-n^2}a_1 y^2 y' + \tfrac{1}{2(36-n^2)}a_1^2 y^4], \qquad n \neq 1, 6.
\end{aligned} \tag{3.69}$$

It should be noted that as $n \to \infty$ the simplified equation reduces to (3.23).

$$\begin{aligned}
\text{III.b.i: } y_{01} &= -\tfrac{2}{a_1}: \ (r_{11}, r_{12}) = (2, 3), \\
y_{02} &= -\tfrac{1}{a_1}(1 - \tfrac{n}{6}): \ (r_{21}, r_{22}) = (6, n/6), \\
y_{03} &= -\tfrac{1}{a_1}(1 + \tfrac{n}{6}): \ (r_{31}, r_{32}) = (6, -n/6), \\
y''' &= a_1[yy'' + \tfrac{468-n^2}{36-n^2}y'^2 - \tfrac{432}{36-n^2}a_1 y^2 y' + \tfrac{108}{36-n^2}a_1^2 y^4], \qquad n \neq 6, 36.
\end{aligned} \tag{3.70}$$

The canonical form of the equations corresponding to the above cases can be obtained by adding the non-dominant terms with the analytic coefficients A_k, $k = 1, ...7$.

III.b.i: By using the transformation (2.6), one can set $A_3 = A_4 = 0$ and $a_1 = 2$. The compatibility conditions at the resonances imply that all the coefficients are zero except A_6 and A_7 which remain arbitrary for $n = 2$. For $n = 3$, A_7 is arbitrary and all the other coefficients are zero. For $n = 4, 5, 6$, all the coefficients A_k are zero. However it was proved in [22], for $n \geq 4$ the equation does not admit the non-dominant terms. The canonical form of the equations for $n = 2$ and $n = 3$ are

$$y''' = 2yy'' + \frac{3}{2}y'^2 - \frac{3}{2}y^2y' + \frac{1}{8}y^4 + A_6y + A_7 \tag{3.71}$$

$$y''' = 2yy'' + \frac{7}{3}y'^2 - \frac{16}{9}y^2y' + \frac{4}{27}y^4 + A_7 \tag{3.72}$$

respectively. (3.71) and (3.72) were also given in [5], [14], and both can be linearized by letting $y = -2u'/u$ and $y = -3u'/2u$ respectively.

III.b.ii: The linear transformation (2.6) and the compatibility conditions at the resonances of the first and second branches give the following canonical form of the equation

$$\begin{aligned} y''' = &-2yy'' + \tfrac{26-2m^2}{m^2-1}y'^2 + \tfrac{24}{m^2-1}(2y' + y^2)y^2 + A_5(y' + y^2) \\ &- \tfrac{m^2-1}{48}(A_5'' - \tfrac{1}{2}A_5^2) + c_1z + c_2, \end{aligned} \tag{3.73}$$

where $m = 6/n,$, $m \neq 1, 6$ and c_1, c_2 are arbitrary constants and A_5 is an arbitrary function of z. (3.73) was also given in [5], [14] and equivalent

$$y' + y^2 = \frac{m^2-1}{48}A_5 - \frac{m^2-1}{4}u, \qquad u'' = 6u^2 + \frac{1}{4(m^2-1)}(c_1z + c_2). \tag{3.74}$$

3.2 Fourth order equations: $P_{II}^{(4)}$

Differentiating (3.3) with respect to z gives the terms $y^{(4)}, yy''', y'y'', y^2y'', yy'^2$ and y^3y', all of which are of order -5 for $\alpha = -1$, as $z \to z_0$. Adding the term y^5 which is also of order -5, gives the following simplified equation

$$y^{(4)} = a_1yy''' + a_2y'y'' + a_3y^2y'' + a_4yy'^2 + a_5y^3y' + a_6y^5, \tag{3.75}$$

where a_i, $i = 1, ..., 6$ are constants. Substituting (3.4) into (3.75) gives the following equations for resonance r and for y_0 respectively,

$$\begin{aligned} Q(r) = (r+1)\{&r^3 - (11 + a_1y_0)r^2 - [a_3y_0^2 - (7a_1 + a_2)y_0 - 46]r \\ &- a_5y_0^3 + 2(2a_3 + a_4)y_0^2 - 6(3a_1 + a_2)y_0 - 96\} = 0, \\ a_6y_0^4 - a_5y_0^3 &+ (2a_3 + a_4)y_0^2 - 2(3a_1 + a_2)y_0 - 24 = 0. \end{aligned} \tag{3.76}$$

Equation (3.76.b) implies that in general there are four branches of Painlevé expansion, if $a_6 \neq 0$, corresponding to the roots y_{0j}, $j = 1, 2, 3, 4$. Now, one should determine y_{0j} and a_i such that at least one of the branches is the principal branch. Depending on the number of branches there are four cases. Each case should be considered separately.

Case I: $a_5 = a_6 = 0, \;\; 2a_3 + a_4 = 0$: In this case there is only one branch which should be the principal branch. There are following two sub cases which will be considered separately.

I.a: $a_1 = 0$: In this case the equation (3.76.a) gives that the resonances (r_1, r_2, r_3) (additional to $r_0 = -1$) satisfy that $\sum_{i=1}^{3} r_i = 11, \;\; \prod_{i=1}^{3} r_i = 24$. Under these conditions only possible distinct positive integer resonances are $(r_1, r_2, r_3) = (1, 4, 6)$. Then (3.76) implies that $a_3 = 0$ and $y_0 = -12/a_2$. Therefore, the simplified equation is

$$y^{(4)} = a_2 y' y''. \tag{3.77}$$

To obtain the canonical form of the equation, one should add the non-dominant terms, y''', yy'', y'', y'^2, y^2y', yy', y', y^4, y^3, y^2, y, 1, that is the terms of order greater than -5 as $z \to z_0$ with the coefficients $A_k(z)$, $k = 1, ..., 12$ which are analytic functions of z. The coefficients A_k can be determined by using the linear transformation (2.6) and the compatibility conditions at the resonances. One can choose $a_2 = -12$, $A_2 = 0$ and $2A_3 - A_6 + A_9 = 0$ by using the linear transformation (2.6). The compatibility conditions, that is, the arbitrariness of $y_1, \; y_4, y_6$ give that

$$\begin{aligned}
&A_3'' - A_3^2 = 0 \\
&A_1' + A_1^2 = A_3/3, \quad A_4 = 6A_1, \quad A_5 = A_8 = A_9 = 0 \\
&A_6 = 2A_3, \quad A_7'' - A_3 A_7 = 2A_1 A_3 A_1' + 2A_1^2 A_3' \\
&A_{10} = A_3' - A_1 A_3, \quad A_{11} = (A_7 - A_{10})' - A_1(A_7 - A_{10}), \\
&A_{12}' + A_1 A_{12} = \tfrac{1}{6}(A_7 - A_{10})^2.
\end{aligned} \tag{3.78}$$

According to the solution of (3.78.a), there are following three cases:

I.a.i: $A_3 = 0$, $A_1 = 0$: Then the canonical form of the equation is

$$y^{(4)} = -12y'y'' + (c_1 z + c_2)y' + c_1 y + \frac{1}{18c_1}(c_1 z + c_2)^3 + c_3 \tag{3.79}$$

where c_i, $i = 1, 2, 3$ are arbitrary constants. Integrating (3.79) once gives

$$y''' = -6y'^2 + (c_1 z + c_2)y + \frac{1}{72c_1^2}(c_1 z + c_2)^4 + c_3 z + c_4 \tag{3.80}$$

where c_4 is an integration constant. If $c_1 \neq 0$, $c_2 = 0$, then the equation (3.80) takes the form of (3.13). For $c_1 = 0$, $c_2 \neq 0$, (3.80) yields (3.12).

I.a.ii: $A_3 = 0$. $A_1 = 1/(z-c)$: Without loss of generality one can choose the integration constant c as zero. Then the canonical form of the equation is

$$\begin{aligned} y^{(4)} = & -12y'y'' + \tfrac{1}{z}y''' + \tfrac{6}{z}y'^2 + (c_1 z - c_2)y' + \tfrac{c_2}{z}y + \tfrac{1}{24}c_1^2 z^3 \\ & - \tfrac{1}{9}c_1 c_2 z^2 + \tfrac{1}{12}c_2^2 z + \tfrac{c_3}{z}. \end{aligned} \tag{3.81}$$

If $c_1 = c_2 = 0$ then (3.81) is equivalent to

$$u' = \frac{1}{z}(u + c_3), \qquad y''' = -6y'^2 + u \tag{3.82}$$

If $c_1 = 0,\ c_2 \neq 0$, after replacing z by γz, y by βy, such that $\beta\gamma = 1,\ c_2\gamma^3 = 6$ the equation (3.81) takes the form of

$$y^{(4)} = -12y'y'' + \frac{1}{z}y''' + \frac{6}{z}(y'^2 + y) - 6y' + 3z + \frac{\tilde{c}_3}{z}, \tag{3.83}$$

where $\tilde{c}_3 = c_3\gamma^4$. If $c_1 \neq 0,\ c_2 = 0$, then the equation (3.81) takes the form of

$$y^{(4)} = -12y'y'' + \frac{1}{z}(y''' + 6y'^2) + 12zy' + 6z^3 + \frac{\tilde{c}_3}{z}, \tag{3.84}$$

where $\tilde{c}_3$ is an arbitrary constant.

I.a.iii: $A_3 = 6/(z-c)^2$: For simplicity let $c = 0$. Then the canonical form of the equation is

$$y^{(4)} = -12y'y'' + A_1(y''' + 6y'^2) + \frac{6}{z^2}(y'' + 2yy') + A_7 y' + A_{10}y^2 + A_{11}y + A_{12}, \tag{3.85}$$

where

$$\begin{aligned} A_1 &= \tfrac{2c_1 z^3 - c_2}{z(c_1 z^3 + c_2)}, \\ A_7 &= \tfrac{1}{c_1 z^3 + c_2}\left(\tfrac{1}{5}c_1 c_3 z^6 + \tfrac{1}{5}c_2 c_3 z^3 + c_1 c_4 z - 24c_1 + c_2 c_4 z^{-2} - 6c_2 z^{-3}\right) \\ A_{10} &= -\tfrac{12}{z^3} - \tfrac{6}{c_1 z^3 + c_2}(2c_1 - c_2 z^{-3}), \\ A_{11} &= \tfrac{1}{c_1 z^3 + c_2}\left(\tfrac{c_1 c_3^2}{1350}z^{10} + \tfrac{c_2 c_3^2}{900}z^7 + \tfrac{c_1 c_3 c_4}{60}z^5 + \tfrac{c_2 c_3 c_5}{15}z^2 + c_5 z - \tfrac{c_1 c_4^2}{6} - \tfrac{c_2 c_4^2}{24}z^{-3}\right) \\ A_{12} &= \tfrac{-1}{(c_1 z^4 + c_2 z)^2}\Big[c_1^2 c_3 z^{10} - 48c_1^2 z^8 + \tfrac{4c_1 c_2 c_3}{5}z^7 + 5c_1^2 c_4 z^5 - \tfrac{c_2^2 c_3}{5}z^4 + 4c_1 c_2 c_4 z^2 \\ &\quad - 42c_1 c_2 z - c_2^2 c_3 z^{-1} + 6c_2^2 z^{-2} \\ &\quad - (c_1 z^4 + c_2 z)\left(\tfrac{6c_1 c_3}{5}z^6 + \tfrac{3c_2 c_3}{5}z^3 - 48c_1 + c_1 c_4 - 2c_2 c_4 z^{-2} + 6c_2 z^{-3}\right)\Big]. \end{aligned} \tag{3.86}$$

where $c_i, i = 1, \ldots, 5$ are constants. The equations (3.79), (3.81), (3.83), (3.84) and (3.85) were also considered in [7, 11, 15].

I.b: $a_1 \neq 0$: The equation (3.76.a) implies that $r_1 r_2 r_3 = 24$. Under this condition there are four possible cases of (r_1, r_2, r_3) such that $r_i > 0$ and distinct integers. But, there is only the following case out of the four cases such that the compatibility conditions at the resonances for the simplified equations are identically satisfied and $y_0 \neq 0$.

$$(r_1, r_2, r_3) = (2, 3, 4), \qquad y_0 = -2/a_1, \quad a_2 = 3a_1, \quad a_3 = a_4 = 0, \qquad (3.87)$$

By adding the non-dominant terms to the simplified equation, using the linear transformation (2.6) and the compatibility conditions one finds the canonical form of the equation as follows:

$$y^{(4)} = -2yy''' - 6y'y'' + A_1(y''' + 2yy'' + 2y'^2) + A_3(y'' + 2yy') \\ +A_7(y' + y^2) + A_{12}, \qquad (3.88)$$

where A_1, A_3, A_7, A_{12} are arbitrary functions of z. If one lets $u = y^2 + y'$ then the equation (3.88) can be linearized. (3.88) was also considered in [7, 15].

Case II: $a_5 = a_6 = 0$: In this case there are two branches corresponding to $(-1, y_{0j})$, $j = 1, 2$ where y_{0j} are the roots of (3.76.b) and

$$y_{01} + y_{02} = \frac{2(3a_1 + a_2)}{2a_3 + a_4}, \quad y_{01}y_{02} = -\frac{24}{2a_3 + a_4} \qquad (3.89)$$

Let (r_{j1}, r_{j2}, r_{j3}) be the roots (additional to $r_0 = -1$) of the resonance equation (3.76.a) corresponding to y_{0j}. By setting

$$P(y_{0j}) = -2(2a_3 + a_4)y_{0j}^2 + 6(3a_1 + a_2)y_{0j} + 96, \quad j = 1, 2 \qquad (3.90)$$

then (3.76.a) implies that

$$\prod_{i=1}^{3} r_{ji} = P(y_{0j}) = p_j, \quad j = 1, 2 \qquad (3.91)$$

where p_j are integers and at least one of them is positive integer in order to have the principal branch. Let the branch corresponding to y_{01} be the principal branch, that is $p_1 > 0$. The equations (3.89) and (3.90) give

$$P(y_{01}) = 24(1 - \frac{y_{01}}{y_{02}}) = p_1, \quad P(y_{02}) = 24(1 - \frac{y_{02}}{y_{01}}) = p_2 \qquad (3.92)$$

Hence p_j satisfy the following Diophantine equation if $p_1p_2 \neq 0$,

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{24} \qquad (3.93)$$

There are 21 integer solutions (p_1, p_2) of (3.93) such that one of the p_j is positive. Once p_1 is known, for each p_1 one can write possible distinct positive integer (r_{11}, r_{12}, r_{13}) such that $\prod_{i=1}^{3} r_{1i} = p_1$. Then for each set of (r_{11}, r_{12}, r_{13}), a_k, $k = 2, 3, 4$ and y_{0j} can be determined in terms of a_1 by using

$$\sum_{i=1}^{3} r_{ji} = 11 + a_1 y_{0j}, \qquad \sum_{i \neq k} r_{ji}r_{jk} = -a_3 y_{0j}^2 + (7a_1 + a_2)y_{0j} + 46, \qquad (3.94)$$

for $j = 1$, and the equation (3.89). Then for these values of a_k and y_{0j} one should check that whether the resonance equation (3.76.a) has the distinct integer roots r_{2i} corresponding to y_{02}. Only for the following cases a) $(p_1, p_2) = (12, -24)$ and b) $(p_1, p_2) = (20, -120)$ all the resonances are distinct integers for both branches and one of which is the principal branch. The resonances and the simplified equations for these cases are as follows:

$$\begin{aligned} \text{II.a:} \quad & (p_1, p_2) = (12, -24): \\ & y_{01} = -\tfrac{3}{a_1}: \ (r_{11}, r_{12}, r_{13}) = (1, 3, 4), \\ & y_{02} = -\tfrac{6}{a_1}: \ (r_{21}, r_{22}, r_{23}) = (-2, 3, 4) \\ y^{(4)} &= a_1(yy''' + 3y'y'' - \tfrac{1}{3}a_1 y^2 y'' - \tfrac{2}{3}a_1 yy'^2) \end{aligned} \tag{3.95}$$

$$\begin{aligned} \text{II.b:} \quad & (p_1, p_2) = (20, -120): \\ & y_{01} = -\tfrac{1}{a_1}: \ (r_{11}, r_{12}, r_{13}) = (1, 4, 5), \\ & y_{02} = -\tfrac{6}{a_1}: \ (r_{21}, r_{22}, r_{23}) = (-5, 4, 6) \\ y^{(4)} &= a_1(yy''' + 11y'y'' - a_1 y^2 y'' - 2a_1 yy'^2) \end{aligned} \tag{3.96}$$

For the case II.a the compatibility conditions at the resonances of the simplified equation are identically satisfied. For the case II.b the compatibility condition at the resonance $r_{13} = 5$ implies that $y_4 = 0$ which contradicts with the arbitrariness of y_4. Moreover, in the case II.b, if one lets $y = \lambda u$ such that $\lambda a_1 = 1$ and integration of the simplified equation once yields

$$u''' = uu'' + 5u'^2 - u^2 u' + c \tag{3.97}$$

where c is an arbitrary integration constant. Equation (3.97) is not a Painlevé type equation unless $c = 0$ and was also studied in [7, 11]. Hence, we will consider the case II.a. Adding the non-dominant terms to the simplified equation and by using the linear transformation (2.6) and the compatibility conditions of the first branch, the coefficients $A_k(z)$ of the non-dominant terms can be determined. The canonical form of the equation for the case II.a is as follows:

$$\begin{aligned} y^{(4)} &= -3yy''' - 9y'y'' - 3y^2y'' - 6yy'^2 + Ry'' + 2R'y' + R''y \\ &\quad + A_9(y^3 + 3yy' + y'' - Ry) + A_{12}, \end{aligned} \tag{3.98}$$

where $R(z) = A_3(z) - A_9(z)$ and A_3, A_9 are arbitrary analytic functions of z. If one lets

$$u = y'' + 3yy' + y^3 - Ry \tag{3.99}$$

then the equation (3.98) can be reduced to a linear equation for u. (3.98) was also considered in [7, 15].

Case III: $a_6 = 0$: There are three branches corresponding to y_{0j}, $j = 1, 2, 3$ which are the roots of the equation (3.76.b). If one lets

$$\prod_{i=1}^{3} r_{ji} = p_j = P(y_{0j}) = a_5 y_{0j}^3 - 2(2a_3 + a_4)y_{0j}^2 + 6(3a_1 + a_2)y_{0j} + 96, \quad j = 1, 2, 3 \tag{3.100}$$

where p_j are integers and at least one of them is positive. By using the same procedure which was carried in the previous case, p_j satisfy the following Diophantine equation:

$$\sum_{j=1}^{3} \frac{1}{p_j} = \frac{1}{24}. \tag{3.101}$$

if $p_1 p_2 p_3 \neq 0$, and if $a_1 \neq 0$

$$\prod_{j=1}^{3} p_j = -\frac{24^3}{(y_{01}y_{02}y_{03})^2}(y_{01} - y_{02})^2(y_{01} - y_{03})^2(y_{02} - y_{03})^2 \tag{3.102}$$

Hence, let $p_1, p_2 > 0$ and $p_3 < 0$. If (r_{j1}, r_{j2}, r_{j3}) are the resonances corresponding to y_{0j} respectively, then they satisfy the equation (3.94) for $j = 1, 2, 3$. There are following two cases which should be considered separately.

III.a: $a_1 = 0$: Equation (3.94.a) for $j = 1$ implies that there are five possible values of (r_{11}, r_{12}, r_{13}) and hence five possible values of p_1. For each value of p_1 one can solve (3.101) such that $p_2 > 0$, $p_3 < 0$ and integers. Then for each (p_1, p_2, p_3), the equations

$$\begin{aligned} p_1 &= 24(1 - \frac{y_{01}}{y_{02}})(1 - \frac{y_{01}}{y_{03}}), \\ p_2 &= 24(1 - \frac{y_{02}}{y_{01}})(1 - \frac{y_{02}}{y_{03}}), \\ p_3 &= 24(1 - \frac{y_{03}}{y_{01}})(1 - \frac{y_{03}}{y_{02}}), \end{aligned} \tag{3.103}$$

give the equations (3.54) for y_{0j} for

$$k = \frac{24}{y_{01}y_{02}y_{03}}(y_{01} - y_{02})(y_{02} - y_{03})(y_{01} - y_{03}). \tag{3.104}$$

The system (3.54) has non-trivial solution if $k^2 = -(p_1p_2 + p_1p_3 + p_2p_3)$. For each value of k, one can find y_{0j} and a_i, $i = 3, 4, 5$ in terms of a_2. Once the coefficients of the resonance equation (3.76.a) are known for all branches, one should look at the cases such that the roots of (3.76.a) are distinct integers for the second and third branches. There is only one case, $(p_1, p_2, p_3) = (40, 40, -120)$, and $k = 40\sqrt{5}$. y_{0j}, the resonances and the simplified equation for this case are as follows:

$$y_{01} = \tfrac{4}{a_2}(1 - \sqrt{5}) : \ (r_{11}, r_{12}, r_{13}) = (2, 4, 5),$$

$$y_{02} = \tfrac{4}{a_2}(1+\sqrt{5}) : \ (r_{21}, r_{22}, r_{23}) = (2, 4, 5),$$
$$y_{03} = \tfrac{24}{a_2} : \ (r_{31}, r_{32}, r_{33}) = (-3, 4, 10),$$
$$y^{(4)} = a_2(y'y'' + \tfrac{1}{8}a_2y^2y'' + \tfrac{1}{4}a_2yy'^2 + \tfrac{1}{64}a_2^2y^3y') \qquad (3.105)$$

The compatibility conditions are identically satisfied for the simplified equation. To obtain the canonical form of the equation one should add the non-dominant terms with analytic coefficients $A_k(z)$, $k = 1, ..., 12$. The linear transformation (2.6) and the compatibility conditions at the resonances of the first and second branches give the following equation

$$y^{(4)} = 24y'y'' + 72y^2y'' + 144yy'^2 + 216y^3y'^2 \qquad (3.106)$$

Integrating (3.106) once gives (3.57).

III.b: $a_1 \neq 0$: In this case the resonances (r_{j1}, r_{j2}, r_{j3}) and y_{0j} satisfy (3.94) for $j = 1, 2, 3$ and

$$\sum_{i=1}^{3} y_{0j} = \frac{1}{a_5}(2a_3+a_4), \quad \sum_{j\neq k} y_{0j}y_{0k} = -\frac{2}{a_5}(3a_1+a_2), \quad \prod_{i=1}^{3} y_{0j} = -\frac{24}{a_5}, \quad (3.107)$$

respectively. $p_j = \prod_{j=1}^{3} r_{ji}$ satisfy the Diophantine equation (3.101). If one lets

$$n^2 = \frac{24^2}{(y_{01}y_{02}y_{03})^2}(y_{01}-y_{02})^2(y_{01}-y_{03})^2(y_{02}-y_{03})^2 \qquad (3.108)$$

then (3.102) gives

$$p_1p_2\hat{p}_3 = 24n^2 \qquad (3.109)$$

where $\hat{p}_3 = -p_3$. $p_1 < 48$. If one follows the procedure given in the previous section, (3.101) and (3.109) give that

$$(p_1\hat{p}_3)^2 = n^2[24p_1 - (24-p_1)\hat{p}_3], \qquad (p_1p_2)^2 = n^2[24p_1 + (24-p_1)p_2] \quad (3.110)$$

for $p_1 < 24$ and for $24 < p_1 < 48$ respectively. So, the right hand side of both equations in (3.110) must be complete square. Based on these conditions on p_i, $i = 1, 2, 3$, there are 71 integer solutions (p_1, p_2, p_3) of the Diophantine equation (3.101). For each solution (p_1, p_2, p_3), one can find y_{0j} by solving the system of equations (3.54). Then one can write possible resonances (r_{11}, r_{12}, r_{13}) for each p_1 provided that

$$a_1y_{01} = \sum_{i=1}^{3} r_{1i} - 11 \qquad (3.111)$$

are all integers. There are following three cases such that all the resonances of all three branches are distinct integers.

III.b.i: $(p_1, p_2, p_3) = (15, 60, -24)$

$$\begin{aligned} &y_{01} = -\tfrac{2}{a_1}: \quad (r_{11}, r_{12}, r_{13}) = (1, 3, 5), \\ &y_{02} = -\tfrac{12}{a_1}: \quad (r_{21}, r_{22}, r_{23}) = (-2, -5, 6), \\ &y_{03} = -\tfrac{8}{a_1}: \quad (r_{31}, r_{32}, r_{33}) = (-4, 1, 6), \\ &a_2 = \tfrac{11}{2}a_1, \quad a_3 = -\tfrac{1}{2}a_1^2, \quad a_4 = -\tfrac{7}{4}a_1^2, \quad a_5 = \tfrac{1}{8}a_1^3. \end{aligned} \tag{3.112}$$

III.b.ii: $(p_1, p_2, p_3) = (24, n, -n)$

$$\begin{aligned} &y_{01} = -\tfrac{2}{a_1}: \ (r_{11}, r_{12}, r_{13}) = (2, 3, 4), \\ &y_{02} = -\tfrac{1}{a_1}(1 - \tfrac{n}{24}): \ (r_{21}, r_{22}, r_{23}) = (4, 6, \tfrac{n}{24}), \\ &y_{03} = -\tfrac{1}{a_1}(1 + \tfrac{n}{24}): \ (r_{31}, r_{32}, r_{33}) = (4, 6, -\tfrac{n}{24}), \\ &a_2 = \tfrac{15552-3n^2}{576-n^2}a_1, \ a_3 = -\tfrac{6912}{576-n^2}a_1^2, \ a_4 = -\tfrac{13824}{576-n^2}a_1^2, \ a_5 = \tfrac{6912}{576-n^2}a_1^3. \end{aligned} \tag{3.113}$$

III.b.iii: $(p_1, p_2, p_3) = (24, n, -n), \quad n > 0, \quad n \neq 4, 24$

$$\begin{aligned} &y_{01} = -\tfrac{12}{a_1}: \ (r_{11}, r_{12}, r_{13}) = (-2, -3, 4), \\ &y_{02} = -\tfrac{1}{a_1}(6 - \tfrac{n}{4}): \ (r_{21}, r_{22}, r_{23}) = (1, 4, \tfrac{n}{4}), \\ &y_{03} = -\tfrac{1}{a_1}(6 + \tfrac{n}{4}): \ (r_{31}, r_{32}, r_{33}) = (1, 4, -\tfrac{n}{4}), \\ &a_2 = \tfrac{1152+2n^2}{576-n^2}a_1, \ a_3 = -\tfrac{192}{576-n^2}a_1^2, \ a_4 = -\tfrac{384}{576-n^2}a_1^2, \ a_5 = \tfrac{32}{576-n^2}a_1^3. \end{aligned} \tag{3.114}$$

For all three cases, the simplified equations pass the Painlevé test. To obtain the canonical form of the equation one should add the non-dominant terms with the coefficients $A_k(z)$, $k = 1, ..., 12$. The linear transformation (2.6) and the compatibility conditions at the resonances give the following equations:

III.b.i:

$$\begin{aligned} y^{(4)} = &-2yy''' - 11y'y'' - 2y^2y'' - 7yy'^2 - y^3y'^2 + A_6(y'' + yy') \\ &+ \tfrac{1}{3}A_6'(y^2 + 4y') + \tfrac{1}{3}A_6''' - \tfrac{2}{9}A_6A_6', \end{aligned} \tag{3.115}$$

where A_6 is an arbitrary function of z. (3.115) was also given in [15].

III.b.ii: Since the compatibility condition at the resonance $r = 6$ for the third branch gives that

$$A_1' + A_1^2 = 0. \tag{3.116}$$

The following two cases should be considered separately.

III.b.ii.1: $A_1 = 0$: The canonical form of the equation is

$$\begin{aligned} y^{(4)} = &-2yy''' - \tfrac{6}{m^2-1}[(m^2 - 9)y'y'' - 8y^2y'' - 16(yy'^2 + y^3y')] \\ &+ A_3(y'' + 2yy') + (A_3' + c_1)(y' + y^2) + A_{12}, \end{aligned} \tag{3.117}$$

where $m = n/24$, $m \neq 1, 4, 6$, A_3 is an arbitrary function of z and

$$A_{12} = \frac{m^2-1}{48}(A_3''' - A_3A_3' - c_1A_3 + 2c_1^2z + c_2, \qquad c_1, c_2 = \text{constant}. \quad (3.118)$$

(3.117) was also given in [15].

III.b.ii.2: $A_1 = 1/(z-c)$: Without loss of generality, one can set $c = 0$. The canonical form of the equation is

$$\begin{aligned} y^{(4)} = &-2yy''' + \tfrac{1}{m^2-1}[(54-6m^2)y'y'' + 48y^2y'' + 96(yy'^2 + y^3y')] \\ &+ \tfrac{1}{z}[y''' + 2yy'' - \tfrac{1}{m^2-1}[(26-2m^2)y'^2 + 48y^2y' + 24y^4] \\ &+ A_3(y'' + 2yy') + (A_3' - A_3\tfrac{1}{z} + c_1z)(y' + y^2) + A_{12}] \end{aligned} \quad (3.119)$$

A_3 is an arbitrary function of z and

$$\begin{aligned} &A_{12} = -\tfrac{m^2-1}{48}(A_3''' - \tfrac{1}{z}A_3'' - A_3A_3' + \tfrac{1}{2z}A_3^2 - c_1zA_3 + \tfrac{1}{2}c_1^2z^3) + \tfrac{c_2}{z}, \\ &c_1, c_2 = \text{constant}. \end{aligned} \quad (3.120)$$

(3.119) was also given in [15].

III.b.iii: If we let $m = n/4$, $m \neq 1, 4, 6$ then the canonical form of the equation for $m = 2$ is

$$\begin{aligned} &y^{(4)} = 2yy''' + 5y'y'' - \tfrac{3}{2}y^2y'' - 3yy'^2 + \tfrac{1}{2}y^3y' \\ &+A_1[y''' - 2yy'' - \tfrac{3}{2}y'^2 + \tfrac{3}{2}y^2y' - \tfrac{1}{8}y^4 - A_7y] + A_7y' + A_7'y + A_{12}, \end{aligned} \quad (3.121)$$

If one lets

$$u = y''' - 2yy'' - \frac{3}{2}y'^2 + \frac{3}{2}y^2y' - \frac{1}{8}y^4 - A_7y, \quad (3.122)$$

then (3.121) can be reduced to a linear equation for u. It should be noted that (3.122) belongs to $\mathrm{P}_{II}^{(3)}$ and was given in (3.71). For $m = 3$

$$\begin{aligned} y^{(4)} = &2yy''' + \tfrac{20}{3}y'y'' - \tfrac{16}{9}y^2y'' - \tfrac{32}{9}yy'^2 + \tfrac{16}{27}y^3y' + A_1(y''' - 2yy'' - \tfrac{7}{3}y'^2 \\ &+ \tfrac{16}{9}y^2y' - \tfrac{4}{27}y^4) + A_{12}, \end{aligned} \quad (3.123)$$

where A_1 and A_{12} are arbitrary functions of z. If one lets

$$u = y''' - 2yy'' - \frac{7}{3}y'^2 + \frac{16}{9}y^2y' - \frac{4}{27}y^4, \quad (3.124)$$

then (3.123) can be reduced to a linear equation. (3.124) belongs to $\mathrm{P}_{II}^{(3)}$ and was given in (3.72). (3.121) and (3.123) were also given in [15]. It should be noted that for $m \geq 4$, integration of the simplified equation once gives the simplified equation of the case given in (3.70) with an additional integration constant c. Thus, for $m > 4$ the simplified equation is not of Painlevé type if $c \neq 0$.

Case IV: $a_6 \neq 0$: In this case there are four branches corresponding to $(-1, y_{0j})$, $j = 1,2,3,4$. If (r_{j1}, r_{j2}, r_{j3}) are the resonances corresponding to the branches, then $\prod_{i=1}^{3} r_{ji} = p_j$ such that p_j are integers and at least one of them is positive. Then (3.76.a) implies that

$$P(y_{0j}) = a_5 y_{0j}^3 - 2(2a_3 + a_4) y_{0j}^2 + 6(3a_1 + a_2) y_{0j} + 96 = p_j, \quad j = 1,2,3,4. \quad (3.125)$$

On the other hand (3.76.b) implies that

$$\sum_{j=1}^{4} y_{0j} = \frac{a_5}{a_6}, \quad \sum_{j \neq i} y_{0j} y_{0i} = \frac{2a_3 + a_4}{a_6},$$
$$\sum_{j \neq i \neq k} y_{0j} y_{0i} y_{0k} = \frac{2(3a_1 + a_2)}{a_6}, \quad \prod_{j=1}^{4} y_{0j} = -\frac{24}{a_6} \quad (3.126)$$

Then (3.125) yields

$$p_j = P(y_{0j}) = 24 \prod_{j \neq k} \left(1 - \frac{y_{0j}}{y_{0k}}\right), \quad j = 1,2,3,4 \quad (3.127)$$

Therefore p_j satisfy the following Diophantine equation

$$\sum_{j=1}^{4} \frac{1}{p_j} = \frac{1}{24}. \quad (3.128)$$

To find the simplified equation, one should proceed the following steps: **a)** Find all integer solutions (p_1, p_2, p_3, p_4) of the Diophantine equation (3.128). **b)** For each pair (p_1, p_2) from the solution set of the Diophantine equation, write all possible (r_{j1}, r_{j2}, r_{j3}) such that $\prod_{i=1}^{3} r_{ji} = p_j$, $j = 1,2$. **c)** Determine y_{01} and y_{02} in terms of a_1, if $a_1 \neq 0$, by using the equation (3.94.a) for $j = 1,2$. **d)** Use (3.127) to find y_{03} and y_{04} in terms of a_1. **e)** Eliminate the cases (r_{j1}, r_{j2}, r_{j3}) $j = 1,2$ such that $a_1 y_{0k}$, $k = 3,4$ are not integers (see the equation (3.94.a)). **f)** Find a_i, $i = 2, ..., 6$ in terms of a_1 by using the (3.125) and (3.126). Once all the coefficients of the equation (3.76.a) are known, look at the cases such that the roots of (3.76.a) are distinct integers for y_{03} and y_{04}. There are four cases such that all the resonances are distinct integers for all branches. These cases and the corresponding simplified equations are as follows:

IV.a: $(p_1, p_2, p_3, p_4) = (6, -4, 6, -24)$:

$$\begin{aligned} y_{01} &= -\tfrac{5}{a_1} : \ (r_{11}, r_{12}, r_{13}) = (1,2,3), \\ y_{02} &= -\tfrac{10}{a_1} : \ (r_{21}, r_{22}, r_{23}) = (-2,1,2), \\ y_{03} &= -\tfrac{15}{a_1} : \ (r_{31}, r_{32}, r_{33}) = (-3,-2,1), \\ y_{04} &= -\tfrac{20}{a_1} : \ (r_{41}, r_{42}, r_{43}) = (-4,-3,-2), \end{aligned} \quad (3.129)$$
$$y^{(4)} = a_1 (y y''' + 2y'y'' - \tfrac{2}{5} a_1 y^2 y'' - \tfrac{3}{5} a_1 y y'^2 + \tfrac{2}{25} a_1^2 y^3 y' - \tfrac{1}{625} a_1^3 y^5)$$

IV.b: $(p_1, p_2, p_3, p_4) = (36, 36, -84, -504)$:

$$
\begin{aligned}
y_{01} &= -\tfrac{5}{a_2} : \ (r_{11}, r_{12}, r_{13}) = (2, 3, 6), \\
y_{02} &= \tfrac{10}{a_2} : \ (r_{21}, r_{22}, r_{23}) = (2, 3, 6), \\
y_{03} &= \tfrac{15}{a_2} : \ (r_{31}, r_{32}, r_{33}) = (-2, 6, 7), \\
y_{04} &= -\tfrac{20}{a_2} : \ (r_{41}, r_{42}, r_{43}) = (-7, 6, 12), \\
y^{(4)} &= a_2[y'y'' + \tfrac{1}{5}a_2(y^2y'' + yy'^2 - \tfrac{1}{125}a_2^2y^5)]
\end{aligned}
\tag{3.130}
$$

IV.c: $(p_1, p_2, p_3, p_4) = (36, 36, -144, -144)$:

$$
\begin{aligned}
y_{01}^2 &= \tfrac{10}{a_3} : \ (r_{11}, r_{12}, r_{13}) = (2, 3, 6), \\
y_{02} &= -y_{01} : \ (r_{21}, r_{22}, r_{23}) = (2, 3, 6), \\
y_{03}^2 &= \tfrac{40}{a_3} : \ (r_{31}, r_{32}, r_{33}) = (-3, 6, 8), \\
y_{04} &= -y_{03} : \ (r_{41}, r_{42}, r_{43}) = (-3, 6, 8), \\
y^{(4)} &= a_3(y^2y'' + yy'^2 - \tfrac{3}{50}a_3y^5)
\end{aligned}
\tag{3.131}
$$

IV.d: $(p_1, p_2, p_3, p_4) = (20, -120, -60, 60)$:

$$
\begin{aligned}
y_{01} &= \tfrac{2}{a_1} : \ (r_{11}, r_{12}, r_{13}) = (1, 2, 10), \\
y_{02} &= -\tfrac{8}{a_1} : \ (r_{21}, r_{22}, r_{23}) = (-10, 1, 12), \\
y_{03} &= \tfrac{4}{a_1} : \ (r_{31}, r_{32}, r_{33}) = (-2, 2, 15), \\
y_{04} &= -\tfrac{6}{a_1} : \ (r_{41}, r_{42}, r_{43}) = (-3, -2, 10), \\
y^{(4)} &= a_1(yy''' - \tfrac{17}{2}y'y'' + \tfrac{11}{4}a_1y^2y'' - \tfrac{15}{4}a_1yy'^2 + \tfrac{1}{2}a_1^2y^3y' - \tfrac{1}{16}a_1^3y^5)
\end{aligned}
\tag{3.132}
$$

The simplified equation for the case IV.d does not pass the Painlevé test. So this case will not be considered. The canonical forms for the other cases can be obtained by adding the non-dominant terms with the coefficients $A_k(z)$, $k = 1, ..., 12$ to the simplified equations. All the coefficients A_k can be obtained by using the linear transformation (2.6) and the compatibility conditions at the resonances. The canonical forms are as follows:

IV.a:

$$
\begin{aligned}
y^{(4)} = &-5yy''' - 10(y'y'' + y^2y'' + y^3y') - 15yy'^2 - y^5 \\
&+ A_1(y''' + 4yy'' + 3y'^2 + 6y^2y' + y^4) + A_3(y'' + 3yy' + y^3) \\
&+ A_7(y' + y^2) + A_{11}y + A_{12}.
\end{aligned}
\tag{3.133}
$$

If one lets $y = u'/u$ then (3.133) gives the fifth order linear equation for u. (3.133) was also given in [15].

IV.b:

$$
y^{(4)} = -5y'y'' + 5y^2y'' + 5yy'^2 - y^5 + (c_1z + c_2)y + c_3 \tag{3.134}
$$

where c_i are constants and (3.134) was also given in [15].

IV.c:

$$y^{(4)} = 10y^2y'' + 10yy'^2 - 6y^5 + c_1(y'' - 2y^3) + (c_2z + c_3)y + c_4 \tag{3.135}$$

where c_i are constants and (3.135) was also given in [15, 16].

3.3 Fifth order equations: $\mathrm{P}_{II}^{(5)}$

Differentiating (3.75) and adding the term y^6 which is also of order -6 as $z \to z_0$ gives the following simplified equation of order five

$$\begin{aligned} y^{(5)} &= a_1yy^{(4)} + a_2y'y''' + a_3y''^2 + a_4y^2y''' + a_5yy'y'' \\ &+ a_6y'^3 + a_7y^3y'' + a_8y^2y'^2 + a_9y^4y' + a_{10}y^6, \end{aligned} \tag{3.136}$$

where a_i, $i = 1, ..., 10$ are constants. Substituting (3.4) into (3.136) into above equation gives the following equations for resonance r and for y_0 respectively,

$$\begin{aligned} Q(r) &= (r+1)\{r^4 - (16 + a_1y_0)r^3 - [a_4y_0^2 - (11a_1 + a_2)y_0 - 101]r^2 \\ &- [a_7y_0^3 - (7a_4 + a_5)y_0 + (46a_1 + 7a_2 + 4a_3)y_0 + 326]r \\ &- [a_9y_0^4 - 2(2a_7 + a_8)y_0^3 + 3(6a_4 + 2a_5 + a_6)y_0^2 \\ &- 8(12a_1 + 3a_2 + 2a_3)y_0 - 600] = 0, \\ a_{10}y_0^5 &- a_9y_0^4 + (2a_7 + a_8)y_0^3 - (6a_4 + 2a_5 + a_6)y_0^2 \\ &+ 2(12a_1 + 3a_2 + 2a_3)y_0 + 120 = 0. \end{aligned} \tag{3.137}$$

(3.137.b) implies that there are five branches, if $a_6 \neq 0$. If $(r_{j1}, r_{j2}, r_{j3}, r_{j4})$, $j = 1, ..., 5$ are the distinct integer resonances corresponding to $(-1, y_{0j})$, and if $\prod_{i=1}^{4} r_{ji} = p_j$ where p_j are integers and at least one of them is positive, then p_j satisfy the following Diophantine equation,

$$\sum_{j=1}^{5} \frac{1}{p_j} = \frac{1}{120}. \tag{3.138}$$

Finding the solution of the Diophantine equation is quite difficult and has large number of solutions, including infinite families. So, for the sake of completeness, in this section we will present the cases when we have one, two, three and four branches. Since the procedure to obtain the canonical form of the differential equations is the same as described in the previous sections, we will only give the canonical form of the differential equations for each cases.

The canonical form of the equation can be obtained by adding the non-dominant terms $y^{(4)}$, yy''', $y'y''$, y^2y'', yy'^3, y^3y', y^5, y''', yy'', y'^2, y^2y', y^4, y'',

yy', y^3, y', y^2, y, 1 with the coefficients $A_k(z)$, $k = 1, ..., 19$ which are analytic functions of z, respectively.

Case I: If $a_l = 0$, $l = 4, ..., 10$, then there is only one branch, and there are two cases such that the resonances are distinct positive integers.

I.a:

$$\begin{aligned} &y_{01} = -2/a_1 : \qquad (r_1, r_2, r_3, r_4) = (2, 3, 4, 5) \\ &y^{(5)} = -2yy^{(4)} - 8y'y''' - 6y''^2 + A_1(y^4 + 2yy'' + 6y'y'') \\ &+A_8(y'' + 2yy'' + 2y'^2) + A_{13}(y'' + 2yy') + A_{16}(y' + y^2) + A_{19}, \end{aligned} \tag{3.139}$$

where $A_1, A_8, A_{13}, A_{16}, A_{19}$ are arbitrary analytic functions of z. (3.139) can be linearized by letting $u = y' + y^2$.

I.b: $y_{01} = -12/a_2$: $(r_1, r_2, r_3, r_4) = (1, 4, 5, 6)$.
In this case the linear transformation and the compatibility conditions give $A_i = 0$, $i = 1, ..., 7$, $A_{11} = A_{12} = A_{15} = 0$ and

$$A_9'' - \frac{1}{2}A_9^2 = 0 \tag{3.140}$$

Depending on the solution of (3.140) there are following two sub cases.

I.b.i: $A_9 = 0$. The canonical form of the equation is

$$y^{(5)} = -12y'y''' - 12y''^2 + (c_1 z + c_2)y'' + 2c_1 y' + \frac{1}{6}(c_1 z + c_2)^2, \tag{3.141}$$

where c_1, c_2 are constants. If $c_1 \neq 0$, (3.141) can be reduced to (3.13). If $c_1 = 0$ (3.141) can be reduced to a third order equation which belongs to the hierarchy of the first Painlevé equation, $P_I^{(3)}$ [19], by integrating once and letting $y = u'$.
I.b.ii: $A_9 = 12/z^2$. The canonical form of the equation is

$$\begin{aligned} y^{(5)} &= -12y'y''' - 12y''^2 + \tfrac{12}{z^2}(\tfrac{3}{2}y''' + yy'' + 2y'^2) + (c_1 z^3 + \tfrac{c_2}{z^2} - \tfrac{24}{z^3})y''' \\ &-\tfrac{48}{z^3}yy' + (6c_1 z^2 - \tfrac{4c_2}{z^3} + \tfrac{24}{z^4})y' + (4c_1 + \tfrac{4c_2}{z^4})y + \tfrac{24}{z^4}y^2 + \tfrac{1}{6}(c_1 z^3 + \tfrac{c_2}{z^2})^2 \end{aligned} \tag{3.142}$$

where c_1, c_2 are constants.

Case II: $a_7 = ... = a_{10} = 0$: In this case there are two branches. The resonances and the canonical form of the equation is

$$\begin{aligned} &y_{01} = -\tfrac{3}{a_1} : (r_{11}, r_{12}, r_{13}, r_{14}) = (1, 3, 4, 5), \\ &\quad y_{02} = -\tfrac{6}{a_1} : (r_{21}, r_{22}, r_{23}, r_{24}) = (-2, 3, 4, 5) \\ &y^{(5)} = -3yy^{(4)} - 12y'y''' - 9y''^2 - 18yy'y'' - 6y'^3 - 3y^2y'' + (Ry)''' + \\ &\tfrac{1}{3}A_{10}[y''' + 3yy'' + 3y^2y' + 3y'^2 - (Ry)'] + A_{15}(y'' + 3yy' + y^3 - Ry) + A_{19} \end{aligned} \tag{3.143}$$

where $R = A_8 - A_9/3$ and A_8, A_9, A_{10}, A_{15} and A_{19} are arbitrary analytic functions of z. (3.143) can be linearized, if one lets

$$u = y'' + 3yy' + y^3 - Ry \tag{3.144}$$

It should be noted that (3.144) is of Painlevé type.

Case III: $a_9 = a_{10} = 0$: In this case there are three branches. The resonances and the canonical form of the equations are as follows:

III.a:

$$\begin{aligned}
&y_{01} = \tfrac{-2}{a_1}: \ (r_{11}, r_{12}, r_{13}, r_{14}) = (2, 3, 4, 5),\\
&y_{02} = -\tfrac{1-n}{a_1}: \ (r_{21}, r_{22}, r_{23}, r_{24}) = (4, 5, 6, n),\\
&y_{03} = -\tfrac{1+n}{a_1}: \ (r_{31}, r_{32}, r_{33}, r_{34}) = (4, 5, 6, -n),\\
&y^{(5)} = -2yy^{(4)} + \tfrac{1}{n^2-1}[(56 - 8n^2)y'y''' + (54 - 6n^2)y''^2 + 48y^2y''\\
&+288yy'y'' + 96(y'^3 + y^3y'') + 288y^2y'^2] + A_8(y''' + 2yy'' + y'^2)\\
&+(2A_8' + c_1z + c_2)(y'' + 2yy') + (A_8'' + 2c_1)(y' + y^2) + A_{19},
\end{aligned} \tag{3.145}$$

where

$$A_{19} = -\frac{n^2-1}{48}[A_8''' - A_8A_8'' - A_8'^2 - A_8'(c_1z + c_2) - 2c_1A_8 + 2(c_1z + c_2)^2] \tag{3.146}$$

and c_1, c_2 are constants , n is a positive integer $\neq 1, 4, 5, 6$. If $c_1 = c_2 = 0$, twice integration of (3.145) yields (3.73).

III.b: The resonances are

$$\begin{aligned}
y_{01} &= -\tfrac{6-n}{a_1}: (r_{11}, r_{12}, r_{13}, r_{14}) = (1, 4, 5, n),\\
y_{02} &= -\tfrac{6+n}{a_1}: (r_{21}, r_{22}, r_{23}, r_{24}) = (1, 4, 5, -n),\\
y_{03} &= -\tfrac{12}{a_1}: \ (r_{31}, r_{32}, r_{33}, r_{34}) = (-3, -2, 4, 5).
\end{aligned} \tag{3.147}$$

where n is a positive integer $\neq 1, 4, 5$. It should be noted that, when $n \geq 6$ the twice integration of the simplified equation gives the third order equation (3.69) with the additional term $(c_1z + c_2)$. Therefore, the simplified equation is not of Painlevé type if $c_1, c_2 \neq 0$. Hence, we will only consider the cases for $n = 2, 3$. The canonical form of the equation for $n = 2$ is

$$\begin{aligned}
&y^{(5)} = 2yy^{(4)} + 7y'y''' + 5y''^2 - \tfrac{3}{2}y^2y''' - 9yy'y'' - 3y'^3 + \tfrac{1}{2}y^3y'' + \tfrac{3}{2}y^2y'^2\\
&+A_1(y^{(4)} - 2yy''' - 5y'y'' + \tfrac{3}{2}y^2y' + 3yy'^2 - \tfrac{1}{2}y^3y')\\
&+A_8(y''' - 2yy'' - \tfrac{3}{2}y'^2 + \tfrac{3}{2}y^2y' - \tfrac{1}{8}y^4) + A_{13}y'' + (2A_{13}' - A_1A_{13})y'\\
&+(A_{13}'' - A_1A_{13}' - A_8A_{13})y + A_{19},
\end{aligned} \tag{3.148}$$

where A_1, A_8, A_{13}, A_{19} are arbitrary analytic functions of z. Twice integration of (3.148) yields (3.71).

For $n = 3$

$$\begin{aligned} y^{(5)} &= 2yy^{(4)} + \tfrac{1}{3}(26y'y''' + 20y''^2 - \tfrac{16}{3}y^2y''' - 32yy'y'' - \tfrac{32}{3}y'^3 + \tfrac{16}{9}y^3y'' \\ &+ \tfrac{16}{3}y^2y'^2 + A_1[y^{(4)} - 2yy''' - \tfrac{2}{3}(10y'y'' - \tfrac{8}{3}y^2y'' - \tfrac{16}{3}yy'^2 + \tfrac{8}{9}y^3y')] \\ &+ A_8(y''' - 2yy'' - \tfrac{7}{3}y'^2 + \tfrac{16}{9}y^2y' - \tfrac{4}{27}y^4) + A_{19}, \end{aligned} \tag{3.149}$$

Where A_1, A_8, A_{19} are arbitrary analytic functions of z. If one lets

$$u = y''' - 2yy'' - \frac{7}{3}y'^2 + \frac{16}{9}y^2y' - \frac{4}{27}y^4 \tag{3.150}$$

then (3.149) can be reduced to a linear equation for u. It should be noted that (3.150) belongs to $P_{II}^{(3)}$ and given by the equation (3.72).

Case IV: $a_{10} = 0$: In this case there are four branches. The resonances and the canonical form of the equations are as follows:

IV.a:

$$\begin{aligned} y_{01} &= \tfrac{-5}{a_1} : (r_{11}, r_{12}, r_{13}, r_{14}) = (1, 2, 3, 5), \\ y_{02} &= \tfrac{-10}{a_1} : (r_{21}, r_{22}, r_{23}, r_{24}) = (-2, 1, 2, 5), \\ y_{03} &= \tfrac{-15}{a_1} : (r_{31}, r_{32}, r_{33}, r_{34}) = (-3, -2, 1, 5), \\ y_{04} &= \tfrac{-20}{a_1} : (r_{41}, r_{42}, r_{43}, r_{44}) = (-2, -3, -4, 5), \\ y^{(5)} &= -5(yy^{(4)} + 3y'y''' + 2y''^2 + 2y^2y''' + 10yy'y'' + 3y'^3 + 2y^3y'' \\ &+ 6y^2y'^2 + y^4y') + A_1(y^{(4)} + 5yy''' + 10y'y'' + 10y^2y' + 15yy'^2 \\ &+ 10y^3y' + y^5) + A_{13}(y'' + 2yy') + A_{16}y' + (A_{13}' - A_{13}A_1)y^2 \\ &+ (A_{16}' - A_{13}'' + A_1'A_{13} + 2A_1A_{13}' - A_1A_{16} - A_1^2A_{13})y + A_{19} \end{aligned} \tag{3.151}$$

where $A_1, A_{13}, A_{16}, A_{19}$ are arbitrary analytic functions of z. Integrating (3.151) once gives the special case of (3.133).

IV.b:

$$\begin{aligned} y_{01} &= -\tfrac{5}{a_2} : (r_{11}, r_{12}, r_{13}, r_{14}) = (2, 3, 5, 6), \\ y_{02} &= \tfrac{10}{a_2} : (r_{21}, r_{22}, r_{23}, r_{24}) = (2, 3, 5, 6), \\ y_{03} &= \tfrac{15}{a_2} : (r_{31}, r_{32}, r_{33}, r_{34}) = (-2, 5, 6, 7), \\ y_{04} &= -\tfrac{20}{a_2} : (r_{41}, r_{42}, r_{43}, r_{44} = (-7, 5, 6, 12), \\ y^{(5)} &= -5y'y''' - 5(y'')^2 + 5y^2y''' + 20yy'y'' + 5(y')^3 - 5y^4y' \\ &+ (c_1z + c_2)y' + c_1y \end{aligned} \tag{3.152}$$

Integrating (3.152) once gives (3.134)

IV.c:

$$\begin{aligned}
y_{01}^2 &= \tfrac{10}{a_4}: \quad y_{02} = -y_{01}; \quad (r_{i1}, r_{i2}, r_{i3}, r_{i4}) = (2, 3, 5, 6); \quad i = 1, 2\\
y_{03}^2 &= \tfrac{40}{a_4}: \quad y_{04} = -y_{03}; \quad (r_{j1}, r_{j2}, r_{j3}, r_{j4}) = (-3, 5, 6, 8); \quad j = 3, 4\\
y^{(5)} &= 10y^2y''' + 40yy'y'' + 10(y')^3 - 30y^4y' + c_1(y''' - 6y^2y')\\
&\quad + (c_2z + c_2)y' + c_2y
\end{aligned} \tag{3.153}$$

3.4 Sixth order equations: $\mathrm{P}_{II}^{(6)}$

Differentiating equation (3.136) and adding the term y^7 which is of order -7 as $z \to z_0$ give the following simplified equation of order six

$$\begin{aligned}
y^{(6)} &= a_1yy^{(5)} + a_2y'y^{(4)} + a_3y''y''' + a_4y^2y^{(4)} + a_5yy'y'''\\
&\quad + a_6y(y'')^2 + a_7(y')^2y'' + a_8y^3y''' + a_9y^2y'y'' + a_{10}y(y')^3\\
&\quad + a_{11}y^4y'' + a_{12}y^3(y')^2 + a_{13}y^5y' + a_{14}y^7,
\end{aligned} \tag{3.154}$$

where a_i, $i = 1, 2, ..., 14$ are constants. Substituting (3.4) into (3.154) gives the following equations for the resonances r and y_0 respectively,

$$\begin{aligned}
Q(r) &= (r+1)[r^5 - (22 + a_1y_0) + [197 + (16a_1 + a_2)y_0 - dy_0^2]r^3 + [-932\\
&\quad - (101a_1 - 11a_2 - 2a_3)y_0 + (11a_4 + a_5)y_0^2 - a_8y_0^3]r^2 + [2556\\
&\quad + (326a_1 + 46a_2 + 20a_3)y_0 - (46a_4 + 7a_5 + 4a_6)y_0^2 + (7a_8 + a_9)y_0^3\\
&\quad - a_{11}y_0^4]r - 4320 - (600a_1 + 120a_2 + 60a_3)y_0 + (96a_4 + 24a_5 + 16a_6\\
&\quad + 8a_7)y_0^2 - (18a_8 + 6a_9 + 3a_{10})y_0^3 + (4a_{11} + 2a_{12})y_0^4 - a_{13}y_0^5] = 0\\
&a_{14}y_0^6 - a_{13}y_0^5 + (a_{12} + 2a_{11})y_0^4 - (a_{10} + 2a_9 + a_8)y_0^3 + (2a_7 + 4a_6\\
&+6a_5 + 24a_4) - (12a_3 + 24a_2 + 120a_1) - 720 = 0.
\end{aligned} \tag{3.155}$$

(3.155.b) implies that we have five branches if $a_{14} = 0$ and $a_{13\neq 0}$ and six branches if $a_{14} \neq 0$. If $(r_{j1}, r_{j2}, r_{j3}, r_{j4}, r_{j5})$, $j = 1, ...6$ are the distinct integer resonances corresponding to $(-1, y_{0j})$, and if $\prod_{i=1}^{6} r_{ji} = p_j$, where p_j are integers and at least one of them is positive, then p_j, $j = 1, ...5$ satisfy the following Diophantine equation

$$\sum_{j=1}^{5} \frac{1}{p_j} = \frac{1}{720}, \tag{3.156}$$

when $a_{14} = 0$ and $a_{13} \neq 0$, and

$$\sum_{i=1}^{6} \frac{1}{p_j} = \frac{1}{720}, \tag{3.157}$$

when $a_{14} \neq 0$. Finding the solutions of the Diophantine equations (3.156) and (3.157) is quite difficult. So, for the sake of completeness we will present the cases when we have one, two, three and four branches. Since the procedure to obtain the canonical form of the differential equations of order six is the same as the one described in the previous sections, we will only give the canonical form of the differential equations and their corresponding resonances.

The canonical form of the equations can be obtained by adding the non-dominant terms $y^{(5)}, yy^{(4)}, y'y'', y''^2, y^2y''', yy'y'', y'^3, y^3y'', y^2y'^2, y^4y', y^6, y^{(4)}, yy''', y'y'', y^2y'', yy'^2, y^3y', y^5, y''', yy'', y'^2, y^2y', y^4, y'', yy', y^3, y', y^2, y, 1$,with the cofficients A_k, $k = 1, ..., 30$ which are analytic functions in z, respectively.

Case I: If $a_l = 0$, $l = 4, ...14$, then there is only one branch. The resonances and the canonical form of the equation is

$$\begin{aligned}
y_{01} &= -\tfrac{2}{a_1}: \qquad (r_1, r_2, r_3, r_4, r_5) = (2, 3, 4, 5, 6) \\
y^{(6)} &= -2yy^{(5)} - 10y'y^{(4)} - 20y''y''' + A_1(y^{(5)} + 2yy^{(4)} + 8y'y''' + (y'')^2) \\
&\quad + A_{12}(y^{(4)} + 2yy''' + 6y'y'') + A_{19}(y''' + 2yy'' + 2y'^2) + A_{24}(y'' + yy') \\
&\quad + A_{27}(y' + y^2) + A_{30},
\end{aligned} \tag{3.158}$$

where $A_1, A_{12}, A_{19}, A_{24}, A_{27}, A_{30}$ are arbitrary analytic functions of z. (3.158) can be linearized by letting $u = y' + y^2$.

Case II: If $a_l = 0$, $l = 8, ...14$, then we have two branches. The resonances and the canonical form of the equation are

$$\begin{aligned}
y_{01} &= -\tfrac{3}{a_1}: \qquad (r_{11}, r_{12}, r_{13}, r_{14}, r_{15}) = (1, 3, 4, 5, 6) \\
y_{02} &= -\tfrac{6}{a_1}: \qquad (r_{21}, r_{22}, r_{23}, r_{24}, r_{25}) = (-2, 3, 4, 5, 6) \\
y^{(6)} &= -3yy^{(5)} - 15y'y^{(4)} - 30y''y''' - 3y^2y^{(4)} - 24yy'y''' - 18y(y'')^2 - 36(y')^2y'' \\
&\quad + A_{13}[y^{(4)} + 3yy''' + 9y'y'' + 3y^2y'' + 6y(y')^2 - ((A_{12} - A_{13})y)''] \\
&\quad + A_{20}[y''' + 3yy'' + 3(y')^2 + 3y^2y' - ((A_{12} - A_{13})y)'] \\
&\quad + A_{26}[y'' + 3yy' + y^3 - (A_{12} - A_{13})y] + A_{30},
\end{aligned} \tag{3.159}$$

where $A_{12}, A_{13}, A_{20}, A_{26}, A_{30}$ are arbitrary functions. (3.159) can be linearized under the transformation

$$u = y'' + 3yy' + y^3 - (A_{12} - A_{13})y \tag{3.160}$$

Case III: If $a_l = 0$, $l = 11, ..., 14$, then we have three branches. The resonances of the equation are

$$\begin{array}{ll} y_{01} = -\frac{6-n}{a_1}: & (r_{11}, r_{12}, r_{13}, r_{14}, r_{15}) = (1, 4, 5, 6, n) \\ y_{02} = -\frac{6+n}{a_1}: & (r_{21}, r_{22}, r_{23}, r_{24}, r_{25}) = (1, 4, 5, 6, -n) \\ y_{03} = -\frac{12}{a_1}: & (r_{31}, r_{32}, r_{33}, r_{34}, r_{35}) = (-3, -2, 4, 5, 6) \end{array} \tag{3.161}$$

where n is a positive integer such that $n \neq 1, 4, 5, 6$. It should be noted that integration three times of the simplified equation gives the third order equation (3.69) with the additional term $(c_1 z^2 + c_2 z + c_3)$. Therefore, the simplified equation is not of Painlevé type unless $c_1 = c_2 = c_3 = 0$. Hence we will consider the cases for $n = 2, 3$. The canonical form of the equation when $n = 2$ is

$$\begin{aligned} y^{(6)} = & \, 2yy^{(5)} + 9y'y^{(4)} + 17y''y''' - \tfrac{3}{2}y^2y^{(4)} - 9y(y'')^2 - 18y''(y')^2 \\ & - 12yy'y''' + \tfrac{1}{2}y^3y''' + \tfrac{9}{2}y^2y'y'' + 3y(y')^3 \\ & + A_1[y^{(5)} + 2yy^{(4)} - 7y'y'' - 5(y'')^2 + \tfrac{3}{2}y^2y''' + 9yy'y'' + 3(y')^3 - \tfrac{1}{2}y^3y'' \\ & - \tfrac{3}{2}y^2(y')^2] + A_{12}[y^{(4)} - 2yy''' - 5y'y'' + \tfrac{3}{2}y^2y'' + 3y(y')^2 - \tfrac{1}{2}y^3y'] \\ & + \tfrac{1}{2}(2A_{19} + A_{20})y''' - \tfrac{1}{2}A_{20}(y''' - 2yy'' - \tfrac{3}{2}(y')^2 - \tfrac{3}{2}y^2y' - \tfrac{1}{8}y^4) \\ & + [\tfrac{3}{2}(2A_{19} + A_{20})' - \tfrac{1}{2}A_1(2A_{19} + A_{20})]y'' \\ & + [\tfrac{3}{2}(2A_{19} + A_{20})'' - A_1(2A_{19} + A_{20})' - \tfrac{1}{2}A_{12}(2A_{19} + A_{20})]y' \\ & + [\tfrac{1}{2}(2A_{19} + A_{20})''' + \tfrac{1}{4}A_{20}(2A_{19} + A_{20}) \\ & - \tfrac{1}{2}A_1(2A_{19} + A_{20})'' - \tfrac{1}{2}A_{12}(2A_{19} + A_{20})']y + A_{30}, \end{aligned} \tag{3.162}$$

where $A_1, A_{12}, A_{19}, A_{20}, A_{30}$ are arbitrary analytic functions. Integrating (3.162) three times gives an equation of the form of the equation (3.71).
For $n = 3$

$$\begin{aligned} y^{(6)} = & \, 2yy^{(5)} + \tfrac{32}{3}y'y^{(4)} + 22y''y''' - \tfrac{16}{9}y^2y^{(4)} - \tfrac{128}{9}yy'y''' \\ & - \tfrac{32}{3}y(y'')^2 - \tfrac{64}{3}y''(y')^2 + \tfrac{16}{27}y^3y''' + \tfrac{16}{3}y^2y'y'' + \tfrac{32}{9}y(y')^3 \\ & + A_1[y^{(5)} - 2yy^{(4)} - \tfrac{26}{3}y'y''' - \tfrac{20}{3}(y'')^2 + \tfrac{16}{9}y^2y''' + \tfrac{32}{3}yy'y'' + \tfrac{32}{9}(y')^3 \\ & - \tfrac{16}{27}y^3y'' - \tfrac{16}{9}y^2(y')^2] + A_{12}[y^{(4)} - 2yy''' - \tfrac{20}{3}y'y'' + \tfrac{32}{9}y(y')^2 - \tfrac{16}{27}y^3y'] \\ & + A_{19}[y''' - 2yy'' - \tfrac{7}{3}(y')^2 + \tfrac{16}{9}y^2y' - \tfrac{4}{27}y^4] + A_{30}, \end{aligned} \tag{3.163}$$

where $A_1, A_{12}, A_{19}, A_{30}$ are arbitrary analytic function in z. (3.163) is linearizable under the transformation (3.150).

Case IV: If $a_{13} = a_{14} = 0$, then there are four branches. The resonances and the canonical form of the equation are as follows:

$$
\begin{aligned}
y_{01} &= -\tfrac{5}{a_1}: \quad (r_{11}, r_{12}, r_{13}, r_{14}, r_{15}) = (1, 2, 3, 5, 6) \\
y_{02} &= -\tfrac{10}{a_1}: \quad (r_{21}, r_{22}, r_{23}, r_{24}, r_{25}) = (-2, 1, 2, 5, 6) \\
y_{03} &= -\tfrac{15}{a_1}: \quad (r_{31}, r_{32}, r_{33}, r_{34}, r_{35}) = (-3, -2, 1, 5, 6) \\
y_{04} &= -\tfrac{20}{a_1}: \quad (r_{41}, r_{42}, r_{43}, r_{44}, r_{45}) = (-4, -3, -2, 5, 6) \\
y^{(6)} &= -5yy^{(5)} - 20y'y^{(4)} - 35y''y''' - 10y^2y^{(4)} - 70yy'y''' - 50y(y'')^2 \\
&\quad - 95y''(y')^2 - 10y^3y''' - 90y^2y'y'' - 60y(y')^3 - 5y^4y'' - 20y^3(y')^2 \\
&\quad + A_1[y^{(5)} + 5yy^{(4)} + 10(y'')^2 + 10y^2y''' + 50yy'y'' + 15(y')^3 + 10y^3y'' \\
&\quad + 30y^2(y')^2 + 5y^4y'] + A_{12}[y^{(4)} + 5yy''' + 10y'y'' + 10y^2y'' + 15y(y')^2 \\
&\quad + 10y^3y' + y^5] + (4A'_{19} - 2A_1A_{19})yy' + A_{19}[y''' + 2yy'' + 2(y')^2] + A_{24}y'' \\
&\quad + [-3A''_{19} + 3A_1A'_{19} + 2A'_{24} - A_{12}A_{19} + 2A'_1A_{19} - A_1A_{24} - A_1^2A_{19}]y' \\
&\quad + [A''_{19} - A_1A'_{19} - A_{12}A_{19}]y^2 + [A''_{24} - 2A'''_{19} + A''_1A_{19} + 2A'_1A'_{19} \\
&\quad + 3A_1A''_{19} - A_1A'_{24} - A_1A'_1A_{19} - A_1^2A'_{19} - A_{12}A_{24} + 2A'_{19}A_{12} \\
&\quad - A_1A_{12}A_{19}]y + A_{30},
\end{aligned} \tag{3.164}
$$

where $A_1, A_{12}, A_{19}, A_{24}, A_{30}$ are arbitrary analytic functions in z. Integrating (3.164) twice gives

$$
\begin{aligned}
y^{(4)} &= -5yy''' - 10y'y'' - 10y^2y'' - 15y(y')^2 - 10y^3y' - y^5 \\
&\quad + A_{19}(y' + y^2) + (A_1A_{19} - 2A'_{19} + A_{24})y + B(z),
\end{aligned} \tag{3.165}
$$

where $B(z)$ satisfies the equation

$$
B'' = A_1B' + A_{12}B + A_{30} \tag{3.166}
$$

(3.165) is of the form of equation (3.133) and linearizable under the transformation $y = \frac{u'}{u}$.

Chapter 4

The third Painlevé hierarchy

In this chapter we apply the procedure to the third Painlevé equation and obtain non polynomial Painlevé type differential equations of order three.

4.1 Third order equations: $P_{III}^{(3)}$

The third Painlevé equation P_{III} is

$$y'' = \frac{(y')^2}{y} - \frac{y'}{z} + \frac{\mu y^2 + \nu}{z} + \gamma y^3 + \frac{\tau}{y} \tag{4.1}$$

The Painlevé test gives that there are two branches with common resonances and

$$(\alpha, \gamma y_0^2) = (-1, 1), \qquad Q(r) = (r+1)(r-2) \tag{4.2}$$

If one applies the transformation $z = z_0 + \epsilon t$ to equation (4.1) and then take the limit as $\epsilon \to 0$ one gets

$$\ddot{y} = \frac{\dot{y}^2}{y} \tag{4.3}$$

where $\dot{} = \frac{d}{dt}$. The only values of y in (4.1) for which the general existence theorem of Cauchy does not apply are $0, \infty$. The dominant terms of P_{III} are y'', $\frac{(y')^2}{y}$ and γy^3 which are of weight -3 as $z \to z_0$. Taking the derivative of the simplified equation gives

$$y''' = 2\frac{y'y''}{y} - \frac{(y')^3}{y^2} + 3\gamma y^2 y' \tag{4.4}$$

The leading order is -1 and the leading terms are of weight -4. Adding the dominant terms of weight -4 with constant coefficients such that the only values of y for which the Cauchy general existence theorem does not apply are $0, \infty$ and these dominant terms vanish as $\epsilon \to 0$ when we make the transformation $z = z_0 + \epsilon t$ give

$$y''' = c_1 \frac{y'y''}{y} + c_2 \frac{(y')^3}{y^2} + a_1 y y'' + a_2 (y')^2 + a_3 y^2 y' + a_4 y^4, \tag{4.5}$$

where $c_1^2 + c_2^2 \neq 0$. If $c_1 = c_2 = 0$, then (4.5) reduces to equation (3.3). Applying the transformation $z = z_0 + \epsilon t$ to equation (4.5) and taking the limit as $\epsilon \to 0$ give

$$\dddot{y} = c_1 \frac{\dot{y}\ddot{y}}{y} + c_2 \frac{\dot{y}^3}{y^2} \tag{4.6}$$

Equation (4.6) is of Painlevé type if its solution can be written as

$$y = \sum_{i=0}^{\infty} y_i (t - t_0)^{i+\alpha}, \tag{4.7}$$

where $\alpha \in \mathbb{Z}$ and y is single-valued. Substituting (4.7) in (4.6) gives

$$(c_1 + c_2 - 1)\alpha^2 + (3 - c_1)\alpha - 2 = 0 \tag{4.8}$$

Substituting $y = u^\alpha$, which preserves the Painlevé property, in (4.6) and using (4.8) give the equation

$$u\dddot{u} = (c_1\alpha - 3\alpha + 3)\dot{u}\ddot{u} \tag{4.9}$$

Integrating (4.9) once yields

$$\ddot{u} = k_1 u^{c_1\alpha - 3\alpha + 3}, \tag{4.10}$$

where k_1 is an integration constant. Equation (4.10) can be solved in terms of the Weiestrass elliptic function, if

$$c_1\alpha - 3\alpha + 3 = \beta, \quad \beta = 0, 1, 2, 3 \tag{4.11}$$

From equations (4.8) and (4.11), one can have

$$\begin{gathered} c_1 = \tfrac{\beta}{\alpha} + 3(1 - \tfrac{1}{\alpha}) \\ c_2 = -2 - \tfrac{1}{\alpha}(3 - \beta) - \tfrac{1}{\alpha^2}(1 + \beta) \\ \beta = 0, 1, 2, 3 \end{gathered} \tag{4.12}$$

One should note that when $\alpha \to \pm\infty$, $(c_1, c_2) = (3, -2)$, and (4.6) has the solution

$$y = k_3 exp[\frac{(k_1 t + k_2)^2}{2}], \tag{4.13}$$

where k_i, $i = 1, 2, 3$ are integration constants.

(4.5) was considered by Exton [10] without giving the general expression (4.12) of c_1 and c_2. Moreover, he had mistakes in applying the method of finding the necessary conditions for the equations in the canonical forms to be of Painlevé type. Martynov [12] considered (1.13), which reduces to (4.5) when $\nu = 1$. But he only investigated the case $a_4 \neq 0$, and he did not consider the situations when (4.5) attains recessive terms.

Substituting

$$y = y_0(z - z_0)^{-1} + \delta(z - z_0)^{r-1} \tag{4.14}$$

into (4.5) give the following equations for the resonances r and y_0, respectively

$$\begin{aligned} &Q(r) = (r+1)[r^2 - (7 + a_1 y_0)r + 3(6 - 2c_1 - c_2) + 2(2a_1 + a_2)y_0 - a_3 y_0^2] = 0 \\ &a_4 y_0^3 - a_3 y_0^2 + (2a_1 + a_2)y_0 + 6 - 2c_1 - c_2 = 0 \end{aligned} \tag{4.15}$$

Equation (4.15.b) implies that, in general, there are three branches if $a_4 \neq 0$. According to the number of branches, the following cases have to be considered

Case I: $a_4 = a_3 = 0$. In this case there is one branch. Then if r_1 and r_2 are the resonances, (4.15.b) implies

$$-(2a_1 + a_2)y_0 = r_1 r_2 = 6 - 2c_1 - c_2, \qquad r_1 + r_2 = a y_0 + 7 - c_1. \tag{4.16}$$

Therefore one has $(6 - 2c_1 - c_2) \in \mathbb{Z} - \{0\}$ i.e $2c_1 + c_2 \in \mathbb{Z} - \{6\}$. For each value of β one may have the following such cases

I.a: $\beta = 0$; then $(c_1, c_2) = (3 - \frac{3}{\alpha}, -2 + \frac{3}{\alpha} - \frac{1}{\alpha^2})$.
Since $2c_1 + c_2 = 4 - \frac{3}{\alpha} - \frac{1}{\alpha^2}$ is an integer, then $\alpha = \pm 1$
I.a.i: $\alpha = 1$, then $(c_1, c_2) = (0, 0)$.
I.a.ii: $\alpha = -1$, then $6 - 2c_1 - c_2 = 0$.

I.b: $\beta = 1$; then $(c_1, c_2) = (3 - \frac{2}{\alpha}, -2 + \frac{2}{\alpha})$.
Since $2c_1 + c_2 = 4 - \frac{2}{\alpha}$ is an integer, then $\alpha = \pm 1, \pm 2$
I.b.i: $\alpha = -1$, then $6 - 2c_1 - c_2 = 0$.
I.b.ii: $\alpha = 1$, then $(c_1, c_2) = (1, 0)$ and (4.16) gives $r_1 r_2 = 4$. Then the resonances and the simplified equation are

$$\begin{aligned} &y_0 = -\tfrac{1}{a_1}: \qquad (r_1, r_2) = (1, 4) \\ &y''' = \tfrac{y'y''}{y} + a_1 y y'' + 2a_1 (y')^2 \end{aligned} \tag{4.17}$$

Replacing y by λy such that $a_1 \lambda = -1$ and applying the transformation $y = \frac{1}{u}$, which preserves the Painlevé property, to (4.17.b) give

$$u^2 u''' = 5uu'u'' - 4(u')^3 - uu'' + 4(u')^2 \tag{4.18}$$

Painlevé analysis of (4.18) gives that the leading order is -1 and the resonances are $(\hat{r}_0, \hat{r}_1, \hat{r}_2) = (-1, 0, 2)$ with dominant terms $uu''', uu'u'', (u')^3$. Substituting the series

$$\sum_{i=0}^{\infty} u_i (z - z_0)^{i-1} \tag{4.19}$$

into (4.18) yields that the compatibility condition at the resonance $\hat{r}_2 = 2$ is not satisfied identically and hence (4.18), consequently (4.17), is not of Painlevé type.

I.b.iii: $\alpha = 2$, then $(c_1, c_2) = (2, 1)$ and $r_1 r_2 = 3$. Then the resonances and the simplified equation are

$$\begin{array}{ll} y_0 = -\frac{1}{a_1} : & (r_1, r_2) = (1, 3) \\ y''' = 2\frac{y'y''}{y} - \frac{(y')^3}{y^2} + a_1[yy'' + (y')^2] \end{array} \tag{4.20}$$

(4.20.b) has the first integral

$$y'' = \frac{(y')^2}{y} + a_1 y y' + k, \tag{4.21}$$

where k is an integration constant. (4.21) is of Painlevé type [6], [3] .

I.b.iv: $\alpha = -2$, then $(c_1, c_2) = (4, -3)$ and $r_1 r_2 = 1$ i.e $r_1 = r_2 = \pm 1$. That is one has a double resonance at ± 1.

I.c: $\beta = 2$; then $(c_1, c_2) = (3 - \frac{1}{\alpha}, -2 + \frac{1}{\alpha} + \frac{1}{\alpha^2})$

Since $2c_1 + c_2 = 4 - \frac{1}{\alpha} + \frac{1}{\alpha^2}$ is an integer, then $\alpha = \pm 1$

I.c.i: $\alpha = -1$, then $6 - 2c_1 - c_2 = 0$.

I.c.ii: $\alpha = 1$, then $(c_1, c_2) = (2, 0)$ and $r_1 r_2 = 2$. Then the resonances and the simplified equation are

$$\begin{array}{ll} y_0 = -\frac{2}{a_1} : & (r_1, r_2) = (1, 2) \\ y''' = 2\frac{y'y''}{y} + a_1[yy'' - (y')^2] \end{array} \tag{4.22}$$

Equation (4.22.b) does not pass the Painlevé test since the compatibility condition at the resonance $r_2 = 2$ is not satisfied identically.

I.d: $\beta = 3$; then $(c_1, c_2) = (3, -2 + \frac{2}{\alpha^2})$.

Since $2c_1 + c_2 = 4 + \frac{2}{\alpha^2}$ is an integer, then $\alpha = \pm 1$. But then for both values of α, one has $6 - 2c_1 - c_2 = 0$.

Case II: $a_4 = 0, a_3 \neq 0$. In this case there are two branches. If y_{0j} are the roots of (4.15.b) such that $y_{01} \neq y_{02}$, by setting

$$P(y_{0j}) = 3(6 - 2c_1 - c_2) + 2(2a_1 + a_2)y_{0j} - a_3 y_{0j}^2, \quad j = 1, 2 \tag{4.23}$$

and if (r_{j1}, r_{j2}) are the resonances corresponding to y_{0j}, then one has

$$r_{j1}r_{j2} = P(y_{0j}) = p_j, \quad j = 1, 2 \tag{4.24}$$

where $p_j \in \mathbb{Z}$ and such that at least one of them is positive. Equation (4.15.b) gives

$$a_3 = -\frac{6 - 2c_1 - c_2}{y_{01}y_{02}}, \quad 2a_1 + a_2 = a_3(y_{01} + y_{02}). \tag{4.25}$$

Then (4.23) can be written as

$$P(y_{01}) = (6 - 2c_1 - c_2)(1 - \frac{y_{01}}{y_{02}}), \quad P(y_{02}) = (6 - 2c_1 - c_2)(1 - \frac{y_{01}}{y_{02}}). \tag{4.26}$$

For $p_1p_2 \neq 0$ and $6 - 2c_1 - c_2 \neq 0$, p_j satisfy the Diophantine equation

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{6 - 2c_1 - c_2} \tag{4.27}$$

For each (p_1, p_2), one should find (r_{j1}, r_{j2}) such that r_{ji} are distinct integers and $r_{j1}r_{j2} = p_j$. Then y_{0j} and a_i can be obtained from (4.25), (4.26) and

$$r_{j1} + r_{j2} = a_1 y_{0j} + 7 - c_1 \tag{4.28}$$

For each value of β, one may have the following cases

II.a: $\beta = 0$, then the Diopahantine equation takes the form

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{\alpha^2}{2\alpha^2 + 3\alpha + 1}. \tag{4.29}$$

Since it is not possible to find the integer solutions of (4.29) for all α, we will look for the integer solutions of (4.29) when $\alpha = \pm 2, \pm 3$. One should note that when $\alpha = -1$ one has $6 - 2c_1 - c_2 = 0$ and when $\alpha = 1$ one has $c_1 = c_2 = 0$.
II.a.i: $\alpha = 2$; then the Diophantine equation is

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{4}{15} \tag{4.30}$$

Equation (4.30) has the following integer solutions

$$(p_1, p_2) = (3, -15), (4, 60), (5, 15), (6, 10), \tag{4.31}$$

There is only one case $(p_1, p_2) = (3, -15)$, such that all the resonances are distinct integers for both branches. The resonances and the simplified equation of this case are

$$\begin{aligned} y_{01} &= -\tfrac{3}{2a_1}: \quad (r_{11}, r_{12}) = (1, 3) \\ y_{02} &= -\tfrac{15}{4a_1}: \quad (r_{21}, r_{22} = (-5, 3) \\ y''' &= \tfrac{3}{2}\tfrac{y'y''}{y} - \tfrac{3}{4}\tfrac{(y')^3}{y^2} + a_1[yy'' + (y')^2] - \tfrac{1}{3}a_1^2 y^2 y' \end{aligned} \tag{4.32}$$

If one replaces y by λy such that $a_1\lambda = -\frac{1}{3}$, then (4.32) has the first integral

$$y'' = \frac{3}{4}\frac{(y')^2}{y} - \frac{3}{2}yy' - \frac{1}{4}y^4 + k, \tag{4.33}$$

where k is an integration constant. (4.33) is of Painlevé type [6] [3].

II.a.ii: $\alpha = -2$; then (4.29) has the only integer solution $(p_1, p_2) = (1, 3)$ but then there will a double resonance at ± 1.

II.a.iii: $\alpha = -3$; then (4.29) has the only integer solution $(p_1, p_2) = (1, -10)$ but then there will be a double resonance at ± 1.

II.a.iv: $\alpha = 3$; then (4.29) has the only integer solution $(p_1, p_2) = (4, 14)$. Using (4.28) one can obtain that $(r_{11}, r_{12}) = (1, 4)$ and that the resonances r_{2i} satisfy the equation $r_{2i}^2 - 5r_{2i} + 14 = 0$ which has non-integer roots.

II.b: $\beta = 1$; then the Diophantine equation takes the form

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{\alpha}{2(\alpha+1)} \tag{4.34}$$

Equation (4.34) always has the particular solution $(p_1, p_2) = (2, -2\alpha - 2)$. In this case the resonances and the simplified equations are

$$\begin{aligned} &y_{01} = -\tfrac{\alpha+2}{a_1\alpha} : \quad (r_{11}, r_{12}) = (1, 2), \\ &y_{02} = -\tfrac{(\alpha+2)(\alpha+1)}{a_1\alpha} : \ (r_{21}, r_{22}) = (-1-\alpha, 2), \\ &y''' = (3 - \tfrac{2}{\alpha})\tfrac{y'y''}{y} + (-2 + \tfrac{2}{\alpha})\tfrac{(y')^3}{y^2} + a_1 yy'' - \tfrac{2\alpha}{(\alpha+2)^2}a_1^2y^2y' \\ &\alpha \neq 0, -1, -3 \end{aligned} \tag{4.35}$$

Substituting $y = \frac{u'}{u}$ gives

$$u^{(4)} = (3 - \frac{2}{\alpha})\frac{u''u'''}{u'} + (-2 + \frac{2}{\alpha})\frac{(u'')^3}{(u')^2} \tag{4.36}$$

Substituting $u' = v^\alpha$ in (4.36) gives the following differential equation in v

$$vv''' = v'v'' \tag{4.37}$$

Integrating (4.37) once gives $v'' = kv$, where k is an integration constant, which has the solution $v = k_1 z + k_2$ if $k = 0$, or $v = k_1 e^{\sqrt{k}z} + k_2 e^{-\sqrt{k}z}$ if $k \neq 0$. The simple zeros of v might be singularities of u. Then one can easily show that for u' not to contain the term $\frac{1}{z-z_0}$, i.e u and consequently y is of Painlevé type, it is necessary and sufficient that $\alpha \neq -2m - 1$ where $m \in \mathbb{Z}_+$.

Since it is not possible to solve equation (4.34) for all α we will cover the cases $\alpha = 1, \pm 2$. One should note that when $\alpha = -1$ one has $6 - 2c_1 - c_2 = 0$.

II.b.i: $\alpha = 1$, then the Diophantine equation has the form

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{4} \tag{4.38}$$

(4.38) has the solutions

$$(p_1, p_2) = (2, -4), (3, -12), (5, 20), (8, 8), (6, 12) \tag{4.39}$$

There are three cases $(p_1, p_2) = (2, -4), (8, 8), (6, 12)$ such that all the resonances are distinct integers for both branches. The resonances and the simplified equations for these cases are

II.b.i.1: $(p_1, p_2) = (2, -4)$

$$\begin{aligned} y_{01} &= -\tfrac{3}{a_1}: \quad (r_{11}, r_{12}) = (1, 2), \\ y_{02} &= -\tfrac{6}{a_1}: \quad (r_{21}, r_{22}) = (-2, 2), \\ y''' &= \tfrac{y'y''}{y} + a_1(yy'' - \tfrac{2}{9}a_1 y^2 y'). \end{aligned} \tag{4.40}$$

Equation (4.40.c) is nothing but equation (4.35) when $\alpha = 1$. If one replaces y by λy such that $a_1\lambda = -3$, then (4.40.c) has the first integral

$$y'' = -3yy' - y^3 + ky, \tag{4.41}$$

where k is an integration constant. (4.41) is of Painlevé type [6] [3].

II.b.i.2: $(p_1, p_2) = (8, 8)$

$$\begin{aligned} y_{01}^2 &= \tfrac{4}{a_3}: \quad y_{02} = -y_{01}, \quad (r_{j1}, r_{j2}) = (2, 4), \quad j = 1, 2 \\ y''' &= \tfrac{y'y''}{y} + a_3 y^2 y' \end{aligned} \tag{4.42}$$

If one replaces y by λy such that $a_3\lambda = 4$, then (4.42.b) has the first integral

$$y'' = 2y^3 + ky \tag{4.43}$$

where k is an integration constant. (4.43) is of Painlevé type [6] [3].

II.b.i.3: $(p_1, p_2) = (6, 12)$

$$\begin{aligned} y_{01} &= -\tfrac{1}{a_1}: \quad (r_{11}, r_{12}) = (2, 3), \\ y_{02} &= \tfrac{6}{a_1}: \quad (r_{21}, r_{22}) = (2, 6), \\ y''' &= \tfrac{y'y''}{y} + a_1(yy'' + 2a_1 y^2 y'). \end{aligned} \tag{4.44}$$

If one replaces y by λy such that $a_1\lambda = -2$, then (4.44.c) has the first integral

$$y'' = -yy' + y^3 + ky, \tag{4.45}$$

where k is an integration constant. (4.45) is of Painlevé type [6] [3].

II.b.ii: $\alpha = 2$; then the Diophantine equation is of the form

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{3} \tag{4.46}$$

(4.46) has the solutions

$$(p_1,p_2) = (2,-6),(4,-12),(6,6), \tag{4.47}$$

among which only the cases $(2,-6),(6,6)$ are such that the resonances are distinct integers for both branches. The resonances and the simplified equations are as follows

II.b.ii.1: $(p_1,p_2) = (2,-6)$

$$\begin{array}{ll} y_{01} = -\frac{2}{a_1}: & (r_{11},r_{12}) = (1,2), \\ y_{02} = -\frac{6}{a_1}: & (r_{21},r_{22}) = (-3,2), \\ y''' = 2\frac{y'y''}{y} - \frac{(y')^3}{y^2} + a_1(yy'' - \frac{1}{4}a_1y^2y'). & \end{array} \tag{4.48}$$

(4.48.c) is nothing but equation (4.35) when $\alpha = 2$. If one replaces y by λy, then (4.48.c) has the first integral

$$y'' = \frac{1}{2}\frac{(y')^2}{y} - 2yy' - \frac{1}{2}y^3 + ky \tag{4.49}$$

where k is an integration constant. (4.49) is of Painlevé type [6] [3].

II.b.ii.2: $(p_1,p_2) = (6,6)$

$$\begin{array}{l} y_{01}^2 = \frac{3}{a_3}: \quad y_{02} = -y_{01}; \quad (r_{j1},r_{j2}) = (2,3), \quad j = 1,2 \\ y''' = 2\frac{y'y''}{y} - \frac{(y')^3}{y^2} + a_3y^2y' \end{array} \tag{4.50}$$

(4.50.b) has the first integral

$$y'' = \frac{(y')^2}{y} + \frac{a_3}{3}y^3 + k, \tag{4.51}$$

where k is an integration constant. (4.51) is of Painlevé type [6] [3].
II.b.iii: $\alpha = -2$; then the Diophantine equation has the form

$$\frac{1}{p_1} + \frac{1}{p_2} = 1 \tag{4.52}$$

which has the only solution $(p_1,p_2) = (2,2)$. The resonances and the simplified equations in this case are

$$\begin{array}{l} y_{01}^2 = \frac{1}{a_3}: \quad y_{02} = -y_{01}; \quad (r_{j1},r_{j2}) = (1,2), \quad j = 1,2 \\ y''' = 4\frac{y'y''}{y} - 3\frac{(y')^3}{y^2} + a_3y^2y'. \end{array} \tag{4.53}$$

Note that (4.35) reduces to (4.53) if $\alpha = -2$. (4.53) has the first integral

$$y'' = \frac{(y')^2}{y} + cy^3 + ky^2 \tag{4.54}$$

where k is an integration constant. (4.54) is of Painlevé type [6] [3].

II.c: $\beta = 2$; then the Diophantine equation is

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{\alpha^2}{2\alpha^2 + \alpha + 1} \tag{4.55}$$

When $\alpha = -1$, one has $6 - 2c_1 - c_2 = 0$. When $\alpha = \pm 2, \pm 3$, the integer solutions of (4.55) lead to equations with non-integer resonances. When $\alpha = 1$ equation (4.55) has the solutions

$$(p_1, p_2) = (1, -2), (3, 6), (4, 4) \tag{4.56}$$

There are two cases such that the resonances for both branches are distinct integers. The resonances and the simplified equations for these cases are

II.c.1: $(p_1, p_2) = (3, 6)$

$$\begin{array}{ll} y_{01} = -\frac{1}{a_1}: & (r_{11}, r_{12}) = (1, 3) \\ y_{02} = \frac{2}{a_1}: & (r_{21}, r_{22}) = (1, 6) \\ y''' = 2\frac{y'y''}{y} + a_1[yy'' - (y')^2 + a_1 y^2 y'] & \end{array} \tag{4.57}$$

(4.57) does not pass the Painlevé test since the compatibility conditions at the resonance $r_{13} = 3$ is not satisfied identically.

II.c.2: $(p_1, p_2) = (4, 4)$

$$\begin{array}{l} y_{01}^2 = \frac{2}{a_3}: \quad y_{02} = -y_{01}; \quad (r_{j1}, r_{j2}) = (1, 4), \quad j = 1, 2 \\ y''' = 2\frac{y'y''}{y} + a_3 y^2 y'. \end{array} \tag{4.58}$$

(4.58.b) has the first integral

$$y'' = cy^3 + ky^2, \tag{4.59}$$

where k is an integration constant. (4.59) is of Painlevé type [6] [3].

II.d: $\beta = 3$; then the Diophantine equation is

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{\alpha^2}{2(\alpha^2 - 1)} \tag{4.60}$$

Since it is not possible to solve (4.60) for all α will cover the case $\alpha = \pm 2$. One should note that when $\alpha = \pm 1$ one has $6 - 2c_1 - c_2 = 0$ and when $\alpha = \pm 3$ the integer solutions of (4.60) gives non-integer resonances. when $\alpha = 2$ the Diophantine equation has the solutions

$$(p_1, p_2) = (1, -3), (2, 6), (3, 3) \tag{4.61}$$

There are two cases such that all the resonances for both branches are distinct integers. The resonances and the simplified equations for these cases are

II.d.1: $(p_1, p_2) = (2, 6)$

$$\begin{array}{ll} y_{01} = -\frac{1}{a_1}: & (r_{11}, r_{12}) = (1, 2), \\ y_{02} = \frac{3}{a_1}: & (r_{21}, r_{22}) = (1, 6), \\ y''' = 3\frac{y'y''}{y} - \frac{3}{2}\frac{(y')^3}{y^2} + a_1[yy'' - (y')^2 + \frac{1}{2}a_1 y^2 y'] & \end{array} \tag{4.62}$$

(4.62.c) does not pass the Painlevé test since the compatibility condition at the resonance $r_{12} = 2$ is not satisfied identically.

II.d.2: $(p_1, p_2) = (3, 3)$

$$\begin{array}{l} y_{01}^2 = \frac{3}{2a_3}: \quad y_{02} = -y_{01}; \quad (r_{j1}, r_{j2}) = (1, 3), \quad j = 1, 2 \\ y''' = 3\frac{y'y''}{y} - \frac{3}{2}\frac{(y')^3}{y^2} + a_3 y^2 y' \end{array} \tag{4.63}$$

(4.63.b) has the first integral

$$y'' = \frac{1}{2}\frac{(y')^2}{y} + a_3 y^3 + k y^2, \tag{4.64}$$

where k is an integration constant. (4.64) is of Painlevé type [6] [3].

Case III: $a_4 \neq 0$. In this case there are three branches corresponding to $(-1, y_{0j})$, $j = 1, 2, 3$, where y_{0j} are the roots of (4.15.b). (4.15.b) implies that

$$\prod_{j=1}^{3} y_{0j} = -\frac{6 - 2c_1 - c_2}{a_4}, \quad \sum_{i \neq j} y_{0i} y_{0j} = \frac{1}{a_4}(2a_1 + a_2), \quad \sum_{j=1}^{3} y_{0j} = \frac{a_3}{a_4}. \tag{4.65}$$

If the resonances (except $r_0 = -1$ which is common for all branches) are r_{ji}, $i = 1, 2$ corresponding to y_{0j}, and if one sets

$$P(y_{0j}) = 3(6 - 2c_1 - c_2) + 2(2a_1 + a_2) y_{0j} - a_3 y_{0j}^2, \quad j = 1, 2, 3 \tag{4.66}$$

then (4.15.a) implies that

$$\prod_{i=1}^{2} r_{ji} = P(y_{oj}) = p_j, \tag{4.67}$$

where p_j are integers and in order to have a principal branch, at least one of them is positive. Equations (4.65) and (4.66) give

$$p_j = (6 - 2c_1 - c_2) \prod_{l=1,\ l \neq j}^{3} (1 - \frac{y_{0j}}{y_{0l}}), \quad j = 1, 2, 3 \tag{4.68}$$

and hence p_j satisfy the following Diophantine equation

$$\sum_{j=1}^{3} p_j = \frac{1}{6 - 2c_1 - c_2} \tag{4.69}$$

where $\prod_{j=1}^{3} p_j \neq 0$, $6 - 2c_1 - c_2 \neq 0$ and from (4.68) one has the system

$$p_1(y_{02} - y_{03}) = ky_{01}, \quad p_2(y_{03} - y_{01}) = ky_{02}, \quad p_3(y_{01} - y_{02}) = ky_{03}, \tag{4.70}$$

where

$$k = \frac{6 - 2c_1 - c_2}{y_{01}y_{02}y_{03}}(y_{01} - y_{02})(y_{02} - y_{03})(y_{01} - y_{03}) \tag{4.71}$$

Moreover it can be deduced from (4.12) if $6 - 2c_1 - c_2 \neq 0$, then for all $\alpha \in \mathbb{Z}$ and $\beta = 0, 1, 2, 3$, one has $6 - 2c_1 - c_2 > 0$. Then (4.68) gives that

$$\prod_{j=1}^{3} p_j = -\frac{(6 - 2c_1 - c_2)^3}{(y_{01}y_{02}y_{03})^2}(y_{01} - y_{02})^2(y_{01} - y_{03})^2(y_{02} - y_{03})^2 \tag{4.72}$$

Thus from (4.72) if $a_1 \neq 0$, then $\prod_{j=1}^{3} p_j < 0$. That is $p_1 > 0$, and either p_2 or p_3 is a negative integer. So one should consider the case $a_1 = 0$ separately.

III.1: $a_1 = 0$. Then (4.15.a) gives

$$r_{j1} + r_{j2} = 7 - c_1 \tag{4.73}$$

Thus c_1 is an integer and since

$$(r_{j1} - r_{j2})^2 = (r_{j1} + r_{j2})^2 - 4r_{j1}r_{j2}, \tag{4.74}$$

one has that $(7 - c_1)^2 - 4p_j$ is a perfect square. Then one can determine p_j and then by using the system (4.70) and (4.65), one can obtain y_{0j} and a_m, $m = 2, 3, 4$. For each value of β one can have the following cases

III.1.a: $\beta = 0$. Since $c_1 = 3(1 - \frac{1}{\alpha})$ is an integer, then $\alpha = \pm1, \pm3$. There is only one case, $\alpha = -3$, such that $6 - 2c_1 - c_2 \neq 0$, $c_1^2 + c_2^2 \neq 0$ and the resonances of all branches are distinct integers. The resonances and the simplified equation for this case are

$$\begin{aligned}
&y_{01} = -\tfrac{1}{3a_2} : \quad (r_{11}, r_{12}) = (1, 2),\\
&y_{02} = \tfrac{2}{3a_2} : \quad (r_{21}, r_{22}) = (1, 2),\\
&y_{03} = \tfrac{5}{3a_2} : \quad (r_{31}, r_{32}) = (-2, 5),\\
&y''' = 4\tfrac{y'y''}{y} - \tfrac{28}{9}\tfrac{(y')^3}{y^2} + a_2[(y')^2 + 6a_2y^2y' + 3a_2^2y^4]
\end{aligned} \tag{4.75}$$

(4.75 d) does not pass the Painlevé test since the compatibility conditions are not satisfied identically.

III.1.b: $\beta = 1$. Since $c_1 = 3 - \frac{2}{\alpha}$ is an integer, then $\alpha = \pm 1, \pm 2$. No cases such that the resonancses of all branches are distinct integer.

III.1.c: $\beta = 2$. Since $c_1 = 3 - \frac{1}{\alpha}$ is an integer then $\alpha = \pm 1$. When $\alpha = -1$, one has $6 - 2c_1 - c_3 = 0$. The case $\alpha = 1$ leads to the following resonances and simplified equation

$$y_{0j}^3 = -\tfrac{2}{a_4}: \qquad (r_{j1}, r_{j2}) = (2, 3), \quad j = 1, 2, 3 \qquad y''' = 2\tfrac{y'y''}{y} + a_4 y^4 \tag{4.76}$$

Replacing y by λy such that $a_4\lambda^3 = 2$ (4.76.b) becomes

$$y''' = 2\frac{y'y''}{y} + 2y^4 \tag{4.77}$$

(4.77) was considered by Martynov [12].

III.1.d: $\beta = 3$. Then $c_1 = 3$. Since $(7 - c_1)^2 - 4p_1 = 16 - 4p_1^2$ is a perfect square and $p_1 > 0$, then $p_1 = 3$. (4.69) gives that p_2 and p_3 satisfy

$$\frac{1}{p_2} + \frac{1}{p_3} = \frac{\alpha^2 + 2}{6(\alpha^2 - 1)} > 0 \tag{4.78}$$

From (4.78), one can deduce that one of p_2 and p_3, say p_2 is positive. Since $16 - 4p_2$ is a perfect square , one has $p_2 = 3$. Then (4.78) gives $p_3 = \frac{6(\alpha^2-1)}{4-\alpha^2} = -6 + \frac{18}{4-\alpha^2}$. Since p_3 is an integer, one has $\alpha = \pm 1$. But then one has $6 - 2c_1 - c_2 = 0$.

III.2: $a_1 \neq 0$. Then after solving (4.69) for $p_j = r_{j1}r_{j2}$, y_{0j} and a_i, $i = 1, 2, 3, 4$ can be determined from equations (4.70), (4.65) and

$$r_{j1} + r_{j2} = 7 - c_1 + a_1 y_{oj}, \quad j = 1, 2, 3 \tag{4.79}$$

For each value of β, one can have the following cases

III.2.a: $\beta = 0$. Then Diophantine equation takes the form

$$\sum_{j=0}^{3} p_j = \frac{\alpha^2}{2\alpha^2 + 3\alpha + 1} \tag{4.80}$$

$(p_1, p_2, p_3) = (2, 4\alpha + 2, -\alpha - 1)$ is a particular solution of (4.80), but not all the solutions are of this form. For this particular solution the system (4.70) gives $k = \pm 2\alpha$. There is only one case , $k = 2\alpha$, such that the resonances for all branches are distinct integers. The resonances and the simplified equation for this case are

$$\begin{aligned}
y_{01} &= -\tfrac{\alpha+3}{a_1\alpha} : \ (r_{11}, r_{12}) = (1,2),\\
y_{02} &= -\tfrac{(\alpha+3)(2\alpha+1)}{a_1\alpha} : \ (r_{21}, r_{22}) = (-2\alpha-1,-2),\\
y_{03} &= -\tfrac{(\alpha+3)(\alpha+1)}{a_1\alpha} : \ (r_{31}, r_{32}) = (-\alpha-1,1),\\
y''' &= 3(1-\tfrac{1}{\alpha})\tfrac{y'y''}{y} + (-2+\tfrac{3}{\alpha}-\tfrac{1}{\alpha^2})\tfrac{(y')^3}{y^2} + a_1[yy'' + \tfrac{3}{\alpha(\alpha+3)}(y')^2 - \tfrac{3(\alpha+1)}{(\alpha+3)^2}a_1y^2y'\\
&\quad + \tfrac{\alpha}{(\alpha+3)^3}a_1^2y^4], \qquad \alpha \neq -1,-2,-3.
\end{aligned} \tag{4.81}$$

Note that when $\alpha = -3$,(4.81) reduces to (4.75) which is not of Painlevé type. Substituting $y = \frac{u'}{u}$ in (4.81) gives

$$u^{(4)} = 3(1-\frac{1}{\alpha})\frac{u''u'''}{u'} + (-2+\frac{3}{\alpha}-\frac{1}{\alpha^2})\frac{(u'')^3}{(u')^2} \tag{4.82}$$

Substituting $u' = v^\alpha$ in (4.83) gives

$$v''' = 0 \tag{4.83}$$

(4.83) has the solution $v(z) = k_1z^2 + k_2z + k_3$. The zeros z_0 of v are singularities of u' when $\alpha < 0$. For u' not to contain the term $\frac{1}{z-z_0}$, i.e for u and consequently y to be of Painlevé type, it is necessary and sufficient that $\alpha > 0$.

In particular, if $\alpha = 2$, then the only solution of (4.80) such that the resonances are distinct integers is $(p_1, p_2, p_3) = (2, 15, -3)$. The simplified equation and the resonances for this case are as follows

$$\begin{aligned}
y_{01} &= -\tfrac{5}{2a_1} : \quad (r_{11}, r_{21}) = (1,2),\\
y_{02} &= -\tfrac{25}{2a_1} : \quad (r_{21}, r_{22}) = (-5,-3),\\
y_{03} &= -\tfrac{15}{2a_1} : \quad (r_{31}, r_{32}) = (-3,1),\\
y''' &= \tfrac{3}{2}\tfrac{y'y''}{y} - \tfrac{3}{4}\tfrac{(y')^3}{y^2} + a_1[yy'' + \tfrac{3}{10}(y')^2 - \tfrac{9}{25}a_1y^2y' + \tfrac{2}{125}a_1^2y^4]
\end{aligned} \tag{4.84}$$

III.2.b: $\beta = 1$. Then (4.69) has the form

$$\sum_{j=1}^{3} p_j = \frac{\alpha}{2(\alpha+1)} \tag{4.85}$$

Since it is not possible to solve (4.85), for all α, we will consider α say $1, \pm 2$.
III.2.b.i: $\alpha = 1$. Then equation (4.85) has the solutions

$$\begin{aligned}
(p_1, p_2, p_3) = &(3,24,-8), (3,132,-11), (4,-n,n), (5,16,-80), (5,19,-380),\\
&(6,10,-60), (7,8,-56); n \in \mathbb{Z}_+
\end{aligned} \tag{4.86}$$

Only for the following cases out of 7 cases given in (4.86) one has distinct integer resonanes

III.2.b.i.1: $(p_1, p_2, p_3) = (3, 24, -8)$

$$\begin{array}{ll} y_{01} = -\frac{2}{a_1}: & (r_{11}, r_{12}) = (1, 3), \\ y_{02} = \frac{4}{a_1}: & (r_{21}, r_{22}) = (4, 6), \\ y_{03} = -\frac{4}{a_1}: & (r_{31}, r_{32}) = (-2, 4), \\ y''' = \frac{y'y''}{y} + a_1(yy'' + \frac{1}{4}a_1 y^2 y' - \frac{1}{8}a_1^2 y^4). & \end{array} \quad (4.87)$$

Replacing y by λy such that $a_1\lambda = -2$, then (4.87.d) has one of the following first integrals

$$y'' = \frac{1}{2}y^3, \quad y'' = -yy' + y^3, \quad y'' = -3yy' - y^3 \quad (4.88)$$

which are of Painlevé type [6] [3].

III.2.b.i.2: $(p_1, p_2, p_3) = (4, -n, n)$.

Since $p_1 = 4$, one has $(r_{11}, r_{12}) = (1, 4)$ and hence $a_1 y_{01} = -1$. On using the system (4.70), one finds that $a_1 y_{02} = \frac{n-4}{8}$ and $a_1 y_{03} = -\frac{n+4}{8}$. So that the resonances r_{2i} and r_{3i} satisfy the following equations

$$r_{2i}^2 - \frac{44+n}{8} r_{2i} + n = 0 \quad (4.89)$$

$$r_{3i}^2 - \frac{44-n}{8} r_{3i} - n = 0 \quad (4.90)$$

respectively. The simplified equation has the form

$$y''' = \frac{y'y''}{y} + a_1 yy'' - 2\frac{n^2-144}{16-n^2} a_1 (y')^2 - \frac{512}{16-n^2} a^2 y^2 y' + \frac{256}{16-n^2} a_1^3 y^4 \quad (4.91)$$

(4.91) does not pass the Painlevé test unless $n = 12$ since the compatibility condition at $r_{12} = 4$ is not satisfied identically unless $n = 12$. Then (4.89) and (4.90) give that $(r_{21}, r_{22}) = (3, 4)$ and $(r_{31}, r_{32}) = (-2, 6)$ respectively. Thus one has the equation

$$\begin{array}{ll} y_{01} = -\frac{1}{a_1}: & (r_{11}, r_{12}) = (1, 4), \\ y_{02} = \frac{1}{a_1}: & (r_{21}, r_{22}) = (3, 4), \\ y_{03} = -\frac{2}{a_1}: & (r_{31}, r_{32}) = (-2, 6), \\ y''' = \frac{y'y''}{y} + a_1(yy'' + 4a_1 y^2 y' - 2a_1^2 y^4) & \end{array} \quad (4.92)$$

Replacing y by λy such that $a_1\lambda = -1$, then (4.92) has one of the following first integrals

$$y'' = 2y^3, \quad y'' = yy' + y^3, \quad y'' = -3yy' - y^3 \quad (4.93)$$

which are of Painlevé type [6] [3].

III.2.b.ii: $\alpha = -2$. Then (4.85) takes the form $\sum_{j=1}^{3} p_j = 1$ which has the only solution $(p_1, p_2, p_3) = (1, n, -n)$ such that $p_3 < 0$. But then one has $r_{11} = r_{12} = \pm 1$ that is one has double resonanse at ± 1.

III.2.b.iii: $\alpha = 2$. Then (4.85) has the form

$$\sum_{j=1}^{3} p_j = \frac{1}{3} \tag{4.94}$$

The only solution of (4.94) that might yield an equation with distinct resonances is $(p_1, p_2, p_3) = (3, n, -n)$, where $n \in \mathbb{Z}_+$. The resonances of the first branch are $(r_{11}, r_{12}) = (1, 3)$ and the equation is of the form

$$y''' = 2\frac{y'y''}{y} - \frac{(y')^3}{y^2} + a_1 yy'' - \frac{n^2 - 117}{9 - n^2}(y')^2 - \frac{216}{9 - n^2}a_1^2 y^2 y' + \frac{108}{9 - n^2}a_1^3 y^4 \tag{4.95}$$

(4.95) does not pass the Painlevé test since the compatibility condition at $r_{12} = 3$ is not satisfied identically for any value n.

III.2.c: $\beta = 2$. Then (4.69) takes the form

$$\sum_{j=1}^{3} p_j = \frac{\alpha^2}{2\alpha^2 + \alpha - 1} \tag{4.96}$$

$(p_1, p_2, p_3) = (2, 12\alpha - 6, -3\alpha - 3)$ is a particular solution of (4.96). For this triple, one has $k = \pm 6\alpha$ both of which yield the same simplified equation such that the resonances of all branches are distinct integer. The resonances and the simplified equation are

$$\begin{aligned}
&y_{01} = -\tfrac{\alpha+1}{a_1\alpha}: \qquad (r_{11}, r_{12}) = (1, 2)\\
&y_{02} = -\tfrac{(\alpha+1)^2}{a_1\alpha}: \qquad (r_{21}, r_{22}) = (3, -\alpha - 1)\\
&y_{02} = -\tfrac{(\alpha+1)(2\alpha-1)}{a_1\alpha}: \quad (r_{31}, r_{32}) = (6, 2\alpha - 1)\\
&y''' = (3 - \tfrac{1}{\alpha})\tfrac{y'y''}{y} + (-2 + \tfrac{1}{\alpha} + \tfrac{1}{\alpha^2})\tfrac{(y')^3}{y^2} + a_1 yy'' - \tfrac{3}{\alpha(\alpha+1)}a_1(y')^2 + \tfrac{3-\alpha}{(\alpha+1)^2}a_1^2 y^2 y'\\
&\quad - \tfrac{\alpha}{(\alpha+1)^3}a_1^3 y^4; \qquad \alpha \neq 0, -1, -4.
\end{aligned} \tag{4.97}$$

Substituting $y = \frac{u'}{u}$ in (4.97) gives

$$u^{(4)} = (3 - \frac{1}{\alpha})\frac{u''u'''}{u'} + (-2 + \frac{1}{\alpha} + \frac{1}{\alpha^2})\frac{(u'')^3}{(u')^2} \tag{4.98}$$

Substituting $u' = v^\alpha$, gives the following equation for v

$$vv''' = 2v'v'' \tag{4.99}$$

Integrating (4.99) gives $v'' = k_1 v^2$. Thus $v = k_2 z + k_2$ if $k_1 = 0$, or $v = \sum_{i=0}^{\infty} v_{6i}(z - z_0)^{6i-2}$, where z_0 is a double pole of v. Since

$u' = v^\alpha$, u' does not contain the term $\frac{1}{z-z_0}$. That is u, and consequently y, is of Painlevé type if and only if $\alpha \neq 0, -1, -4$.

Since it not possible to solve (4.96) for all α, we will cover the case $\alpha = 1$. When $\alpha = 1$ (4.96) has the solutions $(p_1, p_2, p_3) = (3, 5, -30), (2, n, -n)$ where $n \in \mathbb{Z}_+$. When $(p_1, p_2, p_3) = (3, 5, -30)$, one has $k = \pm 15$. There is one case , $k = -15$, such that the resonances for all branches are distinct integers. The resonances and the simplified equation for this case are

III.2.c.1: $(p_1, p_2, p_3) = (3, 5, -30)$

$$\begin{aligned} y_{01} &= -\tfrac{1}{a_1}: \quad (r_{11}, r_{12}) = (1, 3), \\ y_{02} &= \tfrac{1}{a_1}: \quad (r_{21}, r_{22}) = (1, 5), \\ y_{03} &= -\tfrac{4}{a_1}: \quad (r_{31}, r_{32}) = (-5, 6), \\ y''' &= 2\tfrac{y'y''}{y} + a_1[yy'' - \tfrac{3}{2}(y')^2 + a_1 y^2 y' - \tfrac{1}{2}a_1^2 y^4]. \end{aligned} \tag{4.100}$$

Replacing y by λy such that $a_1\lambda = -1$, then(4.100.d) has the first integral

$$y'' = \frac{3}{2}\frac{(y')^2}{y} + \frac{1}{2}, \tag{4.101}$$

which can be integrated in terms of elliptic functions [6].

III.2.c.2: $(p_1, p_2, p_3) = (2, n, n)$ The case $k = n$ gives that $y_{01} = 0$. The case $k = -n$ gives the following equation

$$y''' = 2\frac{y'y''}{y} + a_1 yy'' + \frac{n^2+12}{4-n^2}a_1(y')^2 - \frac{16}{4-n^2}a_1^2 y^2 y' + \frac{4}{4-n^2}a_1^3 y^4, \tag{4.102}$$

where $a_1 y_{01} = -2$, $a_1 y_{02} = \frac{n-2}{2}$, $a_1 y_{03} = -\frac{n+2}{2}$, $(r_{11}, r_{12}) = (1, 2)$ and the resonances of the second and the third branches satisfy the following equations respectively

$$r_{2i}^2 - \frac{n+8}{2}r_{2i} + n = 0 \tag{4.103}$$

$$r_{3i}^2 + \frac{8-n}{2}r_{3i} - n = 0 \tag{4.104}$$

The compatibility condition at $r_{12} = 2$ is not satisfied identically unless $n = 6$. Then the roots of (4.103) and (4.104) are $(r_{21}, r_{22}) = (1, 6)$ and $(r_{31}, r_{32}) = (-2, 3)$ respectively. Thus the resonances and the simplified equation become

$$\begin{aligned} y_{01} &= -\tfrac{2}{a_1}: \quad (r_{11}, r_{12}) = (1, 2), \\ y_{02} &= \tfrac{2}{a_1}: \quad (r_{21}, r_{22}) = (1, 6), \\ y_{03} &= -\tfrac{4}{a_1}: \quad (r_{31}, r_{32}) = (-2, 3), \\ y''' &= 2\tfrac{y'y''}{y} + a_1[yy'' - \tfrac{3}{2}(y')^2 - \tfrac{1}{2}a_1 y^2 y' - \tfrac{1}{8}y^4] \end{aligned} \tag{4.105}$$

III.2.d: $\beta = 3$. The Diophantine equation (4.69) becomes

$$\sum_{j=1}^{3} p_j = \frac{\alpha^2}{2(\alpha^2-1)} \tag{4.106}$$

$(p_1, p_2, p_3) = (2, 4\alpha - 4, -4\alpha - 4)$ is a particular solution of equation (4.106). For this particular one has $k = \pm 4\alpha$. There is only one case, $k = -4\alpha$, such that the resonances of all branches are distinct integers. The resonances and the simplified equation for this case are

$$\begin{array}{ll} y_{01} = -\frac{1}{a_1}: & (r_{11}, r_{12}) = (1,2), \\ y_{02} = \frac{\alpha-1}{a_1}: & (r_{21}, r_{22}) = (4, \alpha-1), \\ y_{03} = -\frac{\alpha+1}{a_1}: & (r_{31}, r_{32} == (4, -\alpha-1), \\ \multicolumn{2}{l}{y''' = 3\frac{y'y''}{y} - \frac{2(\alpha^2-1)}{\alpha^2} + a_1 yy'' - \frac{6}{\alpha^2}a_1(y')^2 + \frac{6}{\alpha^2}a_1^2y^2y' - \frac{2}{\alpha^2}a_1^3y^4} \end{array} \tag{4.107}$$

Substituting $y = \frac{u'}{u}$ in (4.107) gives that

$$u^{(4)} = 3\frac{u''u'''}{u'} - \frac{2(\alpha^2-1)}{\alpha^2}\frac{(u'')^3}{(u')^2} \tag{4.108}$$

Substituting $u' = v^\alpha$ in (4.108) gives the following equation for v

$$vv''' = 3v'v'' \tag{4.109}$$

Integrating (4.109) gives $v'' = k_1 v^3$. Then either $v = k_2 z + k_3$ if $k_1 = 0$, or $v = \sum_{i=0}^{\infty} v_{4i}(z-z_0)^{4i-1}$, where z_0 is a simple pole of v. Since $u' = v^\alpha$, then in order that u,and consequently y, be of Painlevé type, it is necessary and sufficient that u' does not contain the term $\frac{1}{z-z_0}$. That is $\alpha \neq 0, \pm(1+4m)$ where $m \in \mathbb{Z}_+$.

Particularly if $\alpha = \pm 2$, then equation (4.106) becomes $\sum_{j=1}^{3} p_j = \frac{2}{3}$, which has the only solution $(p_1, p_2, p_3) = (2, 4, -12)$. This solution yields the particular case of (4.106) when $\alpha = \pm 2$. That is

$$\begin{array}{ll} y_{01} = -\frac{1}{a_1}: & (r_{11}, r_{12}) = (1,2), \\ y_{02} = -\frac{3}{a_1}: & (r_{21}, r_{22}) = (-3, 4), \\ y_{03} = \frac{1}{a_1}: & (r_{31}, r_{32}) = (1, 4), \\ \multicolumn{2}{l}{y''' = 3\frac{y'y''}{y} - \frac{3}{2}\frac{(y')^3}{y^2} + a_1[yy'' - \frac{3}{2}(y')^2 + \frac{3}{2}a_1y^2y' - \frac{1}{2}a_1^2y^4]} \end{array} \tag{4.110}$$

To find the canonical forms of the equations one should add non-dominant terms with coefficients that are locally analytic functions of z. When $c_2 = 0$ multiply both sides of (4.5) by y and add the non dominant terms of weight greater than -5. That is , one should consider the following equation when $c_2 = 0$

$$yy''' = c_1 y'y'' + a_1 y^2 y'' + a_2 y(y')^2 + a_3 y^3 y' + a_4 y^5 + A_1(z) yy'' + A_2(z)(y')^2 + A_3(z) y^2 y' + A_4(z) y^4 + A_5(z) y'' + A_6(z) yy' + A_7(z) y^3 + A_8(z) y' + A_9(z) y^2 + A_{10}(z) y + A_{11}(z). \tag{4.111}$$

When $c_2 \neq 0$ multiply both sides (4.5) by y^2 and add the non-dominant terms of weight -6. That is one should consider this equation

$$y^2 y''' = c_1 yy'y'' + c_2(y')^3 + a_1 y^3 y'' + a_1 y^2 (y')^2 + a_3 y^4 y' + a_4 y^6 + A_1(z) y^2 y'' + A_2(z) y(y')^2 + A_3(z) y^3 y' + A_4(z) y^5 + A_5(z) yy'' + A_6(z)(y')^2 + A_7(z) y^2 y' + A_8(z) y^4 + A_9(z) y'' + A_{10}(z) yy' + A_{11} y^3 + A_{12}(z) y' + A_{13}(z) y^2 + A_{14}(z) y + A_{15}(z). \tag{4.112}$$

The coefficients A_i can be determined by using the compatibility conditions at the resonances r_{ij} and the compatibility conditions corresponding to parametric zeros; that is the compatibility conditions at the resonances of the equations obtained by the transformation $y = \frac{1}{u}$.

I.b.iii: The transformation

$$y = \mu(z)\tilde{y}(x), \qquad x = \rho(z) \tag{4.113}$$

allows one to take

$$a_1 = 1, \quad A_1 + A_2 = 0 \tag{4.114}$$

The compatibility conditions at the resonances $r_1 = 1$ and $r_2 = 3$ give that

$$\begin{aligned} &2A_1 + A_2 - A_3 + A_4 = 0, \\ &2A_1 + A_2 - A_3 - \tfrac{5}{7} A_4 = 0, \\ &A_5 + A_6 - A_8 = 0, \\ &2A_5' + A_6' - A_7' + A_8' + 2A_9 - A_{10} + A_{11} - A_1(2A_5 + A_6 - A_7 + A_8) = 0. \end{aligned} \tag{4.115}$$

(4.114) and (4.115.a-b) give that $A_4 = 0$, $A_3 = -A_2 = A_1$. To find the conditions corresponding to movable zeros one has to substitute $y = \frac{1}{u}$ in (4.112) to get

$$u^2 u''' = 4uu'u'' - 3(u')^3 - uu'' + 3(u')^2 + A_1(u^2 u'' - u(u')^2 + uu') + A_5 u^3 u'' + A_7 u^2 u' - (2A_5 + A_6) u^2 (u')^2 - A_8 u^2 + A_9(u^4 u'' - 2u^3 (u')^2) + A_{10} u^3 u' - A_{11} u^3 + A_{12} u^4 u' - A_{13} u^4 - A_{14} u^5 - A_{15} u^6. \tag{4.116}$$

Substituting

$$u = \sum_{i=0}^{\infty} u_i (z - z_0)^{i+p}, \qquad p \in \mathbb{Z}_- \tag{4.117}$$

in (4.116), give that the term $A_9(u^4 u'' - 2u^3(u')^2)$ is dominant for all $p < 0$. Then $A_9 = 0$. There are two possibilities for the leading order p

(a) $p = -1$. Then $A_{12} \neq 0$, $A_5 = A_6 = A_{15} = 0$ and

$$A_{12}(z_0) u_0^2 = 1, \qquad (\tilde{r}_{j1}, \tilde{r}_{j2}) = (1, 2), \quad j = 1, 2 \tag{4.118}$$

Then the compatibility conditions at the resonances $\tilde{r}_{ji}$, $i = 1, 2$, $j = 1, 2$, give that

$$\begin{aligned} &A_{10} = 0, \quad A'_{12} = 2A_1 A_1 2, \quad A_{14} = -A_1 A_{12}, \quad A_{13} = A_{12}, \\ &A_7 = A_1, \quad A'_{11} = A''_1 - A_1 A'_1 \end{aligned} \tag{4.119}$$

The canonical form of the equation in this case is

$$\begin{aligned} y^2 y''' = {} & 2yy'y'' - (y')^3 - y^3 y'' - y^2 (y')^2 + A_1(y^2 y'' - y(y')^2 + y^3 y') + A'_1 y^2 y' \\ & + (A''_1 - A_1 A'_1) y^3 + A_{12}(y' + y^2) - A_1 A_{12}, \end{aligned} \tag{4.120}$$

where $A'_{12} = 2A_1 A_{12}$ and A_1, is an arbitrary analytic function of z. (4.120) has the first integral

$$y'' = \frac{(y')^2}{y} - yy' + A'_1 y - \frac{A_{12}}{y} + B, \tag{4.121}$$

where $B' - A_1 B = A_{12}$. (4.121) is of Painlevé type if and only if $A_1 = 0$. That is, $A_{12} = k_1$ and $B = k_1 z + k_2$. Replacing y by μy, z by νz such that $\mu\nu = 1$ and $k_1 \nu^4 = 1$, (4.121) becomes of the form of an equation considered by Bureau [6]. That is

$$y'' = \frac{(y')^2}{y} - yy' - \frac{1}{y} + z \tag{4.122}$$

(b) $p = -2$. Then $A_5 = A_6 = A_{10} = A_{12} = A_{15} = 0$. u_0 is arbitrary and the resonances are $(\tilde{r}_1, \tilde{r}_2) = (0, 2)$. Then the compatibility condition at $\tilde{r}_2 = 2$ gives that

$$A_7 = A'_1, \qquad A_8 = 0, \quad A_{11} = A''_1 - A'_1 A_1, \quad A_{13} = 0 \tag{4.123}$$

The canonical form in this case is

$$\begin{aligned} y^2 y''' = {} & 2yy'y'' - (y')^3 - y^3 y'' - y^2 (y')^2 + A_1(y^2 y'' - y(y')^2 + y^3 y') + A'_1 y^2 y' \\ & + (A''_1 - A'_1 A_1) y^3, \end{aligned} \tag{4.124}$$

where A_1 is an arbitrary function of z. (4.124 has the first integral

$$y'' = \frac{(y')^2}{y} - yy' + A'_1 y + B, \tag{4.125}$$

where $B' + A_1 B = 0$. (4.125) is of Painlev'e type if and only if $A_1' = 0$. That is $A_1 = k_1$ and $B = k_2 e^{k_1 z}$. Then applying the transformation $y = \frac{1}{u}$ gives an equation of the form

$$u'' = \frac{(u')^2}{u} - \frac{u'}{u} - k_2 e^{k_1 z} u^2 \tag{4.126}$$

(4.126) was considered in [6].

II.a.i: The transformation (4.113) allows one to assume that

$$a_1 = -\frac{3}{2}, \quad 2A_1 + A_2 - 5A_3 + 25A_4 = 0. \tag{4.127}$$

The compatibility conditions at $r_{11} = 1$, $r_{12} = 3$ and $r_{22} = 3$ and (4.127) give that

$$\begin{gathered} A_2 = -\tfrac{3}{4}A_1, \quad A_3 = \tfrac{3}{2}A_1, \quad A_4 = \tfrac{1}{4}A_1, \quad A_8 = A_5 + A_6, \\ (2A_5 + A_6 - A_7 + A_8)' - A_1(2A_5 + A_6 - A_7 + A_8) + 2A_9 - A_{10} + A_{11} = 0 \end{gathered} \tag{4.128}$$

To find the conditions produced by the movable zeros, one should substitute $y = \frac{1}{u}$ in (4.112) to get

$$\begin{aligned} u^2 u''' = {} & \tfrac{9}{2}uu'u'' - \tfrac{15}{4}(u')^3 - \tfrac{3}{2}(uu'' - 3(u')^2) - \tfrac{3}{4}u' + A_1(u^2u'' - \tfrac{5}{4}u(u')^2 + \tfrac{3}{2}uu' \\ & - \tfrac{1}{4}u) + A_5 u^3 u'' - (2A_5 + A_6)u^2(u')^2 + A_7 u^2 u' - A_8 u^2 + A_9(u^4 u'' \\ & - 2u^3(u')^2) + A_{10}u^3 u' - A_{11}u^3 + A_{12}u^4 u' - A_{13}u^4 - A_{14}u^5 - A_{15}u^6 \end{aligned} \tag{4.129}$$

Substituting (4.117) in (4.129) implies that there is only one possibility for the leading order $p = -2$ such that

$$A_5 = A_6 = A_9 = A_{10} = A_{12} = A_{14} = A_{15} = 0, \quad (\tilde{r}_1, \tilde{r}_2) = (0, 1), \tag{4.130}$$

where u_0 is arbitrary. The compatibility condition at $\tilde{r}_2 = 1$ together with , equations (4.128) and (4.130) give that

$$A_1 = A_8 = A_{13} = 0, \qquad A_{11} = A_7' \tag{4.131}$$

The canonical form of the equation in this case is

$$y^2 y''' = \frac{3}{2}yy'y'' - \frac{3}{4}(y')^3 - \frac{3}{2}(y^3 y'' + y^2(y')^2) - \frac{3}{4}y^4 y' + A_7 y^2 y' + A_7' y^3, \tag{4.132}$$

where A_7 is an arbitrary analytic function of z . (4.132) has the first integral

$$y'' = \frac{3}{4}\frac{(y')^2}{y} - \frac{3}{2}yy' - \frac{1}{4}y^3 + A_7 y + k_1, \tag{4.133}$$

where k_1 is an integration constant. (4.133) possesses the Painlevé property [6] [3].

II.b.i.1: The transformation (4.113) allows one to assume

$$a_1 = -3, \qquad 2A_1 + A_2 - 2A_3 + 4A_4 = 0. \tag{4.134}$$

The compatibility conditions at $r_{11} = 1$, $r_{12} = 2$ and $r_{22} = 2$ give

$$\begin{aligned} &2A_1 + A_2 - A_3 + A_4 = 0, \quad A_1 - A_3 + 2A_4 = 0, \quad 2A_5 - A_6 + A_7 = 0 \\ &A_5 - A_6 + A_7 = 0 \end{aligned} \tag{4.135}$$

To find the conditions corresponding to movable zeros, one should substitute $y = \frac{1}{u}$ in (4.111) to get

$$\begin{aligned} u^2u''' &= 5uu'u'' - 4(u')^3 - 3(uu'' - 2(u')^2) - 2u' + A_1u^2u'' - (2A_1 + A_2)u(u')^2 \\ &+ A_3uu' - A_4u + A_5(u^3u'' - 2u^2(u')^2) + A_6u^2u' - A_7u^2 + A_8u^3u' \\ &- A_9u^3 - A_{10}u^4 - A_{11}u^5. \end{aligned} \tag{4.136}$$

Substituting (4.117) in (4.136) gives that $p = -1$ is a possible leading order with

$$A_5 = 0, \quad (\tilde{r}_1, \tilde{r}_2) = (0, 2) \tag{4.137}$$

where u_0 is arbitrary. The compatibility condition at $\tilde{r}_2 = 2$ gives

$$(A_1 + A_2)A_8 - A_{10} - A_8' = 0, \quad A_2(A_1 + A_2) - A_2' - A_6 = 0, \quad A_{11} = 0. \tag{4.138}$$

(4.134), (4.135), (4.137) and (4.138) imply that

$$A_2 = 0, \ A_3 = 3A_1, \ A_4 = A_1, \ A_5 = A_6 = A_7 = 0, \ A_{10} = A_1A_8 - A_8', \ A_{11} = 0. \tag{4.139}$$

The canonical form of the equation for this case is

$$yy''' = y'y'' - 3y^2(y')^2 - 2y^3y' + A_1(yy'' + 3y^2y' + y^4) + A_8y' + A_9y^2 + (A_1A_8 - A_8')y, \tag{4.140}$$

where A_1, A_8 and A_9 are arbitrary functions of z. (4.140) has the first integral

$$y'' = -3yy' - y^3 - A_8 + By; \quad \text{where } B' - A_1B = A_9 \tag{4.141}$$

(4.141) is of Painlevé type [6] [3].

II.b.i.2: The transformation (4.113) allows one to take

$$a_1 = 4, \qquad 2A_1 + A_2 + A_4 = 0. \tag{4.142}$$

The compatibility conditions at $r_{j1} = 2$ and $r_{j2} = 2$, $j = 1, 2$ give

$$
\begin{aligned}
&-A_3' + \tfrac{A_3}{6}(6A_1 + 4A_2) + 2A_5 + A_7 = 0, \quad A_6 = 0\\
&-A_3' + 3A_7 + (5A_1 + 4A_)A_3 = 0, \quad A_2' + A_1' + (A_1 + A_2)(A_1 + 2A_2) = 0,\\
&\tfrac{1}{6}A_3A_3'' - \tfrac{1}{6}(2A_1 + A_2)'A_3^2 - \tfrac{1}{2}A_3A_8' + A_9' + \tfrac{1}{3}A_3A_3'(A_1 + 2A_2)\\
&-\tfrac{A_3^2}{6}(A_1 + A_2)(2A_1 + A_2) - \tfrac{1}{2}A_3A_7(A_1 + 2A_2) + A_9(A_1 + 2A_2) = 0,\\
&-\tfrac{A_3'''}{6} - \tfrac{1}{6}A_3^2A_3' + \tfrac{A_3}{6}(3A_1'' + 2A_2'') + \tfrac{A_3^3}{108}(2A_1 + A_2) + A_5''\\
&+\tfrac{1}{2}A_7'' - A_8' - \tfrac{1}{3}A_3A_9 + A_{10} + \tfrac{1}{12}A_3^2A_7 - \tfrac{A_3''}{2}(A_1 + 2A_2)\\
&+\tfrac{A_3}{6}(A_1 + 2A_2)(6A_1' + 4A_2') + (2A_5' + A_7' - A_8)(A_1 + 2A_2) = 0.
\end{aligned}
\tag{4.143}
$$

To find the compatibility conditions corresponding to movable zeros one should substitute $y = \frac{1}{u}$ to in (4.111) get equation with the same simplified equation of (4.136) with the same possible leading order $p = -1$, resonances and compatibility conditions (4.138). Equations (4.138), (4.142) and (4.143) give that

$$
\begin{aligned}
&A_2'' + A_2A_2' = 0,\\
&A_1' + A_1^2 = 0, \quad \text{if} \quad A_2 = 0,\\
&A_1 = \tfrac{A_2' - A_2^2}{A_2}, \quad \text{if} \quad A_2 \neq 0.
\end{aligned}
\tag{4.144}
$$

The following cases can be considered

(a) $A_1 = A_2 = 0$: One has

$$
\begin{aligned}
&A_4 = A_5 = A_6 = A_7 = A_{11} = 0, \quad A_3 = k_1, \quad A_8 = -\tfrac{1}{6}k_1k_2z + k_3,\\
&A_9 = k_2, \quad A_{10} = \tfrac{1}{6}k_1k_2
\end{aligned}
\tag{4.145}
$$

The canonical form of the equation in this case is

$$
yy''' = y'y'' + 4y^3y' + k_1y^2y' + (-\frac{k_1k_2}{6}z + k_3)y' + k_2y^2 + \frac{k_1k_2}{6} \tag{4.146}
$$

(4.146) has the first integral

$$
y'' = 2y^3 + k_1y^2 + (k_2z + k_4)y + \frac{k_1k_2}{6}z + k_3 \tag{4.147}
$$

Replacing y by $y - \frac{k_1}{6}$ (4.147) can be reduced to an equation of the form

$$
y'' = 2y^3 + (\hat{k}_1z + \hat{k}_2)y + \hat{k}_3, \tag{4.148}
$$

which is of Painlevé type [6] [3].

(b) $A_2 = 0$, $A_1 = \frac{1}{z}$: One has

$$
\begin{aligned}
&A_3 = \tfrac{k_1}{z}, \quad A_4 = -\tfrac{2}{z}, \quad A_5 = A_6 = A_{11} = 0, \quad A_7 = -\tfrac{2k_1}{z^2}, \quad A_9 = -\tfrac{k_1^2}{2z^3} + \tfrac{k_2}{z},\\
&A_8 = \tfrac{k_1}{3z^3} + \tfrac{k_1k_2}{6z} - \tfrac{k_1^3}{108z^3} + k_3, \quad A_{10} = \tfrac{4k_1}{3z^4} + \tfrac{k_1k_2}{3z^2} - \tfrac{k_1^3}{27z^4} + \tfrac{k_3}{z}.
\end{aligned}
\tag{4.149}
$$

The canonical form of the equation in this case is

$$yy''' = y'y'' + 4y^3y' + \tfrac{1}{z}(yy'' - 2y^4) + \tfrac{k_1}{z}y^2y' - \tfrac{2k_1}{z^2}y^3 + (-\tfrac{k_1^2}{2z^3} + \tfrac{k_2}{z})y^2 + (\tfrac{k_1}{3z^3} + \tfrac{k_1k_2}{6z} - \tfrac{k_1^3}{108z^3} + k_3)y' + (\tfrac{4k_1}{3z^4} + \tfrac{k_1k_2}{6z} - \tfrac{k_1^3}{27z^4} + \tfrac{k_3}{z})y. \tag{4.150}$$

(4.150) has the first integral

$$y'' = 2y^3 + \frac{k_1}{z}y^2 + (\frac{k_1^2}{6z^2} - k_2 + k_4z)y - \frac{k_1}{3z^3} - \frac{k_1k_2}{6z} + \frac{k_1^3}{108z^3} - k_3, \tag{4.151}$$

which can be transformed to an equation of the form (4.148) if one replaces y by $y - \frac{k_1}{6z}$.

(c) $A_2 = \frac{2}{z}$: One has

$$A_1 = -\tfrac{3}{z},\quad A_3 = k_1z,\quad A_4 = \tfrac{4}{z},\quad A_5 = A_6 = A_{11} = 0,\quad A_7 = \tfrac{8}{3}k_1, \quad A_9 = \tfrac{k_1^2}{2}z + \tfrac{k_2}{z},\quad A_8 = -\tfrac{k_1^3}{144}z^3 - \tfrac{k_1k_2}{12}z + \tfrac{k_3}{z},\quad A_{10} = \tfrac{k_1^3}{36}z^2 + \tfrac{k_1k_2}{6}. \tag{4.152}$$

The canonical form of the equation in this case is

$$yy''' = y'y'' + 4y^3y' + \tfrac{1}{z}(-3yy'' + 2(y')^2 + k_1z^2y^2y' + 4y^4 + \tfrac{8}{3}k_1zy^3) + (\tfrac{k_1^2}{2}z + \tfrac{k_2}{z})y^2 + (-\tfrac{k_1^3}{144}z^3 - \tfrac{k_1k_2}{12}z + \tfrac{k_3}{z})y' + (\tfrac{k_1^3}{36}z^2 + \tfrac{k_1k_2}{6})y \tag{4.153}$$

II.b.i.3: Using the transformation (4.113) one can assume that

$$a_1 = -1, \qquad 2A_1 + A_2 - A_3 + A_4 = 0. \tag{4.154}$$

If one applies the transformation $y = \frac{1}{u}$ to (4.11) then u satisfies an equation with the same simplified equation as in (4.136) with the same possible leading order, resonances and compatibility conditions (4.138). Then the compatibility conditions at the resonances $r_{11} = 2$ and $r_{12} = 3$ give

$$A_6 = A_7, \quad A_8 = A_9, \quad 2A_1 - 2A_2 - A_3 + 4A_4 = 0 \tag{4.155}$$

Using the conditions (4.138), (4.154) and (4.155), the compatibility conditions at the resonances $r_{21} = 2$ and $r_{22} = 6$ give that $A_i = 0$, $i = 1, 2, ..., 11$. That is the canonical form of the equation is

$$yy''' = y'y'' - y^2y'' + 2y^3y' \tag{4.156}$$

(4.156) has the first integral

$$y'' = -yy' + y^3 + k_1y, \tag{4.157}$$

where k_1 is an integration constant. (4.157) is of Painlevé type [6] [3].

II.b.ii.1: The transformation (4.113) allows one to assume that

$$a_1 = -2, \quad 2A_1 + A_2 - 3A_3 + 9A_4 = 0 \tag{4.158}$$

The compatibility condition at the resonances $r_{11} = 1$, $r_{12} = 2$ and $r_{22} = 2$ give that

$$\begin{aligned} &4A_1 + A_2 - 3A_3 + 5A_4 = 0, \quad 2A_5 + A_6 - A_7 + A_8 = 0, \\ &2A_1 + A_2 - A_3 + A_4 = 0, \quad 2A_5 + A_6 - A_7 + A_8 = 0. \end{aligned} \tag{4.159}$$

To find the compatibility conditions corresponding to parametric zeros, one should substitute $y = \frac{1}{u}$ in (4.112) to get the equation

$$\begin{aligned} u^2u''' = {} & 4uu'u'' - 3(u')^3 - 2(uu'' - 2(u')^2) - u' + A_1u^2u'' - (2A_1 + A_2)u(u')^2 \\ & + A_3uu' - A_4u + A_5u^3u'' - (2A_5 + A_6)u^2(u')^2 + A_7u^2u' - A_8u^2 \\ & + A_9(u^4u'' - 2u^2(u')^2) + A_{10}u^3u' - A11u^3 + A_{12}u^4u' - A_{13}u^4 \\ & - A_{14}u^5 - A_{15}u^6. \end{aligned} \tag{4.160}$$

Painlevé analysis of (4.160) gives that $p = -1$ is a possible leading order such that

$$\begin{aligned} &A_5 = A_6 = A_9 = A_{15} = 0, \quad A_{12} \neq 0. \\ &A_{12}(z_0)u_0^2 = 1, \quad (\tilde{r}_{i1}, \tilde{r}_{i2}) = (1, 2), \quad i = 1, 2 \end{aligned} \tag{4.161}$$

The compatibility conditions at the resonances $\tilde{r}_{i1} = 1$, $\tilde{r}_{i2}$, $i = 1, 2$ on using the conditions (4.158), (4.159) and (4.161) give that

$$\begin{aligned} &A_2 = -\tfrac{1}{2}A_1, \quad A_3 = 2A_1, \quad ,A_4 = \tfrac{1}{2}A_1, \quad A_7 = A_8 = A_{10} = A_{13} = 0, \\ &A_{12} = k_1(\neq 0), \quad A_{14} = \tfrac{1}{2}k_1A_1 \end{aligned} \tag{4.162}$$

The canonical form of the equation in this case is

$$\begin{aligned} y^2y''' = {} & 2yy'y'' - (y')^3 - 2y^3y'' - y^4y' + A_1(y^2y'' - \tfrac{1}{2}y(y')^2 + 2y^3y' + \tfrac{1}{2}y^5) \\ & + A_{11}y^3 + k_1y' + \tfrac{k_1}{2}A_1, \end{aligned} \tag{4.163}$$

where A_1 and A_{11} are arbitrary functions of z. If $A_1 = A_1^2$, then (4.163) has the first integral

$$y'' = \frac{1}{2}\frac{(y')^2}{y} - 2yy' - \frac{y^3}{2} + 2A_1y^2 - \frac{k_1}{2y} + By, \tag{4.164}$$

where $B' - A_1B = A_{11}$. (4.164) possesses the Painlevé property if and only if $A_1 = 0$ [6] [3]

II.b.ii.2: The transformation (4.113) allows one to assume that

$$a_3 = 3, \quad 2A_1 + A_2 + A_4 = 0 \tag{4.165}$$

The compatibility conditions at the resonances $r_{j1} = 2$ and $r_{j2} = 3$, $j = 1, 2$ and (4.165) give that

$$A_2 = A_4 = -A_1, \quad A_7 = 0, \quad 2A_5 + A_6 + A_8 - A_3' + \tfrac{1}{2}A_3A_1 = 0,$$
$$-\tfrac{1}{2}A_3'' + \tfrac{1}{2}A_3A_1' + 2A_5' + A_6' - A_{10} = 0, \tag{4.166}$$
$$\tfrac{3}{4}A_3A_3' - \tfrac{1}{2}A_3^2A_1 - \tfrac{1}{2}A_3A_5 - A_3A_8 + 2A_9 + A10.$$

Substituting $y = \frac{1}{u}$ in (4.112) gives the equation

$$u^2u''' = 4uu'u'' - 3(u')^3 + 3u' + A_1(u^2u'' - u(u')^2 + u) + A_3uu' + A_5u^3u''$$
$$- (2A_5 + A_6)u^2(u')^2 - A_8u^2 + A_9(u^4u'' - 2u^3(u')^2) + A_{10}u^3u'$$
$$- A_{11}u^3 + A_{12}u^4u' - A_{13}u^4 - A_{14}u^5 - A_{15}u^6. \tag{4.167}$$

Painlevé analysis of (4.167) gives that $p = -1$ is a possible leading order with

$$A_5 = A_6 = A_9 = A_{15} = 0, \quad A_{12} \neq 0,$$
$$A_{12}(z_0)u_0^2 = 1, \quad (\tilde{r}_{i1}, \tilde{r}_{i2}) = (1, 2), \quad i = 1, 2 \tag{4.168}$$

The compatibility conditions corresponding to parametric zeros by using (4.166) and (4.168) give

$$A_1 = k_1, \quad A_{10} = 0, \quad A_{12}' = 2A_1A_{12}, \quad A_{13} = 0, \quad A_{14} = -A_1A_{12}. \tag{4.169}$$

Thre following cases have to be considered

(a) $k_1 = 0$; then one has

$$A_3 = k_3z + k_4, \quad A_8 = k_3, \quad A_{11} = \frac{k_3^2}{4}z + \frac{1}{4}k_3k_4, \quad A_{12} = k_2(\neq 0) \tag{4.170}$$

The canonical form of the equation in this case is

$$y^2y''' = 2yy'y'' - (y')^3 + 3y^4y' + (k_3z + k_4)y^3y' + k_3y^4 + \frac{k_3}{4}(k_3z + k_4)y^3 + k_2y' \tag{4.171}$$

If $k_3 = 0$, then (4.171) has the first integral

$$y'' = \frac{(y')^2}{y} + y^3 + \frac{k_4}{2}y^2 - \frac{k_2}{y} + k_5, \tag{4.172}$$

which is of Painlevé type [6] [3].

(b) $k_1 \neq 0$; then one has

$$A_3 = -\tfrac{k_3}{k_1} + k_4e^{k_1z}, \quad A_8 = \tfrac{k_3}{2} + \tfrac{k_1k_4}{e}^{2k_1z},$$
$$A_{11} = \tfrac{k_4}{4}e^{k_1z}(-k_3 + K_1k_4e^{k_1z}), \quad A_{12} = k_2e^{2k_1z}, \quad A_{14} = -k_1k_2e^{2k_1z}. \tag{4.173}$$

The canonical form of the equation in this case becomes

$$y^2y''' = 2yy'y'' - (y')^3 + 3y^4y' + k_1(y^2y'' - y(y')^2 - y^5) + (k_4e^{k_1z} - \tfrac{k_3}{k_1})y^3y'$$
$$+ (\tfrac{k_3}{2} + \tfrac{k_1k_4}{2}e^{2k_1z})y^4 + \tfrac{k_4}{4}e^{k_1z}(-k_3 + k_1k_2e^{k_1z})y^3 + k_2e^{2k_1z}y' - k_1k_2e^{2k_1z}. \tag{4.174}$$

If $k_4 = 0$, then (4.174) has the first integral

$$y'' = \frac{(y')^2}{y} + y^3 - \frac{k_3}{2k_1}y^2 + k_5 e^{k_1 z} - \frac{k_2 e^{2k_1 z}}{y}, \tag{4.175}$$

which, within the transformation $y = e^{\frac{k_1}{2}z} v(\frac{2}{k_1} e^{\frac{k_1}{2}z})$, becomes

$$\ddot{v} = \frac{\dot{v}^2}{v} - \frac{1}{t}\dot{v} + v^3 - \frac{2k_3}{k_1 t}v^2 - \frac{k_2}{v} + \frac{2k_5}{k_1 t}, \tag{4.176}$$

where $t = \frac{2}{k_1} e^{\frac{k_1}{2}z}$. (4.176) has a special form of the third Painlevé equation P_{III}.

II.b.iii: The transformation (4.113) allows one to assume that

$$a_3 = 1, \quad A_1 = 0 \tag{4.177}$$

The compatibility conditions at the resonances $r_{j1} = 1$ and $r_{j2} = 2$, $j = 1, 2$ and (4.177) give that,

$$A_1 = A_2 = A_3 = A_4 = A_7 = 0, \quad 2A_5 + A_6 + A_8 = 0. \tag{4.178}$$

Substituting $y = \frac{1}{u}$ in (4.112) give the equation

$$\begin{aligned} u^2 u''' = {} & 2uu'u'' - (u')^3 + u' + A_5 u^3 u'' - (2A_5 + A_6)u^2(u')^2 - A_8 u^2 \\ & + A_9(u^4 u'' - 2u^3(u')^2) + A10 u^3 u' - A_{11}u^3 + A_{12}u^4 u' - A_{13}u^4 \\ & - A14 u^5 - A_{15}u^6 \end{aligned} \tag{4.179}$$

Painlevé analysis of (4.179) gives that there are three possibilities according to the number of Painlevé branches

(a) The leading order is $p = -1$ with

$$\begin{gathered} A_9 = A_{12} = A_{15} = 0, \quad A_6 = -3A_5, \quad A_5 \neq 0, \\ A_5(z_0)u_0 = -1 : \quad (\tilde{r}_1, \tilde{r}_1) = (1, 3). \end{gathered} \tag{4.180}$$

The compatibility conditions corresponding to movable zeros at the resonances $\tilde{r}_1$ and $\tilde{r}_2$, (4.178) and (4.180) give that

$$A_8 = A_5, \quad A_{10} = 3A_5', \quad A_{13} = -A_5'', \quad A_{11} = A_{14} = 0 \tag{4.181}$$

The canonical form of the equation in this case is

$$y^2 y''' = 4yy'y'' - 3(y')^3 + y^4 y' + A_5(yy'' - 3(y')^2 + y^4) + 3A_5' yy' - A_5'' y^2, \tag{4.182}$$

where A_5 is an arbitrary function of z.

(b) The leading order is $p = -1$ with two branches

$$\begin{gathered} A_5 = A_6 = A_9 = A_{15} = 0, \quad A_{12} \neq 0 \\ A_{12}(z_0)u_0^2 = 3 : \quad (\tilde{r}_{i1}, \tilde{r}_{i2}) = (2, 3), \quad i = 1, 2 \end{gathered} \tag{4.183}$$

The compatibility conditions corresponding to movable zeros at the resonances $\tilde{r}_{i1}$, $\tilde{r}_{i2}$, $i = 1, 2$, (4.178) and (4.183) give that

$$\begin{array}{l} A_8 = 0, \quad A_{12}A_{12}'' = (A_{12}')^2, \quad A_{14} = -\frac{1}{3}A_{12}', \quad A_{13} = -A_{10}' + \frac{A_{12}'}{4A_{12}}A_{10} \\ A_{10}'' - \frac{3}{2}\frac{A_{12}'}{A_{12}}A_{10}' + \frac{1}{2}(\frac{A_{12}'}{A_{12}})^2 A_{10} = 0, \quad A_{11} = -\frac{3}{4}\frac{A_{10}A_{10}'}{A_{12}} + \frac{3}{8}\frac{A_{12}'}{A_{12}}A_{10}^2. \end{array} \tag{4.184}$$

One should consider the following cases:

(i) $A_{12} = k_1 (\neq 0)$. (4.184) gives that

$$A_{14} = 0, \quad A_{13} = -K_3, \quad A_{10} = k_3 z + K_4, \quad A_{11} = -\frac{3k_3}{4k_1}(k_3 z + k_4). \tag{4.185}$$

The canonical form of the equation in this case is

$$\begin{array}{l} y^2 y''' = 4yy'y'' - 3(y')^3 + y^4 y' + (k_3 z + k_4)yy' + k_1 y' \\ \quad - \frac{3k_3}{4k_1}(k_3 z + k_4)y^3 - k_3 y^2. \end{array} \tag{4.186}$$

If $k_3 = 0$, then (4.186) has the first integral

$$y'' = \frac{(y')^2}{y} - \frac{1}{2}k_4 - \frac{k_1}{3y} + k_5 y^2 + y^3. \tag{4.187}$$

(4.187) is of Painlevé type [6] [3].

(ii) $A_{12} = k_2 e^{k_1 z}$; $k_1 k_2 \neq 0$. (4.185) gives that

$$\begin{array}{l} A_{14} = -\frac{k_1 k_2}{3}e^{k_1 z}, \quad A_{11} = -\frac{3k_1 k_3}{8k_2}(k_3 e^{k_1 z} + k_4 e^{\frac{k_1}{2}z}), \\ A_{10} = k_3 e^{k_1 z} + k_4 e^{\frac{k_1}{2}z}, \quad A_{13} = -\frac{k_1}{4}(3k_3 e^{k_z} + k_4 e^{\frac{k_1}{2}z}). \end{array} \tag{4.188}$$

The canonical form of the equation in this case is

$$\begin{array}{l} y^2 y''' = 4yy'y'' - 3(y')^3 + (k_3 e^{k_1 z} + k_4 e^{\frac{k_1}{2}z})yy' + k_2 e^{k_1 z}y' - \frac{k_1 k_2}{e}{}^{k_1 z}y \\ \quad - \frac{3k_1 k_3}{8k_2}(k_3 e^{k_1 z} + k_4 e^{\frac{k_1}{2}z})y^3 - \frac{k_1}{4}(3k_3 e^{k_1 z} + k_4 e^{\frac{k_1}{2}z})y^2 \end{array} \tag{4.189}$$

If $k_3 = 0$, then (4.189) has the first integral

$$y'' = \frac{(y')^2}{y} - \frac{k_4}{2}e^{\frac{k_1}{2}z} - \frac{k_2}{3}\frac{e^{k_1 z}}{y} + y^3 + k_5 y^2, \tag{4.190}$$

which, under the transformation $e^{\frac{k_1}{4}z}v(t)$, $t = \frac{4}{k_1}e^{\frac{k_1}{4}z}$ becomes

$$\ddot{v} = \frac{\dot{v}^2}{v} - \frac{\dot{v}}{t} + v^3 - \frac{k_3}{3v} + \frac{4k_5 v^2 - 2k_4}{k_1 t} \tag{4.191}$$

(4.191) is of the form of the third Painlevé equation P_{III}.

(c) The leading order is $p = -1$ with

$$\begin{array}{c} A_9 = A_{15} = 0, \quad A_6 = -2A_5, \quad A_{12} = -\frac{1}{2}A_5, \quad A_5 \neq 0, \\ A_5(z_0)u_{01} = -2: \quad (\tilde{r}_{11}, \tilde{r}_{12}) = (1, 2), \\ A_5(z_0)u_{02} = -6: \quad (\tilde{r}_{21}, \tilde{r}_{22}) = (-3, 2). \end{array} \tag{4.192}$$

The compatibility conditions corresponding to movable zeros at $\tilde{r}_{11} = 1,\ \tilde{r}_{12} = 2,\ \tilde{r}_{22} = 2$ give by using (4.178) and (4.192) that

$$A_8 = 0, \quad A_{10} = \frac{3}{2}A_5', \quad A_{13} = -\frac{1}{2}A_5'', \quad A_{14} = \frac{1}{4}A_5A_5' \tag{4.193}$$

The canonical form of the equation in this case is

$$\begin{aligned} y^2y''' &= 4yy'y'' - 3(y')^3 + y^4y' + A_5(yy'' - 2y(y')^2) + \tfrac{3}{2}A_5'yy' + A_{11}y^3 - \tfrac{10}{4}A_5^2y' \\ &- \tfrac{1}{2}A_5''y^2 + \tfrac{1}{4}A_5A_5'y, \end{aligned} \tag{4.194}$$

where A_5, A_{11} are arbitrary functions of z. If $A_{11} = BB'$, where $B' = -\frac{1}{2}A_5$, then (4.194) has the first integral

$$y'' = \frac{(y')^2}{y} - B'\frac{y'}{y} + y^3 + By^2 - B'' \tag{4.195}$$

Replacing y by $-y(-z)$, (4.195) becomes

$$y'' = \frac{(y')^2}{y} - B'\frac{y'}{y} + y^3 - By^2 + B'' \tag{4.196}$$

(4.196) is of Painlevé type [6] [3].

II.c.2: The transformation (4.113) allows one to take

$$a_3 = 2, \qquad A_1 = 0. \tag{4.197}$$

The compatibility conditions at $r_{j1} = 1,\ r_{j2} = 4,\ j = 1, 2$ by using (4.197) give that

$$A_2 = A_3 = A_4 = A_7 = 0, \quad A_9 = \frac{1}{2}A_6', \quad A_5'' + A_{10} = A_8 \tag{4.198}$$

Substituting $\frac{1}{u}$ in (4.111) gives

$$\begin{aligned} u^2u''' &= 4uu'u'' - 2(u')^3 + 2u' + A_5(u^3u'' - 2u^2(u')^2) - A_6u^2u' \\ &+ A_8u^3u' - A_9u^3 - A_{10}u^4 - A_{11}u^5 \end{aligned} \tag{4.199}$$

Painlevé analysis of (4.199) gives that $p = -1$ is a possible leading order with

$$A_5 = 0, \quad u_0 \ \text{ arbitrary}, \quad (\tilde{r}_1, \tilde{r}_2) = (0, 3) \tag{4.200}$$

Then the compatibility conditions corresponding to movable zeros at $\tilde{r}_2 = 3$ by using (4.198) and (4.200) give

$$A_8 = A_9 = A_{10} = A_{11} = 0, \quad A_6 = k_1 \tag{4.201}$$

The canonical form of the equation in this case is

$$yy''' = 2y'y'' + 2y^3y' + k_1yy'. \tag{4.202}$$

(4.202) has the first integral

$$y'' = 2y^3 + k_2 y^2 - \frac{k_1}{2}, \tag{4.203}$$

where k_2 is an integration constant. (4.203) can be solved in terms of elliptic functions [6] [3].

II.d.2: The transformation (4.113) allows one to assume

$$a_3 = \frac{3}{2}, \quad A_1 = 0. \tag{4.204}$$

The compatibility conditions at the resonances $r_{j1} = 1,\ r_{j2} = 3,\ j = 1,2$ on using (4.204) give

$$A_2 = A_3 = A_4 = 0,\ A_8 = A_5 + A_6,\ A_{10} = A_5' + A_6' + A_8',\ A_7' = 2A_9 + A_{11}. \tag{4.205}$$

Substituting $y = \frac{1}{u}$ in (4.112) gives

$$\begin{aligned} u^2 u''' = {} & 3uu'u'' - \tfrac{3}{2}(u')^3 + \tfrac{3}{2}u' + A_5 u^3 u'' - (2A_5 + A_6)u^2(u')^2 \\ & + A_7 u^2 u' - A_8 u^2 + A_9(u^4 u'' - 2u^3(u')^2) + A_{10}u^3 u' - A_{11}u^3 \\ & + A_{12}u^4 u' - A_{13}u^4 - A_{14}u^5 - A_{15}u^6. \end{aligned} \tag{4.206}$$

Painlevé analysis of (4.206) implies the following possible cases:

(a) $p = -1$ is a leading order with

$$\begin{aligned} & A_5 = A_6 = A_9 = A_{15} = 0, \quad A_{12} \neq 0 \\ & A_{12}(z_0)u_0^2 = \tfrac{3}{2}: \quad (\tilde{r}_{i1}, \tilde{r}_{i2}) = (1,3),\ i = 1,2 \end{aligned} \tag{4.207}$$

The compatibility conditions corresponding to movable zeros at $\tilde{r}_{i1} = 1,\ \tilde{r}_{i2} = 2,\ i = 1,2$ by using (4.205) and (4.207) give

$$A_8 = A_{10} = A_{11} = A_{13} = A_{14} = 0,\ \ A_7 = k_2,\ \ A_{12} = k_1(\neq 0). \tag{4.208}$$

The canonical form of the equation in this case is

$$y^2 y''' = 3yy'y'' - \frac{3}{2}(y')^3 + \frac{3}{2}y^4 y' + k_1 y' + k_2 y^2 y' \tag{4.209}$$

(4.209) has the first integral

$$y'' = \frac{(y')^2}{2y} + \frac{3}{2}y^3 + k_3 y^2 - k_2 y - \frac{k_1}{3y}. \tag{4.210}$$

(4.210) is of Painlevé type [6] [3].

(b) $p = -1$ is a leading order with

$$\begin{aligned} & A_9 = 0,\ \ A_6 = -\tfrac{1}{2}A_5,\ \ A_{12} = \tfrac{3}{2}A_5^2,\ \ A_{15} = -\tfrac{1}{2}A_5^3, \\ & A_5(z_0)u_{01} = -1: \quad (\tilde{r}_{11}, \tilde{r}_{12}) = (1,2), \\ & A_5(z_0)u_{02} = 1: \quad (\tilde{r}_{21}, \tilde{r}_{22}) = (1,4), \\ & A_5(z_0)u_{03} = -3: \quad (\tilde{r}_{31}, \tilde{r}_{32}) = (-3,4) \end{aligned} \tag{4.211}$$

Then the compatibility conditions corresponding to movable zeros by using (4.205) and (4.211) give that

$$A_m = 0, \quad m = 1, 2, ...15 \tag{4.212}$$

That is the equation attains only dominant terms.

III.1.c: The transformation (4.113) allows one to assume that

$$a_4 = 2, \quad 2A_1 + A_2 + A_3 + A_4 = 0. \tag{4.213}$$

The compatibility conditions at the resonances $r_{j1} = 2,\ r_{j2} = 3,\ j = 1, 2, 3$ and (4.213) give

$$A_m = 0,\ m = 1, 2, ..., 9. \tag{4.214}$$

Substituting $y = \frac{1}{u}$ in (4.111) give the equation

$$u^2 u''' = 4uu'u'' - 2(u')^3 + 2u - A_{10}u^4 - A_{11}u^5. \tag{4.215}$$

Painlevé analysis (4.215) implies that $p = -1$ is a possible leading order with

$$u_0 \ \text{arbitrary}, (\tilde{r}_1, \tilde{r}_2) = (0, 3). \tag{4.216}$$

Then the compatibility condition at $\tilde{r}_1 = 3$ gives

$$A_{10} = A_{11} = 0. \tag{4.217}$$

The canonical form of the equation in this case is

$$yy''' = 2y'y'' + 2y^5 \tag{4.218}$$

(4.218) was considered by Martynov [12].

III.2.a: The transformation (4.113) allows one to assume that

$$a_1 = -\frac{5}{2}, \quad A_1 = 0. \tag{4.219}$$

The compatibility conditions at the resonances $r_{11} = 1, r_{12} = 2, r_{31} = 1$ give that

$$A_2 = A_3 = A_4 = 0, \ \ 2A_5 + A_6 - A_7 + A_8 = 0. \tag{4.220}$$

Substituting $y = \frac{1}{u}$ in (4.112) gives the equation

$$\begin{aligned} u^2u''' = {} & \tfrac{9}{2}uu'u'' - \tfrac{15}{4}(u')^3 - \tfrac{5}{2}uu'' + \tfrac{23}{4}(u')^2 - \tfrac{9}{4}u' + \tfrac{1}{4} + A_5u^3u'' \\ & - (2A_5 + A_6)u^2(u')^2 + A_7u^2u' - A_8u^2 + A_9(u^4u'' - 2u^2(u')^2) \\ & + A_{10}u^3u' - A_{11}u^3 + A_{12}u^4u' - A_{13}u^4 - A_{14}u^5 - A_{15}u^6. \end{aligned} \tag{4.221}$$

Painlevé analysis of (4.221) implies that $p = -2$ is the only possible leading order such that the resonances are distinct integers with

$$\begin{gathered} A_5 = A_6 = A_9 = A_{10} = A_{12} = A_{14} = A_{15} = 0, \\ u_0 \text{ arbitrary}, \quad (\tilde{r}_1, \tilde{r}_2) = (0, 1). \end{gathered} \tag{4.222}$$

The compatibility condition corresponding to movable zeros at the resonance on using $\tilde{r}_2 = 1$ (4.220) and (4.222) gives that

$$A_7 = A_8, \quad A_{13} = 0. \tag{4.223}$$

The canonical form of the equation in this case is

$$y^2 y''' = \frac{3}{2} y y' y'' - \frac{3}{4}(y')^3 - \frac{5}{2} y^3 y'' - \frac{3}{4} y^2 (y')^2 - \frac{9}{4} y^4 y' - \frac{1}{4} y^6 + A_7(y^2 y' + y^4) + A_{11} y^3 \tag{4.224}$$

where A_7, A_{11} are arbitrary functions of z. If $A'_{11} = A''_7$, then (4.224) has the first integral

$$y'' = \frac{3}{4}\frac{(y')^2}{y} - \frac{3}{2} y y' - \frac{1}{4} y^4 + A_7 y + B, \tag{4.225}$$

where $B = A_{11} - A'_7$. (4.225) is of Painlevé type [6] [3].

III.2.b.i.1: The transformation (4.113) allows one to assume that

$$a_1 = -2, \quad 2A_1 + A_2 + 2A_3 + 4A_4 = 0. \tag{4.226}$$

For the sake of simplicity, one first finds the compatibility conditions corresponding to movable zeros. Substituting $y = \frac{1}{u}$ in (4.111) gives an equation with the same simplified equation as of (4.136) with the same possible leading order $p = -1$, resonances (4.137) and compatibility conditions (4.138). Then the compatibility conditions at the resonances $r_{11} = 1$, $r_{12} = 3$, $r_{21} = 4$, $r_{22} = 6$ by using (4.137), (4.138) and (4.226) give that

$$\begin{aligned} &A_m = 0, \quad m = 1, 2..., 6, \quad A_9 = k_1 \text{ (constant)}, \quad A_{10} = -A'_8, \\ &A_8 = A'_7 + k_1, \quad A''''_7 + A_7 A''_7 + (A'_7 - k_1)(A'_7 + 2k_1) = 0 \end{aligned} \tag{4.227}$$

One should note that the equation which A_7 satisfies, is a special form of (2.52) a member of P^4_I. One may consider the following cases:

(a) $k_1 = 0, A_7 = -\frac{12}{z^2}$. Then $A_8 = \frac{24}{z^3}$, $A_{10} = \frac{72}{z^4}$. The canonical form of the equation in this case is

$$y y''' = y' y'' - 2y^2 y'' + y^3 y' + y^5 - \frac{12}{z^2} y^3 + \frac{24}{z^3} y' + \frac{72}{z^4} y. \tag{4.228}$$

(4.228) has the first integral

$$y'' = -y y' + y^3 - \frac{12}{z^2} y - \frac{24}{z^3}, \tag{4.229}$$

which is of Painlevé type [6] [3].
(b) $A_7'' = 0,\ A_7' = k_1$. Then $A_8 = 2k_1,\ A_{10} = 0$. The canonical form of the equation in this case

$$yy''' = y'y'' - 2y^2y'' + y^3y' + y^5 + (k_1z + k_2)y^3 + 2k_1y' + k_1y^2. \qquad (4.230)$$

(4.230) has the first integral

$$y'' = -3yy - y^3 - (k_1z + k_2)y - 2k_1, \qquad (4.231)$$

which is of Painlevé type [6] [3].
(c) $A_7'' = 0,\ A_7' = -2k_1$. Then $A_8 = -k_1,\ A_{10} = 0$. The canonical form of the equation in this case is

$$yy''' = y'y'' - 2y^2y'' + y^3y' + y^5 + (-2k_1z + k_2)y^3 - k_1y' + k_1y^2. \qquad (4.232)$$

(4.232) has the first integral

$$y'' = \frac{1}{2}y^3 + (-k_1z + \frac{k_2}{2})y + k_1, \qquad (4.233)$$

which can be solved in terms of elliptic functions if $k_1 = 0$, or can be transformed to the second Painlevé equation P_{II} if $k_1 \neq 0$ [6] [3].

III.2.b.i.2: The transformation (4.113) allows one to assume

$$a_1 = -1, \quad 2A_1 + A_2 + A_3 + A_4 = 0. \qquad (4.234)$$

For the sake of simplicity one first obtains the compatibility conditions corresponding to movable zeros. Substituting $y = \frac{1}{u}$ in (4.111) gives an equation with the same simplified equation as of (4.136), the same possible leading order $p = -1$, the same resonances (4.137) and same compatibility conditions (4.138). Then the compatibility conditions at the resonances $r_{11} = 1,\ r_{12} = 4,\ r_{21} = 3,\ r_{22} = 4$ on using (4.137), (4.138) and (4.234) give that

$$A_{10} = A_{11} = A_m = 0,\ m = 1, 2, ..., 6,\ \ A_8 = A_9 = k_1,\ \ A_7 = 2k_1z + k_2, \qquad (4.235)$$

where k_1, k_2 are constants of integration. The canonical form of the equation in this case is

$$yy''' = y'y'' - y^2y'' + 4y^3y' + 2y^5 + (2k_1z + k_2)y^3 + k_1(y' + y^2). \qquad (4.236)$$

(4.236) has the first integral

$$y'' = 2y^3 + (2k_1z + k_2)y - k_1, \qquad (4.237)$$

which can be solved in terms of elliptic functions if $k_1 = 0$ or it can be transformed to the second Painlevé equation P_{II} if $k_1 \neq 0$.

III.2.c.1: The transformation (4.113) allows one to take

$$a_1 = -1, \quad A_2 = 0. \tag{4.238}$$

The compatibility conditions at the resonances $r_{11} = 1$ $r_{12} = 3,\ r_{21} = 1,\ r_{22} = 5$ give

$$\begin{aligned}
&A_2 = A_3 = A_4 = 0,\ A_5 - A_6 + 2A_7 = 0,\ \tfrac{7A_5}{24} + \tfrac{17A_6}{6} - \tfrac{19A_7}{36} = 0,\\
&103A_6' - 89A_7' - 103A_8 - 55A_9 = 0,\\
&(A_8 + A_9 - 2A_5' - A_6' - A_7')(-\tfrac{A_5}{36} - \tfrac{5A_6}{36} + \tfrac{A_7}{9}) - \tfrac{A_7'}{3}(2A_5 + A_6 + A_7)\\
&+\tfrac{1}{9}(2A_9 - A_8)(2A_5 + A_6 + A_7) - \tfrac{A_5'''}{3} - \tfrac{A_6'''}{6} - \tfrac{A_7'''}{6} + \tfrac{A_8''}{2} + \tfrac{A_9''}{2} - A_{10}' + A_{10} = 0.
\end{aligned} \tag{4.239}$$

To find the compatibility conditions corresponding produced by the movable zeros one should substitute $y = \frac{1}{u}$ in (4.111) to get the equation

$$\begin{aligned}
u^2u''' = {}& 4uu'u'' - 2(u')^3 - uu'' - \tfrac{1}{2}(u')^2 + 2u' - \tfrac{1}{2} + A_5(u^3u'' - 2u(u')^2)\\
&+ A_6u^2u' - A_7u^2 + A_8u^3u' - A_9u^3 - A_{10}u^4 - A_{11}u^5
\end{aligned} \tag{4.240}$$

Painlevé analysis of (4.240) implies that $p = -1$ is a possible leading order with

$$A_5 = 0, \quad u_0 \ \text{arbitrary}, \quad (\tilde{r}_1, \tilde{r}_2) = (0, 3). \tag{4.241}$$

The compatibility condition at $\tilde{r}_2 = 3$ by using (4.239) and (4.241) gives

$$A_6 = A_7 = A_8 = A_9 = A_{11} = 0, \quad A_{10} = k_1 \ \text{(constant)}. \tag{4.242}$$

The canonical form of the equation in this case is

$$yy''' = 2y'y'' - y^2y'' + \frac{3}{2}y(y')^2 + 2y^3y' + \frac{1}{2}y^5 + k_1y. \tag{4.243}$$

(4.243) has the first integral

$$y'' = \frac{3}{2}\frac{(y')^2}{y} + \frac{1}{2}y^3 + \frac{k_1}{y}, \tag{4.244}$$

which can be solved in terms of elliptic functions [6].

III.2.c.2: The transformation (4.113) allows one to assume

$$a_1 = -2, \quad A_2 = 0. \tag{4.245}$$

The compatibility conditions at $r_{11} = 1,\ r_{12} = 2,\ r_{21} = 1$ give

$$A_1 = A_3 = A_4 = 0, \quad 2A_5 - A_6 + A_7 = 0 \tag{4.246}$$

For the sake of simplicity one can find the compatibility conditions corresponding to movable zeros before obtaining the compatibility condition corresponding to movable poles at $r_{22} = 6$. Substituting $y = \frac{1}{u}$ in (4.111) gives the equation

$$u^2u''' = 4uu'u'' - 2(u')^3 - uu'' + (u')^2 + 2u' - 1 + A_5(u^3u'' - u^2(u')^2) \\ + A_6u^2u' - A_7u^2 + A_8u^3u' - A_9u^3 - A_{10}u^4 - A_{11}u^5. \tag{4.247}$$

Painlevé analysis of (4.247) implies that $p = -1$ is a possible leading order with

$$A_5 = 0, \quad u_0 \text{ arbitrary}, \quad (\tilde{r}_1, \tilde{r}_2) = (0, 3). \tag{4.248}$$

The compatibility condition produced by movable zeros at the resonance on using (4.246) and (4.248) give

$$A_6 = A_7, \quad A_8 = A_{11} = 0, \quad A_9 = -A_6', \quad A_{10} = k_1 \text{ (constant)}. \tag{4.249}$$

Then the compatibility condition at the resonance $r_{22} = 6$ on using (4.249) gives

$$A_6 = A_7 = A_9 = 0 \tag{4.250}$$

The canonical form of the equation in this case is

$$yy''' = 2y'y'' - 2y^2y'' + 3y(y')^2 + 2y^3y' + y^5 + k_1y. \tag{4.251}$$

(4.251) has the first integral

$$y'' = \frac{3}{2}\frac{(y')^2}{y} + \frac{1}{2}y^3 + \frac{k_1}{2y}, \tag{4.252}$$

which can be solved in terms of elliptic functions [6].

III.2.d The transformation (4.113) allows one to assume that

$$a_1 = -1, \quad A_1 = 0 \tag{4.253}$$

The compatibility conditions at $r_{11} = 1,\ r_{12} = 2,\ r_{21} = 1,\ r_{22} = 4$ give that

$$\begin{aligned} &A_2 = A_3 = A_4 = 0, \quad 2A_5 + A_6 - A_7 + A_8 = 0, \\ &(4A_5 + 3A_6 + A_7 - A_8)' - 6A_9 - 2A_{10} = 0, \\ &\tfrac{1}{2}(2A_5 + A_6 + A_7 + A_8)'' - (2A_9 + A_{10} - A_{11})' - \tfrac{5}{12}(2A_5 + A_6 + A_7 + A_8)^2 \\ &+\tfrac{1}{6}(2A_5 + A_6 + A_7 + A_8)(2A_5 - 2A_6 + A_7 + 4A_8) = 0 \end{aligned} \tag{4.254}$$

Substituting $y = \frac{1}{u}$ in (4.112) gives the equation

$$\begin{aligned} u^2u''' = {}& 3uu'u'' - \tfrac{3}{2}(u')^3 - uu'' + \tfrac{1}{2}(u')^2 + \tfrac{3}{2}u' - \tfrac{1}{2} + A_5u^3u'' \\ & - (2A_5 + A_6)u^2(u')^2 + A_7u^2u' - A_8u^2 + A_9(u^4u'' - 2u^3(u')^2) \\ & + A_{10}u^3u' - A_{11}u^3 + A_{12}u^4u' - A_{13}u^4 - A_{14}u^5 - A_{15}u^6. \end{aligned} \tag{4.255}$$

Painlevé analysis of equation (4.255) shows that $p = -1$ is a possible leading order with one of the following choices

(a)

$$A_5 = A_6 = A_9 A_{15} = 0, \quad A_{12} \neq 0, A_{12}(z_0) u_0^2 = \tfrac{3}{2} : \quad (\tilde{r}_{i1}, \tilde{r}_{i2}) = (1,3), \quad i = 1,2 \tag{4.256}$$

where u_0 is the leading coefficient of the series (4.117). The compatibility conditions corresponding to movable zeros at the resonances $\tilde{r}_{i1} = 1$, $\tilde{r}_{i2} = 2$, $i = 1,2$ by using (4.254) and (4.256) give

$$\begin{aligned} A_7 = A_8 = \tfrac{k_1}{3} z^2 + k_2 z + k_3, \quad A_{10} = A_{14} = 0, \\ A_{11} = -(\tfrac{2k_1}{3} z + k_2), \quad A_{12} = k_1, \quad A_{13} = \tfrac{k_1}{3}, \end{aligned} \tag{4.257}$$

where $k_1 \neq 0$, k_2, k_3 are constant of integration. The canonical form of the equation in this case is

$$\begin{aligned} y^2 y''' = 3yy'y'' - \tfrac{3}{2}(y')^3 - y^3 y'' + \tfrac{3}{2} y^2 (y')^2 + \tfrac{3}{2} y^4 y' + \tfrac{1}{2} y^6 \\ + (\tfrac{k_1}{3} z^2 + k_2 z + k_3)(y^2 y' + y^4) - (\tfrac{2k_1}{3} z + k_2) y^3 + k_1 y' + \tfrac{k_1}{3} y. \end{aligned} \tag{4.258}$$

(4.258) has the first integral

$$y'' = \frac{1}{2} \frac{(y')^2}{y} - 2yy' - \frac{1}{2} y^3 - (\frac{k_1}{3} z^2 + k_2 z + k_3) y - \frac{k_1}{3y}, \tag{4.259}$$

which possesses the Painlevé property [6] [3].

(b)

$$\begin{aligned} A_6 = -\tfrac{1}{2} A_5, \quad A_{12} = \tfrac{3}{2} A_5^2, \quad A_{15} = -\tfrac{1}{2} A_5, \quad A_5 \neq 0, \\ A_5(z_0) u_{01} = -1 : \quad (\tilde{r}_{11}, \tilde{r}_{12}) = (1,2), \\ A_5(z_0) u_{02} = 1 : \quad (\tilde{r}_{21}, \tilde{r}_{22}) = (1,4), \\ A_5(z_0) u_{03} = -3 : \quad (\tilde{r}_{31}, \tilde{r}_{32}) = (-3,4), \end{aligned} \tag{4.260}$$

where u_0 is the leading coefficient in the series (4.117). The compatibility conditions at the resonances $\tilde{r}_{11} = 1$, $\tilde{r}_{12} = 2$, $\tilde{r}_{21} = 1$ on using (4.254) and (4.260) give

$$A_5 = k_1 \text{ (constant)}, A_{10} = A_{14} = 0, \quad \frac{3}{2} k_1^2 - k_1 A_7 + A_{13} = 0. \tag{4.261}$$

But then the compatibility condition at the resonance $\tilde{r}_{22} = 4$ gives $k_1 = 0$. That is the equation attains the dominant terms only.

Chapter 5

Conclusion

In the procedure followed to obtain higher order Painlevé-type ordinary differential equations, we have imposed the existence of at least one principal branch for the sake of applicability of the singular point analysis. However, the compatibility conditions at the positive resonances of the second branches are identically satisfied in all cases. Besides, following this procedure one can also obtain equations with negative resonances only like Chazy equation (3.23), which has three negative distinct integer resonances.

Since the simplified versions of P_I and P_{II} are constant coefficient polynomial-type equations, starting from these two equations higher-order polynomial-type simplified equations with constant coefficients were considered. However, non polynomial-type simplified equations with constant coefficient were obtained starting from the constant coefficient non polynomial-type simplified equation of P_{III}. One can also obtain non polynomial-type higher order equations having the Painlevé property if one follows the procedure starting from P_V and P_{VI}.

REFERENCES

[1] Painlevé P 1900 *Bull. Soc. Math. Fr.* **28** 214
Painlevé P 1912 *Acta. Math.* **25** 1

[2] Gambier B 1909 *Acta. Math.* **33** 1

[3] Ince E L 1956 *Ordinary Differential Equations* (New York: Dover).

[4] Fuchs R 1907 *Math. Ann.*

[5] Chazy J 1911 *Acta Math.* **34** 317

[6] Bureau F 1964 *Ann. Math.* **64** 229

[7] Bureau F 1964 *Ann. Math.* **66** 1

[8] Bureau F 1972 *Ann. Math.* **91** 163

[9] Exton H 1971 *Rendiconti di Matematica* **4** 385

[10] Exton H 1973 *Bull. Sci. Royal Belg.* **55** 419

[11] Martynov I P 1973 *Differential Equations* **9** 1368

[12] Martynov I P 1985 *Differential Equations* **21** 623
Martynov I P 1985 *Differential Equations* **21** 512

[13] Cosgrove C M and Scoufis G, Stud. Appl. Math. 88 (1993) 25.

[14] Cosgrove C M 2000 *Stud. Appl. Math.* **104** 171.

[15] Cosgrove C M *Stud. Appl. Math.* **104** 1
Cosgrove C M *"Higher order Painlevé equations in the polynomial class II, Bureau symbol PI"* Preprint.

[16] Kudryashov N A 1997 *Phys. Lett. A* **224** 353

[17] Clarkson P A, Joshi N and Pickering A (1999) *Inverse Problems* **15** 175

[18] Sakka A and Muğan U 1997 *J. Phys. A:Math. Gen.* **30** 5159
Sakka A and Muğan U 1998 *J. Phys. A:Math. Gen.* **31** 2471
Muğan U and Sakka A 1999 *J. Math. Phys.* **40** 3569

[19] Muğan U and Jrad F 1999 *J. Phys. A:Math. Gen.* **32** 7933

[20] Ablowitz M J, Ramani A and Segur H 1978 *Lett. Nuovo Cim.* **23** 333
Ablowitz M J, Ramani A and Segur H 1980 *J. Math. Phys.* **21** 715

[21] Fordy A P and Pickering A 1992 *Chaotic Dynamics: Theory and Practice* ed T Bountis (New York: Plenum)

[22] Conte R, Fordy A P and Pickering A 1993 *Physica D* **69** 33

[23] Weiss J,Tabor M and Carnavale G, J. Math. Phys. 24 (1983) 522.

[24] Fordy A P and Pickering A, Phys. Lett. A 160 (1991) 347.

[25] Weiss J, J. Math. Phys. 25 (1984) 13.

VITA

Fahd Jrad was born in Nahr-El-Bared Palestinian refugees camp, Tripoli, Lebanon on July 25, 1971. He received his B.S. degree from the Department of Mathematics, Middle East Technical University, Ankara, Turkey in 1993. After that, he continued his studies in the Department of Mathematics, Bilkent University. He got his M.S. degree in July 1996 under the supervision of Assoc. Prof. Uĝurhan Muĝan by the thesis entitled "Partial Differential equations having the Painlevé property". His research interests include integrability of partial and ordinary differential equations .