STRUCTURAL ANALYSIS OF POLE ASSIGNMENT AND STABILIZATION IN DYNAMIC SYSTEMS

A THESIS
SUBMITTED TO THE DEPARTMENT OF ELECTRICAL AND ELECTRONICS ENGINEERING
AND THE INSTITUTE OF ENGINEERING AND SCIENCES
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

By
Ayla Şefik
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To Ali and Mehmet Şefik

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

Prof. Dr. M. Erol Sezer(Principal Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

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ABSTRACT

STRUCTURAL ANALYSIS OF POLE ASSIGNMENT AND STABILIZATION IN DYNAMIC SYSTEMS

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Ph.D. in Electrical and Electronics Engineering
Supervisor: Prof. Dr. M. Erol Sezer
April, 1989

Motivated by the need for qualitative investigation of general system properties such as controllability, observability, existence of fixed modes, etc. as the complement of the quantitative approach in analysis, especially of large-scale systems, the problems of pole assignability and stabilizability are considered from the structural point of view. The study is based on the definition of a generic property as a property that holds for almost all values of the nonzero system parameters. Structured matrices and digraphs are used for system description. Both problems are first formulated in an algebraic setting and then translated to a structural framework by means of several graph-theoretic results which give sufficient conditions for solvability, in terms of the existence of particular cycle families in the digraph. Following a similar approach, a graphical investigation of structural observability is presented. Lastly, genericity of several results are reconsidered in the light of these graphical characterizations.

Keywords: Qualitative approach, algebraic approach, pole assignment, stabilization, observability, structural property, genericity, structured matrix, digraph.

ÖZET

DEVİNİK SİSTEMLERDE KUTUP YERLEŞTİRME VE KARARLILAŞTIRMA PROBLEMLERİNİN YAPISAL ÇÖZÜMLEMESİ

Ayla Şefik
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Tez Yöneticisi: Prof. Dr. M. Erol Sezer
Nisan, 1989

Denetlenirlik, gözlenirlik, değişmez özdeğerlerin varlığı, vb. gibi genel sistem özelliklerinin nitel veya yapısal anlamda incelenmesinin, özellikle büyük çaplı sistemler için, nicel yaklaşımın tümleri olarak gerektiği bilinmektedir. Tezde, bu gerçekten yola çıkılarak, yapısal açıdan kutup yerleştirme ve kararlılaştırma problemleri ele alınmıştır. Bu çalışma, 'jenerik' (generic) özelliğin, sistemde sıfır olmayan parametrelerin hemen tüm değerleri için bulunan özellik olarak tanımını temel almaktadır. Sistem modellemesi için yapı matrisleri ve yönlü çizgeler kullanılmıştır. Her iki problem de önce cebirsel olarak tanımlanmış, daha sonra çözüm için yeterli koşulları veren çizgesel sonuçlar aracılığıyla yapısal bir çerçeveye oturtulmuştur. Benzer bir yaklaşım kullanılarak, yapısal gözlenirliğin çizgesel incelemesi gerçekleştirilmiştir. Son olarak yapısal yaklaşımdan çıkan gözlemler ışığında, bilinen bazı sonuçlar 'jenerisite' (genericity) açısından, yeniden ele alınmıştır.

Anahtar sözcükler: Nitel yaklaşım, cebirsel yaklaşım, kutup yerleştirme, kararlılaştırma, gözlenirlik, yapısal özellik, jenerisite, yapı matrisi, yönlü çizge.

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Contents

List of Figures

List of Tables

Chapter 1

INTRODUCTION

In systems theory, a traditional approach in analysis is to transform the equations describing the system in order to obtain a standard representation, such as Kalman's or Luenberger's canonical forms or the standard block diagram configuration. Once this is accomplished, long-established and well-tested methods are employed to treat the problem on hand. This is a quantitative analysis in which every step depends completely on the corresponding numerical data.

Frequently, however, there arise complications, especially when dealing with dynamic systems such as electric power systems, aerospace systems, economic systems, process control systems in chemical and petroleum industries, ecological systems, etc.. One possible cause of complication is dimensionality: The system may comprise a large number of variables making it impossible or uneconomical to analyze it as a whole. Uncertainity in system parameters may also be a reason: In such a case, it is impossible to obtain an exact mathematical model of the system. Information structure constraint is another possibility: Restriction on what goes where in information distribution, especially in interconnected systems, makes the traditional control and estimation methods difficult to apply to dynamic systems even with smaller dimensions. (A system possessing any one of these characteristics is termed as a complex dynamic system [1]).

On the other hand, it is well-known that a way out through many problems and complications arising in various branches of mathematics and engineering sciences can be established after sufficient insight into their structures has been gained. An insight into the system structure in its original form would yield information on effects of individual system components, subsystems, subloops, trade-off information between various subsystems and interconnecting structure; which may often be of great value to the analyst and to the designer.

This need for dealing with system structures is met by the qualitative analysis of systems. The qualitative analysis is concerned with the general properties of systems such as controllability, observability, stability, existence of fixed modes, etc.. Analogous to the term potential energy used in classical mechanics to describe the latent capacity of a system for doing mechanical work, these properties may be viewed as potential properties in the sense that they represent latent qualities that are determined by the structure of the system [2]. In the rest of the thesis, we shall refer to such properties as qualitative properties or structural properties.

The general tool that combines the qualitative properties of a system with the system structure is the structural modeling [3] based on the axiomatic theory of directed graphs [4]. Structural information is, in general, binary in nature and hence directed graphs (digraphs) serve as excellent mathematical models in this respect. In a structural description by a digraph,system variables are associated with vertices, and oriented edges correspond to the interaction between the variables. Signs or weights may be assigned to the vertices or the edges when it is necessary to represent some of the quantitative properties of the system.

The computational simplifications offered via graph-theory have resulted in the applications of structural modeling in many areas of engineering and societal problems [5-11]. There have been a considerable number of results that exploited the theory of digraphs for the stability, optimality, and reliability analysis of large-scale systems [12-28].

A system is said to have a structural property in the generic sense if that property holds for almost all values of the nonzero system parameters. For example, in a structurally controllable system a possible loss of controllability can occur only in pathological cases when there is an exact matching of system parameters. In that case, a slight change in the value of some of the parameters can restore the property. Conversely, if the uncontrollability is due to a special structure of the system, then no matter how much the parameters are perturbed, the property can not be regained. From the physical point of view, only the latter case is important because it is not possible to know whether such a matching occurs in a given system. This concept of structural property is consistent with physical reality also because of the fact that system parameter values are never known precisely with the exception of zeros that are fixed by coordinatization or by the nonexistence of physical connections between certain parts of a system. (Note that digital computers work with 'true' zeros and 'fuzzy' numbers justifying the need for investigating the system properties independently of the numerical data.)

It was Lin [29] who first introduced the concept. He developed a purely graph-theoretical characterization of structural controllability for single-input systems. Shields and Pearson [30] extended his results to multi-input systems but on a purely algebraic basis. The algebraic approach due to Shields and Pearson was simplified considerably by Glover and Silverman who used Boolean matrix algebra [31]. Davison [32] generalized the approach to observability where he switched back to Lin's graph-theoretic point of view and interpreted the Boolean operations of [31] in terms of the reachability properties of a digraph. Later, Lin [33] defined minimal structural controllability and gave a characterization for structurally controllable multi input systems in terms of structured matrices and digraphs.

After the introduction of the concept of fixed modes by Wang and Davison [34] in their systematic approach to the decentralized stabilization problem, Sezer and Šiljak [35] recognized that the existence of fixed modes was a structural property in the context of the ideas and results due to Lin [29], Shields and Pearson [30] and Glover and Silverman [31]. Similar to the occurrence of structural uncontrollability and unobservability, the existence of fixed modes is either a consequence of an exact matching of system parameters, which is quite unlikely to occur, or is due to a special structure of the system. Motivated by this fact, Pichai, Sezer and Šiljak [36], defined structurally fixed modes and obtained a graph-theoretic characterization for the existence structurally fixed modes. All almost at the same time, Reinschke [37], and Papadimitriou and Tsitsiklis [38] gave alternative graph-theoretic criteria for the existence of fixed modes.

Reinschke did considerable work related to the structural properties of dynamic systems and obtained purely graph-theoretic formulations. In an early paper [39], he formulated structural completeness of systems. Later, he developed another criterion for structural completeness in terms of the existence of certain cycles in an appropriately chosen digraph [40]. In [37], he provided a result which relates the coefficients of the characteristic polynomial of a system to the cycle families in the digraph associated with the system, and based on this result, derived his graph-theoretic criterion for the existence of structurally fixed modes. He utilized this approach of characterization of structural properties by means of cycle families in investigating the problem of pole assignability. In one of his recent papers [41], he dealt with the explicit nonlinear dependencies between the coefficients of the closed-loop characteristic polynomial and the output feedback gains and gave a graph-theoretical interpretation of the relation.

The main motivation of the thesis, which is concerned with a qualitative analysis of arbitrary pole assignability and stabilizability as potential system properties, comes from the benefits and the simplicity of the structural insights, especially in the context of the ideas and result due to Reinschke.

In Chapter 2, we introduce the structural framework for our qualitative approach. Here, we review tools of structural modeling and structural description of systems. We also discuss some well-known structural properties, namely, structural controllability and the existence of fixed modes.

Chapters 3 and 4 consider the structural pole assignability and stabilizability problems, respectively, on a purely graph-theoretical basis. In both chapters, we first present an algebraic formulation of the problem, based on the characterizations and results due to Reinschke [37,41]. We then establish sufficient algebraic conditions for generic pole assignability and stabilizability, respectively. In the next step, the algebraic characterization of the problem is carried to a structural setting, and several results are stated and proved. For Chapter 3, the main result which is stated in the form of two theorems is translated to an algorithm.

In Chapter 5, we present a graphical investigation of structural observability, the inspiration for which came from the close study of the system digraph, during the analyses given in Chapters 3 and 4. The structural observability matrix is interpreted in terms of paths from the state vertices to output vertices in the system digraph, and a result, which characterizes structural observability in connection with the existence of such particular paths is derived. Generic observability index is defined and lower and upper bounds are provided for it in terms of the system digraph.

Chapter 6 is an account of an algebraic study on the genericity of some results on pole assignability and stabilizability. Here, we use an algebraic approach, in combination with the insight provided by the results of the preceding chapters, and reconsider some well-known results on pole assignability and stabilizability of certain classes of systems.

Finally, Chapter 7 includes a summary of the results, with emphasis on the contribution made by the thesis, and on points requiring further research.

Chapter 2

STRUCTURAL REPRESENTATION OF DYNAMIC SYSTEMS

In this chapter, we introduce the structural framework for the analysis of various qualitative properties of systems. We start with an introduction to the mathematical tools of structural modeling, namely, structured matrices and directed graphs (digraphs). Structured matrices and the related concept of genericity are taken mainly from Shields and Pearson [30], whose formulations are connected to König's theorem [42]. A summary of the standard material on digraphs, which can be found in books such as those of Harary, Norman and Cartwright [4] and Deo [6], is followed by a review concerning a special digraph structure, called cactus, first introduced by Lin [29,33].

After an account on the description of dynamic systems via system structure matrices and digraphs, as done by Šiljak [27], a discussion on the two important qualitative properties of systems, namely, structural controllability (observability) [29,33] and existence of structurally fixed modes [36], is presented. Characterizations of these two properties are crucial, as structural pole assignability and stabilizability are defined in the same context in this thesis.

A concise collection of the preliminary material presented in this chapter, together with a list of related references, is presented by Jamshidi [43].

2.1 STRUCTURED MATRICES AND GENERICITY

Two matrices M_1,$M_2 \in \mathcal{R}^{p\times q}$ are said to be structurally equivalent if there is a one-to-one correspondence between the locations of their nonzero entries. The equivalence class of structurally equivalent matrices in $\mathcal{R}^{p\times q}$ can be represented by a $p \times q$ structured matrix $\mathbf{M}$, whose entries are either fixed zeros or algebraically independent parameters in $\mathcal{R}$. If the number of nonzero elements of $\mathbf{M}$ is μ, then we can define a parameter space $\mathcal{R}^{\mu}$ associated with $\mathbf{M}$ such that for every $d \in \mathcal{R}^{\mu}$, $\mathbf{M}(d)$ defines a fixed matrix in the equivalence class that $\mathbf{M}$ represents. A fixed matrix M is said to be admissible with respect to $\mathbf{M}$, denoted as $M \in \mathbf{M}$, if $M = \mathbf{M}(d)$ for some $d \in \mathcal{R}^{\mu}$. If, for an admissible $M = \mathbf{M}(d)$, some elements of d are zeros, then $\mathbf{M}$ is said to be structurally reduced to M.

Let Π be a property that may be asserted about the structured matrix $\mathbf{M}$. Then Π is a mapping $\Pi : \mathcal{R}^{\mu} \to \{0, 1\}$, where

$$\Pi(d) = \begin{cases} 1 & , \text{ if } \Pi \text{ holds for } \mathbf{M}(d) \\ 0 & , \text{ otherwise} \end{cases}$$

Consider a polynomial $\Phi(d)$ in $d = (d_1, ..., d_{\mu})$ with real coefficients. The set

$$\Gamma = \{d \in \mathcal{R}^{\mu} | \; \Phi(d) = 0\},$$

is called a variety in $\mathcal{R}^{\mu}$. Γ is said to be proper if $\Gamma \neq \mathcal{R}^{\mu}$ and non-trivial if $\Gamma \neq \emptyset$. The property Π is said to be generic if there exists a proper variety Γ such that $ker\Pi \subset \Gamma$.

The implications of genericity are based upon the fact that if a variety $\Gamma \subset \mathcal{R}^{\mu}$ is proper and nontrivial, then it is a closed set. Thus, a property which is generic relative to Π holds at any point $d' \in \Gamma^c$, the complement of Γ, and in a sufficiently small neighborhood of d'. Also, if $d \in \Gamma$ with Γ proper and nontrivial, then almost all points in a sufficiently small neighborhood of d are in Γ^c. Therefore, all the points at which a generic property fails to hold lie on a hypersurface in $\mathcal{R}^{\mu}$, and can be suitably perturbed so that the property holds. In other words, a generic property is expected to hold almost everywhere in $\mathcal{R}^{\mu}$.

For a structured matrix **M**, we define the generic rank, denoted by $\bar{\rho}(\mathbf{M})$, as the maximal rank $\mathbf{M}(d)$ can attain in $\mathcal{R}^{\mu}$. It can be shown that the set $\{d \in \mathcal{R}^{\mu} | \ rank \ \mathbf{M}(d) < \bar{\rho}(\mathbf{M})\}$ is a proper variety in $\mathcal{R}^{\mu}$. Therefore, almost all fixed matrices $\mathbf{M}(d)$ have rank $\bar{\rho}(\mathbf{M})$. Note that in a structured matrix, due to the algebraic independence of the nonzero entries, generic rank equals term rank. Indeed, it has been shown in [30] that for some $r \leq min(p, q)$, generic rank of **M** is r if and only if **M** has r independent nonzero entries (i.e., no two parameters lie on the same row or column).

2.2 DIGRAPHS

A digraph can be represented by an ordered pair $\mathcal{D} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ and $\mathcal{E}$ are the finite sets of **vertices** and **oriented edges**, respectively. An edge oriented from $v_j \in \mathcal{V}$ to $v_i \in \mathcal{V}$ is denoted by the ordered pair (v_j, v_i). Then v_j is called the **tail** and v_i the **head** of the edge.

If $(v_j, v_i) \in \mathcal{E}$, then v_j is said to be **adjacent** to v_i, and v_i adjacent from v_j. This adjacency relationship between the vertices of a digraph is described by a square binary matrix, $\mathbf{R} = (\mathbf{r_{ij}})$ called the **adjacency matrix**, where $\mathbf{r_{ij}} = 1$ if and only if $(v_j, v_i) \in \mathcal{E}$. **R** characterizes the structure of $\mathcal{D}$ completely. This relationship can be used to define an equivalence relation called connectedness on $\mathcal{D}$ as follows:

(i) Adjacent vertices are connected.

(ii) Any two vertices connected separately to a third one are connected.

Maximal subgraphs that contain connected vertices are called connected components of $\mathcal{D}$. If all vertices in $\mathcal{D}$ are connected, then the digraph is said to be connected.

A sequence of edges $\{(v_1, v_2), (v_2, v_3), \cdots, (v_{k-1}, v_k)\}$ where all vertices are distinct is called a **path** from v_1 to v_k, denoted by $\overline{(v_1, v_k)}$. In this case, v_k is said to be **reachable** from v_1. This relationship can be represented by a matrix $\bar{\mathbf{R}} = (\bar{\mathbf{r}}_{\mathbf{ij}})$ where $\bar{\mathbf{r}}_{\mathbf{ij}} = 1$ if and only if v_j reaches v_i. Thus the adjacency matrix $\mathbf{R}$ can be interpreted as the one step reachability matrix. $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$, where all the multiplications and additions are Boolean, represents the two step reachability . With $\mathbf{R}^k = \mathbf{R}^{k-1} \times \mathbf{R}$, the reachability matrix of the digraph $\mathcal{D}$ can be written as $\bar{\mathbf{R}} = \mathbf{I} + \mathbf{R} + \mathbf{R}^2 + \cdots$. Note that, since $\mathcal{D}$ has a finite number of vertices, say n, any vertex reaches another one in at most n-1 steps, so that to compute $\bar{\mathbf{R}}$ it suffices to take only the first n terms of the infinite series above. Reachability defines another equivalence relation, namely strong connectedness, on $\mathcal{D}$. Two vertices are said to be **strongly connected** if they are mutually reachable from each other. A maximal subgraph containing strongly connected vertices is called a **strong component** of $\mathcal{D}$.

A sequence of edges $\{(v_1, v_2), (v_2, v_3), \cdots, (v_{k-1}, v_k)\}$ where $v_k = v_1$ with the remaining vertices distinct is called a **cycle**. The path that remains after the removal of an edge of a cycle is called the complementary path of that edge with respect to the cycle. Any two cycles are said to be disjoint if they have no common vertices. A collection of disjoint cycles is called a **cycle family**.

We now define some special structure digraphs which are characterized by Lin [29]:

A digraph $\mathcal{D}_s = (\mathcal{V}_s, \mathcal{E}_s)$, with a vertex set $\mathcal{V}_s = \{v_0, v_1, \cdots, v_t\}$ and the edge set $\mathcal{E}_s = \{(v_0, v_1), (v_1, v_2), \cdots, (v_{t-1}, v_t)\}$, is called a **stem**. Vertices v_0 and v_t are the **origin** and the **tip** of the stem, respectively.

A digraph $\mathcal{D}_b = (\mathcal{V}_b, \mathcal{E}_b)$, with $\mathcal{V}_b = \{v_0, v_1, \cdots, v_t\}$ and $\mathcal{E}_b = \{(v_0, v_1), \cdots, (v_{t-1}, v_t), (v_t, v_1)\}$, is called a **bud**. Vertex v_0 is the origin and edge (v_0, v_1) is called the **distinguished edge** of the bud. Clearly, if the edge (v_t, v_1) is deleted from $\mathcal{D}_b$, then it becomes a stem.

A digraph $\mathcal{D}_c = \mathcal{D}_s \cup \mathcal{D}_{b1} \cup \mathcal{D}_{b2} \cdots \cup \mathcal{D}_{bk}$, where $\mathcal{D}_s$ is a stem with origin v_o and tip v_t; and $\mathcal{D}_{bi}$ are buds with origins $v_i \neq v_t$ such that v_i is the only vertex common to $\mathcal{D}_s \cup \mathcal{D}_{b1} \cup \mathcal{D}_{b2} \cup \cdots \cup \mathcal{D}_{b,i-1}$ and $\mathcal{D}_{bi}$, $i = 1, \cdots, k$, is called a **cactus**. Origin v_o and tip v_t of $\mathcal{D}_s$ are also the origin and the tip of $\mathcal{D}_c$, respectively. If $\mathcal{D}_s$ above is replaced by a bud, then the digraph becomes a **precactus**,denoted by $\mathcal{D}_p$. Again, by deleting an appropriate edge of a precactus, it can be reduced to a cactus. Illustrations of these structures are given in Figure 2.1.

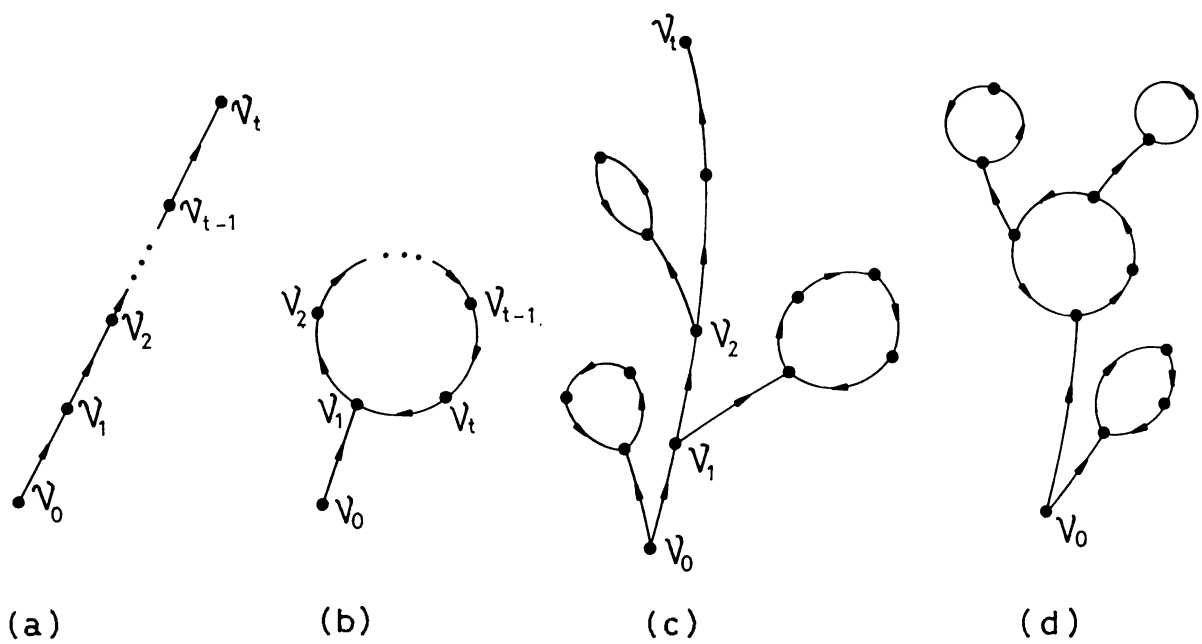


Figure 2.1. Illustrations of (a) a stem, (b) a bud, (c) a cactus, and (d) a precactus.

In a cactus $\mathcal{D}_c = (\mathcal{V}_c, \mathcal{E}_c)$, every vertex is reachable from the origin through a unique path. Let $v_1, v_2, \cdots, v_q$ be the vertices that are adjacent from the origin v_0. Then the sets $\mathcal{V}_i = \{v \in \mathcal{V} | \ v \text{ is reachable from } v_i\}$ are disjoint and $\mathcal{V}_c = \{v_0\} \cup \mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_q$. Each of the subgraphs of $\mathcal{D}_c$ defined by one of the vertex sets $\{v_0\} \cup \mathcal{V}_i$ is called a **bunch** of the cactus. The bunch that contains the tip of the cactus is called the **terminal bunch**, and the others (if any) **nonterminal bunches**. Thus a terminal bunch is a cactus itself and a nonterminal bunch is a precactus.

2.3 SYSTEM STRUCTURE MATRIX AND SYSTEM DIGRAPH

Consider a linear, time-invariant dynamic system with the state equations

$$\mathcal{S}: \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{2.1}$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}^m$ and $y \in \mathcal{R}^r$ denote the states, inputs and the outputs of $\mathcal{S}$, respectively, and A, B and C are real, constant matrices of appropriate dimensions.

Associated with this system, we define a square structured matrix $\mathbf{S}$ as

$$\mathbf{S} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{C} & \mathbf{O} & \mathbf{O} \end{bmatrix} \tag{2.2}$$

where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are structured matrices that correspond to A, B and C, respectively. $\mathbf{S}$ is called the system structure matrix. Viewing the matrix $\mathbf{S}$ as a binary matrix with zero and nonzero elements, we define the digraph $\mathcal{D} = (\mathcal{V}, \mathcal{E})$ which assumes $\mathbf{S}$ as its adjacency matrix to be the digraph of the system $\mathcal{S}$. For convenience, the vertex set of $\mathcal{D}$ can be partitioned as $\mathcal{V} = \mathcal{U} \cup \mathcal{X} \cup \mathcal{Y}$, where $\mathcal{U}$, $\mathcal{X}$ and $\mathcal{Y}$ are the sets of input, state and output variables, respectively. Digraph $\mathcal{D}$ completely characterizes the structure of system $\mathcal{S}$ of (2.1)

We say that two dynamic systems, represented by the triples (A_i, B_i, C_i), $i = 1, 2$, are structurally equivalent if

(a) their digraphs are the same up to an enumeration of their vertices, or equivalently,

(b) there exists a permutation of states, inputs and outputs after which A_1, B_1 and C_1 becomes structurally equivalent to A_2, B_2 and C_2, respectively; that is, there exist permutation matrices $\mathbf{P}_x$, $\mathbf{P}_u$ and $\mathbf{P}_y$ such that

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{C}_1 & \mathbf{O} & \mathbf{O} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_x^T\mathbf{A}_2\mathbf{P}_x & \mathbf{P}_x^T B_2 \mathbf{P}_u & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{P}_y\mathbf{C}_2\mathbf{P}_x & \mathbf{O} & \mathbf{O} \end{bmatrix}$$

The digraph $\mathcal{D}$ associated with these systems defines an equivalence class of structurally equivalent systems. Then a property is a structural property of a system if it is a property of the associated digraph.

For a treatment of the structural properties of the pair (A,B) of $\mathcal{S}$ of (2.1), one can use the subgraph $D_{ux} = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{ux})$ obtained by removing, from the associated digraph $\mathcal{D}$ of $\mathcal{S}$, the output vertices and the edges connected to them. D_{ux} is called the output truncated system digraph and corresponds to the system structure matrix,

$$\mathbf{S}_{ux} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \tag{2.3}$$

Subgraph $\mathcal{D}_{xy}$ for the pair (A, C) can be defined, similarly.

Let $\mathbf{F} = (\mathrm{f_{ij}})$ be an $m \times r$ structured matrix with $\nu \leq m.r$ nonzero elements. Suppose a feedback of form

$$\mathcal{F}: \quad u = Fy, \tag{2.4}$$

where F is a matrix admissible with respect to $\mathbf{F}$, is applied to system $\mathcal{S}$ of (2.1). The resulting closed loop-system represented by

$$\mathcal{S}(\mathcal{F}): \quad \dot{x} = (A + BFC)x \tag{2.5}$$

has the system structure matrix

$$\mathrm{S}(\mathbf{F}) = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{F} \\ \mathbf{C} & \mathbf{O} & \mathbf{O} \end{bmatrix}. \tag{2.6}$$

The associated system digraph then becomes $\mathcal{D}(\mathcal{F}) = (\mathcal{V}, \mathcal{E} \cup \mathcal{E}_{\mathcal{F}})$, where $\mathcal{E}_{\mathcal{F}} = \{(y_j, u_i) |\ \mathbf{f_{ij}} \neq 0\}$ is the set of feedback edges.

For convenience, the edges in $\mathcal{E}$ are called the d-edges and those in $\mathcal{E}_{\mathcal{F}}$ the f-edges. Accordingly, a cycle is called an f-cycle if it contains at least one f-edge and a d-cycle otherwise. Similarly, a cycle family is called an f-cycle family if it contains at least one f-edge, a simple f-cycle family if it contains one and only one f-edge, and a d-cycle family otherwise. Note that if a feedback variable f_{ij} is given a fixed nonzero value, then the corresponding f-edge (y_j, u_i) becomes a d-edge as f_{ij} is no more different from a nonzero parameter of A, B or C.

2.4 STRUCTURAL CONTROLLABILITY (OBSERVABILITY)

In a linear, time-invariant system represented by the triple (A,B,C), a possible loss of controllability (observability) may occur in the following two different ways:

(i) It may be due to an exact matching of the system parameters, e.g., as in the system represented by the triple,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

which is obviously both uncontrollable and unobservable. We know, however, that except for the fixed zeros that come by coordinatization or by absence of physical connections between some parts of the system, system parameter values are never precise. Hence, an investigation of the system properties, with some parameter values slightly perturbed, is justifiable. Indeed, if the above triple is reconsidered with the A matrix slightly perturbed as

$$A = \begin{bmatrix} 0 & 1+\epsilon \\ 1 & 0 \end{bmatrix},$$

it turns out to be both controllable and observable.

(ii) Loss of controllability (observability) which is due to the special structure of the system represents the other case. Here, no matter how much the parameters are perturbed, controllability (observability) cannot be restored. For example, in the above triple with the A matrix as

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

this is the situation.

It is obvious that (i) represents pathological cases while (ii) is of primary importance, especially when dealing with an actual physical system. The distinction between these two cases is provided in the concept of structural controllability (structural observability):

Definition 2.1 *A system S of (2.1) is structurally controllable (S.C.) if there exists a controllable system structurally equivalent to S.*

Structural observability can be defined similarly.

Both algebraic and graph-theoretical characterizations of structural controllability have been given by Lin [29,33] and Shields and Pearson [30]. The following two theorems summarize these results.

Lemma 2.1 *A system S of (2.1) is S.C. if and only if*

(a) $\bar{\rho}\,[\mathbf{A}\ \ \mathbf{B}] = n$, *and*

(b) the system digraph is input reachable, i.e., each state vertex is reachable from an input vertex.

Lemma 2.2 *The following are equivalent:*

(a) The system S of (2.1) is S.C.

(b) *The output truncated system digraph $\mathcal{D}_{ux}$ is spanned by a family of disjoint cacti, $\mathcal{D}_{ci} = (\mathcal{V}_{ci}, \mathcal{E}_{ci})$ with $\mathcal{V}_{ci} = \{u_{k_i}\} \cup \mathcal{X}_i$ and $\mathcal{E}_{ci} \subset \mathcal{E}_{ux}$ such that $\cup \mathcal{X}_i = \mathcal{X}$.*

Structural observability can be characterized by dual statements.

It is obvious from these characterizations that structural controllability (observability) is a generic property of the system.

2.5 STRUCTURALLY FIXED MODES

Consider the system $\mathcal{S}$ of (2.1) and a feedback $\mathcal{F}$ of (2.4) specified by the structured matrix $\mathbf{F}$, applied to $\mathcal{S}$. The set of fixed modes of $\mathcal{S}$ with respect to $\mathcal{F}$ is defined by

$$\Lambda_{\mathbf{F}} = \bigcap_{F \in \mathbf{F}} \Lambda(A + BFC),$$

where $\Lambda(\cdot)$ denotes the set of eigenvalues of $(\cdot)$, and the intersection is over all F admissible with respect to $\mathbf{F}$.

As in the case of loss of controllability (observability), a fixed mode either originates from an exact matching of system parameters or is due to the special structure the system. This fact allowed Sezer and Šiljak [36] to employ the ideas and results developed in the context of structural controllability, in characterizing the existence of structurally fixed modes as a generic property of the system. According to this, a system is said to have structurally fixed modes with respect to a feedback structure constraint $\mathcal{F}$ if every system structurally equivalent to $\mathcal{S}$ has fixed modes with respect to $\mathcal{F}$. The following lemma, which is due to Sezer and Šiljak gives necessary and sufficient conditions for the existence of structurally fixed modes in terms of system digraph.

Lemma 2.3 *A system $\mathcal{S}$ of (2.1) has no structurally fixed modes with respect to a feedback $\mathcal{F}$ of (2.4) if and only if both of the following conditions hold:*

(*i*) *Each state vertex in $\mathcal{X}$ is contained in a strong component of $\mathcal{D}(\mathcal{F})$ which includes an edge from $\mathcal{E}_{\mathcal{F}}$.*

(*ii*) *There exists a cycle family in $\mathcal{D}(\mathcal{F})$ which covers all the state vertices.*

Chapter 3

THE POLE ASSIGNMENT PROBLEM: A STRUCTURAL APPROACH

In this chapter, we present a qualitative analysis of the pole assignment problem based on the structure of the pair $(\mathcal{S}, \mathcal{F})$. We start with a discussion on an algebraic formulation of the problem, as has been done by Reinschke in [44]. Based on this and structural interpretation of characteristic polynomial (also due to Reinschke [37]), we derive purely graph-theoretical conditions for structural pole assignability. We then provide a search algorithm to detect these conditions. Finally, we consider some examples of structurally pole assignable systems to demonstrate nontriviality of our conditions.

3.1 ALGEBRAIC FORMULATION OF THE POLE ASSIGNMENT PROBLEM

Consider the system $\mathcal{S}$ of (2.1) with a feedback $\mathcal{F}$ given by (2.4) applied to it. Then, the closed loop system $\mathcal{S}(\mathcal{F})$ of (2.5) has a characteristic polynomial

$$p(s) = det(sI - A - BFC) = s^n + p_1 s^{n-1} + \cdots + p_{n-1}s + p_n \tag{3.1}$$

The pole assignability problem is that of arbitrary assignability of the closed loop characteristic polynomial coefficients p_k by a proper choice of the nonzero elements of F.

Let the nonzero elements of F and the coefficients of the characteristic polynomial $p(s)$ in (3.1) be represented as points $f = (f_1, f_2, \cdots, f_\nu)$ in $\mathcal{R}^\nu$ and $p = (p_1, p_2, \cdots, p_n)$ in $\mathcal{R}^n$, respectively. From (3.1), p and f are related by a smooth mapping $g : \mathcal{R}^\nu \rightarrow \mathcal{N}$ defined as

$$p = g(f) \tag{3.2}$$

where $\mathcal{N}$ is a smooth manifold in $\mathcal{R}^n$. Therefore, the concern of the pole assignment problem is the existence of a solution $f \in \mathcal{R}^\nu$ of (3.2) for every given $p \in \mathcal{R}^n$. To provide conditions for the solvability of (3.2), we recall few concepts from differential geometry[45]:

Suppose that $\nu \geq n$, and let $\bar{g} : \mathcal{R}^n \rightarrow \mathcal{R}^n$ denote the restriction of g to $\mathcal{R}^n$. Let the derivative of $\bar{g}$ at a point $x \in \mathcal{R}^n$ be denoted by $\bar{g}_x(x)$, that is $d\bar{g} = \bar{g}_x \, dx$.

The mapping $\bar{g}$ defines a homeomorphism between $\mathcal{R}^n$ and $\bar{g}(\mathcal{R}^n)$ if and only if $\bar{g}$ is one-to-one, and $\bar{g}$ and $\bar{g}^{-1}$ are continuous on $\mathcal{R}^n$ and $\bar{g}(\mathcal{R}^n)$, respectively. Following is a well-known theorem on homeomorphic mappings.

Lemma 3.1 (Hadamard Theorem) *Assume that $\bar{g} : \mathcal{R}^n \rightarrow \mathcal{R}^n$ is continuously differentiable on $\mathcal{R}^n$ and that $\| \bar{g}_x^{-1} \|$ is bounded on $\mathcal{R}^n$. Then $\bar{g}$ is a homeomorphism of $\mathcal{R}^n$ onto $\mathcal{R}^n$.*

We now return to the pole assignment problem and consider (3.2). It is clear that a necessary condition for solvability of (3.2) for all $p \in \mathcal{R}^n$ is that $\nu \geq n$. We assume that in our investigations this is always the case and partition the feedback variables $f_1, f_2, \cdots, f_\nu$ into two disjoint subsets f_v and f_c containing n and $\nu - n$ elements, respectively. If we fix the variables in f_c at particular real values, then p depends only on f_v, i.e., (3.2) is reduced to

$$p = \bar{g}(f_v) \tag{3.3}$$

where $\bar{g} : \mathcal{R}^n \to \mathcal{R}^n$ is obviously a restriction of g to $\mathcal{R}^n$. This leads us to the following result [44], which gives a sufficient condition for pole assignability:

Lemma 3.2 *Assume $n \leq \nu \leq mr$. If there exists a partitioning of the feedback variables $f_1, f_2, \cdots, f_\nu$ into two disjoint sets f_v and f_c containing n and $\nu - n$ elements such that after appropriately fixing those in f_c, the derivative $\bar{g}_{f_v}$ is unimodular, then the system $\mathcal{S}$ is arbitrarily pole assignable by the feedback $\mathcal{F}$.*

This result depends on the fact that when $\bar{g}_{f_v}$ is unimodular, then *det* $\bar{g}_{f_v}$ is a constant, so that $\bar{g}$ is a homeomorphism by Lemma 3.1. Hence, for every $p \in \mathcal{R}^n$, there exists a unique $f_v \in \mathcal{R}^n$ satisfying $\bar{g}(f_v) = g(f_v, f_c) = p$.

Example 3.1 To demonstrate the result of Lemma (3.2), consider the following system.

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x \end{aligned} \tag{3.4}$$

controlled by the feedback

$$u = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} y. \tag{3.5}$$

Let us conveniently take $f_{11} = f_1$, $f_{12} = f_2$, $f_{21} = f_3$ and $f_{22} = f_4$. Then, the characteristic polynomial of the closed loop system is given by

$$p(s) = s^3 - (f_1 + f_4)s^2 - (f_3 + f_4 + f_2 f_3 - f_1 f_4)s + f_1 f_4 - f_2 f_3.$$

With $f = (f_1, f_2, f_3, f_4)$, we have

$$p = g(f) = \begin{bmatrix} -f_1 - f_4 \\ -f_3 - f_4 - f_2 f_3 + f_1 f_4 \\ f_1 f_4 - f_2 f_3 \end{bmatrix}$$

Let us partition f as $f = f_v \cup f_c$ where $f_v = (f_1, f_2, f_4)$ and $f_c = f_3$. Fixing $f_3 = 1$, we get the mapping

$$p = \bar{g}(f_v) = \begin{bmatrix} -f_1 - f_4 \\ -1 - f_4 - f_2 + f_1 f_4 \\ f_1 f_4 - f_2 \end{bmatrix}. \tag{3.6}$$

The derivative is given by

$$\bar{g}_{f_v}(f_v) = \begin{bmatrix} -1 & 0 & -1 \\ f_4 & -1 & -1 + f_1 \\ f_4 & -1 & f_1 \end{bmatrix}$$

for which $\det \bar{g}_{f_v} = 1$. So, by Lemma 3.2 this system is arbitrarily pole assignable with the feedback of (3.5). Indeed, (3.6) can be written as

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_4 \\ f_2 - f_1 f_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

which is clearly solvable for f_v, for every $p \in \mathcal{R}^3$.

3.2 THE STRUCTURAL POLE ASSIGNMENT PROBLEM

3.2.1 Problem Formulation

Imitating the definitions of structural controllability and existence of structurally fixed modes given in Sections 2.4 and 2.5, we define a structurally pole assignable system as follows:

Definition 3.1 *A system $\mathcal{S}$ of (2.1) is said to be structurally pole assignable by a feedback $\mathcal{F}$ of (2.4) if there exists a system structurally equivalent to $\mathcal{S}$ which is pole assignable by $\mathcal{F}$.*

Let us assume, as in an analysis of structural controllability that the nonzero parameters of the system structure matrix **S** in (2.2) are algebraically independent, and thus correspond to a data point $d \in \mathcal{R}^{\mu}$. Then, the mapping g between p and f of (3.2) depends also on the system parameters, and this dependence can be indicated by expressing (3.2) as

$$p = g(d, f). \tag{3.7}$$

It is clear from Definition 3.1 that structural pole assignability is equivalent to the existence of a particular data point $d^* \in \mathcal{R}^{\mu}$ for which the equation

$$p = g(d^*, f) = g^*(f) \tag{3.8}$$

has a solution for every given $p \in \mathcal{R}^n$.

It is important to note that solvability of (3.8) does not readily imply solvability of (3.7) for almost all $d \in \mathcal{R}^n$. This is due to the fact that (3.7) is, in general, a non-linear equation, solvability of which cannot easily be reduced to a condition involving only the parameter vector d. Therefore, unlike structural controllability, structural pole assignability is not a generic property, or at least can not easily be proved to be a generic property. In our analysis, however, we do aim at obtaining structural conditions in terms of the system digraph, which guarantee genericity of structural pole assignability.

In order to complete the establishment of the framework needed for our structural approach, we refer to Reinschke's [37] graph-theoretic formulation of the characteristic polynomial which is summarized below:

Consider the closed loop system digraph $\mathcal{D}(\mathcal{F}) = (\mathcal{V}, \mathcal{E} \cup \mathcal{E}_F)$ associated with the system structure matrix S(F) of (2.6). By assigning a weight to every edge, $\mathcal{D}(\mathcal{F})$ becomes a weighted digraph. The weight of a d-edge is the corresponding nonzero parameter value of A, B or C, and the weight of an f-edge is the corresponding variable feedback gain. In the thesis, the associated weight also refers to the edge. Accordingly, a path or a cycle is represented by a sequence of the weights of the edges it contains as $\{\cdot\}$, and a cycle family by a collection of the cycles involved as $\{\ \{\cdot\}\ \}$. The weight of a path, a cycle or a cycle family is the product of weights of all edges involved. Denoting the number of cycles in a cycle family $\mathcal{CF}$ by $\sigma(\mathcal{CF})$, and defining the width $\gamma(\mathcal{CF})$ of $\mathcal{CF}$ to be the total number of state vertices covered by $\mathcal{CF}$, Reinschke proved the following:

Lemma 3.3 *The coefficients $p_k = g_k(f)$, $k = 1, 2, \cdots, n$, of the closed loop characteristic polynomial are given as*

$$g_k(f) = \sum_{\gamma(\mathcal{CF})=k} (-1)^{\sigma(\mathcal{CF})} \omega(\mathcal{CF}) \tag{3.9}$$

where $\omega(\mathcal{CF})$ denotes the weight of $\mathcal{CF}$, and the summation is carried over all cycle families of width k.

An immediate application of this lemma is that a feedback variable appears in a coefficient p_k of the closed loop characteristic polynomial only if it takes part in a cycle family of width k, as illustrated by the following example.

Example 3.2 Consider again the system of Example 3.1 for which the system structure matrix is

$$\mathbf{S(F)} = \left[\begin{array}{ccc|cc|cc} 0 & d_1 & 0 & d_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_3 & 0 & 0 \\ 0 & d_4 & 0 & 0 & d_5 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & f_1 & f_2 \\ 0 & 0 & 0 & 0 & 0 & f_3 & f_4 \\ \hline d_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_7 & 0 & 0 & 0 & 0 \end{array}\right].$$

The digraph $\mathcal{D}(\mathcal{F}) = (\mathcal{V}, \mathcal{E} \cup \mathcal{E}_F)$ with $\mathcal{V} = \{x_1, x_2, x_3, u_1, u_2, y_1, y_2\}$, associated with $\mathbf{S}(\mathbf{F})$ is shown in Figure 3.1. The f-cycle families $\mathcal{CF}_{ks}$ of width k, $1 \leq k \leq 3$, in $\mathcal{D}(\mathcal{F})$ are listed in Table 3.1.

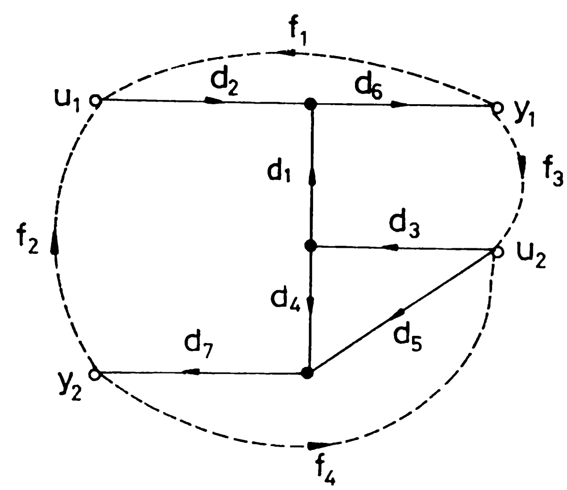


Figure 3.1. $\mathcal{D}(\mathcal{F})$ of Example 3.2.

k	$\mathcal{CF}_{ks}$
1	$\{d_6, d_2, f_1\}$
	$\{d_7, d_5, f_4\}$
2	$\{d_6, d_1, d_3, f_3\}$
	$\{d_7, d_4, d_3, f_4\}$
	$\{d_7, d_5, f_3, d_6, d_2, f_2\}$
	$\{ \{d_6, d_2, f_1\}, \{d_7, d_5, f_4\} \}$
3	$\{ \{d_6, d_2, f_1\}, \{d_7, d_4, d_3, f_4\} \}$
	$\{d_7, d_4, d_3, f_3, d_6, d_2 f_2\}$

Table 3.1. F-cycle families in $\mathcal{D}(\mathcal{F})$ of Figure 3.1.

Then, applying Lemma 3.3, we obtain

$$g(f) = \begin{bmatrix} -d_6 d_2 f_1 - d_7 d_5 f_4 \\ -d_6 d_1 d_3 f_3 - d_7 d_4 d_3 f_4 - d_7 d_5 f_3 d_6 d_2 f_2 + (d_6 d_2 f_1)(d_7 d_5 f_4) \\ (d_6 d_2 f_1)(d_7 d_4 d_3 f_4) - d_7 d_4 d_3 f_3 d_6 d_2 f_2 \end{bmatrix}. \quad (3.10)$$

Observe that (3.10) reduces to (3.6) when values of the elements of the parameter vector $d = (d_1, d_4, d_2, d_3, d_5, d_6, d_7)$ and f_3 are all fixed at unity.

Lemmas 3.2 and 3.3 provide the basis for deriving sufficient conditions for structural, but at the same time, generic pole assignability.

3.2.2 Conditions For Structural Pole Assignability

In the following, we first prove a result which is a special case of Lemma 3.2:

Corollary 3.1 *Let f_v and f_c be as defined in Lemma 3.2, with the feedback variables in f_v renumbered as $f_1, f_2, \cdots, f_n$. For a partitioning $\mathcal{N} = \mathcal{I} \cup (\mathcal{N} - \mathcal{I})$ with $\mathcal{I} \neq \emptyset$, of the index set $\mathcal{N} = \{1, 2, \cdots, n\}$, define auxiliary variables $\tilde{f}_k$ as*

$$\tilde{f}_k = \begin{cases} f_k & , \quad k \in \mathcal{I} \\ \theta_k f_k + \psi_k & , \quad k \in \mathcal{N} - \mathcal{I} \end{cases} \tag{3.11}$$

where $\theta_k = \theta_k(d)$ is a nonzero polynomial in d, and $\psi_k = \psi_k(d, f_{\mathcal{I}})$ is a polynomial in f_l, $l \in \mathcal{I}$, with coefficients being polynomials in d. Suppose that the restriction $\bar{g}$ of g in (3.3) to $\mathcal{R}^n$ is given by

$$\bar{g}_k(d, f_v) = \tilde{g}_k(d; \tilde{f}) = \alpha_k + \sum_{l=1}^{n} e_{kl} \tilde{f}_l, \quad k = 1, 2, \cdots, n \tag{3.12}$$

where $\alpha_k = \alpha_k(d)$ and $e_{kl} = e_{kl}(d)$. Then, $\mathcal{S}$ is structurally pole assignable by $\mathcal{F}$ if the coefficient matrix $E = E(d) = (e_{kl})$ has full generic rank.

Proof: The derivative of $\bar{g}$ is computed as

$$\bar{g}_{f_v} = E(d) \Xi(d, f_{\mathcal{I}}),$$

where $\Xi = (\xi_{kl})$ has elements

$$\xi_{kl} = \begin{cases} 1 & , \ k \in \mathcal{I}, l = k \\ 0 & , \ k \in \mathcal{I}, l \neq k \\ \theta_k & , \ k \in \mathcal{N} - \mathcal{I}, l = k \\ 0 & , \ k, l \in \mathcal{N} - \mathcal{I}, l \neq k \\ \partial\psi_k/\partial f_l & , \ k \in \mathcal{N} - \mathcal{I}, l \in \mathcal{I} \end{cases}$$

It follows that Ξ can be permuted into

$$\begin{bmatrix} I_{\mathcal{I}} & 0 \\ \partial\Psi/\partial f_{\mathcal{I}} & \Theta_{\mathcal{N}-\mathcal{I}} \end{bmatrix},$$

where $\Theta_{\mathcal{N}-\mathcal{I}} = diag.\{\theta_k, k \in \mathcal{N} - \mathcal{I}\}$, and $\partial\Psi/\partial f_{\mathcal{I}} = (\partial\psi_k/\partial f_l)$, $k \in \mathcal{N} - \mathcal{I}, l \in \mathcal{I}$. Thus $\Xi(d, f)$ is generically unimodular, and the proof follows from Lemma 3.2.

It is easily seen from the proof that a structurally pole assignable system by Corollary 3.1 is also generically pole assignable.

We note that under conditions of Corollary 3.1, the mapping $\bar{g}$ can be decomposed as $\bar{g} = \tilde{g} \circ h$, where $\tilde{g} : \mathcal{R}^n \to \mathcal{R}^n$ is the affine mapping defined in (3.12), and $h : \mathcal{R}^n \to \mathcal{R}^n$ is defined in (3.11), both mappings being homeomorphisms. The significance of Corollary 3.1 lies in the fact that its assumptions and the full generic rank condition on the matrix E can be characterized, with the help of Lemma 3.3, in terms of the weighted closed loop digraph $\mathcal{D}(\mathcal{F})$. This leads us to two main results which we state and prove below.

Theorem 3.1 *Suppose that in $\mathcal{D}(\mathcal{F})$ there exists a choice of n distinct f-edges, renumbered conveniently as $f_1, f_2, \cdots, f_n$, which after converting the remaining f-edges into d-edges by fixing their weights at arbitrary values, satisfy the following conditions:*

(*i*) *No two f-edges occur in the same cycle;*

(*ii*) *All f-cycles have a vertex in common;*

(*iii*) *For* $k = 1, 2, \cdots, n$, *there exist particular simple f-cycle families of width* k, *denoted by* $\mathcal{CF}_k^*$, *such that*

(*a*) $f_k \in \mathcal{CF}_k^*$, *and*

(*b*) *any other simple f-cycle family of width k which contains an f-edge* f_l, $l \leq k$, *also contains a d-edge which appears in no* $\mathcal{CF}_j^*$, $j \leq k$.

Then $\mathcal{S}$ *is generically pole assignable with* $\mathcal{F}$.

Proof: Conditions (*i*) and (*ii*) guarantee that every f-cycle family is a simple f-cycle family, so that each product term $\omega(\mathcal{CF})$ in (3.9) contains at most one variable weight. In other words, each g_k in (3.9) is an affine function of $f_1, f_2, \cdots, f_n$ as in (3.12), so that $\bar{g}$ has the structure in Corollary 3.1 with $\tilde{f}_k = f_k$, $k \in \mathcal{N}$, that is with $\mathcal{I} = \mathcal{N}$. Therefore, it suffices to show that the coefficient matrix $E = (e_{kl})$ in Corollary 3.1 is generically nonsingular. For this, we first note that condition $(iii - a)$ implies that each e_{kk}, $k \in \mathcal{N}$, contains at least one nonzero product corresponding to $\mathcal{CF}_k^*$, which we denote by e_{kk}^*. We now define $d_n = d$, $E_n(d_n) = E(d)$, and partition E_n as

$$E_n(d_n) = \left[\begin{array}{c|c} E_{n-1}(d_n) & e_{jn}(d_n)'s \\ \hline e_{nl}(d_n)'s & e_{nn}^*(d_n) + e_{nn}(d_n) \end{array}\right], \tag{3.13}$$

where, for convenience, we denote what is left from e_{nn} after separating e_{nn}^* again by e_{nn} (if there remains any). For a fixed $l \leq n$, either f_l appears in no cycle family of width n, in which case $e_{nl} \equiv 0$ or if it does, then by condition $(iii - b)$, the corresponding product term contains the weight of a d-edge, which occurs in no e_{kk}^*, $k \leq n$. Let d_{n-1} denote the parameter vector after all parameters corresponding to such d-edges are set to zero. Then $E_n(d_{n-1})$

is of the form

$$E_n(d_{n-1}) = \left[\begin{array}{c|c} E_{n-1}(d_{n-1}) & e_{jn}(d_{n-1})'s \\ \hline 0 & e^*_{nn}(d_{n-1}) \end{array}\right],$$

where $e^*_{nn}(d_{n-1})$ consists of a single nonzero product term, and each diagonal element $e_{kk}(d_{n-1})$ of $E_{n-1}(d_{n-1})$ still contains the product term $e^*_{kk}(d_{n-1}) = e^*_{kk}(d_n)$, $k = 1, 2, \cdots, n-1$. Obviously, $E_n(d_n)$ is generically nonsingular if $E_n(d_{n-1})$ is. On the other hand, $E_n(d_{n-1})$ is generically nonsingular if and only if $E_{n-1}(d_{n-1})$ is. Now, replacing d_n and $E_n(d_n)$ by d_{n-1} and $E_{n-1}(d_{n-1})$ and repeating the argument above, we come to the conclusion that $E_n(d_n)$ is generically nonsingular if $E_1(d_1) = e^*_{11}(d)$ is nonzero, which is guaranteed by condition $(iii - a)$. This completes the proof. □

We demonstrate the result of Theorem 3.1 in the following example.

Example 3.3 Consider a system whose closed-loop digraph, $\mathcal{D}(\mathcal{F})$, corresponding to

$$\mathcal{F}: \quad u = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ 0 & f_{22} & f_{23} \end{bmatrix} y$$

is as given in Figure 3.2.

Let us fix $f_{22} = 1$, $f_{23} = 0$, and renumber the remaining nonzero feedback edges as $f_1 = f_{11}$, $f_2 = f_{12}$, and $f_3 = f_{13}$.Then, the resulting f-cycle families

$\mathcal{CF}_{ks}$ of width k, $1 \leq k \leq 3$, in $\mathcal{D}(\mathcal{F})$ are as listed in Table 3.2.

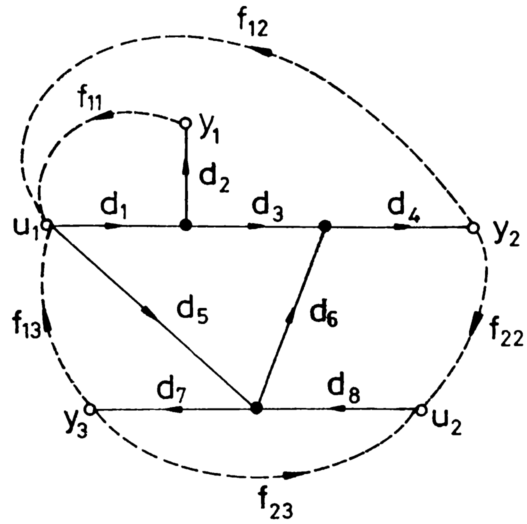


Figure 3.2. $\mathcal{D}(\mathcal{F})$ of Example 3.3.

k	$\mathcal{CF}_{ks}$
1	$\{d_2, d_1, f_1\}$
	$\{d_7, d_5, f_3\}$
2	$\{d_4, d_3, d_1, f_2\}$
	$\{d_4, d_6, d_5, f_2\}$
3	$\{d_4, d_3, d_1, f_3, d_7, d_8\}$
	$\{\ \{d_2, d_1, f_1\}, \{d_4, d_6, d_8\}\ \}$

Table 3.2. F-cycle families in $\mathcal{D}(\mathcal{F})$ of Figure 3.2.

Consider the following choice of $\mathcal{CF}_k^*$, $k = 1, 2, 3$:

$$\mathcal{CF}_1^* = \mathcal{CF}_{11} = \{d_2, d_1, f_1\}$$

$$\mathcal{CF}_2^* = \mathcal{CF}_{21} = \{d_4, d_3, d_1, f_2\}$$

$$\mathcal{CF}_3^* = \mathcal{CF}_{31} = \{d_7, d_8, d_4, d_3, d_1, f_3\}$$

Clearly conditions (i), (ii) and $(iii-a)$ of Theorem 3.1 are satisfied. Let us test condition $(iii-b)$: For $k=2$, $\mathcal{CF}_{22}$ is the only f-cycle family of width 2, other than $\mathcal{CF}_2^*$ and it contains f_2. But it also contains d_5 and d_6, both of which are d-edges that do not occur in $\mathcal{CF}_1^*$ or $\mathcal{CF}_2^*$, thus satisfying condition $(iii-b)$. For $k=3$, there is $\mathcal{CF}_{32}$ as the only f-cycle family of width 3, other than $\mathcal{CF}_3^*$, which contains f_1, but also d_6 which appears in no $\mathcal{CF}_j^*$, $j \leq 3$, again satisfying condition $(iii-b)$. Therefore, the system is generically pole assignable. Indeed, the coefficients of the closed-loop characteristic polynomial can be expressed as,

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \bar{g}_1(f_v) \\ \bar{g}_2(f_v) \\ \bar{g}_3(f_v) \end{bmatrix} = \begin{bmatrix} d_2d_1 & 0 & d_7d_5 \\ 0 & d_4(d_3d_1+d_6d_5) & 0 \\ d_2d_1d_4d_6d_8 & 0 & d_4d_3d_1d_7d_8 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

which is generically solvable for all $p=(p_1,p_2,p_3)$ as $det\ \bar{g}_{f_v} = d_4(d_3d_1 + d_6d_5)[d_2d_1d_4d_7d_8(d_3d_1-d_6d_5)]$.

A more general result, which makes full use of Corollary 3.1 is given by the following:

Theorem 3.2 *The result of Theorem 3.1 remains valid if condition* (ii) *is replaced by*

$(ii)'$ *To any two f-edges* f_p *and* f_q *that appear in disjoint cycles there corresponds a unique pair of edges* f_r *and* d_r *such that*

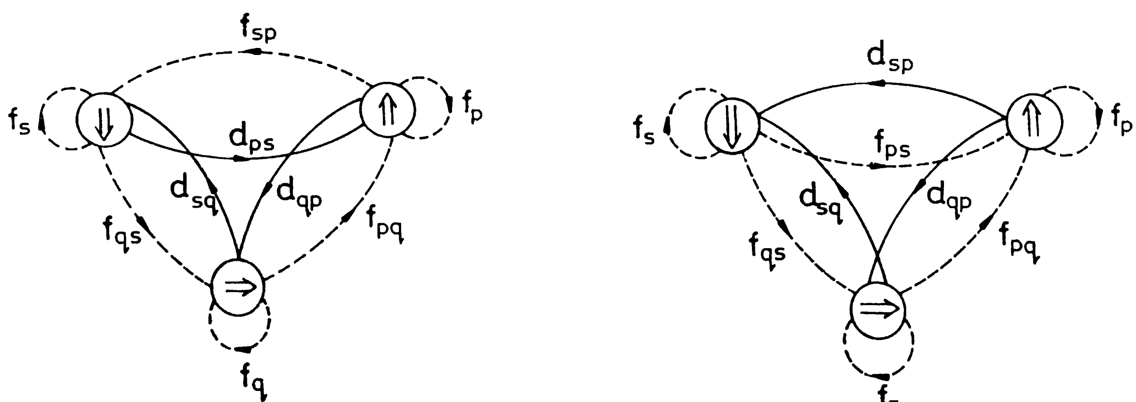
(a) d_r *appears in every cycle of* f_r *but in no cycle of* f_p *or* f_q*, and*

(b) to any two disjoint cycles C_p *and* C_q *of* f_p *and* f_q *there corresponds a cycle* C_r *of* f_r *which covers exactly the same state vertices as* C_p *and* C_q *cover, and vice versa.*

Proof: The proof is based on the following facts:

Fact 3.1 $\mathcal{D}(\mathcal{F})$ does not contain more than two pairwise disjoint f-cycles.

Proof of Fact 3.1: Suppose that $\mathcal{D}(\mathcal{F})$ contains three pairwise disjoint f-cycles formed by the f-edges f_p, f_q and f_s. Let us denote, for convenience, the pair of edges f_r and d_r associated with each pair (f_i, f_j), $i, j = p, q, s$, $i \neq j$, by f_{ij} and d_{ij}. Then, condition $(ii)'$ implies that $\mathcal{D}(\mathcal{F})$ contains a subgraph which is isomorphic to one of the basic structures shown in Figure 3.3. (There are eight possible combinations of different orientations of the edges f_{ij}, $i, j = p, q, s$, $i \neq j$, but six of these are essentially the same as one of the other two except for a relabeling of p,q and s.) However, each of these subgrahs contradicts condition (i), the one in Figure 3.3(a) containing a cycle which includes three f-edges f_{pq}, f_{sp} and f_{qs}, and the one in Figure 3.3(b) containing a cycle which includes two f-edges f_{pq} and f_{qs}. Therefore, $\mathcal{D}(\mathcal{F})$ cannot contain three disjoint f-cycles. It cannot contain four or more pairwise disjoint f-cycles either, because this necessarily includes the existence of three pairwise disjoint f-cycles. This completes the proof of Fact 3.1.



(a) (b)

Figure 3.3. The two basic structures mentioned in the proof of Fact 3.1.

Fact 3.2 The correspondence between (f_r, d_r)'s and the pair (f_p, f_q)'s in the statement of condition $(ii)'$ is one-to-one.

Proof of Fact 3.2: If (f_r, d_r) corresponds to two distinct pairs (f_p, f_q) and $(f_{p'}, f_{q'})$ then either all cycles formed by f_p and $f_{p'}$ or all cycles formed by f_q and $f_{q'}$ should cover the same state vertices. Suppose, without loss of generality, that the former is true and that $p < p'$. Since $f_{p'}$ appears in $\mathcal{CF}^*_{p'}$, which is of width p', then so does f_p in some $\mathcal{CF}_{p'}$ of width p'. However, every d-edge in $\mathcal{CF}_{p'}$ appears either in $\mathcal{CF}^*_{p'}$ or in $\mathcal{CF}^*_p$, which violates condition $(iii - b)$. The situation is illustrated in Figure 3.4, where $p = 1$, $p' = 2$, $\mathcal{CF}^*_p = \{d_2, d_1, f_p\}$, $\mathcal{CF}^*_{p'} = \{d_5, d_4, d_3, f_{p'}\}$ and $\mathcal{CF}_{p'} = \{d_5, d_4, d_1, f_p\}$.

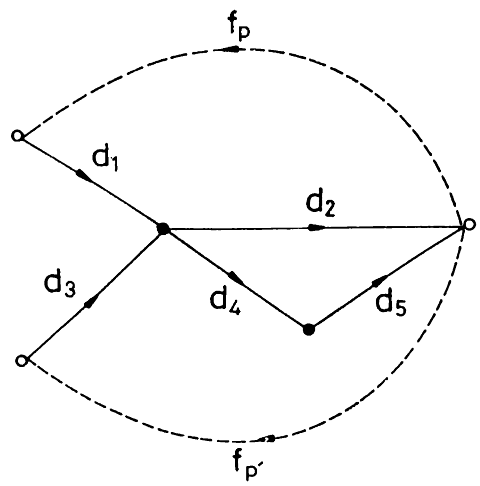


Figure 3.4. Illustration of the situation mentioned in the proof of Fact 3.2.

Fact 3.3 Suppose the pair (f_r, d_r) corresponds to the (unique) pair (f_p, f_q). If f_r appears in a product term in some $g_k(f)$ of (3.9), then so does the product $f_p f_q$, and vice versa. Moreover, all the product terms that contain f_r in any $g_k(f)$ are of the form $e_{kr}(e_r f_r + e_{pq} f_p f_q)$, where e_{kr}, e_r, and e_{pq} are polynomials in d with e_r and e_{pq} being the same in all such expressions.

Proof of Fact 3.3: Let $\mathcal{C}_{r1}$, $\mathcal{C}_{r2}$,..., denote all simple f-cycles formed by f_r; and for each i, let $\mathcal{CF}_{di1}, \mathcal{CF}_{di2}$,..., denote all d-cycle families which have no vertex in common with $\mathcal{C}_{ri}$. Then, any simple f-cycle family containing f_r is of the form $\mathcal{CF}_r = \mathcal{C}_{ri} \cup \mathcal{C}_{dij}$ for some i and j, so that $\omega(\mathcal{CF}_r) = \omega(\mathcal{C}_{ri}) \cdot \omega(\mathcal{C}_{dij}) = e_{ri} f_r e_{dij}$. By condition $(ii)'$, to every $\mathcal{C}_{ri}$ there correspond disjoint simple f-cycles $\mathcal{C}_{pi}$ and $\mathcal{C}_{qi}$ formed by f_p and f_q, which are also disjoint from all $\mathcal{C}_{dij}$.

Therefore, they form an f-cycle family $\mathcal{CF}_{pq} = \mathcal{C}_{pi} \cup \mathcal{C}_{qi} \cup \mathcal{C}_{dij}$ of the same width as that of $\mathcal{CF}_r$, and having the weight $\omega(\mathcal{CF}_{pq}) = e_{pi} f_p \cdot e_{qi} f_q \cdot e_{dij}$. This shows the existence of the product $f_p f_q$ wherever f_r appears. The converse is also true, and the proof of the first part is complete. Now, let e_r be the product of the weights of the d-edges which are common to all $\mathcal{C}_{ri}$, and which does not occur in some $\mathcal{C}_{pi} \cup \mathcal{C}_{qi}$ (obviously, d_r appears in e_r), so that $e_{ri} = e'_{ri} \cdot e_p$. Also define e_p and e_q to be the products of the weights of the d-edges which are common to all $\mathcal{C}_{pi}$ and $\mathcal{C}_{qi}$, respectively, and which do not appear in some $\mathcal{C}_{ri}$, and therefore write $e_{pi} = e'_{pi} \cdot e_p$ and $e_{qi} = e'_{qi} \cdot e_q$. Since for fixed i, $\mathcal{C}_{pi} \cup \mathcal{C}_{qi}$ and $\mathcal{C}_{ri}$ cover exactly the same same state vertices, then e_p and e_q may only contain weights of d-edges that are adjacent either from the input or to the output associated with f_p and f_q, respectively. Furthermore, $e'_{ri} = e'_{pi} \cdot e'_{qi}$. Then, $\omega(\mathcal{CF}_r) + \omega(\mathcal{CF}_{pq}) = e'_{ri} \cdot e_{dij} \cdot (e_r f_r + e_p \cdot e_q f_p f_q)$ independent of the widths of the cycle families $\mathcal{CF}_r$ and $\mathcal{CF}_{pq}$, and the proof follows.

Now, returning to the proof of Theorem 3.2, Fact 3.1 together with condition (i) imply that each product term $\omega(\mathcal{CF})$ in (3.9) contains at most two variable weights. Also, defining

$$\mathcal{I} = \{k | f_k \text{ forms a cycle which is disjoint from some other f-cycle}\},$$

and $\tilde{f}_r$ as in (3.11) with $\theta_r = e_r$ and $\psi_r = e_{pq} f_p f_q$, Fact 3.3 guarantees the structure in (3.9). Rest of the proof is the same as that of Theorem 3.1. □

The following simple example illustrates this result:

Example 3.4 Consider the digraph $\mathcal{D}(\mathcal{F})$, of Figure 3.5 corresponding to a closed-loop system under feedback of the form:

$$\mathcal{F}: \quad u = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} y.$$

Suppose we fix $f_{12} = a$, for some arbitrary $a \in \mathcal{R}$, and renumber the remaining feedback edges as $f_1 = f_{22}$, $f_2 = f_{11}$ and $f_3 = f_{21}$. This results in

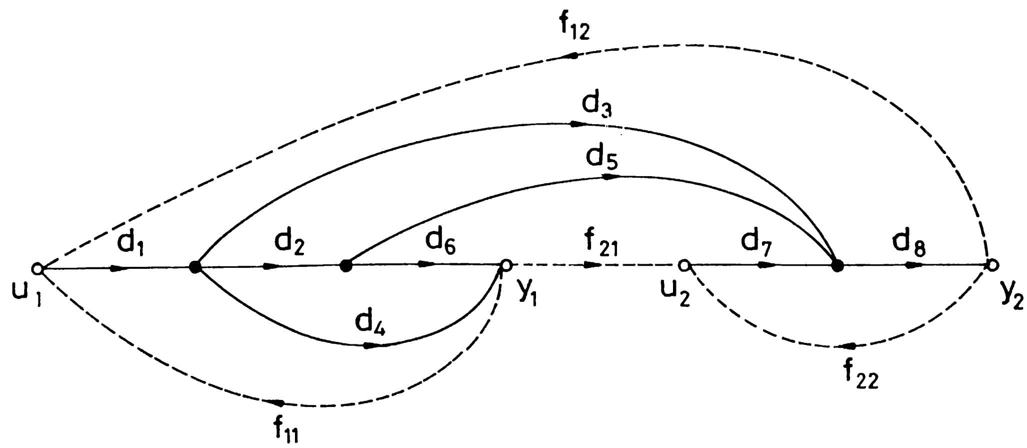


Figure 3.5. $\mathcal{D}(\mathcal{F})$ of Example 3.4.

the f-cycle families of Table 3.3, from where we choose,

$$\begin{aligned} \mathcal{CF}_1^* = \mathcal{CF}_{12} &= \{d_8, d_7, f_1\} \\ \mathcal{CF}_2^* = \mathcal{CF}_{21} &= \{d_6, d_2, d_1, f_2\} \\ \mathcal{CF}_3^* = \mathcal{CF}_{31} &= \{d_8, d_7, f_3, d_6. d_2, d_1, a\} \end{aligned}$$

k	$\mathcal{CF}_{ks}$
1	$\{d_4, d_1, f_2\}$, $\{d_8, d_7, f_1\}$
2	$\{d_6, d_2, d_1, f_2\}$ $\{d_8, d_7, f_3, d_4, d_1, a\}$ $\{\ \{d_4, d_1, f_2\}, \{d_8, d_7, f_1\}\ \}$
3	$\{d_8, d_7, f_3, d_6, d_2, d_1, a\}$ $\{\ \{d_6, d_2, d_1, f_2\}, \{d_8, d_7, f_1\}\ \}$

Table 3.3. F-cycle families in $\mathcal{D}(\mathcal{F})$ of Figure 3.5.

Condition (i) and $(iii-a)$ of Theorem 3.1 are obviously satisfied. On the other hand, we observe that for the f-edges f_1 and f_2, which appear in disjoint cycles, there is the pair of edges f_3 and a, as in condition $(ii)'$ of Theorem 3.2. Hence, the system is generically pole assignable.

The usefulness of Theorem 3.1 and Theorem 3.2 depends largely on the choice of n feedback gains to be included in f_v, as well as on the choice of zero or nonzero fixed values to be assigned to the remaining feedback gains in f_c. An algorithm, which determines whether such a choice of n feedback edges that satisfy the conditions of Theorem 3.2 exists, is given in the the next section.

3.2.3 The Choice Algorithm

In this section we present an algorithm to check the existence of a set of n f-edges $f_1, f_2, ..., f_n$ in $\mathcal{D}(\mathcal{F})$ which satisfies conditions of Theorem 3.2, and to identify one such set if there exists any. The algorithm accepts as input

I1: n, the number of state vertices in $\mathcal{D}(\mathcal{F})$,

I2: $f = (f_1, f_2, ..., f_\nu)$, a set of all f-edges, $\nu \geq n$.

I3: for each $1 \leq k \leq n$, a list of all f-cycle families $\{\mathcal{CF}_{ks}\}$ of width k, each $\mathcal{CF}_{ks}$ being specified as a product of the parametric weights of all the edges appearing in $\mathcal{CF}_{ks}$,

and produces as output

O1: a positive or negative response regarding the existence of a required set of f-edges, and if the response is positive,

O2: the chosen subset $f_v = (f_1^*, f_2^*, \cdots, f_n^*)$ of f (here we use a starred notation for the variable f-edges to distinguish between the orderings of f and f_v), O3: $\{\mathcal{CF}_k^*\}$, the list of particular simple f-cycle families defined by f_k^*, $1 \leq k \leq n$,

O4: the fixed values (0 or 1) assigned to the f-edges in $f_c = f - f_v$.

The algorithm tries to construct an arborescence (a directed tree) $\mathcal{T} = (\mathcal{V}_c \cup \mathcal{V}_f, \mathcal{E}_t)$ having a longest path of length $2n$ by examining all cycle families CF_{ks}, $k = 1, 2, \cdots, n$, $s = 1, 2, \cdots, n_k^c$; and all f-edges f_t^{ks}, $t = 1, 2, \cdots, n_{ks}^c$ appearing in each $\mathcal{CF}_{ks}$. It halts with a positive response as soon as such a longest path is constructed, and with a negative response if no such path can be formed. The vertices of $\mathcal{T}$ are arranged in $n+1$ levels, each of which, except level 0, is further divided into two sublevels. The vertices at the first sublevels constitute $\mathcal{V}_c$, and are called the c-vertices, while the vertices at the second sublevels constitute $\mathcal{V}_f$, and are called the f-vertices. Each c-vertex at level k is the child of some f-vertex at level $k-1$, and corresponds to an f-cycle family $\mathcal{CF}_{ks}$ of width k, while each f-vertex at level k is the child of some c-vertex $\mathcal{CF}_{ks}$ at the same level, and corresponds to an f-edge that occur in $\mathcal{CF}_{ks}$, $1 \leq k \leq n$. Level 0 contains a single f-vertex, denoted by f_0^*, which is the root of $\mathcal{T}$. The algorithm proceeds as follows:

Suppose that a path $\mathcal{P}_{k-1}$ of length $2(k-1)$ is constructed from $f_0{}^*$ to some f-vertex f_{k-1}^* at level $(k-1)$, with some f-edges of $\mathcal{D}(\mathcal{F})$ assigned to the branches and f-vertices on $\mathcal{P}_{k-1}$ as described below. Choose an f-cycle family $\mathcal{CF}_{ks}$ of width k which contains no f-edges that are assigned to the f-vertices of $\mathcal{P}_{k-1}$. If no such $\mathcal{CF}_{ks}$ exists, terminate the path $\mathcal{P}_{k-1}$, and search for an unexplored f-vertex at level $(k-1)$ to replace f_{k-1}^*. If there exist one or more such cycle families, construct a c-vertex for each of them and extend a branch from f_{k-1}^* to these c-vertices. Pick any one of these c-vertices, say $\mathcal{CF}_{ks}$, $s = 1, 2, \cdots, n_k^c$, and label it as $\mathcal{CF}_k^*$. Corresponding to each f-edge that occurs in $\mathcal{CF}_k^* = \mathcal{CF}_{ks}$ construct an f-vertex, f_t^{ks}, $t = 1, 2, \cdots, n_{ks}^c$, extend a branch from $\mathcal{CF}_k^*$ to each f_t^{ks}, and assign all other f-edges in $\mathcal{CF}_k^*$ to the branch $(\mathcal{CF}_k^*, f_t^{ks})$ of $\mathcal{T}$. Pick one of the f-vertices, say f_t^{ks}, and check if the assignment $f_k^* = f_t^{ks}$ violates the conditions of Theorem 3.2. If not, set $f_k^* = f_t^{ks}$, and repeat the whole procedure with $k-1$, f_{k-1}^* and $\mathcal{P}_{k-1}$ replaced by k, f_k^* and $\mathcal{P}_k$. If the assignment $f_k^* = f_t^{ks}$ violates the conditions of Theorem 3.2, terminate the path from f_0^* to f_t^{ks}, and pick another unexplored f-vertex to replace f_t^{ks}. If none of f_t^{ks} can be chosen as f_k^*, go back to the upper sublevel to replace $\mathcal{CF}_k^*$ with another unexplored c-vertex $\mathcal{CF}_{ks}$. If all the paths through all $\mathcal{CF}_{ks}$ are terminated, search for an unexplored f-vertex at level $(k-1)$ to replace f_{k-1}^*. In checking whether the assignment $f_k^* = f_t^{ks}$ violates the conditions of Theorem 3.2, the f-edges of $\mathcal{D}(\mathcal{F})$ that are assigned to any branch of $\mathcal{P}_k{}^*$ are assumed to be fixed at some nonzero value (at 1, for convenience), and all f-edges other than these and f_j^*, $1 \leq j \leq k$ can be fixed or variable, as appropriate.

With this introduction, we now state the choice algorithm to identify $f_v = (f_1^*, f_2^*, \cdots, f_n^*)$, where we adopt the following notation:

k: index to scan the levels of $\mathcal{T}$, $0 \leq k \leq n$,

n_k^c: number of distinct f-cycle families of width k in $\mathcal{D}(\mathcal{F})$,

s_k: index to scan the c-vertices of $\mathcal{T}$ at level k, $0 \leq s_k \leq n_k^c$,

$\mathcal{CF}_k^*$: the c-vertex chosen at level k

n_k^*: number of distinct f-edges of $\mathcal{D}(\mathcal{F})$ that appear in $\mathcal{CF}_k^*$,

t_k: index to scan the f-vertices of $\mathcal{T}$ at level k, $0 \leq t_k \leq n_k^*$,

f_k^*: the f-vertex chosen at level k

The corresponding flowchart is given in Figure 3.6.

The Algorithm

1. Set $k \leftarrow 1$, and construct vertex f_0^* of $\mathcal{T}$
2. Add the c-vertices $\mathcal{CF}_{ks}$ and the branches $(f_{k-1}^*, \mathcal{CF}_{ks})$ to $\mathcal{T}$, $1 \leq s \leq n_k^c$, and set $s_k \leftarrow 0$.
3. Set $s_k \leftarrow s_k + 1$. If $s_k \leq n_k^c$ go to 5.

4. Set $k \leftarrow k-1$. If $k=0$, stop. No choice of f_v is possible. Otherwise, go to 7.

5. If $\mathcal{CF}_{ks_k}$ contains an f-edge corresponding to an f-vertex f_j^* of $\mathcal{T}$, $i \leq j \leq k-1$, terminate the path from f_0^* to $\mathcal{CF}_{ks_k}$, and go to 3. Otherwise, let $\mathcal{CF}_k^* = \mathcal{CF}_{ks_k}$.

6. Add the f-vertices f_{kt} and the branches $(\mathcal{CF}_k^*, f_{kt})$ to $\mathcal{T}$, $1 \leq t \leq n_k^*$, and set $t_k \leftarrow 0$.

7. Set $t_k \leftarrow t_k + 1$. If $t_k > n_k^*$, go to 3.

8. If f_{kt_k} is assigned to any branch $(\mathcal{CF}_j^*, f_j^*)$ of $\mathcal{T}$, $1 \leq j \leq k-1$, terminate the path from f_0^* to f_{kt_k}, and go to 7. Otherwise, assign all the f-edges in $\mathcal{CF}_k^*$, except f_{kt_k}, to the branch $(\mathcal{CF}_k^*, f_{kt_k})$.

9. Delete all f-edges of $\mathcal{D}(\mathcal{F})$ except f_{kt_k}, those that correspond to the f-vertices f_j^*, $1 \leq j \leq k-1$, and those that are assigned to the branches $(\mathcal{CF}_j^*, f_j^*)$, $1 \leq j \leq k-1$. Convert all f-edges that are assigned to the branches $(\mathcal{CF}_j^*, f_j^*)$ to d-edges by choosing their weights to be unity. If $f_1^*, f_2^*, \cdots, f_{k-1}^*$ and f_{kt_k} do not satisfy the conditions of Theorem 3.2 for the remaining subgraph and with n replaced by k, terminate the path from f_0^* to f_{kt_k} and go to 7. Otherwise, let $f_k^* = f_{kt_k}$.

10. If $k < n$, set $k \leftarrow k+1$, and go to 2.

11. Let $f_v = (f_1^*, f_2^*, \cdots, f_n^*)$. Convert all the remaining f-edges of $\mathcal{D}(\mathcal{F})$ into d-edges by fixing their weights to 1 if they are assigned to some branch $(\mathcal{CF}_k^*, f_k^*)$ of $\mathcal{T}$, $1 \leq k \leq n$, and to 0 otherwise. Stop.

The following example demonstrates an application of the choice algorithm. This example also shows the significance of Theorem 3.1 and Theorem 3.2 and hence the usefulness of our choice algorithm. In a classical approach, in order to place all the poles of a system with a digraph $\mathcal{D}(\mathcal{F})$ of the example, as given in Figure 3.7, at desired locations, one would attempt to use a dynamic output compensator , whereas we show below that constant output feedback is sufficient for the job.

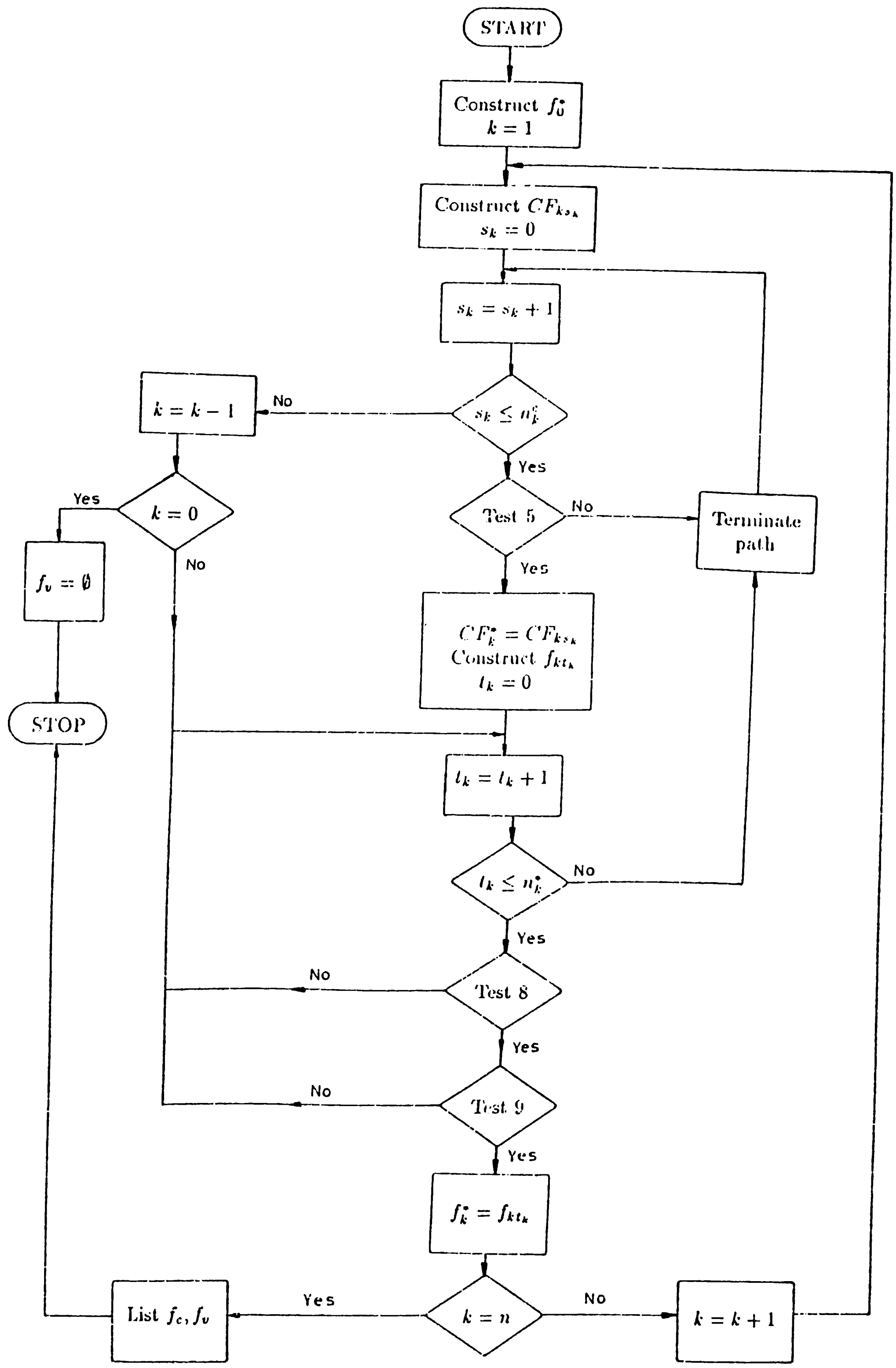


Figure 3.6. Flowchart of the choice algorithm

Example 3.5 Consider the digraph $\mathcal{D}(\mathcal{F})$ of Figure 3.7, in which unity weight is assigned to any d-edge adjacent from an input or to an output vertex. In this example, this causes no loss of generality as every input or output vertex has a unique edge adjacent from or to itself. We want to identify an $f_v = (f_1^*, f_2^*, f_3^*, f_4^*)$, if there exists any. We have, $n = 4$, $f = (f_1, f_2, f_3, f_4, f_5, f_6)$ and the list of all f-cycle families $\{\mathcal{CF}_{ks}\}$ of width k, $1 \leq k \leq n$, is given in Table 3.4.

Let us now apply the choice algorithm. The steps which the algorithm goes through are given below in detail. Figure 3.8 shows the arborescence $\mathcal{T}$ constructed during this application.

1. $k = 1$, $\mathcal{T} = (\{f_0^*\}, \emptyset)$

2. The c-vertex $\mathcal{CF}_{11}$, and the branch $(f_0{}^*, \mathcal{CF}_{11})$ added to $\mathcal{T}$. $s_1 = 0$.

3. $s_1 = 1$ $(= n_1{}^c)$.

5. $\mathcal{CF}_1^* = \mathcal{CF}_{11}$, $n_1^* = 1$.

6. The f-vertex $f_{11} = f_6$, and the branch $(\mathcal{CF}_1^*, f_6)$ added to $\mathcal{T}$. $t_1 = 0$.

7. $t_1 = 1$ $(= n_1^*)$.

8. Pass

9. Satisfied. $f_1^* = f_{11} = f_6$

10. $k = 2$

2. $\mathcal{CF}_{21}$, $\mathcal{CF}_{22}$ and $(f_1^*, \mathcal{CF}_{21})$, $(f_2^*, \mathcal{CF}_{22})$ added to $\mathcal{T}$. $s_2 = 0$.

3. $s_2 = 1$ $(< n_2^c)$

5. $\mathcal{CF}_2^* = \mathcal{CF}_{21}$, $n_2^* = 1$

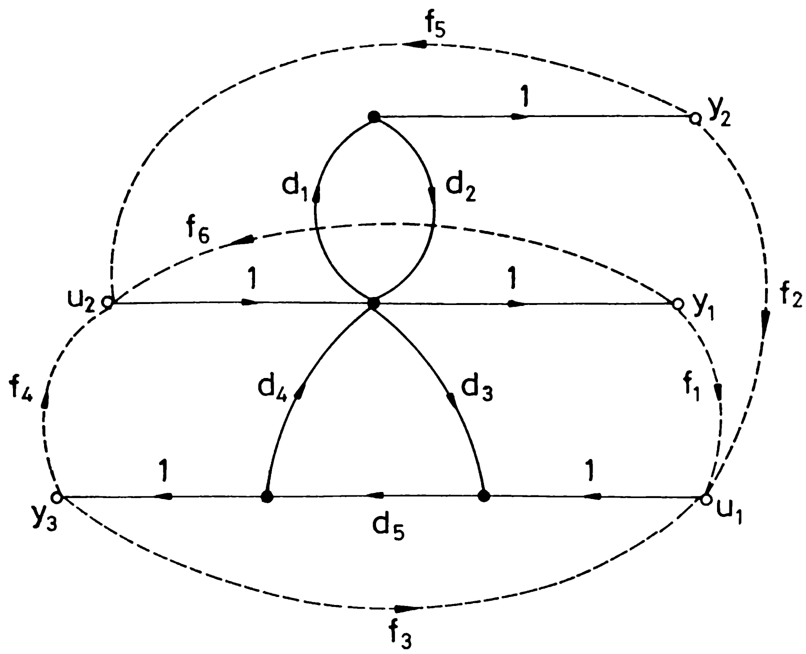


Figure 3.7. $\mathcal{D}(\mathcal{F})$ of Example 3.5.

k	$\mathcal{CF}_{ks}$
1	$\{f_6\}$
2	$\{d_5, f_3\}$
	$\{d_1, f_5\}$
3	$\{\ \{d_5, f_3\}, \{f_6\}\ \}$
	$\{d_5, d_3, f_4\}$
	$\{d_5, f_1, f_4\}$
	$\{d_4, d_5, f_1\}$
4	$\{d_1, d_4, d_5, f_2\}$
	$\{d_1, f_4, d_5, f_2\}$
	$\{\ \{d_5, f_3\}, \{d_1, f_5\}\ \}$

Table 3.4. F-cycle families in $\mathcal{D}(\mathcal{F})$ of Figure 3.7.

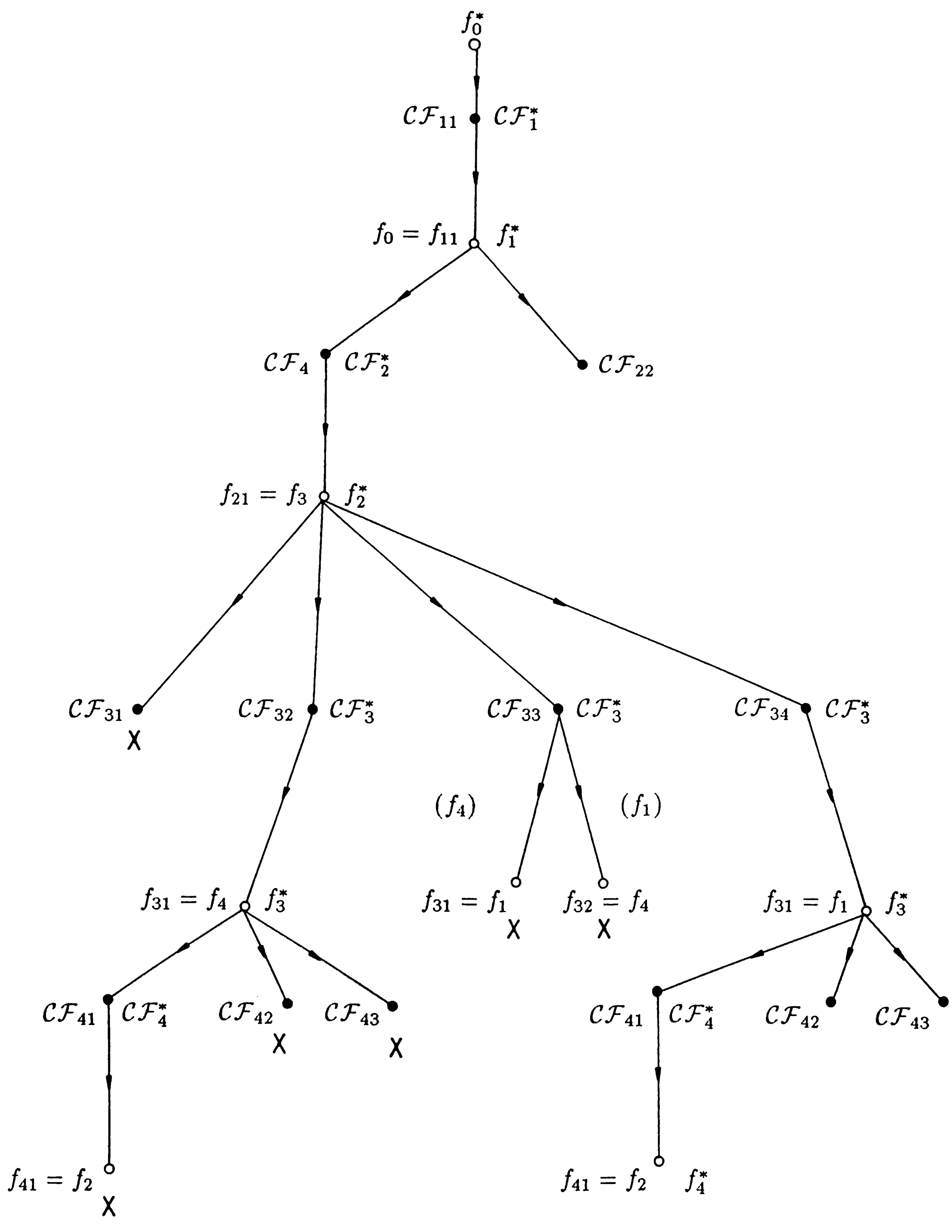


Figure 3.8. The arboresence generated by the choice algorithm in Example 3.10.

6. $f_{21} = f_3$ and $(\mathcal{CF}_2^*, f_3)$ added to $\mathcal{T}$. $t_2 = 0$.

7. $t_2 = 1$ $(= n_2^*)$

8. Pass

9. Satisfied. $f_2^* = f_{21} = f_3$

10. $k = 3$

2. $\mathcal{CF}_{31}$, $\mathcal{CF}_{32}$, $\mathcal{CF}_{33}$, $\mathcal{CF}_{34}$ and $(f_2^*, \mathcal{CF}_{31})$, $(f_2^*, \mathcal{CF}_{32})$,$(f_2^*, \mathcal{CF}_{33})$, $(f_2{}^*, \mathcal{CF}_{34})$ added to $\mathcal{T}$. $s_3 = 0$

3. $s_3 = 1$ $(< n_3^c)$

5. Terminate the path ($\mathcal{CF}_{31}$ contains $f_3 = f_2^*$).

3. $s_3 = 2$ $(< n_3^c)$

5. $\mathcal{CF}_3^* = \mathcal{CF}_{32}$, $n_3^* = 1$

6. f_4, $(\mathcal{CF}_3^*, f_4)$ added to $\mathcal{T}$. $t_3 = 0$

7. $t_3 = 1$ $(= n_3^*)$

8. Pass

9. Satisfied. $f_3^* = f_{31} = f_4$.

10. $k = 4$

2. $\mathcal{CF}_{41}, \mathcal{CF}_{42}, \mathcal{CF}_{43}$, $(f_3^*, \mathcal{CF}_{41})$, $(f_3^*, \mathcal{CF}_{42})$, $(f_3^*, \mathcal{CF}_{43})$ added to $\mathcal{T}$. $s_4 = 0$

3. $s_4 = 1$ $(< n_4^c)$.

5. $\mathcal{CF}_4^* = \mathcal{CF}_{41}$, $n_4^* = 1$

6. f_2, $(\mathcal{CF}_4^*, f_2)$ added to $\mathcal{T}$. $t_4 = 0$

7. $t_4 = 1$ $(= n_4^*)$.

8. Pass

9. Violated. (f_3 and f_6 appear in disjoint cycles, with no corresponding (d_r, f_r) pair). Terminate the path.

7. $t_4 = 2$ $(> n_4^*)$

3. $s_4 = 2$ $(< n_4^c)$

5. Terminate the path. ($\mathcal{CF}_{42}$ contains $f_4 = f_3^*$).

3. $s_4 = 4$ $(> n_4^c)$

4. $k = 3$

7. $t_3 = 2$ $(> n_3^*)$

3. $s_3 = 3$ $(< n_3^c)$

5. $\mathcal{CF}_3^* = \mathcal{CF}_{33}$, $n_3^* = 2$

6. $f_{31} = f_1$, $f_{32} = f_4$, $(\mathcal{CF}_3^*, f_1)$, $(\mathcal{CF}_3^*, f_4)$ added to $\mathcal{T}$. $t_3 = 0$.

7. $t_3 = 1$ $(< n_3^*)$

8. Assign f_4 to branch $(\mathcal{CF}_3^*, f_{31})$.

9. Not satisfied (f_3 and f_6 should be grouped by f_4 (d-edge) and f_1, but f_4 does not appear in every cycle of f_1.). Terminate the path.

7. $t_3 = 2$ $(= n_3^*)$

8. Assign f_1 to $(\mathcal{CF}_3^*, f_{32})$.

9. Violated. Terminate the path.

7. $t_3 = 3$ $(> n_3^*)$

3. $s_3 = 4$ $(= n_3^c)$

5. $\mathcal{CF}_3^* = \mathcal{CF}_{34}$, $n_3^* = 1$.

6. $f_{31} = f_1$ and $(\mathcal{CF}_3^*, f_{31})$ added to $\mathcal{T}$. $t_3 = 0$

7. $t_3 = 1$ $(= n_3^*)$

8. Pass

9. Satisfied. $f_3^* = f_{31}$

10. $k = 4$

2. $\mathcal{CF}_{41}$,$\mathcal{CF}_{42}$, $\mathcal{CF}_{43}$ and $(f_3^*, \mathcal{CF}_{45})$, $s = 1, 2, 3$ added to $\mathcal{T}$. $s_4 = 0$

3. $s_4 = 1$ $(< n_4^c)$.

5. $\mathcal{CF}_4^* = \mathcal{CF}_{41}$, $n_4^* = 1$.

6. Add $f_{41} = f_2$ and $(\mathcal{CF}_4^*, f_{41})$ to $\mathcal{T}$. $t_4 = 0$

7. $t_4 = 1$ $(= n_4^*)$

8. Pass

9. Satisfied. $f_4^* = f_{41} = f_2$

10. Pass

11. $f_v = (f_6, f_3, f_1, f_2)$, $f_4 = f_5 = 0$, $\mathcal{CF}_1^* = \{f_6\}$, $\mathcal{CF}_2^* = \{d_5, f_3\}$, $\mathcal{CF}_3^* = \{d_4, d_5, f_1\}$, $\mathcal{CF}_4^* = \{d_1, d_4, d_5, f_2\}$.

Hence the response is positive and the chosen f_v, the corresponding particular simple f-cycle families, and the fixed values assigned to the remaining edges are as displayed in step 11 above.

3.3 CLASSES OF STRUCTURALLY POLE ASSIGNABLE SYSTEMS

In this section we show that certain classes of systems which are known to be generically pole assignable by state or dynamic output feedback satisfy conditions of Theorem 3.2 and hence demonstrate that Theorem 3.2 characterizes a non-trivial class of pole assignable structures.

3.3.1 Structurally Controllable Systems With State Feedback

Consider a system described by

$$\mathcal{S}: \quad \dot{x} = Ax + Bu, \tag{3.14}$$

and a full state feedback law

$$\mathcal{F}: \quad u = Fx, \tag{3.15}$$

where $x \in \mathcal{R}^n$ and $u \in \mathcal{R}^m$. Since $\mathcal{F}$ is a special case of static output feedback with states considered as outputs, the resulting closed-loop system $\mathcal{S}(\mathcal{F})$ can be represented by the reduced system structure matrix

$$\mathbf{S(F)} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{F} & \mathbf{0} \end{bmatrix} \tag{3.16}$$

Let the corresponding open- and closed-loop system digraphs be $\mathcal{D}_{ux} = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{ux})$ and $\mathcal{D}_{xu}(\mathcal{F}) = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{ux} \cup \mathcal{E}_F)$. We state our main result concerning $\mathcal{S}(\mathcal{F})$ as:

Theorem 3.3 *The following are equivalent.*

(a) $\mathcal{S}$ is structurally controllable.

(b) $\mathcal{S}(\mathcal{F})$ is generically pole-assignable.

(c) There exists a choice of n feedback edges such that when the remaining feedback edges are assigned suitably fixed weights, $\mathcal{D}_{xu}(\mathcal{F})$ satisfies the conditions of Theorem 3.1.

The proof of Theorem 3.3 is based on the following two lemmas.

Lemma 3.4 *Let $\mathcal{D}_c = (\mathcal{X} \cup \{u\}, \mathcal{E})$ be a cactus. Then there exists an enumeration of the state vertices such that*

(a) if x_i is on a non-terminal bunch and x_j is on the terminal bunch, then $i < j$,

(b) if $(x_i, x_j) \in \mathcal{E}$ and x_j is not the tail of the distinguished edge of some bud, then $j = i + k + 1$, where k is the total number of state vertices on the precactus with origin x_i.

Proof: Using a modified depth-first search algorithm [46], scan first the non-terminal bunches (if there are any) in any order, and last the terminal bunch of $\mathcal{D}_c$, and assign the integers $1, 2, \cdots, n$ to the state vertices during the scanning process according to the following simple recursive scheme: Let the current vertex being visited be x_i. If there is a cactus or precactus with origin at x_i, then replace $\mathcal{D}_c$ by this cactus or precactus (with x_i taking the role of u) and repeat. Otherwise, let the unique vertex adjacent from x_i be x^*. If x^* is not yet assigned an integer, let $i \leftarrow i+1$, $x_i = x^*$, and repeat. Otherwise, x^* should be adjacent from the root of the cactus or precactus currently being scanned. Continue with another bunch.

It is obvious that this scanning of $\mathcal{D}_c$ results in an enumeration of the state vertices which satisfies the requirements. To illustrate the scheme, enumeration of the vertices of a simple cactus is shown in Figure 3.9. □

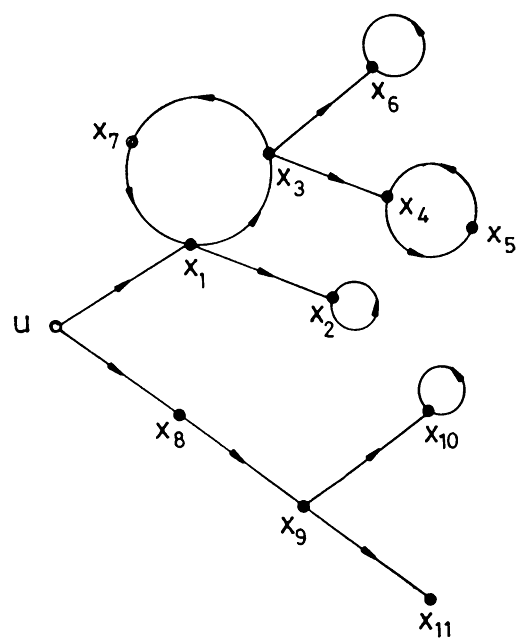


Figure 3.9. Enumeration of the state vertices in a cactus.

Lemma 3.5 *Let* $\mathbf{S} = (\mathbf{A}, \mathbf{B})$ *be structurally controllable. Then there exists a fixed feedback matrix* F_1 *and a column* $\mathbf{b}_i$ *of* $\mathbf{B}$ *such that*

(a) the nonzero elements of $\mathbf{A} + \mathbf{B}F_1$ *and* $\mathbf{b}_i$ *are algebraically independent, and*

(b) the system $\mathbf{S}_1 = (\mathbf{A} + \mathbf{B}F_1, \mathbf{b}_i)$ *is structurally controllable.*

Proof: If $(\mathbf{A}, \mathbf{b_i})$ is structurally controllable for some i, let $F_1 = 0$. Otherwise, let $\mathcal{D}_{xu}$ be spanned by a union of cacti $\mathcal{D}_{c1}, \mathcal{D}_{c2}, \cdots, \mathcal{D}_{ck}$ with roots $u_{i_1}, u_{i_2}, \cdots, u_{i_k}$ and tips $x_{n_1}, x_{n_1+n_2}, \cdots, x_{n_1+\cdots+n_k}$, where $1 \leq k \leq m$, $1 \leq i_1 < \cdots < i_k \leq m$, and $n_1 + n_2 + \cdots + n_k = n$. Let $F_1 = (f_{pq})$ with

$$f_{pq} = \begin{cases} 1, & \text{if } p = i_l,\ q = n_1 + \cdots + n_{l-1}, \text{ for some } 2 \leq l \leq k \\ 0, & otherwise, \end{cases}$$

and let $i = i_1$. Then, since elements of $(\mathbf{A}, \mathbf{B})$ are algebraically independent and nonzero elements of F_1 are fixed as unity, the elements of $(\mathbf{A} + \mathbf{B}F_1, \mathbf{b}_i)$ are also algebraically independent. Moreover, $\mathbf{S}_1$ is spanned by a cactus obtained by coinciding roots of $\mathcal{D}_{c,l+1}$ with $x_{n_1+\cdots+n_l}$, $l = 1, \cdots k-1$. □

Note that Lemma 3.5 is a structural counterpart of the well known algebraic result [47] that if (A, B) is controllable then for almost all F_1, $(A + BF_1, b_i)$ are controllable.

The following example demonstrates the result of Lemma 3.5.

Example 3.6 Consider a system $\mathcal{S} = (A, B)$ given by

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 \\ a_{51} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b_{42} & 0 \\ 0 & b_{52} & 0 \\ 0 & b_{62} & b_{63} \end{bmatrix}$$

The corresponding digraph, $\mathcal{D}$, can be obtained as in Figure 3.10(a)

(A, B) is structurally controllable since $\mathcal{D}$ is spanned by the collection of cacti $\mathcal{D}_{c1}$, $\mathcal{D}_{c2}$, $\mathcal{D}_{c3}$ which are defined by the vertex sets $\mathcal{V}_{c1} = \{u_1, x_1, x_2, x_3, \}$,

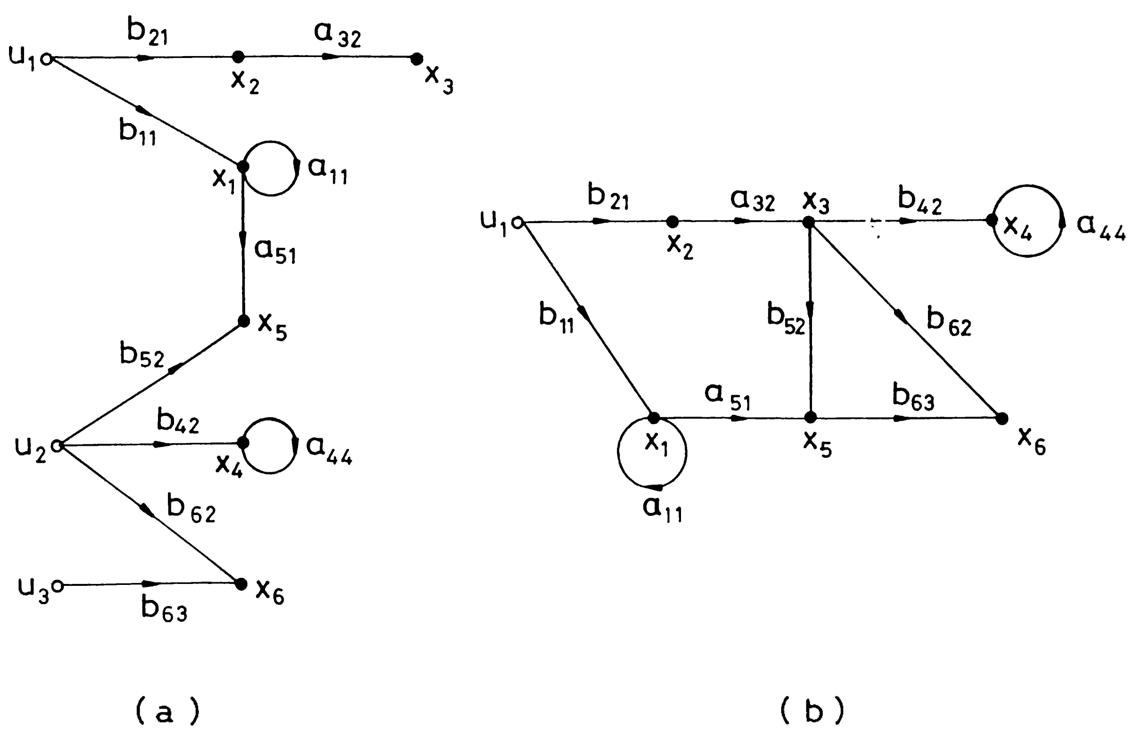


Figure 3.10. $\mathcal{D}(\mathcal{F})$ of Example 3.6.

$\mathcal{V}_{c2} = \{u_2, x_4, x_5,\}$, $\mathcal{V}_{c3} = \{u_3, x_6,\}$, respectively. On the other hand, observe that each (A, b_i) is structurally uncontrollable, $i = 1, 2, 3$.

Let us choose $F_1 = (f_{pq})$ as

$$F_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and consider the system $\mathcal{S}_1 = (A + BF_1, b_1)$ where

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{42} & a_{44} & 0 & 0 \\ a_{51} & 0 & b_{52} & 0 & 0 & 0 \\ 0 & 0 & b_{62} & 0 & b_{63} & 0 \end{bmatrix}, \quad b = \begin{bmatrix} b_{11} \\ b_{21} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Elements of $(A + BF_1, b_1)$ are obviously algebraically independent. The digraph associated with $\mathcal{S}_1$ is as shown in Figure 3.10(b), from where structural controllability of $\mathcal{S}_1$ can easily be concluded.

We now prove Theorem 3.3.

Proof of Theorem 3.3: Due to Lemma 3.5, it suffices to give the proof for single-input case.

(a) $\Leftrightarrow$ (b): Obvious

(c) $\Rightarrow$ (b) : Theorem 3.1

(a) $\Rightarrow$ (c) : Let the system digraph $\mathcal{D}_{xu}$ be spanned by a cactus $\mathcal{D}_c$, whose state vertices are enumerated as in Lemma 3.4. Let the feedback edges be enumerated in the same way so that $f_i = (x_i, u)$, $i = 1, 2, \cdots, n$. Since all f-cycles in $\mathcal{D}_{xu}(\mathcal{F})$ pass through vertex u, conditions (i) and (ii) of Theorem 3.1 are readily satisfied. The enumeration of the state vertices guarantee that for $i = 1, 2, \cdots, n$, any state vertex x_j with $j \leq i$ either lies on the complementary path of f_i in $\mathcal{D}_c(\mathcal{F})$, and hence belongs to the f-cycle defined by f_i, or belongs to a d-cycle in $\mathcal{D}_c(\mathcal{F})$ which has no vertex in common with the complementary path of f_i. Let $\mathcal{CF}_i^*$ denote the union of these cycles in $\mathcal{D}(\mathcal{F})$. Obviously, $\mathcal{CF}_i^*$ is a simple f-cycle family of width i which contains f_i. (For example, referring to Figure 3.10, $\mathcal{CF}_6^*$ consists of the f-cycle $\{(u, x_1), (x_1, x_3), (x_3, x_6), (x_6, u)\}$

and the d-cycles $\{(x_2, x_2)\}$ and $\{(x_4, x_5), (x_5, x_4)\}$). This proves condition $(iii - a)$ of Theorem 3.1. Now, let $\mathcal{CF}_i$ be any simple f-cycle family of width i which includes an f-edge f_j for some $j < i$. If $\mathcal{CF}_i$ contains a d-edge which does not belong to the edge set of $\mathcal{D}_c$, then this edge does not appear in any $\mathcal{CF}_l^*$, and condition $(iii - b)$ of Theorem 3.1 is readily satisfied for $\mathcal{CF}_i$. Suppose all the d-edges of $\mathcal{CF}_i$ belong to $\mathcal{D}_c$. Since $\mathcal{CF}_i$ covers exactly i vertices, it covers a vertex x_k with $k \geq i$. Then, the edge originating from x_k in $\mathcal{CF}_i$ is a d-edge (the only f-edge in $\mathcal{CF}_i$ is f_j which originates from x_j and $j < k$ which does not appear in any $\mathcal{CF}_l^*$, $l \leq k$. Again, $(iii - b)$ is satisfied. This completes the proof. □

3.3.2 A Class of Structurally Controllable and Observable Systems With Dynamic Output Feedback

Consider a single input/single output system

$$\mathcal{S}: \quad \begin{aligned} \dot{x} &= Ax + bu \\ y &= c^T x \end{aligned} \tag{3.17}$$

to be controlled by a dynamic output feedback of the form

$$\hat{\mathcal{S}}: \quad \begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{b}y \\ u &= \hat{c}^T\hat{x} + \hat{f}y \end{aligned} \tag{3.18}$$

where $\hat{x} \in \mathcal{R}^{\hat{n}}$ is the state of the controller $\hat{\mathcal{S}}$. It is well known [48] that the closed-loop system consisting $\mathcal{S}$ and $\hat{\mathcal{S}}$ is the same as the one obtained by applying a constant output feedback of the form

$$\mathcal{F}_a: \quad \begin{bmatrix} u \\ u_c \end{bmatrix} = \begin{bmatrix} \hat{f} & \hat{c}^T \\ \hat{b} & \hat{A} \end{bmatrix} \begin{bmatrix} y \\ y_c \end{bmatrix} \tag{3.19}$$

to an augmented system

$$\mathcal{S}_a: \quad \begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} u \\ u_c \end{bmatrix} \\ \begin{bmatrix} y \\ y_c \end{bmatrix} &= \begin{bmatrix} c^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \end{aligned} \tag{3.20}$$

Thus the pole assignment problem by dynamic output feedback is essentially the same as the pole assignment by constant output feedback, and hence, can be attacked with the graph-theoretic approach of Section 3.2.2.

We assume that $\mathcal{S}$ is structurally controllable and observable, that is, it has no structurally fixed modes. Let $\mathcal{D}(f)$ be the digraph of the closed-loop system consisting of $\mathcal{S}$ and the (scalar) constant output feedback

$$\mathcal{F}: \quad u = fy.$$

Then, $\mathcal{S}$ having no structurally fixed modes is equivalent to the following two conditions [36]:

(a) $\mathcal{D}(f)$ contains a cycle family $\mathcal{CF}$ of width n.

(b) $\mathcal{D}(f)$ is strongly connected, that is, each state vertex reaches every other either in $\mathcal{D}$, or through the feedback edge (y, u).

We further assume that each cycle in $\mathcal{CF}$ has a vertex in common with some input-output path in $\mathcal{D}$. This is a crucial assumption that enables us to define the auxiliary variables $\tilde{f}_k$ in (13) using simple polynomials ψ_k as will become clear in the following development.

We now choose the order of the controller $\hat{\mathcal{S}}$ to be $\hat{n} = n - 1$, and fix its structure as

$$\hat{A} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \hat{a}_{n-1} \\ 1 & 0 & \cdots & 0 & \hat{a}_{n-2} \\ 0 & 1 & \cdots & 0 & \hat{a}_{n-3} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & \hat{a}_1 \end{bmatrix}, \qquad \hat{b} = \begin{bmatrix} \hat{b}_{n-1} \\ \hat{b}_{n-2} \\ \vdots \\ \hat{b}_1 \end{bmatrix} \tag{3.21}$$

$$\hat{c}^T = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \qquad \hat{f} = f$$

where $\hat{a}_i$, $\hat{b}_i$, $i = 1, 2, \cdots, n-1$, and $\hat{f}$ are variable feedback gains. Thus, of the n^2 elements of $\mathcal{F}_a$ in (3.19), $n^2 - (2n-1)$ are fixed at 0 or 1 with the remaining $2n-1$ left as variable parameters.

With $\hat{\mathcal{S}}$ chosen as above, the closed-loop digraph $\mathcal{D}_a(\mathcal{F}_a)$ which corresponds to the system $\mathcal{S}_a(\mathcal{F}_a)$ has the structure shown in Figure 3.11.

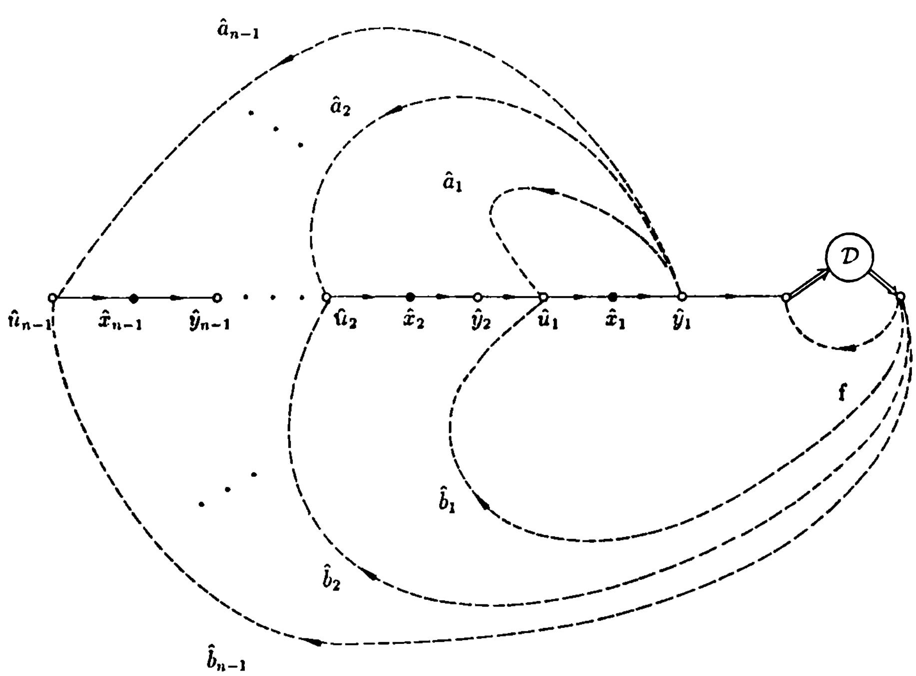


Figure 3.11. Illustration of the closed-loop system digraph $\mathcal{D}_a(\mathcal{F}_a)$.

We now prove the following result about pole assignability of $\mathcal{S}_a(\mathcal{F}_a)$:

Theorem 3.4 *Suppose that $\mathcal{D}(f)$ contains a cycle family of width n, each cycle of which has a vertex in common with some input-output path in $\mathcal{D}$. Then $\mathcal{D}_a(\mathcal{F}_a)$ satisfies the conditions of Theorem 3.2 with n replaced by $n_a = 2n - 1$.*

Proof: Referring to Figure 3.11, we first note that $\mathcal{D}_a(\mathcal{F}_a) = (\mathcal{V} \cup \hat{\mathcal{V}}, \mathcal{E} \cup \hat{\mathcal{E}} \cup \mathcal{E}_f)$, where $\mathcal{D} = (\mathcal{V}, \mathcal{E})$ is the digraph of $\mathcal{S}$, $\hat{\mathcal{D}} = (\hat{\mathcal{V}}, \hat{\mathcal{E}})$ is the digraph associated with the fixed parameters of $\hat{\mathcal{S}}$, and $\mathcal{E}_f$ is the set of (variable) f-edges corresponding to the feedback parameters $\hat{a}_i$, $\hat{b}_i$, $i = 1, 2, \cdots, n-1$; and $\hat{f}$. Thus $\mathcal{D}_a(\mathcal{F}_a)$ has $n_a = 2n - 1$ state vertices, which is exactly the same as the number of f-edges. We will show that these f-edges can be suitably ordered so as to satisfy the conditions of Theorem 3.2.

We first observe that f-cycles in $\mathcal{D}_a(\mathcal{F}_a)$ are of one of the following forms:

$$\begin{aligned} \mathcal{C}_I &= \{(y, u), \overline{(u, y)}\} \\ \mathcal{C}_{II} &= \{(\hat{y}_1, \hat{u}_i), \overline{(\hat{u}_i, \hat{y}_1)}\} \\ \mathcal{C}_{III} &= \{(y, \hat{u}_i), \overline{(\hat{u}_i, \hat{y}_1)}, (\hat{y}_1, u), \overline{(u, y)}\} \end{aligned}$$

where $\overline{(u, y)}$ denotes a path in $\mathcal{D}$ and $\overline{(\hat{u}_i, \hat{y}_1)}$ denotes a path in $\hat{\mathcal{D}}$. Consequently, no f-cycle in $\mathcal{D}_a(\mathcal{F}_a)$ contains more than one f-edge, satisfying condition (i) of Theorem 3.2. Also, only $f_p = (y, u)$ and an f-edge $f_{qi} = (\hat{y}_1, \hat{u}_i)$, $i = 1, 2, \cdots, n-1$, can appear in disjoint f-cycles (of forms $\mathcal{C}_I$ and $\mathcal{C}_{II}$, respectively). It is not difficult to see that for every such pair (f_p, f_{qi}), $d_r = (\hat{y}_1, u)$ and $f_{ri} = (y, \hat{u}_i)$ form a unique pair which satisfies condition $(ii)'$ of Theorem 3.2.

To continue the proof we need the following result:

Fact 3.4 $\mathcal{D}(f)$ has a subgraph $\tilde{\mathcal{D}}$ with the following properties:

(a) $\tilde{\mathcal{D}}$ contains a unique cycle family $\tilde{\mathcal{CF}}$ of width n.

(b) Each cycle in $\tilde{\mathcal{CF}}$ has a vertex in common with some input-output path.

(c) $\tilde{\mathcal{D}}$ is minimal in the sense that removal of any edge violates (a) or (b) above.

Proof of Fact 3.4. Pick an arbitrary cycle family $\tilde{\mathcal{CF}}$ of width n in $\mathcal{D}(f)$, and a minimal set $\tilde{\mathcal{E}}_a$ of additional d-edges such that each cycle in $\tilde{\mathcal{CF}}$ has a vertex in common with some input-output path in the subgraph $\tilde{\mathcal{D}}$, formed by $\tilde{\mathcal{CF}}$ and these additional d-edges. Include the f-edge into $\tilde{\mathcal{D}}$, if not already included. If $\tilde{\mathcal{D}}$ contains another cycle family $\bar{\mathcal{CF}}$ of width n, then one of the cycles in $\tilde{\mathcal{CF}}$ contains a d-edge which is not included in $\bar{\mathcal{CF}}$. The subgraph $\bar{\mathcal{D}}$ of $\tilde{\mathcal{D}}$ obtained by removing this particular d-edge still contains a cycle family of width n each cycle of which has a vertex in common with some input-output path. Replace $\tilde{\mathcal{D}}$ by $\bar{\mathcal{D}}$, $\tilde{\mathcal{CF}}$ by $\bar{\mathcal{CF}}$, and repeat the same argument. Each time by deleting a d-edge from $\tilde{\mathcal{D}}$ and modifying $\tilde{\mathcal{CF}}$, we eventually obtain a subgraph which satisfies properties (a) and (b). Finally, removing some d-edges from $\tilde{\mathcal{E}}_a$ if not needed for (b), minimality of $\tilde{\mathcal{D}}$ with respect to the properties (a) and (b) is guaranteed. This completes the proof. □

We note that in $\tilde{\mathcal{D}}$ in Fact 3.4 may or may not contain the f-edge (y, u). We continue with the proof of Theorem 3.3 by considering the two cases separately:

Case I: $\tilde{\mathcal{D}}$ does not include the f-edge (y, u).

In this case, $\tilde{\mathcal{CF}}$ is a d-cycle family of width n. Let $\tilde{\mathcal{D}}_a(\mathcal{F}_a)$ be the digraph obtained from $\mathcal{D}_a(\mathcal{F}_a)$ by replacing $\mathcal{D}$ with $\tilde{\mathcal{D}}$. Since $\tilde{\mathcal{D}}_a(\mathcal{F}_a)$ is obtained from $\mathcal{D}_a(\mathcal{F}_a)$ by removing some d-edges of $\mathcal{D}$, it suffice to complete the proof for $\tilde{\mathcal{D}}_a(\mathcal{F}_a)$, because $\tilde{\mathcal{D}}_a(\mathcal{F}_a)$ still satisfies conditions (i) and $(ii)'$ of Theorem 3.2, and if it also satisfies condition (iii), then so does $\mathcal{D}_a(\mathcal{F}_a)$.

Let $\{\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \cdots, \tilde{\mathcal{P}}_s\}$ be a family of input-output paths in $\tilde{\mathcal{D}}$ such that any d-cycle in $\tilde{\mathcal{CF}}$ has a vertex in common with some $\tilde{\mathcal{P}}_j$. Define $\tilde{\mathcal{CF}}_j$ to be the subfamily of all cycles in $\tilde{\mathcal{CF}}$ which has no vertex in common with any $\tilde{\mathcal{P}}_l$, $l > j$. The definition of $\tilde{\mathcal{P}}_j$ and $\tilde{\mathcal{CF}}_j$ is illustrated in Figure 3.12 for a simple

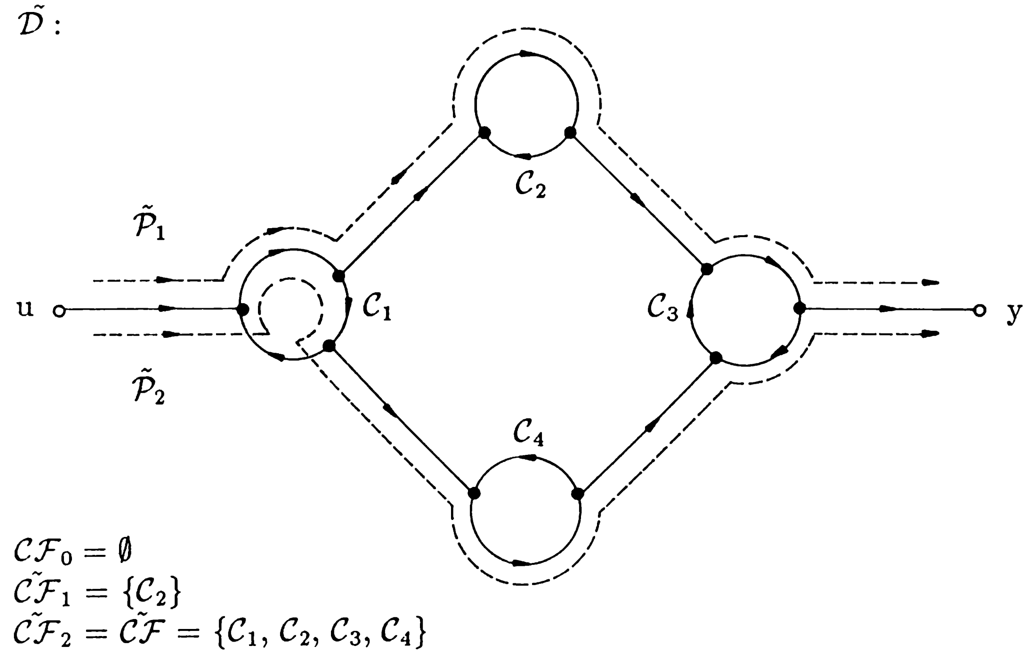


Figure 3.12. Definition of $\tilde{\mathcal{P}}_j$ and $\tilde{\mathcal{CF}}_j$.

digraph $\tilde{\mathcal{D}}$. Note that $\tilde{\mathcal{CF}}_0 = \emptyset$, $\tilde{\mathcal{CF}}_s = \tilde{\mathcal{CF}}$, and $\tilde{\mathcal{CF}}_{j-1} \subset \tilde{\mathcal{CF}}_j$, $j = 1, 2, \cdots, s$. We further define the integers α_j and β_j as the number of state vertices in $\tilde{\mathcal{P}}_j$ and $\tilde{\mathcal{CF}}_j$, that is, $\alpha_j = \gamma(\tilde{\mathcal{P}}_j)$ and $\beta_j = \gamma(\tilde{\mathcal{CF}}_j)$, $j = 1, 2, \cdots, s$, and let $\alpha_{s+1} = \beta_{-1} = \beta_0 = 0$ for convenience. It is easy to see that α_j and β_j satisfy

$$(a) \quad 1 \leq \beta_1 < \beta_2 < \cdots < \beta_s = n$$

$$(b) \quad \alpha_j + \beta_{j-1} \leq \alpha_{j+1} + \beta_j - 1, \quad 1 \leq j \leq s.$$

We partition the integers $\{1, 2, \cdots, n_a = 2n - 1\}$ into two groups at s levels as

Level	Group A	Group B
0	$1, \cdots, \alpha_1 - 1$	$\alpha_1, \cdots, \alpha_1 + \beta_1 - 1$
1	$\alpha_1 + \beta_1, \cdots, \alpha_2 + \beta_1 + \beta_1 - 1$	$\alpha_2 + \beta_1 + \beta_1, \cdots, \alpha_2 + \beta_1 + \beta_2 - 1$
$\vdots$	$\vdots$	
p	$\alpha_p + \beta_{p-1} + \beta_p, \cdots \alpha_{p+1} + \beta_p + \beta_p - 1$	$\alpha_{p+1} + \beta_p + \beta_p, \cdots, \alpha_{p+1} + \beta_p + \beta_{p+1} - 1$
$\vdots$	$\vdots$	$\vdots$
$s-1$	$\alpha_{s-1} + \beta_{s-2} + \beta_{s-1}, \cdots, \alpha_s + \beta_{s-1} + \beta_{s-1} - 1$	$\alpha_s + \beta_{s-1} + \beta_{s-1}, \cdots, \alpha_s + \beta_{s-1} + \beta_s - 1$
s	$\alpha_s + \beta_{s-1} + \beta_s, \cdots, 2n - 1.$	

where Group A/ Level 0 is empty if $\alpha_1 = 1$, and Group A/Level s is empty if $\alpha_s + \beta_{s-1} = n$.

We now define the feedback edges f_k and the associated cycle families $\tilde{\mathcal{CF}}_k^*$, $1 \leq k \leq 2n-1$, for $\tilde{\mathcal{D}}_a(\mathcal{F}_a)$ as follows:

(a) If $k \in$ Group A/ Level p, that is, if

$$\alpha_p + \beta_{p-1} + \beta_p \leq \alpha_{p+1} + \beta_p + \beta_p - 1,$$

then

$$\begin{aligned} f_k &= (\hat{y}_1, \hat{u}_{k-\beta_p}), \\ \tilde{\mathcal{CF}}_k^* &= \{\overline{(\hat{u}_{k-\beta_p}, \hat{y}_1)}, (\hat{y}_1, \hat{u}_{k-\beta_p})\} \cup \tilde{\mathcal{CF}}_p \end{aligned}$$

(b) If $k \in$ Group B / Level p, that is, if

$$\alpha_{p+1} + \beta_p + \beta_p \leq k \leq \alpha_{p+1} + \beta_p + \beta_{p+1} - 1,$$

then

$$\begin{aligned} f_k &= (y, \hat{u}_{k-\alpha_{p+1}-\beta_p}) \\ \tilde{\mathcal{CF}}_k^* &= \{\overline{(\hat{u}_{k-\alpha_{p+1}-\beta_p}, \hat{y}_1)}, (\hat{y}_1, u), \tilde{\mathcal{P}}_{p+1}, (y, \hat{u}_{k-\alpha_{p+1}-\beta_p})\} \cup \tilde{\mathcal{CF}}_p \end{aligned}$$

Note that in case (a)

$$\begin{aligned} \gamma(\tilde{\mathcal{CF}}_k^*) &= \gamma(\overline{(\hat{u}_{k-\beta_p}, \hat{y}_1)}) + \gamma(\tilde{\mathcal{CF}}_p) \\ &= (k - \beta_p) + \beta_p = k, \end{aligned}$$

and in case (b)

$$\begin{aligned}\gamma(\tilde{\mathcal{CF}}_k^*) &= \gamma(\overline{(\hat{u}_{k-\alpha_{p+1}-\beta_p}, \hat{y}_1)}) + \gamma(\tilde{\mathcal{P}}_{p+1}) + \gamma(\tilde{\mathcal{CF}}_p) \\ &= (k - \alpha_{p+1} - \beta_p) + \alpha_{p+1} + \beta_p = k,\end{aligned}$$

so that $\tilde{\mathcal{CF}}_k^*$ is an f-cycle family of width k in $\tilde{\mathcal{D}}_a(\mathcal{F}_a)$. By definition, it includes f_k and no other f-edge, satisfying condition $(iii-a)$ of Theorem 3.2. Finally, to prove condition $(iii-b)$, let $\tilde{\mathcal{CF}}_k$ be a simple f-cycle family of width k, which includes some f-edge f_l with $l < k$. We consider all possibilities for k and l:

1. $k \in$ Group A or B / Level p, $l \in$ Group A/ Level q, $q \leq p$. In this case, $\tilde{\mathcal{CF}}_k = \{\overline{(\hat{u}_{l-\beta_q}, \hat{y}_1)}, (\hat{y}_1, \hat{u}_{l-\beta_k})\} \cup \tilde{\mathcal{CF}}_d$, where $\tilde{\mathcal{CF}}_d$ is a d-cycle family in $\mathcal{D}$. If $\tilde{\mathcal{CF}}_d$ contains a d-edge which does not belong to $\tilde{\mathcal{CF}}_p$ (remember that $k \in$ Level p), then that d-edge does not belong to any $\tilde{\mathcal{CF}}_r$, $r \leq p$, either. Since any $j \leq k$ is at some level $r \leq p$, and $\tilde{\mathcal{CF}}_j^*$ includes $\tilde{\mathcal{CF}}_r$, this particular d-edge appears in no $\tilde{\mathcal{CF}}_j^*$, $j \leq k$, and condition $(iii-b)$ is satisfied. If $\tilde{\mathcal{CF}}_d$ does not contain such a d-edge, then minimality of $\tilde{\mathcal{D}}$ implies $\tilde{\mathcal{CF}}_d \subset \tilde{\mathcal{CF}}_p$. Then, $\gamma(\tilde{\mathcal{CF}}_d) \leq \beta_p$, and we must have $\gamma(\tilde{\mathcal{CF}}_k) = l - \beta_q + \beta_p$, with equality holding only if $\tilde{\mathcal{CF}}_d = \tilde{\mathcal{CF}}_p$. This, however, is impossible because

(a) If $q = p$, then either $\gamma(\tilde{\mathcal{CF}}_k) < l \leq k$ or $\gamma(\tilde{\mathcal{CF}}_k) \leq l < k$ (for $\gamma(\tilde{\mathcal{CF}}_k) = l = k$ can occur only if $\tilde{\mathcal{CF}}_k = \tilde{\mathcal{CF}}_k^*$),

(b) if $q < p$, then $l - \beta_q + \beta_p \leq \alpha_{q+1} + \beta_q + \beta_p - 1 < \alpha_{p+1} + \beta_p + \beta_p - 1 \leq k - 1$, contradicting the assumption that $\gamma(\tilde{\mathcal{CF}}_k) = k$.

2. $k \in$ Group A/ Level p, $l \in$ Group B/ Level q, $q < p$. In this case, $\tilde{\mathcal{CF}}_k = \{\overline{(\hat{u}_{l-\alpha_{q+1}-\beta_q}, \hat{y}_1)}, (\hat{y}_1, u), \tilde{\mathcal{P}}_t, (y, \hat{u}_{l-\alpha_{q+1}-\beta_q})\} \cup \tilde{\mathcal{CF}}_d$ where $\tilde{\mathcal{P}}_t$ is some input-output path in $\tilde{\mathcal{D}}$, and $\tilde{\mathcal{CF}}_d$ is some d-cycle family in $\tilde{\mathcal{D}}$. As in case 1, if $\tilde{\mathcal{CF}}_d$ contains a d-edge which does not occur in $\tilde{\mathcal{CF}}_p$, then condition $(iii-b)$ is satisfied. On the other hand, if $t \leq p+1$, then $\tilde{\mathcal{P}}_t$ contains a d-edge which has no vertex in common with any $\tilde{\mathcal{CF}}_r$, $r \leq p$, and again condition $(iii-b)$ is satisfied. The only remaining possibility is the case when $\tilde{\mathcal{CF}}_d \subset \tilde{\mathcal{CF}}_p$

and $t \leq p$. This case, however, can be shown, as in case 1, to lead to a contradiction that $\gamma(\tilde{\mathcal{CF}}_k) < k$.

3. $k \in$ Group B/ Level p, $l \in$ Group B/ Level q, $q \leq p$. $\tilde{\mathcal{CF}}_k$ is as in case 2. Again, if $\tilde{\mathcal{CF}}_d$ contains a d-edge which does not occur in $\tilde{\mathcal{CF}}_p$, or if $t \geq p+2$, then condition $(iii-b)$ is satisfied. Otherwise, $\tilde{\mathcal{CF}}_d \subset \tilde{\mathcal{CF}}_p$ and $t \leq p+1$, then $\gamma(\tilde{\mathcal{CF}}_k) < k$ unless $l = k$ and $\tilde{\mathcal{CF}}_d = \tilde{\mathcal{CF}}_p$, in which case $\tilde{\mathcal{CF}}_k = \tilde{\mathcal{CF}}_k^*$, and if $t \leq p$, then again $\gamma(\tilde{\mathcal{CF}}_k) < k$, both contradicting the assumption on $\tilde{\mathcal{CF}}_k$.

As a result, condition $(iii-b)$ is also satisfied, and the proof is complete for Case I.

Case II. The f-edge (y,u) is included in $\tilde{\mathcal{D}}$.

In this case, $\tilde{\mathcal{CF}}$ is an f-cycle family of width n, which includes the f-cycle $\{\overline{(u,y)},(y,u)\}$ with $\bar{\mathcal{P}} = (u,y)$ being some input-output path in $\tilde{\mathcal{D}}$. Let the family of the remaining d-cycles of $\tilde{\mathcal{CF}}$ be denoted by $\bar{\mathcal{CF}}$. Let $\{\bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2, \cdots, \bar{\mathcal{P}}_s\}$, $\bar{\mathcal{P}}_i \neq \bar{\mathcal{P}}$, be a minimal family of input-output paths in $\tilde{\mathcal{D}}$ such that any d-cycle in $\bar{\mathcal{CF}}$ has a vertex in common with some $\bar{\mathcal{P}}_j$, and let $\bar{\mathcal{P}}_{s+1} = \bar{\mathcal{P}}$. We now define subfamilies of $\bar{\mathcal{CF}}_j$, $1 \leq j \leq s$, the same way as $\tilde{\mathcal{CF}}_j$'s are defined in Case I, but with respect to $\bar{\mathcal{CF}}$ and $\{\bar{\mathcal{P}}_j\}$ rather than $\tilde{\mathcal{CF}}$ and $\{\tilde{\mathcal{P}}_j\}$, and similarly define integers α_j and β_j, $1 \leq j \leq s$, in terms of $\bar{\mathcal{CF}}_j$ and $\bar{\mathcal{P}}_j$. With these definitions, the proof follows the same lines as the proof of Case I, except that the integers at Level s are modified as

Level	GROUP A	GROUP B
s	$\alpha_s + \beta_{s-1} + \beta_s, \cdots, \alpha_{s+1} + \beta_s + \beta_s - 1$	$\alpha_{s+1} + \beta_s + \beta_s, \cdots, 2n-1,$

where, obviously, $\alpha_{s+1} + \beta_s = n$. This completes the proof of Theorem 3.4. □

The following two examples illustrate the ordering of the feedback edges, and the definition of $\tilde{\mathcal{D}}_k^*$, $1 \leq k \leq 2n-1$, for $\mathcal{D}_a(\mathcal{F}_a)$ of a structurally controllable and observable system controlled by dynamic output feedback, according to the scheme given in the proof of Theorem 3.4.

Example 3.7 The digraph $\mathcal{D}(f)$ associated with a structurally controllable and observable single-input/single-output system under feedback $u = fy$, is given in Figure 3.13(a). Figure 3.13(b) shows a subgraph $\tilde{\mathcal{D}}$ of $\mathcal{D}(f)$, which satisfies Fact 3.4.

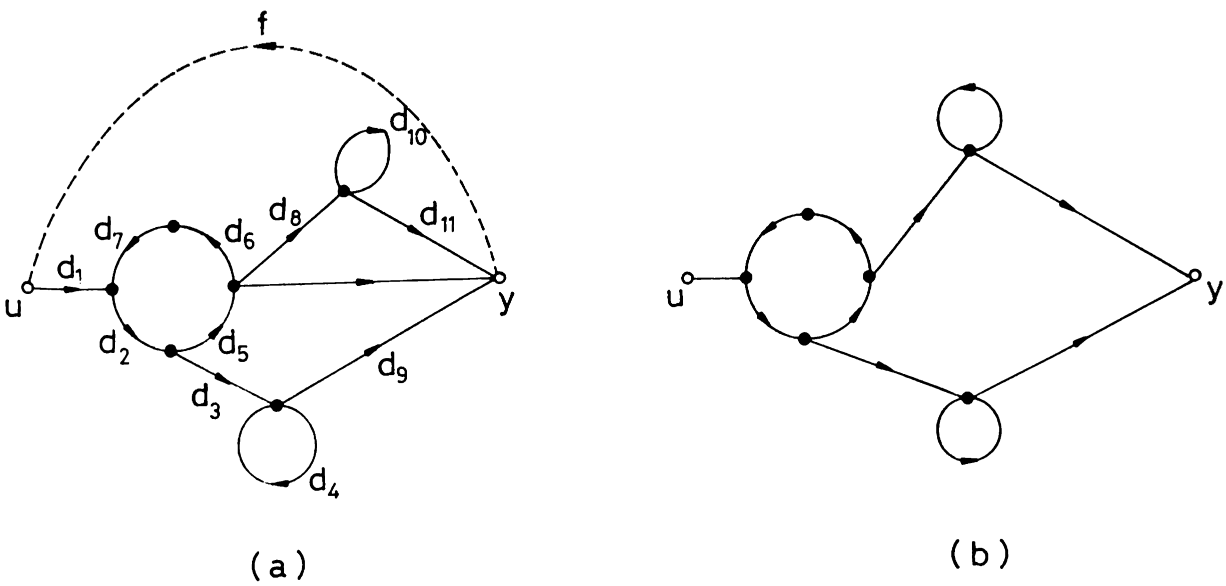


Figure 3.13. $\mathcal{D}(f)$ of Example 3.7.

The corresponding cycle family $\tilde{\mathcal{CF}}$ of width $n = 6$ is a d-cycle family (Case I in the proof of Theorem 3.4) defined as $\tilde{\mathcal{CF}} = \{\ \{d_7, d_6, d_5, d_2\}, \{d_4\}, \{d_{10}\}\ \}$. Then, $\tilde{\mathcal{P}}_1 = \{d_1, d_2, d_3, d_9\}$, $\tilde{\mathcal{P}}_2 = \{d_1, d_2, d_5, d_8, d_{11}\}$ are the input-output paths such that any cycle in $\tilde{\mathcal{CF}}$ has a common vertex with either $\tilde{\mathcal{P}}_1$ or $\tilde{\mathcal{P}}_2$. Thus we have, $\tilde{\mathcal{CF}}_1 = \{d_{10}\}$, $\tilde{\mathcal{CF}}_2 = \tilde{\mathcal{CF}}$, so that $\alpha_1 = 3$, $\alpha_2 = 4$, $\beta_1 = 1$, $\beta_2 = 6$, and the partitioning of the integers $k = 1, 2, \cdots, n_a$, $n_a = 11$, becomes

Level	Group A	Group B
0	1,2	3
1	4,5	6,7,8,9,10
2	11	-

Figure 3.14 shows the corresponding definition of the feedback edges in $\mathcal{D}_a(\mathcal{F}_a)$. Then, our choice of $\tilde{\mathcal{CF}}_k^*$, $1 \leq k \leq 11$, becomes

$$\tilde{\mathcal{CF}}_1^* = \{f_1\}$$

$$\tilde{\mathcal{CF}}_2^* = \{f_2\}$$

$$\tilde{\mathcal{CF}}_3^* = \{d_9, d_3, d_2, d_1, f_3\}$$

$$\tilde{\mathcal{CF}}_4^* = \{f_4\}$$

$$\tilde{\mathcal{CF}}_5^* = \{f_5\}$$

$$\tilde{\mathcal{CF}}_6^* = \{\ \{d_4\}, \{d_{11}, d_8, d_5, d_2, d_1, f_6\}\ \}$$

$$\tilde{\mathcal{CF}}_7^* = \{\ \{d_4\}, \{d_{11}, d_8, d_5, d_2, d_1, f_7\}\ \}$$

$$\tilde{\mathcal{CF}}_8^* = \{\ \{d_4\}, \{d_{11}, d_8, d_5, d_2, d_1, f_8\}\ \}$$

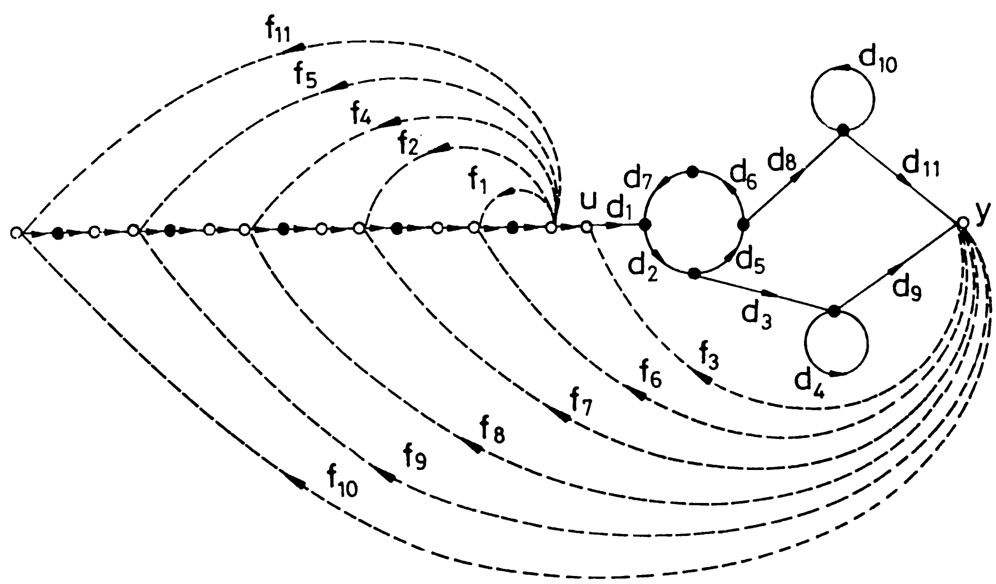


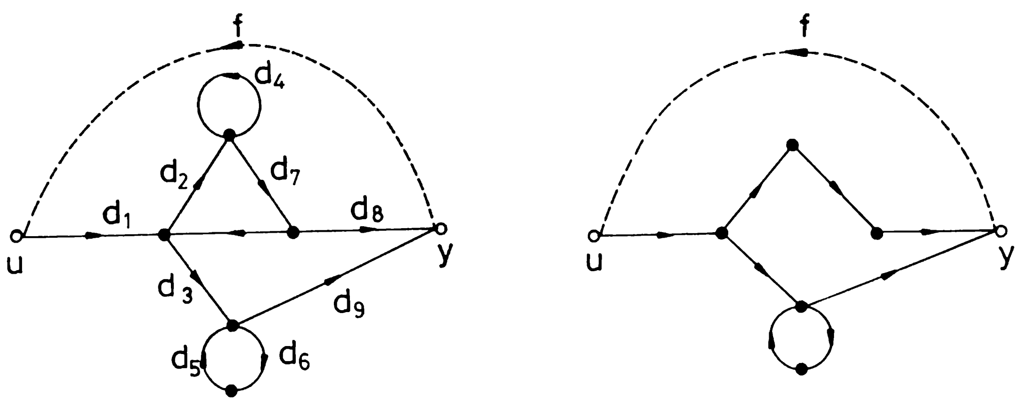
Figure 3.14. $\mathcal{D}_a(\mathcal{F}_a)$ illustrating the ordering of the f-edges corresponding to $\mathcal{D}(f)$ of Figure 3.13(a).

$$\tilde{\mathcal{CF}}^*_9 = \{\ \{d_4\}, \{d_{11}, d_8, d_5, d_2, d_1, f_9\}\ \}$$

$$\tilde{\mathcal{CF}}^*_{10} = \{\ \{d_4\}, \{d_{11}, d_8, d_5, d_2, d_1, f_{10}\}\ \}$$

$$\tilde{\mathcal{CF}}^*_{11} = \{\ \{d_4\}, \{d_{10}\}, \{d_7, d_6, d_5, d_2\}\ \}$$

Example 3.8 Consider the digraph $\mathcal{D}(f)$ given in Figure 3.15(a). The subgraph $\tilde{\mathcal{D}}$ of $\mathcal{D}(f)$ (there exists only one) is shown in Figure 3.15(b).



(a) (b)

Figure 3.15. Illustrations of (a) $\mathcal{D}(f)$, and (b) $\tilde{\mathcal{D}}$ of Example 3.8

$\tilde{\mathcal{CF}}$ is an f-cycle family (Case II in the proof of Theorem 3.3) defined as $\tilde{\mathcal{CF}} = \{\ \{d_8, d_7, d_2, d_1, f\}, \{d_5, d_6\}\ \}$. Then, we have $\bar{\mathcal{P}} = \{d_1, d_2, d_7, d_8\}$, and $\bar{\mathcal{CF}} = \{d_5, d_6\}$. Thus, $\bar{\mathcal{P}}_1 = \{d_1, d_3, d_9\}$, $\bar{\mathcal{P}}_2 = \bar{\mathcal{P}}$, $\bar{\mathcal{CF}}_1 = \bar{\mathcal{CF}}$ and $\alpha_1 = 2$, $\alpha_2 = 3$, $\beta_1 = 2$. This results in a partitioning of the integers $k = 1, \cdots, n_a$, $n_a = 9$, as

Level	Group A	Group B
0	1	2,3
1	4,5,6	7,8,9

Ordering the feedback edges as shown in Figure 3.16, we define $\tilde{\mathcal{CF}}_k^*$, $1 \leq k \leq 9$, as

$$\tilde{\mathcal{CF}}_1^* = \{f_1\}$$

$$\tilde{\mathcal{CF}}_2^* = \{d_9, d_3, d_1, f_2\}$$

$$\tilde{\mathcal{CF}}_3^* = \{d_9, d_3, d_1, f_3\}$$

$$\tilde{\mathcal{CF}}_4^* = \{f_4\}$$

$$\tilde{\mathcal{CF}}_5^* = \{f_5\}$$

$$\tilde{\mathcal{CF}}_6^* = \{f_6\}$$

$$\tilde{\mathcal{CF}}_7^* = \{\ \{d_5, d_6\}, \{d_8, d_7, d_2, d_1, f_7\}\ \}$$

$$\tilde{\mathcal{CF}}_8^* = \{\ \{d_5, d_6\}, \{d_8, d_7, d_2, d_1, f_8\}\ \}$$

$$\tilde{\mathcal{CF}}_9^* = \{\ \{d_5, d_6\}, \{d_8, d_7, d_2, d_1, f_9\}\ \}$$

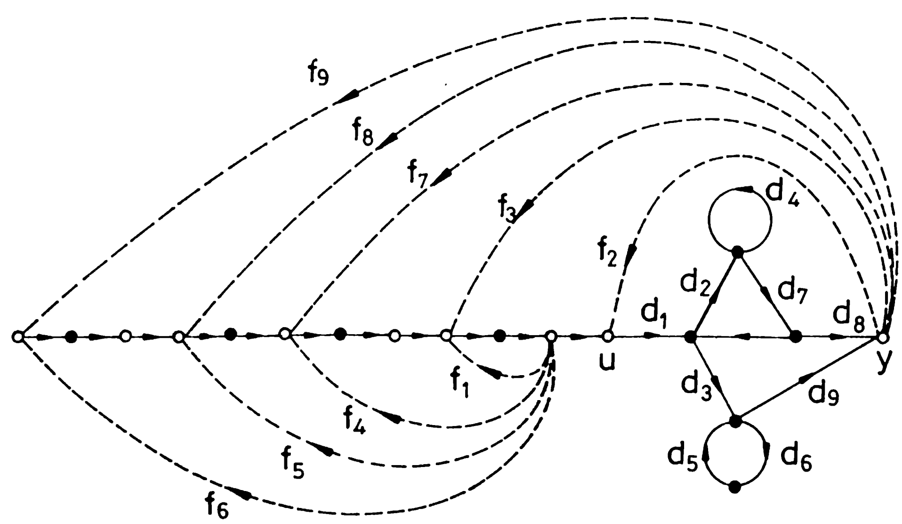


Figure 3.16. $\mathcal{D}_a(\mathcal{F}_a)$ illustrating the ordering of the f-edges corresponding to $\mathcal{D}(f)$ of Figure 3.15(a)

We note that our assumption that each cycle in the cycle family $\tilde{\mathcal{CF}}$ of width n has a vertex in common with some input-output path is obviously not essential for structural pole assignability of $\mathcal{S}$ using a dynamic output feedback controller $\hat{\mathcal{S}}$. However, it is needed for proving generic pole assignability using Theorem 3.2. On the other hand, we have observed through the study of several examples that it might be possible to remove this assumption by modifying Theorem 3.2 to include more general cases when ψ_k of (3.11) contains linear terms in addition to a single quadratic term. We illustrate this situation by the following example.

Example 3.9 Consider the digraph of Figure 3.17. The only cycle family of width $n = 4$ in $\mathcal{D}(f)$ is $\tilde{\mathcal{CF}} = \{ \{d_2, d_1, f\}, \{d_7, d_6, d_5\} \}$ and the cycle $\{d_7, d_6, d_5\}$ in $\tilde{\mathcal{CF}}$ does not have a vertex in common with any input-output path. Therefore, we cannot apply Theorem 3.4 in this case. Let us choose the controller $\hat{\mathcal{S}}$ as before and consider the enumeration of the feedback edges for the resulting $\mathcal{D}_a(\mathcal{F}_a)$, as shown in Figure 3.18.

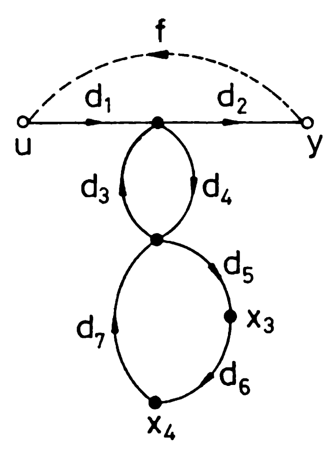


Figure 3.17. $\mathcal{D}(f)$ of Example 3.9.

Obviously, conditions (i) and $(ii)'$ of Theorem 3.2 are satisfied and thus each cycle family contains at most two variable weights. We claim that for the definition of the feedback variables as in Figure 3.18, the system satisfies the conditions of Corollary 3.1. To see this, we first obtain the cycle families of $\mathcal{D}_a(\mathcal{F}_a)$, as given in Table 3.5.

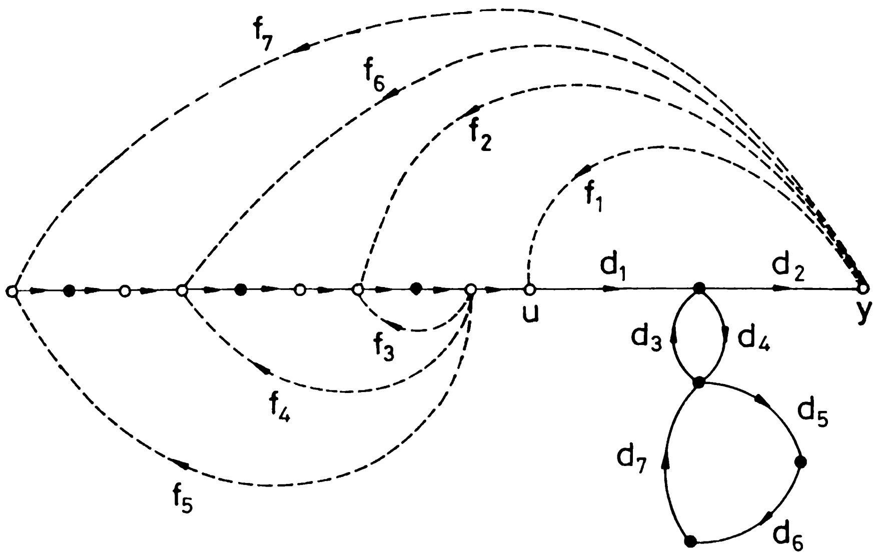


Figure 3.18. $\mathcal{D}_a(\mathcal{F}_a)$ illustrating the ordering of the f-edges for $\mathcal{D}(f)$ of Figure 3.17.

Let us now define $\tilde{f}_1 = f_1$, $\tilde{f}_2 = f_2 + f_1 f_3$, $\tilde{f}_3 = f_3$, $\tilde{f}_4 = f_4$, $\tilde{f}_5 = f_5$, $\tilde{f}_6 = d_2 d_1 f_6 + d_2 d_1 f_1 f_4 + f_5$, $\tilde{f}_7 = f_7 + f_1 f_5$. Then, $\bar{g}$ of (3.12) can be written as

$$\bar{g} = \alpha + E\tilde{f} =$$

$$\begin{bmatrix} 0 \\ d_4 d_3 \\ 0 \\ d_7 d_6 d_5 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} d_2 d_1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & d_2 d_1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & d_4 d_3 & 0 & 1 & 1 & 0 \\ d_7 d_6 d_5 d_2 d_1 & 0 & d_7 d_6 d_5 & d_4 d_3 & 0 & 0 & d_2 d_1 \\ 0 & d_7 d_6 d_5 d_2 d_1 & 0 & d_7 d_6 d_5 & d_4 d_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_7 d_6 d_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_7 d_6 d_5 \end{bmatrix} \tilde{f}$$

k	$\mathcal{CF}_{ks}$
1	$\{d_2, d_1, f_1\}$
	$\{f_3\}$
2	$\{d_4, d_3\}$
	$\{d_2, d_1, f_2\}$
	$\{f_4\}$
	$\{\ \{d_2, d_1, f_1\}, \{f_3\}\ \}$
3	$\{d_7, d_6, d_5\}$
	$\{d_2, d_1, f_6\}$
	$\{\ \{d_4, d_3\}, \{f_3\}\ \}$
	$\{f_5\}$
	$\{\ \{d_2, d_1, f_1\}, \{f_4\}\ \}$
4	$\{d_2, d_1, f_7\}$
	$\{\ \{d_7, d_6, d_5\}, \{d_2, d_1, f_1\}\ \}$
	$\{\ \{d_7, d_6, d_5\}, \{f_3\}\ \}$,
	$\{\ \{d_4, d_3\}, \{f_4\}\ \}$
	$\{\ \{d_2, d_1, f_1\}, \{f_5\}\ \}$
5	$\{\ \{d_7, d_6, d_5\}, \{d_2, d_1, f_2\}\ \}$
	$\{\ \{d_4, d_3\}, \{f_5\}\ \}$
	$\{\ \{d_7, d_6, d_5\}, \{d_2, d_1, f_1\}, \{f_3\}\ \}$
6	$\{\ \{d_7, d_6, d_5\}, \{d_2, d_1, f_6\}\ \}$
	$\{\ \{d_7, d_6, d_5\}, \{f_5\}\ \}$
	$\{\ \{d_7, d_6, d_5\}, \{d_2, d_1, f_1\}, \{f_4\}\ \}$
7	$\{\ \{d_7, d_6, d_5\}, \{d_2, d_1, f_7\}\ \}$
	$\{\ \{d_7, d_6, d_5\}, \{d_2, d_1, f_1\}, \{f_5\}\ \}$

Table 3.5. F-cycle families for $\mathcal{D}_a(\mathcal{F}_a)$ of Figure 3.18.

The diagonal elements in E are nonzero, and an argument similar to that used in showing the generic nonsingularity of the coefficient matrix in the proof of Theorem 3.1 can be used to justify that E is generically nonsingular.

Before closing the section , we finally note that, provided Theorem 3.2 is modified to remove the assumption mentioned in Theorem 3.3, our second assumption which restricts $\mathcal{S}$ to be a single-input/single-output system can easily be relaxed. One way of doing this is to employ preliminary constant output feedback to reduce the system to a single-input/single-output system without destroying structural controllability and observability, and then design $\hat{\mathcal{S}}$. A more efficient way, which also allows generic pole assignment using a smaller order dynamic compensator is to imitate the well-known results of [49,50] in a structural setting. This, however, requires a structural interpretation of controllability and observability indices of $\mathcal{S}$, which is not a straightforward task, as we consider in the Chapter 5.

Chapter 4

STABILIZATION: A STRUCTURAL APPROACH

This chapter is devoted to a qualitative analysis of the stabilization problem, again based on the structure of the pair $(\mathcal{S}, \mathcal{F})$. We first give an algebraic result on stabilizability of $(\mathcal{S}, \mathcal{F})$. Then, based on this result, we develop sufficient conditions for generic stabilizability of $(\mathcal{S}, \mathcal{F})$ in terms of the system digraph, $\mathcal{D}(\mathcal{F})$.

4.1 ALGEBRAIC FORMULATION

Consider the system $\mathcal{S}$ of (2.1) with a feedback $\mathcal{F}$ of (2.4) applied to it. The characteristic polynomial of the resulting closed loop system $\mathcal{S}(\mathcal{F})$ is $p(s)$ given by (3.1). Let the points $f = (f_1, f_2, \cdots, f_\nu)$ and $p = (p_1, p_2, \cdots, p_n)$ be defined as in Section 3.1 . In the following, we propose and prove a result on the stabilizability of $\mathcal{S}$ of (2.1) with $\mathcal{F}$ of (2.4):

Lemma 4.1 *Let f be partitioned into f_v and f_c as in Lemma 3.2, with feedback variables in f_v renumbered as $f_1, f_2, \cdots, f_n$. Suppose that the mapping $\bar{g}$ between p and f can be written in the 'staircase' form*

$$
\begin{array}{llll}
p_1 = \bar{g}_1(f_v) = \alpha_1 + (a_1 f_1 + c_1)b_{11} + \cdots + (a_k f_k + c_k)b_{k1} + \cdots + (a_n f_n + c_n)b_{n1} \\
\vdots \qquad \ddots \qquad \vdots \qquad \vdots \\
p_k = \bar{g}_k(f_v) = \qquad \alpha_k + (a_k f_k + c_k)b_{kk} + \cdots + (a_n f_n + c_n)b_{nk} \\
\vdots \qquad \ddots \qquad \vdots \\
p_n = \bar{g}_n(f_v) = \qquad \alpha_n + (a_n f_n + c_n)b_{nn}
\end{array}
\tag{4.1}
$$

where a_k, b_{kj}, c_k are polynomials in $f_1, \cdots, f_{k-1}$, $1 \le k \le n$, $1 \le j \le k-1$, with $a_k \not\equiv 0 \not\equiv b_{kk}$ and α_k's are constants. Then $\mathcal{S}$ of (2.1) is stabilizable with $\mathcal{F}$ of (2.4).

Proof: Suppose that conditions of the lemma hold, and let $\alpha_k = 0$, $k = 1, 2, \cdots, n$. Then, $p(s)$ can be written in a nested form as

$$p(s) = s[\cdots s[s(s + \tilde{f}_1 q_1(s)) + \tilde{f}_2 q_2(s)] + \cdots \tilde{f}_{n-1} q_{n-1}(s)] + \tilde{f}_n q_n(s) \tag{4.2}$$

where $\tilde{f}_k = a_k f_k + c_k$, and $q_k(s) = b_{k1}s^{k-1} + b_{k2}s^{k-2} + \cdots + b_{kk}$, $1 \le k \le n$. We use induction to show that (4.2) can be stabilized using a recursive root-locus technique. For this, we define

$$p_k(s) = s[\cdots s[s(s + \tilde{f}_1 q_1) + \tilde{f}_2 q_2] + \cdots \tilde{f}_{k-1} q_{k-1}] + \tilde{f}_k q_k.$$

(*i*) For $k = 1$, $p_1(s) = s + \tilde{f}_1 q_1 = s + (a_1 f_1 + c_1)b_{11}$ can be stabilized by choosing f_1 so as to place the only root of $p_1(s)$ on the negative real axis.

(*ii*) Suppose that $p_{k-1}(s)$ is stabilized by a proper choice of $f_1, f_2, \cdots, f_{k-1}$, and consider the root locus of $p_k(s) = sp_{k-1}(s) + \tilde{f}_k q_k(s)$, with respect to $\tilde{f}_k$. Since $sp_{k-1}(s)$ has k roots, all stable except one, which is at the origin, and $deg\,(q_k) \le k-1$, $\tilde{f}_k$ can be chosen to stabilize all the roots of $p_k(s)$. Since $\tilde{f}_k = a_k(f_1, f_2, \cdots, f_{k-1})f_k + c_k(f_1, f_2, \cdots, f_{k-1})$, f_k can be determined uniquely in terms of $\tilde{f}_k$ and $f_1, f_2, \cdots, f_{k-1}$. This completes the proof for the case when $\alpha_k = 0$, $1 \le k \le n$.

We note that , starting with an arbitrarily large $\tilde{f}_1$, and using high gains at each step, $p_k(s)$ can be stabilized with arbitrary degree of stability. In other words, all the roots of $p(s)$ can be placed to the left of the line $Re(s) = -\sigma_0$ in the complex plane for arbitrarily large real σ_0. With this observation in mind, replacing $p_k(s)$ by $p_k(s) - \alpha_k$ in the proof above, we can stabilize $p_k(s) - \alpha_k$ with arbitrary degree of stability (no matter how large α_k are), implying stabilizability of $p_k(s)$, $1 \leq k \leq n$. This completes the proof. □

The examples below demonstrate this result:

Example 4.1 Consider the system given by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x \end{aligned}$$

and controlled by the feedback

$$u = \begin{bmatrix} f_1 & 0 \\ f_2 & f_3 \end{bmatrix} y.$$

The closed loop characteristic polynomial is obtained as

$$p(s) = s^3 - (f_1 + f_3)s^2 - (f_3 + f_2 - f_1 f_3)s + f_1 f_3.$$

Thus, we obtain the mapping between p and f as

$$\begin{aligned} p_1 &= -f_1 - f_3 \\ p_2 &= -f_2 - f_3 + f_1 f_3 \\ p_3 &= f_1 f_3 \end{aligned}$$

which can be written in the staircase form as

$$\begin{aligned} p_1 &= (f_1)(-1) + (f_2)(0) \quad + (f_3)(-1) \\ p_2 &= \qquad\qquad\quad (f_2)(-1) + (f_3)(-1 + f_1) \\ p_3 &= \qquad\qquad\qquad\qquad\qquad (f_3)(f_1) \end{aligned}$$

and hence is stabilizable by Lemma 4.1.

Example 4.2 Let the system equations be given as

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} x \end{aligned}$$

and suppose that the allowed feedback has the form

$$u = \begin{bmatrix} f_1 & f_2 \\ 0 & f_3 \end{bmatrix} y.$$

This results in

$$p(s) = s^3 - (f_1 + f_3)s^2 - (f_2 + f_3 - f_1 f_3)s - (f_2 - f_1 f_3)$$

so that the mapping between p and f becomes

$$\begin{aligned} p_1 &= -f_1 - f_3 \\ p_2 &= -f_2 - f_3 + f_1 f_3 \\ p_3 &= -f_2 + f_1 f_3 \end{aligned}$$

and can be written in the staircase form as

$$\begin{aligned} p_1 &= (f_1)(-1) + (f_3)(-1) + (f_2 - f_1 f_3)(0) \\ p_2 &= \phantom{(f_1)(-1) + {}} (f_3)(-1) + (f_2 - f_1 f_3)(-1) \\ p_3 &= \phantom{(f_1)(-1) + (f_3)(-1) + {}} (f_2 - f_1 f_3)(-1). \end{aligned}$$

Therefore, this system is also stabilizable.

4.2 GENERIC STABILIZABILITY

4.2.1 Problem Formulation

Let us first give a definition of structural stabilizability, following the previous definitions of certain structural properties.

Definition 4.1 *A system $\mathcal{S}$ of (2.1) is said to be structurally stabilizable by a feedback $\mathcal{F}$ of (2.4) if there exists a system structurally equivalent to $\mathcal{S}$ which is stabilizable by $\mathcal{F}$.*

Now, as before, we associate a data point $d \in \mathcal{R}^{\mu}$ with the nonzero parameters of the system structure matrix of $\mathcal{S}$, which are assumed to be algebraically independent. In this case, the relation in (4.1) can be expressed as

$$p = \bar{g}(d, f_v) \tag{4.3}$$

with $a_k = a_k(d, \bar{f}_k)$, $b_{kj} = b_{kj}(d, \bar{f}_k)$ and $c_k = c_k(d, \bar{f}_k)$ polynomials in d and $\bar{f}_k \triangleq (f_1, \cdots, f_{k-1})$, $1 \leq k \leq n$, $1 \leq j \leq k$; and $\alpha_k = \alpha_k(d)$ is a polynomial in d. Then, we have the following straightforward result:

Lemma 4.2 *Let $f_v = (f_1, f_2, \cdots, f_n)$ and f_c be as in Lemma 4.1 and suppose that the closed-loop characteristic polynomial coefficients in (4.3) can be written as in (4.1), with a_k and b_{kk} being nonzero. Then, $\mathcal{S}$ is structurally stabilizable by $\mathcal{F}$.*

Note that, as in the case of structural pole assignability, structural stabilizability is not a generic property, in general. It is clear, however, that structural stabilizability implied by Lemma 4.2 is a generic property.

4.2.2 Graphical Conditions for Generic Stabilizability

We will use Lemma 4.2 in order to develop graphical conditions sufficient for generic stabilizability.

Let $\mathcal{D}(\mathcal{F}) = (\mathcal{V}, \mathcal{E} \cup \mathcal{E}_F)$ be the digraph associated with the closed-loop system $\mathcal{S}(\mathcal{F})$, in the usual way. The definition below allows for a generalization of condition $(ii)'$ of Theorem 3.2 to any pair of edges of the digraph.

Definition 4.2 *In $\mathcal{D}(\mathcal{F})$, consider a pair of edges, denoted by $\{e_p, e_q\}$ which never appear in the same cycle. Suppose that there corresponds to the pair $\{e_p, e_q\}$ a unique ordered pair of edges (e_r, e_s) such that,*

(a) e_s appears in every cycle of e_r but in no cycle of e_p or e_q, and

(b) to any two disjoint cycles C_p and C_q of e_p and e_q, there corresponds a cycle C_r of e_r which covers exactly the same state vertices as C_p and C_q cover, with no input and/or output vertices that occur in $C_p \cup C_q$ taking part in a cycle disjoint from C_r, and vice versa.

Then, we say that $\{e_p, e_q\}$ is a pair biased to (e_r, e_s) and that any cycle family of $\{e_p, e_q\}$ is an accompanying cycle family of e_r.

Note that, as in Fact 3 in the proof of Theorem 3.2, for a pair $\{e_p, e_q\}$ biased to an ordered pair (e_r, e_s), whenever e_r appears in some product term in $\bar{g}(d, f_v)$ of (4.2), so does the product e_pe_q, and vice versa. Moreover, every product term that contains e_r in any $\bar{g}_k(d, f_v)$ can be grouped with another term that contains the product e_pe_q, as $\beta_{kr}(\beta_r e_r + \beta_{pq}e_pe_q)$, with β_r and β_{pq} being the same in all such expressions.

We can now state and prove our first result on stabilization:

Theorem 4.1 *Suppose that in $\mathcal{D}(\mathcal{F})$ there exists a choice of n distinct f-edges, renumbered conveniently as $f_1, f_2, \cdots, f_n$, which after converting the remaining f-edges into d-edges by fixing their weights at arbitrary values, satisfy the conditions listed below. Then, $\mathcal{S}$ is structurally (generically) stabilizable with $\mathcal{F}$.*

There exists an integer $\bar{n}$, $1 \leq \bar{n} \leq n$, such that,

(i) for $k = n, n-1, \cdots, \bar{n}$, there exist particular cycle families of width k, denoted by $\mathcal{CF}_k^*$, such that $f_k \in \mathcal{CF}_k^*$, $f_j \notin \mathcal{CF}_k^*$, $j > k$, and either of the following holds:

(a) *Any cycle family of width $l < k$ which contains f_k either contains some f_j or is an accompanying cycle family of f_j, $j > k$.*

(b) *Any other cycle family of width k which neither contains nor is an accompanying cycle family of any f_j, $j > k$, contains either f_k or a pair of edges $\{e_p, e_q\}$ biased to (f_k, e), for some e such that if $e_p = f_l$ (respectively $e_q = f_l$), then $\mathcal{CF}_l^*$ does not contain e_q (respectively e_p), $l < k$.*

(ii) *With f_k and all $\{e_p, e_q\}$, which are biased to (f_k, e) for some e, removed, $k \geq \bar{n}$, the remaining digraph satisfies Theorem 3.1, with n replaced by $\bar{n} - 1$.*

Proof: For $k = n, n-1, \cdots, \bar{n}$, existence of $\mathcal{CF}_k^*$ as in condition (i) implies that each $\bar{g}_k$ in (4.3) contains an identically nonvanishing term that contains f_k, but no f_j, $j > k$. Let us denote this term by $a_k b_{kk}^* f_k$.

Consider the case $k = n$, and suppose that condition (i-b) holds. If every other f-cycle family of width n contains f_n, then $\bar{g}_n$ can be written as

$$\bar{g}_n(d, f_v) = a_n f_n(b_{nn}^* + b_{nn})$$

where for convenience, we let $a_n f_n b_{nn}$ represent sum of product terms corresponding to other f-cycle families of width n, which contain f_n. Then, the arrangement of the product terms that contain f_n in $\bar{g}_k$, $k = n, n-1, \cdots, 2, 1$, as in (4.1) follows with $c_n \equiv 0$. If, on the other hand, there are f-cycle families of width n which do not contain f_n but which contain a pair of edges $\{e_p, e_q\}$ as in condition (i-b), then every such cycle family is an accompanying cycle family of f_n and corresponds to a term including the product $e_p e_q$ which can be grouped as $\beta_{kn}(\beta_n f_n + \beta_{pq} e_p e_q)$ with β_n and β_{pq} being the same in all such expressions. This, however, defines nothing but the grouping $(a_n f_n + c_n)$ of the staircase form of (4.1), with $c_n = \beta_{pq} e_p e_q$. Note that, this is consistent with the definition of c_n, which is a polynomial in d and $f_1, f_2, \cdots, f_{n-1}$. Moreover, the condition which says that if $e_p = f_l$ (respectively, $e_q = f_l$) then $\mathcal{CF}_l^*$ does not contain e_q (respectively, e_p), guarantees that after this grouping, each $\bar{g}_k$ still contains the product term $a_k b_{kk}^* f_k$, $k = n, \cdots, 2, 1$.

Alternatively, if (i-a) holds for $k = n$, then f_n appears in no $\bar{g}_k$, $k = n, \cdots, 2, 1$, so that every product term in $\bar{g}_k$ is considered in the grouping $(a_n f_n + c_n) b_{nn}$ and we are done since $b_{nl} = 0$, $l = n-1, \cdots, 2, 1$.

For $k = n-1$, if condition (i-b) is satisfied the same argument as above applies. If, on the other hand, (i-a) is satisfied, then every product term in $\bar{g}_k$, $k = n-2, \cdots, 2, 1$, which includes f_{n-1} is a term associated with the grouping of f_n. This implies that, every product term in $\bar{g}_{n-1}$ not associated with f_n can be considered in the grouping of f_{n-1}, i.e., in $(a_{n-1} f_{n-1} + c_{n-1}) b_{n-1,n-1}$ and again we are done as $b_{n-1,l} = 0$, $l = n-2, \cdots, 2, 1$.

The same argument can be repeated for $k = n-2, n-3, \cdots, \bar{n}$ so that we have the following structure:

$$\begin{array}{rc|l}
\bar{g}_1(d, f_v) = & \multirow{3}{*}{$\boxed{(f_1, f_2, \cdots, f_{n-1})}$} & +(a_n f_n + c_n) b_{n1} \quad + \cdots + (a_n f_n + c_n) b_{n1} \\
\vdots & & \qquad\qquad\qquad\qquad \vdots \\
\bar{g}_{n-1}(d, f_v) = & & +(a_n f_n + c_n) b_{n,n-1} + \cdots + (a_n f_n + c_n) b_{n,n-1} \\
\bar{g}_n(d, f_v) = & & \alpha_n + (a_n f_n + c_n) b_{n,n} \quad + \cdots + (a_n f_n + c_n) b_{n,n} \\
\vdots & & \qquad \ddots \qquad\qquad\qquad \vdots \\
\bar{g}_n(d, f_v) = & & \qquad\qquad\qquad\qquad \alpha_n \quad + (a_n f_n + c_n) b_{nn}
\end{array} \tag{4.4}$$

The part appearing in box in (4.4) is exactly the part that satisfies condition (ii) of the theorem; and it is easy to show that by algebraic manipulations we can get,

$$\begin{aligned}
\bar{g}_1(d,f_v)-\alpha_1 &= (a_1f_1+c_1)b_{11}+(a_2f_2+c_2)b_{21}+\cdots+(a_{\bar{n}-1}f_{\bar{n}-1}+c_{\bar{n}-1})b_{\bar{n}-1,1}\\
\bar{g}_2(d,f_v)-\alpha_2 &= \qquad (a_2f_2+c_2)b_{22}+\cdots+(a_{\bar{n}-1}f_{\bar{n}-1}+c_{\bar{n}-1})b_{\bar{n}-1,2}\\
&\ \ \vdots \qquad\qquad \ddots \qquad\qquad \vdots\\
\bar{g}_{\bar{n}-1}(d,f_v)-\alpha_{\bar{n}-1} &= \qquad\qquad (a_{\bar{n}-1}f_{\bar{n}-1}+c_{\bar{n}-1})b_{\bar{n}-1,\bar{n}-1}
\end{aligned}$$

where a_k, b_{kl}, c_k and α_k, $1 \le k \le \bar{n}-1$, $1 \le l \le k$, are polynomials in d. Note that condition (*iii*-b) of Theorem 3.2 guarantees the existence of the term $a_k b^*_{kk} f_k$, for all k, $k = 1, 2, \cdots, \bar{n}-1$. The rest of the proof follows from Lemma 4.2. □

We illustrate Theorem 4.1 by few examples:

Example 4.3 Consider a system whose closed-loop digraph, $\mathcal{D}(\mathcal{F})$, corresponding to

$$\mathcal{F}: \quad u = \begin{bmatrix} f_{11} & 0 & 0 & 0 \\ f_{21} & f_{22} & f_{23} & 0 \\ 0 & f_{32} & f_{33} & f_{34} \\ 0 & 0 & f_{43} & f_{44} \end{bmatrix} y$$

is as given in Figure 4.1.

Let us fix $f_{23} = f_{34} = 0$ and renumber the remaining nonzero feedback variables as $f_1 = f_{11}$, $f_2 = f_{22}$, $f_3 = f_{21}$, $f_4 = f_{23}$, $f_5 = f_{33}$, $f_6 = f_{44}$, $f_7 = f_{43}$. The resulting f-cycle families are listed in Table 4.1. Consider the following choice of $\mathcal{CF}^*$, $k = n, n-1, \cdots, 1$

$$\begin{aligned}
\mathcal{CF}^*_7 &= \mathcal{CF}_{74} = \{\ \{d_2, d_3, d_5, d_4, f_3\}, \{d_{10}, d_9, d_{11}, d_{13}, d_{12}, f_7\}\ \}\\
\mathcal{CF}^*_6 &= \mathcal{CF}_{61} = \{\ \{d_2, d_1, f_1\}, \{d_6, d_7, d_9, d_8, f_4\}, \{d_{14}, d_{13}, d_{12}, f_6\}\ \}\\
\mathcal{CF}^*_5 &= \mathcal{CF}_{53} = \{\ \{d_2, d_3, d_5, d_4, f_3\}, \{d_{10}, d_9, d_8, f_5\}\ \}\\
\mathcal{CF}^*_4 &= \mathcal{CF}_{42} = \{\ \{d_2, d_1, f_1\}, \{d_6, d_7, d_9, d_8, f_4\}\ \}\\
\mathcal{CF}^*_3 &= \mathcal{CF}_{31} = \{d_2, d_3, d_5, d_4, f_3\}\\
\mathcal{CF}^*_2 &= \mathcal{CF}_{21} = \{d_6, d_5, d_4, f_2\}\\
\mathcal{CF}^*_1 &= \mathcal{CF}_{11} = \{d_2, d_1, f_1\}
\end{aligned}$$

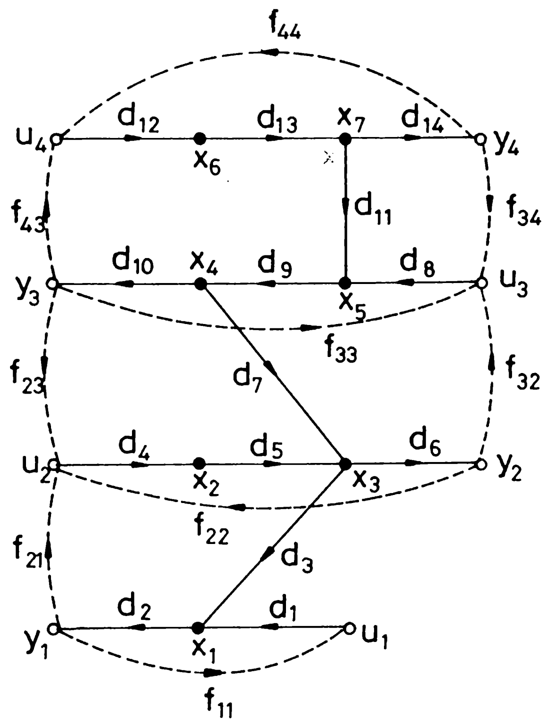


Figure 4.1. $\mathcal{D}(\mathcal{F})$ of Example 4.3.

We observe the following:

For $k = 7$, any f-cycle family of width 7, other than $\mathcal{CF}_7^*$, contains either f_7 or the pair $\{f_5, f_6\}$ biased to (f_7, d_{11}). Moreover, $\mathcal{CF}_5^*$ and $\mathcal{CF}_6^*$ do not contain f_6 and f_5, respectively, so that condition (i-b) of Theorem 4.1 is satisfied.

For $k = 6$, there are two f-cycle families, $\mathcal{CF}_{62}$ and $\mathcal{CF}_{63}$, of width 6, other than $\mathcal{CF}_6^*$, but $\mathcal{CF}_{62}$ is an accompanying cycle family of f_7 and $\mathcal{CF}_{63}$ contains f_7. So again, condition (i-b) is satisfied.

For $k = 5$, there exists only one f-cycle family, $\mathcal{CF}_{51}$, of width 5, which neither contains nor is an accompanying cycle family of f_6 or f_7; but it contains f_5, and hence, condition (i-b) is satisfied.

For $k = 4$, there exists no f-cycle family of width 4, other than $\mathcal{CF}_4^*$, which neither contains nor is an accompanying cycle family of f_5, f_6 or f_7.

Finally, with $\bar{n} = 3$, condition (ii) of Theorem 4.1 is satisfied for

$\gamma(\mathcal{CF})$	$\omega(\mathcal{CF})$
1	$\{d_2, d_1, f_1\}$
2	$\{d_6, d_5, d_4, f_2\}$
	$\{d_{10}, d_9, d_8, f_5\}$
	$\{d_{14}, d_{13}, d_{12}, f_6\}$
3	$\{d_2, d_3, d_5, d_4, f_3\}$
	$\{d_6, d_7, d_9, d_8, f_4\}$
	$\{ \{d_2, d_1, f_1\}, \{d_6, d_5, d_4, f_2\} \}$
	$\{ \{d_2, d_1, f_1\}, \{d_{10}, d_9, d_8, f_5\} \}$
	$\{ \{d_2, d_1, f_1\}, \{d_{14}, d_{13}, d_{12}, f_6\} \}$
4	$\{d_{10}, d_9, d_{11}, d_{13}, d_{12}, f_7\}$
	$\{ \{d_2, d_1, f_1\}, \{d_6, d_7, d_9, d_8, f_4\} \}$
	$\{ \{d_6, d_5, d_4, f_2\}, \{d_{10}, d_9, d_8, f_5\} \}$
	$\{ \{d_6, d_5, d_4, f_2\}, \{d_{14}, d_{13}, d_{12}, f_6\} \}$
	$\{ \{d_{10}, d_9, d_8, f_5\}, \{d_{14}, d_{13}, d_{12}, f_6\} \}$
5	$\{ \{d_2, d_1, f_1\}, \{d_6, d_5, d_4, f_2\}, \{d_{10}, d_9, d_8, f_5\} \}$
	$\{ \{d_2, d_1, f_1\}, \{d_6, d_5, d_4, f_2\}, \{d_{14}, d_{13}, d_{12}, f_6\} \}$
	$\{ \{d_2, d_3, d_5, d_4, f_3\}, \{d_{10}, d_9, d_8, f_5\} \}$
	$\{ \{d_2, d_3, d_5, d_4, f_3\}, \{d_{14}, d_{13}, d_{12}, f_6\} \}$
	$\{ \{d_6, d_7, d_9, d_8, f_4\}, \{d_{14}, d_{13}, d_{12}, f_6\} \}$
	$\{ \{d_2, d_1, f_1\}, \{d_{10}, d_9, d_8, f_5\}, \{d_{14}, d_{13}, d_{12}, f_6\} \}$
	$\{ \{d_2, d_1, f_1\}, \{d_{10}, d_9, d_{11}, d_{13}, d_{12}, f_7\} \}$
6	$\{ \{d_2, d_1, f_1\}, \{d_6, d_7, d_9, d_8, f_4\}, \{d_{14}, d_{13}, d_{12}, f_6\} \}$
	$\{ \{d_6, d_5, d_4, f_2\}, \{d_{10}, d_9, d_8, f_5\}, \{d_{14}, d_{13}, d_{12}, f_6\} \}$
	$\{ \{d_6, d_5, d_4, f_2\}, \{d_{10}, d_9, d_{11}, d_{13}, d_{12}, f_7\} \}$
7	$\{ \{d_2, d_1, f_1\}, \{d_6, d_5, d_4, f_2\}, \{d_{10}, d_9, d_8, f_5\}, \{d_{14}, d_{13}, d_{12}, f_6\} \}$
	$\{ \{d_2, d_3, d_5, d_4, f_3\}, \{d_{10}, d_9, d_8, f_5\}, \{d_{14}, d_{13}, d_{12}, f_6\} \}$
	$\{ \{d_2, d_1, f_1\}, \{d_6, d_5, d_4, f_2\}, \{d_{10}, d_9, d_{11}, d_{13}, d_{12}, f_7\} \}$
	$\{ \{d_2, d_3, d_5, d_4, f_3\}, \{d_{10}, d_9, d_{11}, d_{13}, d_{12}, f_7\} \}$

Table 4.1. F-cycle families in $\mathcal{D}(\mathcal{F})$ of Figure 4.1.

$k = 1, 2, 3$ and hence the system is structurally stabilizable.

Indeed, the components of $\bar{g}$ can be written in the staircase form of (4.1) as

$$\begin{array}{ll}
\bar{g}_1 = & \tilde{f}_1(1) + \tilde{f}_2(0) + \tilde{f}_3(0) + \tilde{f}_4(0) + \tilde{f}_5(0) + \tilde{f}_6(0) + \tilde{f}_7(0) \\
\bar{g}_2 = & \tilde{f}_2(1) + \tilde{f}_3(0) + \tilde{f}_4(0) + \tilde{f}_5(1) + \tilde{f}_6(1) + \tilde{f}_7(0) \\
\bar{g}_3 = & \tilde{f}_3(1) + \tilde{f}_4(1) + \tilde{f}_5(\tilde{f}_1) + \tilde{f}_6(\tilde{f}_1) + \tilde{f}_7(0) \\
\bar{g}_4 = & \tilde{f}_4(\tilde{f}_1) + \tilde{f}_5(\tilde{f}_2) + \tilde{f}_6(\tilde{f}_2) + \tilde{f}_7(1) \\
\bar{g}_5 = & \tilde{f}_5(\tilde{f}_3) + \tilde{f}_6(\tilde{f}_3 + \tilde{f}_4) + \tilde{f}_7(\tilde{f}_1) \\
\bar{g}_6 = & \tilde{f}_6(\tilde{f}_1\tilde{f}_4 + \tilde{f}_2\tilde{f}_5) + \tilde{f}_7(\tilde{f}_2) \\
\bar{g}_7 = & \tilde{f}_7(\tilde{f}_3)
\end{array}$$

with

$$\begin{aligned}
\tilde{f}_1 &= (-d_2 d_1 f_1) \\
\tilde{f}_2 &= (-d_6 d_5 d_4 f_2) \\
\tilde{f}_3 &= (-d_2 d_5 d_7 d_3 f_3 + d_2 d_5 d_7 d_1 d_6 f_1 f_2) \\
\tilde{f}_4 &= (-d_6 d_7 d_9 d_8 f_4) \\
\tilde{f}_5 &= (-d_{10} d_9 d_8 f_5) \\
\tilde{f}_6 &= (-d_{14} d_{13} d_{12} f_6) \\
\tilde{f}_7 &= (-d_{10} d_9 d_{13} d_{12} d_{11} f_7 + d_{10} d_9 d_{13} d_{12} d_8 d_{14} f_5 f_6)
\end{aligned}$$

verifying the result of Theorem 4.1.

Example 4.4 Consider the digraph of Figure 4.2, whose f-cycle families are listed in Table 4.2.

Choosing

$$\begin{aligned}
\mathcal{CF}_4^* &= \mathcal{CF}_{41} = \{\ \{d_2\}, \{d_5\}, \{d_8, d_9, d_{10}, f_4\}\ \} \\
\mathcal{CF}_3^* &= \mathcal{CF}_{31} = \{\ \{d_2\}, \{d_5\}, \{d_8, d_7, f_3\}\ \} \\
\mathcal{CF}_2^* &= \mathcal{CF}_{11} = \{\ \{d_2\}, \{d_6, d_4, f_2\}\ \} \\
\mathcal{CF}_1^* &= \mathcal{CF}_{11} = \{d_3, d_1, f_1\}
\end{aligned}$$

we easily see that conditions of Theorem 4.1 are satisfied, with $\bar{n} = 4$.

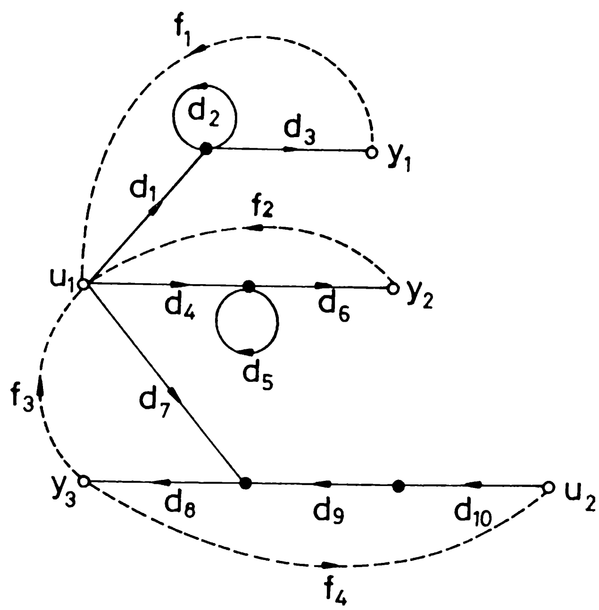


Figure 4.2. $\mathcal{D}(\mathcal{F})$ of Example 4.4.

$\gamma(\mathcal{CF})$	$\omega(\mathcal{CF})$
1	$\{d_3, d_1, f_1\}$
	$\{d_6, d_4, f_2\}$
	$\{d_8, d_7, f_3\}$
2	$\{\ \{d_5\}, \{d_3, d_1, f_1\}\ \}$
	$\{\ \{d_2\}, \{d_6, d_4, f_2\}\ \}$
	$\{\ \{d_2\}, \{d_8, d_7, f_3\}\ \}$
	$\{\ \{d_5\}, \{d_8, d_7, f_3\}\ \}$
	$\{d_8, d_9, d_{10}, f_4\}$
3	$\{\ \{d_2\}, \{d_5\}, \{d_8, d_7, f_3\}\ \}$
	$\{\ \{d_3, d_1, f_1\}, \{d_8, d_9, d_{10}, f_4\}\ \}$
	$\{\ \{d_6, d_4, f_2\}, \{d_8, d_9, d_{10}, f_4\}\ \}$
	$\{\ \{d_2\}, \{d_8, d_9, d_{10}, f_4\}\ \}$
	$\{\ \{d_5\}, \{d_8, d_9, d_{10}, f_4\}\ \}$
4	$\{\ \{d_2\}, \{d_5\}, \{d_8, d_9, d_{10}, f_4\}\ \}$
	$\{\ \{d_5\}, \{d_3, d_1, f_1\}, \{d_8, d_9, d_{10}, f_4\}\ \};$
	$\{\ \{d_2\}, \{d_6, d_4, f_2\}, \{d_8, d_8, d_{10}, f_4\}\ \}$

Table 4.2. F-cycle families in $\mathcal{D}(\mathcal{F})$ of Figure 4.2.

Example 4.5 Consider the digraph $\mathcal{D}(\mathcal{F})$ of Figure 4.3, whose f-cycle families are given in Table 4.3.

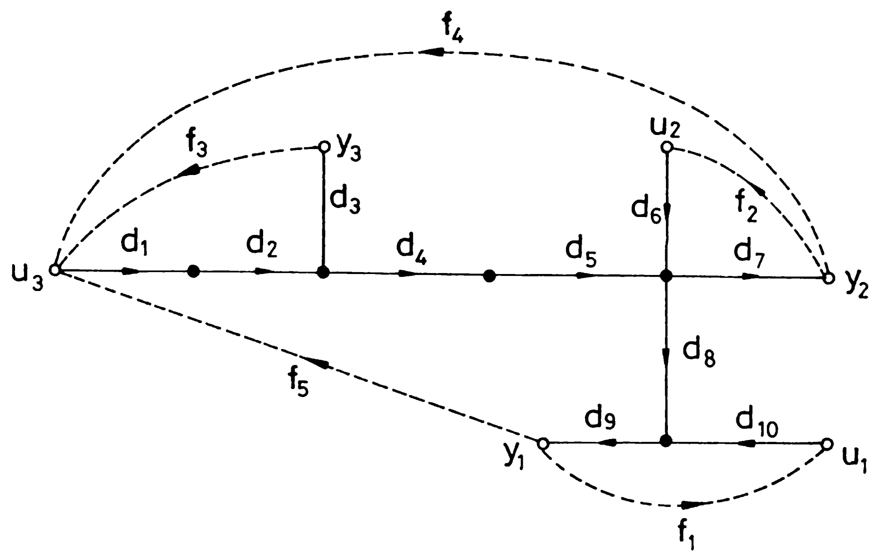


Figure 4.3. $\mathcal{D}(\mathcal{F})$ of Example 4.5.

$\gamma(\mathcal{CF})$	$\omega(\mathcal{CF})$
1	$\{d_9, d_{10}, f_1\}$ $\{d_7, d_6, f_2\}$
2	$\{d_3, d_2, d_1, f_3\}$ $\{ \{d_9, d_{10}, f_1\}, \{d_7, d_6, f_2\} \}$
3	$\{ \{d_9\}, \{d_{10}, f_1\}, \{d_3, d_2, d_1, f_3\} \}$ $\{ \{d_7, d_6, f_2\}, \{d_3, d_2, d_1, f_3\} \}$
4	$\{d_7, d_5, d_4, d_2, d_1, f_4\}$ $\{ \{d_9, d_{10}, f_1\}, \{d_7, d_6, f_2\}, \{d_3, d_2, d_1, f_3\} \}$
5	$\{d_9, d_8, d_5, d_4, d_2, d_1, f_5\}$ $\{ \{d_9, d_{10}, f_1\}, \{d_7, d_5, d_4, d_2, d_1, f_4\} \}$

Table 4.3. F-cycle families in $\mathcal{D}(\mathcal{F})$ of Figure 4.3

Choosing

$$\begin{array}{rcccl} \mathcal{CF}_5^* &=& \mathcal{CF}_{51} &=& \{d_9, d_8, d_5, d_4, d_2, d_1, f_5\} \\ \mathcal{CF}_4^* &=& \mathcal{CF}_{41} &=& \{d_7, d_5, d_4, d_2, d_1, f_4\} \\ \mathcal{CF}_3^* &=& \mathcal{CF}_{31} &=& \{\ \{d_9, d_{10}, f_1\}, \{d_3, d_2, d_1, f_3\}\ \} \\ \mathcal{CF}_2^* &=& \mathcal{CF}_{22} &=& \{\ \{d_9, d_{10}, f_1\}, \{d_7, d_6, f_2\}\ \} \\ \mathcal{CF}_1^* &=& \mathcal{CF}_{11} &=& \{d_9, d_{10}, f_1\} \end{array}$$

Theorem 4.1 is satisfied with $\bar{n} = 2$. Note that condition (i-a) of Theorem 4.1 holds for $k = 5, 4$, and (i-b) for $k = 3$.

We know turn our attention to systems whose characteristic polynomial coefficients are not in the form of (4.1), but can be put effectively into that form with certain modifications. Our desire is motivated by the fact that the stabilization procedure in the proof of Lemma 4.1 involves use of high feedback gains, which suggests that certain system parameters can be neglected to bring the coefficients into the desired form of (4.1). The following results are based on such an asymptotic approach involving use of high gains.

Let us denote by $\#_f(\cdot)$ the number of variable f-edges in $(\cdot)$.

Theorem 4.2 *Suppose that, for $k = 1, 2, \cdots, n$, there exists particular cycle families of width k, denoted by $\mathcal{CF}_k^*$, in $\mathcal{D}(\mathcal{F})$ such that*

(i) $f_k \in \mathcal{CF}_k^*$ and $f_j \notin \mathcal{CF}_k^*$, $j > k$;

(ii) for any other f-cycle family $\mathcal{CF}_k$ of width k, $\#_f(\mathcal{CF}_k) \leq \#_f(\mathcal{CF}_k^*)$, with strict inequality if $\mathcal{CF}_k$ contains no f_j, $j > k$.

Then $\mathcal{S}$ is generically stabilizable by $\mathcal{F}$.

Proof: Let $f_k = \bar{f}_k \rho$, where $\rho > 0$ is an arbitrarily large parameter, and let $\eta_k = max\{\#_f(\mathcal{CF}_k)\}$, where the maximum is taken over all cycle families of width k. Then each characteristic polynomial coefficient has the form

$$p_k = \bar{g}_k(d, f_v) = \rho^{\eta_k} \bar{g}_k(d, \bar{f}_v) + h_k(d, \bar{f}_k, \rho),$$

where $\bar{f}_v = (\bar{f}_1, \bar{f}_2, \cdots, \bar{f}_n)$, and $deg[h_k(d, f_v, \cdot)] < \eta_k$. Thus, for fixed $\bar{f}_v$ and d, as $\rho \to \infty$, roots of $p(s)$ approach the roots of

$$\bar{p}(s) = s^n + \bar{p}_1 s^{n-1} + \cdots + \bar{p}_n,$$

where $\bar{p}_k = \rho^{\eta_k} \bar{g}_k(d, \bar{f}_v)$. The conditions of the theorem guarantee that the cycle families which correspond to the product terms appearing in $\bar{g}_k(d, \bar{f}_v)$ trivially satisfy conditions of Theorem 4.1, and the result follows. □

The example to follow is an illustration of the result of Theorem 4.2.

Example 4.6 The closed-loop digraph, $\mathcal{D}(\mathcal{F})$, associated with a two input-three output system under the feedback

$$\mathcal{F}: \quad u = \begin{bmatrix} f_{11} & f_{12} & 0 \\ f_{21} & 0 & f_{23} \end{bmatrix} y$$

is given in Figure 4.4. Table 4.4 displays a list of the f-cycle families of $\mathcal{D}(\mathcal{F})$ corresponding to a reordering of the feedback variables as $f_1 = f_{11}$, $f_2 = f_{21}$, $f_3 = f_{23}$, $f_4 = f_{12}$.

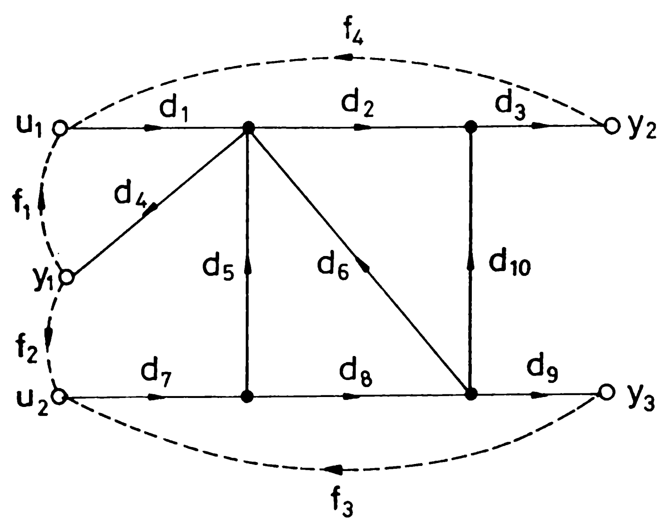


Figure 4.4. $\mathcal{D}(\mathcal{F})$ of Example 4.6.

$\gamma(\mathcal{CF})$	$\omega(\mathcal{CF})$
1	$\{d_4, d_1, f_1\}$
2	$\{d_4, d_5, d_7, f_2\}$ $\{d_9, d_8, d_7, f_3\}$ $\{d_3, d_2, d_1, f_4\}$
3	$\{d_4, d_6, d_8, d_7, f_2\}$ $\{\ \{d_9, d_8, d_7, f_3\}, \{d_4, d_1, f_1\}\ \}$
4	$\{d_3, d_{10}, d_8, d_7, f_2, d_4, d_1, f_4\}$ $\{\ \{d_9, d_8, d_7, f_3\}, \{d_3, d_2, d_1, f_4\}\ \}$

Table 4.4. f-cycle families of $\mathcal{D}(\mathcal{F})$ of Figure 4.4.

We choose

$$\begin{aligned}
\mathcal{CF}_4^* &= \mathcal{CF}_{41} = \{d_3, d_{10}, d_8, d_7, f_2, d_4, d_1, f_4\} &&, \ \eta_4 = 2\\
\mathcal{CF}_3^* &= \mathcal{CF}_{32} = \{\ \{d_9, d_8, d_7, f_3\}, \{d_4, d_1, f_1\}\ \} &&, \ \eta_3 = 2\\
\mathcal{CF}_2^* &= \mathcal{CF}_{21} = \{d_4, d_5, d_7, f_2\} &&, \ \eta_2 = 1\\
\mathcal{CF}_1^* &= \mathcal{CF}_{11} = \{d_4, d_1, f_1\} &&, \ \eta_1 = 1
\end{aligned}$$

Clearly, this choice satisfies Theorem 4.2. For $k = 1, 2, 3, 4$, neglecting those cycle families which contain less than η_k f-edges, the coefficients $\bar{g}_k$ can be written as

$$\begin{aligned}
\bar{g}_1 &= f_1(-d_4 d_1) + f_2(0) + f_3(0) + f_4(0)\\
\bar{g}_2 &= f_2(-d_4 d_5 d_7) + f_3(-d_9 d_8 d_7) + f_4(-d_3 d_2 d_1)\\
\bar{g}_3 &= f_3(d_9 d_8 d_7 d_4 d_2 d_1) + f_4(0)\\
\bar{g}_4 &= f_4(d_8 d_7 f_2 d_1 [-d_3 d_{10} d_4 + d_9 d_3 d_2])
\end{aligned}$$

verifying Theorem 4.2.

As a preparation for our last result on generic stabilizability of $\mathcal{S}(\mathcal{F})$, consider the following recursive reduction process applied to the closed-loop digraph $\mathcal{D}(\mathcal{F})$:

(i) Delete from $\mathcal{D}(\mathcal{F})$ all edges that do not appear in any cycle.

(ii) Let a be a d-edge such that to every simple cycle $\mathcal{C}_l^a$, $l = 1, 2, \cdots$, that contains a, there corresponds a cycle family $\mathcal{CF}_l^a$ with the following properties:

(a) $\mathcal{CF}_l^a$ covers the same state vertices as $\mathcal{C}_l^a$ does,

(b) $\mathcal{CF}_l^a$ covers no input or output vertices which are covered by some f-cycle disjoint from $\mathcal{C}_l^a$,

(c) $\mathcal{CF}_l^a$ includes all the f-edges that appear in $\mathcal{C}_l^a$ and at least one additional f-edge.

Let $\mathcal{E}_l^a(f)$ denote the set of the additional f-edges in $\mathcal{CF}_l^a$, but not in $\mathcal{C}_l^a$. Delete a, and record $\mathcal{E}_l^a(f)$.

Let the digraph obtained from $\mathcal{D}(\mathcal{F})$ by successive application of (i) and (ii) above be denoted by $\overline{\mathcal{D}}(\mathcal{F})$. We state the following:

Theorem 4.3 *Suppose $\overline{\mathcal{D}}(\mathcal{F})$ satisfies either Theorem 4.1 or Theorem 4.2 with at least one f-edge from each $\mathcal{E}_l^a(f)$ included in f_v, $l = 1, 2, \cdots$. Then $\mathcal{S}(\mathcal{F})$ is generically stabilizable.*

Proof: Let $\bar{\mathcal{D}}(\mathcal{F})$ be obtained from $\mathcal{D}(\mathcal{F})$ by deleting a single d-edge satisfying either (i) or (ii) of the reduction process. If the d-edge deleted is one which does not take part in any cycle, then $\bar{\mathcal{S}}(\mathcal{F})$ and $\mathcal{S}(\mathcal{F})$ have the same closed-loop characteristic polynomials, so that stabilizability of $\bar{\mathcal{S}}(\mathcal{F})$ implies stabilizability of $\mathcal{S}(\mathcal{F})$. Suppose that the d-edge deleted is of the second type, i.e., satisfies condition (ii) of the reduction process. Then,

$$p_k = \bar{p}_k + \bar{h}_k(d, f_v), \qquad k = 1, 2, \cdots, n$$

where p_k and $\bar{p}_k$ are the closed-loop charateristic polynomial coefficients of $\mathcal{S}(\mathcal{F})$ and $\bar{\mathcal{S}}(\mathcal{F})$, and $\bar{h}_k$ is a sum of product terms each of which corresponds to a cycle family of width k which includes one of $\mathcal{C}_l^a$, $l = 1, 2, \cdots$. By conditions (ii)(a-c) of the reduction process, corresponding to every such product term, $\bar{g}_k$ contains a product term (due to $\mathcal{CF}_l^a$), which includes more f-variables. Choosing $f_k = \rho \bar{f}_k$, and letting $\rho \to \infty$ as in the proof of Theorem 4.2, we observe that roots of $p(s)$ approach those of $\bar{p}(s)$. Since $\overline{\mathcal{D}}(\mathcal{F})$ is stabilizable by assumption, then so is $\mathcal{D}(\mathcal{F})$. This finishes the proof for a one-step reduction process. Repeating the same argument for every d-edge deleted, the proof is completed. □

We illustrate this result in the example below.

Example 4.7 Figure 4.5 shows the closed-loop system digraph $\mathcal{D}(\mathcal{F})$ to be considered.

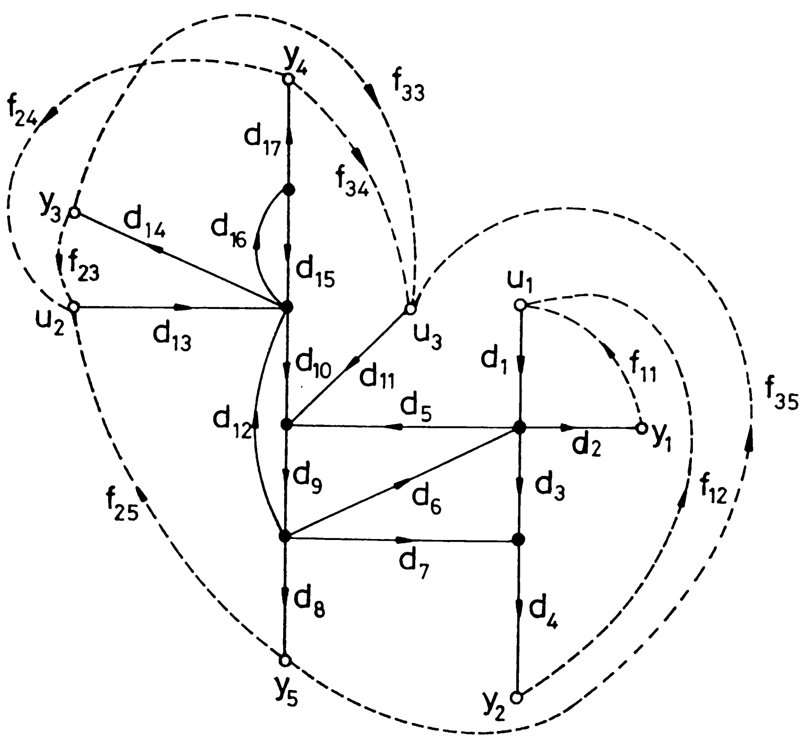


Figure 4.5. $\mathcal{D}(\mathcal{F})$ of Example 4.7.

The reduction process proceeds as follows:

1. For $a = d_5$, the cycles containing a are $C_1^{d_5} = \{d_5, d_9, d_6\}$, and $C_2^{d_5} = \{d_5, d_9, d_7, d_4, f_2, d_1\}$. Corresponding to these cycles, $\mathcal{D}(\mathcal{F})$

contains the cycle families $\mathcal{CF}_1^{d_5} = \{ \{d_1, d_2, f_1\}, \{d_9, d_8, f_{35}, d_{11}\} \}$, and $\mathcal{CF}_2^{d_5} = \{ \{d_1, d_3, d_4, f_2\}, \{d_9, d_8, f_{35}, d_{11}\} \}$, which satisfy conditions (ii)(a-c) of the reduction process. Let $\mathcal{E}_1^{d_5} = \{f_{11}, f_{35}\}$, $\mathcal{E}_2^{d_5} = \{f_{35}\}$, and delete d_5.

2. Delete d_6 and d_7 as they form no cycles.

3. For $a = d_{15}$, the only cycle to be considered is $C_1^{d_{15}} = \{d_{15}, d_{16}\}$, to which there corresponds $\mathcal{CF}_1^{d_{15}} = \{d_{16}, d_{17}, f_{24}, d_{13}\}$, with $\mathcal{E}_1^{d_{15}} = \{f_{24}\}$. Delete d_{15}.

After the reduction process, $\overline{\mathcal{D}}(\mathcal{F})$ consists of two decoupled subgraphs $\overline{\mathcal{D}}_1(\mathcal{F}_1)$ and $\overline{\mathcal{D}}_2(\mathcal{F}_2)$ are as shown in Figure 4.6.

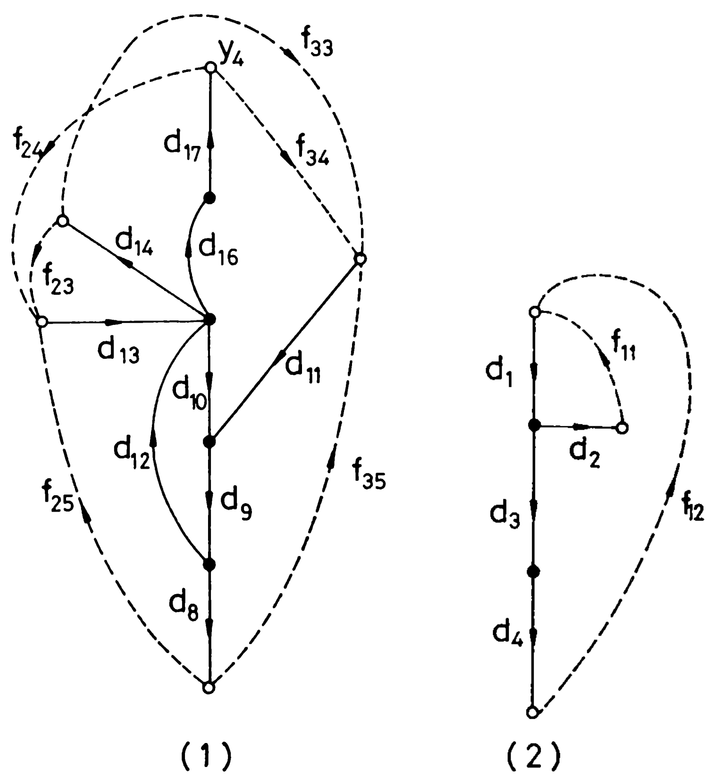


Figure 4.6. Reduced digraph $\overline{\mathcal{D}}(\mathcal{F})$ corresponding to $\mathcal{D}(\mathcal{F})$ of Figure 4.5.

Subsystem $\mathcal{S}_2(\mathcal{F}_2)$ corresponding to $\overline{\mathcal{D}}_2(\mathcal{F}_2)$ is stabilizable (in fact, pole assignable) with the only possible choice of $f_{v1} = (f_1^2, f_2^2)$, where $f_1^2 = f_{11}$ and $f_2^2 = f_{12}$.

Keeping in mind that f_{35} and f_{24} should be chosen as variable f-edges, let us fix $f_{25} = f_{34} = 0$, and renumber the remaining f-edges to get $f_{v1} = (f_1^1, f_2^1, f_3^1, f_4^1)$, where $f_1^1 = f_{23}$, $f_2^1 = f_{24}$, $f_3^1 = f_{33}$, $f_4^1 = f_{35}$. The f-cycle families of $\overline{\mathcal{D}}_1(\mathcal{F}_1)$ are listed in Table 4.5.

$\gamma(\mathcal{CF})$	$\omega(\mathcal{CF})$
1	$\{d_{14}, d_{13}, f_1^1\}$
2	$\{d_{17}, d_{16}, d_{13}, f_2^1\}$ $\{d_8, d_9, d_{11}, f_4^1\}$
3	$\{d_{14}, d_{12}, d_9, d_{11}, f_3^1\}$ $\{\ \{d_8, d_9, d_{11}, f_4^1\}, \{d_{14}, d_{13}, f_1^1\}\ \}$
4	$\{\ \{d_{17}, d_{16}, d_{13}, f_2^1\}, \{d_8, d_9, d_{11}, f_4^1\}\ \}$

Table 4.5. F-cycle families in $\overline{\mathcal{D}}_1(\mathcal{F}_1)$ of Figure 4.6.

Now, choosing

$$\begin{aligned}
\mathcal{CF}_4^* &= \{\ \{d_{17}, d_{16}, d_{13}, f_2^1\}, \{d_8, d_9, d_{11}, f_4^1\}\ \} \\
\mathcal{CF}_3^* &= \{d_{14}, d_{12}, d_9, d_{11}, f_3^1\} \\
\mathcal{CF}_2^* &= \{d_{17}, d_{16}, d_{13}, f_2^1\} \\
\mathcal{CF}_1^* &= \{d_{14}, d_{13}, f_1^1\}
\end{aligned}$$

we observe that conditions of Theorem 4.1 are satisfied so that the subsystem is generically stabilizable. Hence, by Theorem 4.3 the overall system is structurally stabilizable.

As demonstrated by Example 4.7, reduction of $\mathcal{D}(\mathcal{F})$ by deleting certain d-edges provides considerable simplification in the stabilization process especially when the reduced digraph $\overline{\mathcal{D}}(\mathcal{F})$ consists of decoupled components. This shows a parallelism with the decomposition approach to stabilization of large-scale systems. In the following, we take a closer look at how the reduction process can be applied to decentralized stabilizability of a class of interconnected systems.

4.3 A CLASS OF GENERICALLY STABILIZABLE SYSTEMS

In this section, we show that certain class of structures which is known to be stabilizable by decentralized state feedback satisfy the conditions of Theorem 4.3, thus demonstrating the nontriviality of the result of Theorem 4.3.

Consider a system $\mathcal{S}$ composed of N interconnected structurally controllable subsystems described by

$$\mathcal{S}_i: \quad \dot{x}_i = A_i x_i + \sum_{i=1}^{N} A_{ij} x_j + b_i u_i, \quad i = 1, 2, \cdots, N. \tag{4.5}$$

Suppose that local state feedback law

$$\mathcal{F}_i: \quad u_i = f_i^T x_i, \tag{4.6}$$

is applied to the decoupled subsystems

$$\mathcal{S}_i^D: \quad \dot{x}_i = A_i x_i + b_i u_i. \tag{4.7}$$

where $x_i \in \mathcal{R}^{n_i}$ and $u \in \mathcal{R}$, with $\sum_{i=1}^{N} n_i = n$, and $f_i^T = (f_{i1}, f_{i2}, \cdots, f_{in_i})$. By the results of Section 3.3.1 we know that each decoupled subsystem $\mathcal{S}_i^D(\mathcal{F}_i)$ is generically pole assignable. Our aim is to show that the overall system is generically stabilizable under some well-known restrictions on the interconnection structure.

For this, we first note that the closed-loop digraph $\mathcal{D}(\mathcal{F})$ has the structure $\mathcal{D}(\mathcal{F}) = (\mathcal{V}, \mathcal{E}^D \cup \mathcal{E}^C \cup \mathcal{E}_F)$, where $\mathcal{D}^D(\mathcal{F}) = (\mathcal{V}, \mathcal{E}^D \cup \mathcal{E}_F)$ is a collection of disjoint subgraphs $\mathcal{D}_i^D(\mathcal{F}_i) = (\mathcal{V}_i, \mathcal{E}_i^D \cup \mathcal{E}_{F_i})$ associated with the decoupled subsystems $\mathcal{S}_i^D(\mathcal{F}_i)$, and $\mathcal{E}^C$ is the set of interconnecting edges corresponding to the nonzero parameters of A_{ij}.

We assume that each (A_i, b_i) is in controllable canonical form, and that the interaction between the states of the subsystem satisfies the following condition:

$$Im\, A_{ij} \subset Im\, b_i, \quad i \neq j, \quad i, j = 1, 2, \cdots, N \tag{4.8}$$

i.e., the interaction from the states of $\mathcal{S}_j$ to those of $\mathcal{S}_i$ has the same effect on $\mathcal{S}_i$ as the constant input u_i.

Due to the special forms of the pairs (A_i, b_i) and the interconnection terms A_{ij}, the digraph $\mathcal{D}(\mathcal{F})$ has the structure shown in Figure 4.7.

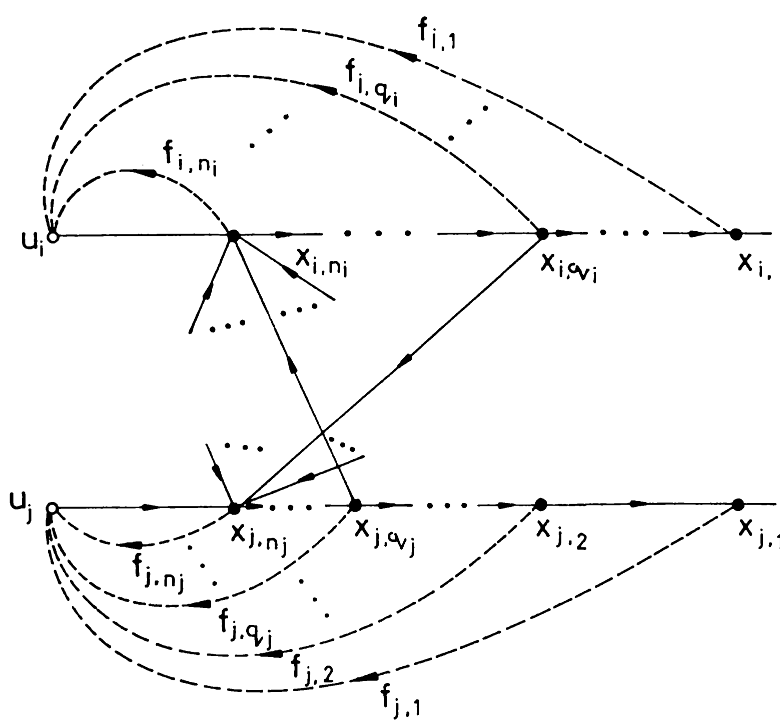


Figure 4.7. The interconnection structure between the subsystems of $\mathcal{D}(\mathcal{F})$ mentioned in Theorem 4.3.

Referring to Figure 4.7 we state the following:

Theorem 4.4 *All the d-edges of $\mathcal{D}(\mathcal{F})$ corresponding to the interconnection matrices A_{ij} of (4.8) can be deleted by the reduction process. The resulting digraph $\overline{\mathcal{D}}(\mathcal{F})$ consists of decoupled components $\overline{\mathcal{D}}_i(\mathcal{F}_i)$ associated with the decoupled systems $\mathcal{S}_i^D(\mathcal{F}_i)$. Since $\mathcal{S}_i^D(\mathcal{F}_i)$ are generically stabilizable by Theorem 3.3, then so is $\mathcal{S}(\mathcal{F})$ by Theorem 4.3.*

Proof: An edge of $\mathcal{D}(\mathcal{F})$ due to a nonzero term of some interconnection matrix A_{ij} is of the form (x_{jq_j}, x_{in_i}), where $1 \le q_j \le n_j$. If such an edge occurs in a cycle covering some state vertices of the subsystems $\mathcal{S}_j$, $\mathcal{S}_i$,$\cdots$,$\mathcal{S}_l$, then this cycle is of the form

$$\mathcal{C}_{ji\cdots l} = \{(x_{jq_j}, x_{in_i}), \overline{(x_{in_i}, x_{iq_i})}, \cdots, \overline{(x_{ln_l}, x_{lq_l})}, (x_{lq_l}, x_{jn_j}), \overline{(x_{jn_j}, x_{jq_j})}\}$$

where $\overline{(x_{in_i}, x_{iq_i})}$, $1 \leq q_i \leq n_i$ denotes the unique path in $\mathcal{D}_i^D$ from x_{in_i} to x_{iq_i}. We note that such a cycle contains no f-edges. Now, the f-cycle family consisting of the cycles

$$\begin{array}{ccccccc}
\{ & (u_j, x_{in_i}) & , & \overline{(x_{in_i}, x_{iq_i})} & , & (x_{iq_i}, u_i) & \} \\
 & \vdots & & \vdots & & \vdots & \\
\{ & (u_l, x_{ln_l}) & , & \overline{(x_{ln_l}, x_{lq_l})} & , & (x_{lq_l}, u_l) & \} \\
\{ & (u_j, x_{jn_j}) & , & \overline{(x_{jn_j}, x_{jq_j})} & , & (x_{jq_j}, u_j) & \}
\end{array}$$

covers exactly the same vertices, and includes the feedback edges $f_{iq_i} = (x_{iq_i}, u_i)$,$\cdots$, $f_{lq_l} = (x_{lq_l}, u_l)$, $f_{jq_j} = (x_{jq_j}, u_j)$. Moreover, none of the input vertices covered by this cycle family, namely, $u_i, \cdots, u_l, u_j$, can be covered by a cycle disjoint from $\mathcal{C}_{ji\cdots l}$. Hence the conditions of the reduction process are satisfied, and the interconnection edge (x_{jq_j}, x_{in_i}) can be deleted from $\mathcal{D}(\mathcal{F})$. Since this is true for all interconnection edges, and since all f-edges are used in stabilization of the resulting decoupled system associated with $\overline{\mathcal{D}}(\mathcal{F})$, the proof follows from Theorem 4.3. □

Chapter 5

A GRAPHICAL INVESTIGATION OF STRUCTURAL OBSERVABILITY

In this chapter, we present a graph-theoretic interpretation of the so-called structural observability matrix and develop graphical conditions for this matrix to have full generic rank. We then show that the digraph of any structurally observable system satisfies these conditions. We also define structural observability index and provide graphical techniques to compute bounds for it. Dual results concerning controllability can easily be obtained.

5.1 STRUCTURAL OBSERVABILITY

Since structural observability is a property of the pair (A, C) of system $\mathcal{S}$ of (2.1) we consider the reduced system structure matrix

$$\mathbf{S}_{xy} = \begin{bmatrix} \mathbf{A} & 0 \\ \mathbf{C} & 0 \end{bmatrix} \tag{5.1}$$

and associated input-truncated digraph $\mathcal{D}_{xy} = (\mathcal{X} \cup \mathcal{Y}, \mathcal{E}_{xy})$ obtained by removing the input vertices and the edges connected to them.

For structural observability it is necessary and sufficient that

(i) every state reaches an output; and

(ii) $\bar{\rho}\begin{bmatrix} A \\ C \end{bmatrix} = n.$

A structural equivalent of these conditions is the existence of a family of disjoint output cacti spanning D_{xy}. On the other hand, let us denote conveniently the vector of reduced system parameters by d and define structural observability matrix as

$$\mathcal{O}_{n-1}(d) = \begin{bmatrix} c_1^T \\ \vdots \\ c_r^T \\ c_1^T A \\ \vdots \\ c_r^T A \\ \vdots \\ c_1^T A^{n-1} \\ \vdots \\ c_r^T A^{n-1} \end{bmatrix} \tag{5.2}$$

where

$$C = \begin{bmatrix} c_1^T \\ \vdots \\ c_r^T \end{bmatrix}$$

Then, it is obvious that the pair (A, C) is structurally observable if and only if $\bar{\rho}[\mathcal{O}_{n-1}(d)] = n$.

5.2 GRAPHICAL INTERPRETATION OF THE OBSERVABILITY MATRIX

Consider the input truncated weighted digraph $\mathcal{D}_{xy}$ in which weight of any edge, denoted by $\omega(v_i, v_j)$, is the parameter value of the corresponding entry of the related system structure matrix $\mathbf{S}_{xy}$. Here, let us generalize our definition of a path so as to allow it go through a vertex more that once and hence also include a multiplicity of some edges (A path which does not go through any vertex more than once will be distinguished, where necessary, by the term simple path).

Recall that $\mathbf{S}_{xy}^{L} = \mathbf{S}_{xy}^{L-1} \times \mathbf{S}_{xy}$ can be interpreted as the L-step reachability matrix where

$$\mathbf{S}_{xy}^{L} = \begin{bmatrix} \mathbf{A}^{L} & \mathbf{O} \\ \mathbf{C}\mathbf{A}^{L-1} & \mathbf{O} \end{bmatrix}.$$

In $\mathbf{S}_{xy}^{L}$, (j,k)-th nonzero entry in the lower block row implies that state vertex x_k reaches output y_j in L-steps, i.e., $\mathcal{D}_{xy}$ contains a path of length L from x_k to y_j, which we denote by $\overline{(x_k, y_j)}_L$. Combining this with the definition of the structural observability matrix $\mathcal{O}_{n-1}(d)$, we conclude that the (j,k)-th entry of the L-th block row of $\mathcal{O}_{n-1}(d)$ is given by $\sum \omega[\overline{(x_k, y_j)}_L]$ where $\omega(\cdot)$ denotes the weight and the sum is over all L-step paths from x_k to y_j.

Let us illustrate this with an example.

Example 5.1 Consider the system digraph of Figure 5.1.

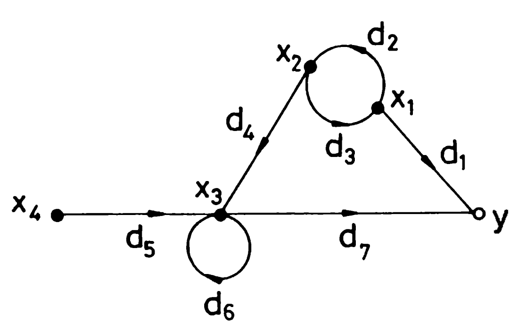


Figure 5.1. $\mathcal{D}_{xy}$ of Example 5.1.

For this system, structural observability matrix can be obtained by inspection of the digraph as

$$\mathcal{O}_3(d) = \begin{bmatrix} d_1 & 0 & d_7 & 0 \\ 0 & d_1d_3 + d_7d_4 & d_7d_6 & d_7d_5 \\ d_1d_3d_2 + d_7d_4d_2 & d_7d_6d_4 & d_7d_6^2 & d_7d_6d_5 \\ d_7d_6d_4d_2 & d_7d_6^2d_4 + d_1d_3d_2d_3 + d_7d_4d_2d_3 & d_7d_6^3 & d_7d_6^2d_5 \end{bmatrix}$$

Structural observability matrix $\mathcal{O}_{n-1}$ is not a structured matrix as its nonzero parameters are not necessarily algebraically independent. Therefore, existence of n nonzero elements lying on independent rows and columns (i.e., no two elements lie on the same row or column), is a necessary but obviously not sufficient condition for $\mathcal{O}_{n-1}$ to have full generic rank. In terms of system structure, this necessary condition is equivalent to having, in $\mathcal{D}_{xy}$ for every distinct x_k, a particular path denoted by $\overline{(x_k, y_{r_k})}^*_{L_k}$, $1 \leq r_k \leq r$, $1 \leq L_k \leq n$, such that for $j \neq k$ either $r_j \neq r_k$ or $L_j \neq L_k$.

In order to guarantee full generic row rank for $\mathcal{O}_{n-1}(d)$, rows corresponding to the n nonzero elements mentioned above should be generically linearly independent, or equivalently the square matrix obtained by taking only the rows and columns that contain these n nonzero elements should be generically nonsingular. Formulation of a structural counterpart of either one of these is not an easy task at all. We derive however, some partial results concerning the second one.

Numerous examples which we have considered reveal a similarity between this case and and the problem of deriving structural conditions for the generic nonsingularity of the coefficient matrix E, dealt with in Theorem 3.1. This leads us to the result stated and proven below:

Theorem 5.1 *Suppose that in $\mathcal{D}_{xy}$, after a suitable enumeration of states, there exists particular paths of length L_k, denoted as $\mathcal{P}_k^* = \overline{(x_k, y_{r_k})}^*_{L_k}$, $1 \leq L_k \leq n$, $k = 1, 2, \cdots, n$, which satisfy the conditions $(i) - (iii)$ below. Then $\mathcal{S}$ is structurally observable.*

(i) For $j < k$, $L_j \leq L_k$ and if $L_j = L_k$ then $r_j < r_k$.

(ii) Any other path $\overline{(x_k, y_{r_k})}_{L_k}$ of length L_k contains an edge which appears in no $\mathcal{P}_l^*$, $l \geq k$.

(iii) For $1 \leq k < j \leq n$, if there exist paths $\overline{(x_k, y_{r_j})}_{L_j}$, then all of these paths, except possibly one, say $\overline{\mathcal{P}}_j$, contain an edge which appears in no $\mathcal{P}_l^*$, $l \geq k$. If $\overline{\mathcal{P}}_j$ exists, then it contains an edge, $\bar{a}_j$, with multiplicity $\bar{\sigma}_j$ such that $\bar{a}_j$ appears in no $\mathcal{P}_l^*$, $l \geq j$; $\mathcal{P}_k^*$ contains $\bar{a}_j$ with multiplicity $< \bar{\sigma}_j$ and if $\bar{a}_j$ appears in any $\mathcal{P}_\alpha^*$, $k < \alpha < j$, then every path $\overline{(x_\alpha, y_{r_k})}_{L_k}$ contains another edge which appears in no $\mathcal{P}_l^*$, $l \geq k$.

Proof: Consider the structural observability matrix $\mathcal{O}_{n-1}(d)$ of the system with digraph $\mathcal{D}_{xy}$. Denote by $T(d)$ the submatrix obtained by taking the rows and columns of $\mathcal{O}_{n-1}(d)$, which contain the product terms corresponding to $\mathcal{P}_k^* = \overline{(x_k, y_{r_k})}_{L_k}^*$, $k = 1, 2, \cdots, n$. Obviously, $\mathcal{O}_{n-1}(d)$ has full generic rank and hence system structurally observable if $T(d)$ is generically nonsingular. Consider now the $n \times n$ matrix $T(d)$. By condition (i), diagonal elements of $T(d)$ are of the form

$$t_{kk}(d) = t_k^*(d) + \hat{t}_{kk}(d),$$

where $t_k^*(d)$ is the weight of $\mathcal{P}_k^*$, and $\hat{t}_{kk}(d)$ is the sum of the weights of all other paths $\overline{(x_k, y_{r_k})}_{L_k}$. Let us define $d_1 = d$, $T_1(d_1) = T(d)$, and partition T_1 as

$$T_1(d_1) = \left[\begin{array}{c|c} t_1^*(d_1) + t_{11}(d_1) & t_{1l}(d_1)'s \\ \hline t_{j1}(d_1)'s & T_2(d_1) \end{array}\right]$$

where, for a fixed $j > 1$, $t_{j1}(d_1)$ is the sum of the weights of all paths $\overline{(x_1, y_{r_j})}_{L_j}$. By condition (ii) every product term in $t_{11}(d_1)$ contains the weight of an edge which occurs in no $\mathcal{P}_k^*$, $k \geq 1$. Let d_1' denote the parameter vector after all parameters corresponding to such edges are set to zero. Then, $T_1(d_1')$ has the form

$$T_1(d_1') = \left[\begin{array}{c|c} t_1^*(d_1') & t_{1l}(d_1')'s \\ \hline t_{j1}(d_1')'s & T_2(d_1') \end{array}\right]$$

where each diagonal term $t_{kk}(d_1')$ of $T_2(d_1')$ still contains the product term $t_k^*(d_1') = t_k^*(d_1)$. If $t_{j1}(d_1') = 0$, $j = 2, \cdots, n$, then let $d_2 = d_1'$. Otherwise, by condition (iii), all product terms in each nonzero $t_{j1}(d_1')$, except possibly those that correspond to $\overline{\mathcal{P}}_j$'s, contain an edge which appears in no $\mathcal{P}_k^*$, $k \geq 1$. Let d_1'' be the parameter vector after all such terms are set to zero. Then $T_1(d_1'')$ has the same structure as $T_1(d_1')$ with $t_{kk}(d_1'')$ still containing $t_k^*(d_1'') = t_k^*(d_1') = t_k^*(d_1)$, and each $t_{j1}(d_1'')$ either being zero, or containing a single nonzero product term due to $\overline{\mathcal{P}}_j$'s. If $t_{j1}(d_1'') = 0$, $j \geq 2$, then let $d_2 = d_1''$. Otherwise, condition (iii) implies that each nonzero t_{j1} is of the form $t_{j1}(d_1'') = \bar{a}_j^{\sigma_j} \bar{t}_{j1}(d_1'')$, with no $t_l^*(d_1'')$, $l \geq j$, containing $\bar{a}_j$; and $t_1^*(d_1'') = \bar{a}_j^{\sigma_1} \bar{t}_1^*(d_1'')$, with $\sigma_1 < \sigma_j$. We can then eliminate all such t_{j1} by subtracting a suitable multiple of the 1-st row from the j-th row. After such operations, $T_1(d_1'')$ becomes

$$T_1(d_1'') = \left[\begin{array}{c|c} t_1^*(d_1'') & t_{1l}(d_1'')'s \\ \hline 0 & \overline{T}_2(d_1'') \end{array}\right],$$

where some elements $\bar{t}_{jl}(d_1'')$ of $\overline{T}_2(d_1'')$, $j, l \geq 2$, are of the form

$$\bar{t}_{jl}(d_1'') = t_{jl}(d_1'') - \bar{a}_j^{\sigma_j - \sigma_1} \bar{t}_{j1}(d_1'') t_{1l}(d_1'') / \bar{t}_1^*(d_1'').$$

Now, if no $t_\alpha^*(d_1'')$, $1 \leq \alpha \leq j$, contains $\bar{a}_j$, let $d_2 = d_1''$. Otherwise, by condition (iii), every product term in each $t_{1\alpha}(d_1'')$ such that $t_\alpha^*(d_1'')$ contains $\bar{a}_j$, contain an edge which appears in no $t_l^*(d_1'')$, $l \geq 1$. Let d_2 be the parameter vector after all parameters corresponding to such edges are set to zero, and consider

$$T_1(d_2) = \left[\begin{array}{c|c} t_1^*(d_2) & t_{1l}(d_2)'s \\ \hline 0 & T_2(d_2) \end{array}\right].$$

Clearly, the elements of $T_2(d_2)$ contain fewer product terms than the corresponding elements of $\overline{T}_2(d_1'')$. Moreover, the diagonal elements still contain the terms $t_k^*(d_2) = t_k^*(d_1)$. Now, if $T_2(d_2)$ has full generic rank, then so does $T_1(d_2)$, and therefore, $T(d) = T_1(d_1)$. Continuing with the same argument with d_1 and $T_1(d_1)$ replaced by d_2 and $T_2(d_2)$, and so on, we finally reach the conclusion that if $T_n(d_n) = t_n^*(d_n) + \hat{t}_{nn}^*(d_n)$ is nonzero, then $T(d)$ is generically nonsingular. The fact that $t_n^*(d_n) = t_n^*(d) \not\equiv 0$ completes the proof. □

L	j	$\overline{(x_1, y_j)_L}$	$\{\overline{(x_2, y_j)_L}\}$	$\{\overline{(x_3, y_j)_L}\}$	$\{\overline{(x_4, y_j)_L}\}$
1	1	$\{d_1\}$	$\{d_2\}$	$\{d_8\}$	
	2			$\{d_5\}$	$\{d_6\}$
2	1	$\{d_7, d_2\}$	$\{d_3, d_8\}$	$\{d_4, d_2\}$	
	2		$\{d_3, d_5\}$		
3	1	$\{d_7, d_3, d_8\}$	$\{d_3, d_4, d_2\}$	$\{d_4, d_3, d_5\}$	
	2	$\{d_7, d_3, d_5\}$		$\{d_4, d_3, d_5\}$	
4	1	$\{d_7, d_3, d_4, d_2\}$	$\{d_3, d_3, d_4, d_8\}$	$\{d_4, d_4, d_3, d_2\}$	
	2		$\{d_3, d_3, d_4, d_5\}$		

Table 5.2. Paths from the state vertices to the output vertices in $\mathcal{D}_{xy}$ of Figure 5.3.

It is easy to see that the choice of

$$
\begin{aligned}
\mathcal{P}_1^* &= \overline{(x_1, y_1)}_1^* = \{d_1\} \\
\mathcal{P}_2^* &= \overline{(x_4, y_2)}_1^* = \{d_6\} \\
\mathcal{P}_3^* &= \overline{(x_3, y_1)}_2^* = \{d_4, d_2\} \\
\mathcal{P}_4^* &= \overline{(x_2, y_2)}_2^* = \{d_3, d_5\}
\end{aligned}
$$

satisfies the conditions of Theorem 5.1, and hence the corresponding system is structurally observable (as expected).

Another choice of paths would be

$$
\begin{aligned}
\mathcal{P}_1^* &= \overline{(x_1, y_1)}_1^* = \{d_1\} \\
\mathcal{P}_2^* &= \overline{(x_4, y_2)}_1^* = \{d_6\} \\
\mathcal{P}_3^* &= \overline{(x_3, y_1)}_2^* = \{d_4, d_2\} \\
\mathcal{P}_4^* &= \overline{(x_2, y_1)}_3^* = \{d_3, d_4, d_2\}
\end{aligned}
$$

which also satisfies Theorem 5.1. Note that by deleting all other edges, $\mathcal{D}_{xy}$ is decomposed into two disjoint cacti, the existence of which is the necessary and sufficient condition for structural observability.

In the following, we study a few examples which illustrate the implications of Theorem 5.1.

Example 5.2 Consider the digraph $\mathcal{D}_{xy}$ of Figure 5.2.

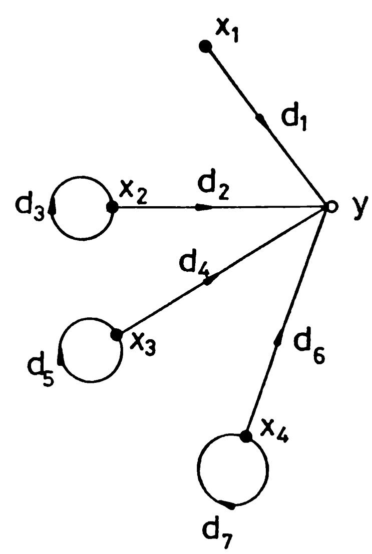


Figure 5.2. $\mathcal{D}_{xy}$ of Example 5.2.

All the paths of length L, $1 \leq L \leq n$, between the state vertices and the output vertex are listed in Table 5.1. (Note the one-to-one correspondence between this table and $\mathcal{O}_{n-1}$).

L	$(x_1, y)_L$	$(x_2, y)_L$	$\{(x_3, y)_L\}$	$\{(x_4, y)_L\}$
1	$\{d_1\}$	$\{d_2\}$	$\{d_4\}$	$\{d_6\}$
2		$\{d_3, d_2\}$	$\{d_5, d_4\}$	$\{d_7, d_6\}$
3		$\{d_3, d_3, d_2\}$	$\{d_5, d_5, d_4\}$	$\{d_7, d_7, d_6\}$

Table 5.1. Paths from the state vertices to the output vertex in $\mathcal{D}_{xy}$ of Figure 5.2.

We choose

$$\begin{aligned}
\mathcal{P}_1^* &= \overline{(x_1, y)}_1^* = \{d_1\} \\
\mathcal{P}_2^* &= \overline{(x_2, y)}_2^* = \{d_3, d_2\} \\
\mathcal{P}_3^* &= \overline{(x_3, y)}_3^* = \{d_5, d_5, d_4\} \\
\mathcal{P}_4^* &= \overline{(x_4, y)}_4^* = \{d_7, d_7, d_7, d_6\}
\end{aligned}$$

which satisfy conditions (i) and (ii) of Theorem 5.2 trivially. For $k = 1$, condition (iii) is also trivially satisfied. Consider $k = 2$. We observe that there exist paths $\overline{(x_2, y)}_3$ and $\overline{(x_2, y)}_4$ which contain d_3 with multiplicity 2 and 3, respectively, and d_3 occurs in $\mathcal{P}_2^*$ with multiplicity 1, while it occurs in no $\mathcal{P}_l^*$, $l > 2$ so that condition (iii) is satisfied. Similar argument applies for $k = 3$. Thus, by Theorem 5.1, the system is structurally observable. This result is also verified by the fact that the corresponding digraph $\mathcal{D}_{xy}$ is a cactus.

Example 5.3 Let us now study a two-output system whose digraph $\mathcal{D}_{xy}$ and the corresponding list of state-output paths are given in Figure 5.3 and Table 5.2.

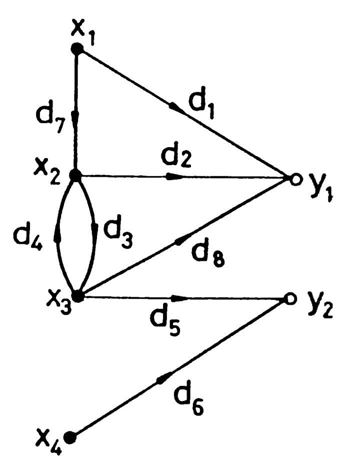


Figure 5.3. $\mathcal{D}_{xy}$ of Example 5.3.

The preceding example leads us to the result which we state and prove below:

Theorem 5.2 *The following are equivalent:*

(a) The system $\mathcal{S}$ in (2.1) is structurally observable.

(b) The input-truncated digraph $\mathcal{D}_{xy}$ associated with $\mathcal{S}$ is spanned by a collection of disjoint output cacti.

(c) $\mathcal{D}_{xy}$ satisfies the conditions of Theorem 5.1.

Proof:

(a)$\Leftrightarrow$(b) : Obvious.

(c)$\Rightarrow$(a) : By Theorem 5.1.

(a)$\Rightarrow$(c) : First consider the case when $r = 1$ (single output) and $\mathcal{D}_{xy}$ is a cactus, with output y. Recall that in the cactus, every state vertex reaches the output vertex along a unique simple path so that for fixed L, there exist one and only one path of length L from any state vertex to the output vertex. If the cactus is just a stem, then Theorem 5.1 is trivially satisfied. Otherwise, denote the stem of $\mathcal{D}_{xy}$ by $\mathcal{B}_0$, and order the buds of $\mathcal{D}_{xy}$ as $\mathcal{B}_1, \mathcal{B}_2, \cdots$, etc. such that for $j < i$, no vertex in $\mathcal{B}_i$ occurs on a simple path from a vertex in $\mathcal{B}_j$ to y. (Note that denoting the stem as $\mathcal{B}_0$ is consistent with this reordering of the buds.) Then, first scan $\mathcal{B}_0$ and label its vertices as $x_1, x_2, \cdots$, etc. such that the length of the unique (simple) path from x_k to y is k. Next, scan the buds $\mathcal{B}_i$, $i = 1, 2, \cdots$, in their order and label their vertices according to the following scheme: Let the bud to be scanned be $\mathcal{B}_i$, and the last vertex in $\mathcal{B}_{i-1}$ which has been labeled be x_k. Suppose that the length of the unique simple path from the tail, x_{ti}, of the distinguished edge of $\mathcal{B}_i$ to y is L_i; where $1 \leq L_i \leq k$ due to the ordering of the buds. Identify in $\mathcal{B}_i$ the unique state vertex that reaches in $\mathcal{B}_i$ to x_{ti} through a path of length $k + 1 - L_i$, and label it x_{k+1}. Once, x_{k+1} is identified, label the remaining vertices of $\mathcal{B}_i$ as $x_{k+2}, x_{k+3}, \cdots$, etc., where x_{k+l+1} is the unique vertex in $\mathcal{B}_i$ that is adjacent to x_{k+l}, $l = 1, 2, \cdots$. The enumeration of the state vertices in the output cactus shown in Figure 5.4 illustrates the scheme.

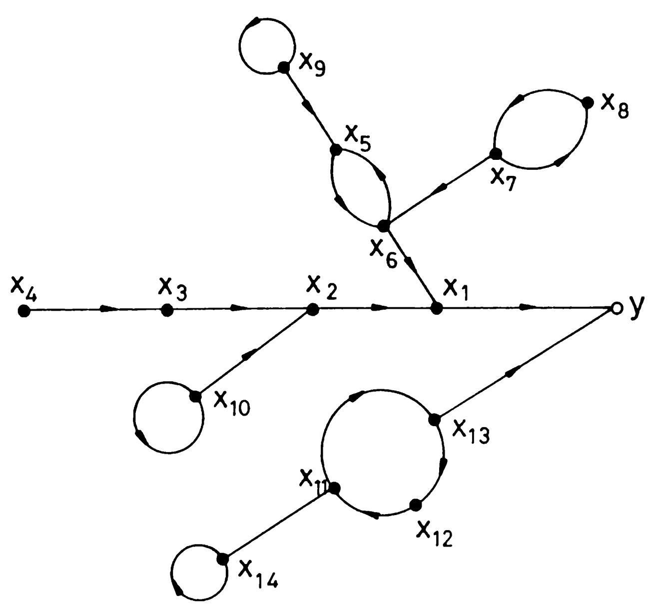


Figure 5.4. Enumeration of the state vertices in a cactus, according to the scheme mentioned in the proof of Theorem 5.2.

With the state vertices of $\mathcal{D}_{xy}$ labeled as above, let the unique path of length k from x_k to y be denoted as $\mathcal{P}_k^* = \overline{(x_k, y)}_k^*$, $1 \le k \le n$. Then, the conditions (i) and (ii) of Theorem 5.2 are readily satisfied. Also, condition (iii) is trivially satisfied for those x_k that belong to $\mathcal{B}_0$ as $\mathcal{P}_k^*$ is the only path from x_k to y. Consider the case when x_k belongs to some bud $\mathcal{B}_i$, $i \ge 1$, and suppose that, for a fixed $j > k$, it reaches y through several paths $\mathcal{P}_{j1}, \mathcal{P}_{j2}, \cdots$, etc., of length $L_j = j$. Then each $\mathcal{P}_{jt}$ should necessarily travel through the cycle of at least one bud $\mathcal{B}_{m_t}$, with $m_t \le i$, at least once. If $\mathcal{P}_{jt}$ travels through the cycle of some $\mathcal{B}_{m_t}$ with $m_t < i$, then it contains an edge (from the cycle of $\mathcal{B}_{m_t}$), which appears in no $\mathcal{P}_l^*$, $l > k$. On the other hand, at most one of $\mathcal{P}_{jt}$'s, denoted by $\overline{\mathcal{P}}_j$, loops in $\mathcal{B}_i$ but in no $\mathcal{B}_m$ with $m < i$ (because, the lengths of such paths differ by an integer multiple of the length or width of the cycle of $\mathcal{B}_i$). A typical situation is illustrated in Figure 5.5, corresponding to the case when $j = k + 3$. Referring to the figure, we identify $\bar{a}_j = (x_{j-1}, x_{j-2})$, which obviously occurs in no $\mathcal{P}_l^*$, $l \geq j$, and the multiplicity of $\bar{a}_j$ in $\mathcal{P}_k^*$ is at least one less than its multiplicity in $\overline{\mathcal{P}}_j$. Moreover, only α, $k < \alpha < j$, such that $\bar{a}_j$ appears in $\mathcal{P}_\alpha^*$ can possibly be $\alpha = j - 1$. Then, any path of length $L_k = k$ from x_{j-1} to y should loop in some $\mathcal{B}_m$, $m < i$, and therefore, should contain an edge which appears in no $\mathcal{P}_l^*$, $l \geq k$. Thus, condition (iii) is satisfied, and the proof is complete for the single output case, and when $\mathcal{D}_{xy}$ is a cactus.

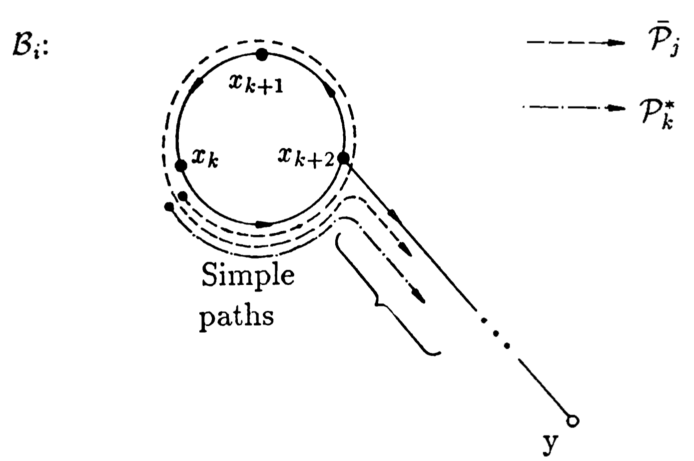


Figure 5.5. Illustration of $\mathcal{P}_k^*$ and $\overline{\mathcal{P}}_j$ for $j = k + 3$.

For the general case, let $\mathcal{D}_{xy} = (\mathcal{X} \cup \mathcal{Y}_C \cup \mathcal{Y}_A, \mathcal{E}_C \cup \mathcal{E}_A)$, where $\mathcal{X} = \cup \mathcal{X}_t$, $\mathcal{Y}_C = \cup\{y_t\}$ and $\mathcal{E}_C = \cup \mathcal{E}_t$, $t = 1, 2, \cdots$, are such that the disjoint subgraphs $\mathcal{D}_t = (\mathcal{X}_t \cup \{y_t\}, \mathcal{E}_t)$ form a family of spanning cacti $\mathcal{D}_C = \cup \mathcal{D}_t = (\mathcal{X} \cup \mathcal{Y}_C, \mathcal{E}_C)$ for $\mathcal{D}_{xy}$. Let the state vertices x_k^t of each individual cactus $\mathcal{D}_t$ be labeled as in the single output case. Then, it is easy to see that $\mathcal{D}_t$'s can be reordered, and the sets $\mathcal{X}_t$'s can be merged to obtain a new ordering of all the state vertices in such a way as to satisfy condition (i) of Theorem 5.2. Also, the paths in the collection $\mathcal{D}_C$ of cacti satisfy the remaining conditions. Noting that any other path in $\mathcal{D}_{xy}$ which does not appear in $\mathcal{D}_C$ is due to the additional edges in $\mathcal{E}_A$, which obviously appear in no $\mathcal{P}_k^*$, the proof is completed. □

The significance of interpreting structural observability in terms of the particular paths mentioned in the statement of Theorem 5.1 lies in its contribution to the structural interpretation of the observability index. Furthermore, we show in the next section that it provides a better upper bound for the estimate of the so-called generic observability index.

5.3 GENERIC OBSERVABILITY INDEX

Generic observability index can be defined as the structural counterpart of observability index as follows:

Definition 5.1 *Let $\mathcal{O}_{L-1}(d)$ denote the L-step structural observability matrix. Then*

$$\bar{L}_o = \min_L\{\bar{\rho}\,[\mathcal{O}_{L-1}(d)] = n\}$$

is defined as the generic observability index.

If we let

$$L_j = \min_L\{\bar{\rho}\begin{bmatrix}\mathcal{O}_{L-1}(d)\\ c_j^T A^L\end{bmatrix} = \bar{\rho}\,[\mathcal{O}_{L-1}(d)]\},$$

then we can easily deduce from this definition that

$$\bar{L}_o = \max_{1\leq j\leq r}(L_j).$$

A characterization of generic observability index can also be given in terms of Rosenbrock's extended observability matrix [51]. Consider the L-step Rosenbrock matrix which is an $[Lr + (L-1)n] \times Ln$ matrix written in terms of the structured matrices $\mathbf{A}$ and $\mathbf{C}$ as

$$\mathbf{R}_L(d) = \begin{bmatrix} \mathbf{C} & & & & \\ \mathbf{I}_n & -\mathbf{A} & & & \\ & \mathbf{C} & & & \\ & \mathbf{I}_n & & & \\ & & \ddots & & \\ & & & -\mathbf{A} & \\ & & & \mathbf{C} & \\ & & & \mathbf{I}_n & -\mathbf{A} \\ & & & & \mathbf{C} \end{bmatrix} \tag{5.3}$$

Generic observability index $\bar{L}_o$ can also be written as

$$\bar{L}_o = \min_L \{\bar{\rho}[\mathbf{R}_L(d)] = Ln\},$$

i.e., it is the minimum L for which $\mathbf{R}_L(d)$ has full generic column rank. In the following, we state two facts that provide bounds for $\bar{L}_o$.

Fact 5.1 *Let $\gamma\overline{(\cdot,\cdot)}$ represent the number of state vertices, equivalently the steps, occuring in $\overline{(\cdot,\cdot)}$. Then,*

$$\bar{L}_o \geq \max_{1 \leq i \leq n} \{ \min_{1 \leq j \leq r} \gamma\overline{(x_i, y_j)} - 1 \}$$

Proof: Let

$$\gamma\overline{(x', y')} = \max_{1 \leq i \leq n} \{ \min_{1 \leq j \leq r} \gamma\overline{(x_i, y_i)} \}$$

and suppose that $\bar{L}_o < \gamma\overline{(x', y')}$. This implies that $\mathcal{O}_{L-1}(d)$ has a zero column, which contradicts the definition of $\bar{L}_o$. □

Note that any shortest path algorithm can be used to determine this lower bound for $\bar{L}_o$.

Next fact is concerned with an upper bound for $\bar{L}_o$.

Fact 5.2 :

$$\bar{L}_o \leq \min\{\max_{1\leq j\leq r}(n_j)\}$$

where the minimization is over all possible decompositions of $\mathcal{D}_{xy}$ into disjoint subgraphs $\mathcal{D}_j = (\mathcal{X}_j \cup \{y_j\}, \mathcal{E}_j)$ with $\gamma(\mathcal{X}_j) = n_j$, each of which is spanned by an output cactus.

Proof: Obvious. □

Example 5.4 Consider the digraph of Figure 5.6:(a). The two possible decompositions of this digraph is as given in Figures 5.6:(b) and (c).

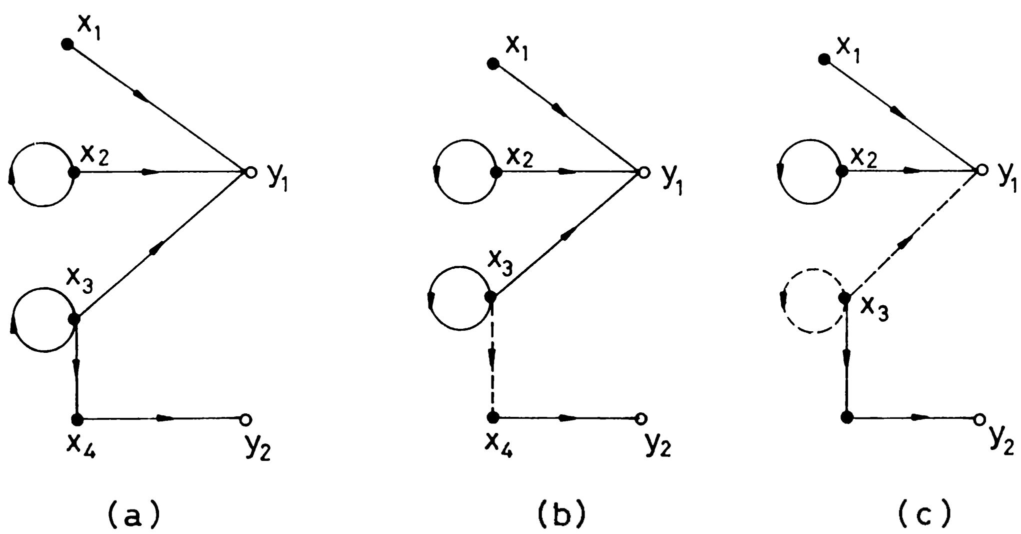


Figure 5.6. $\mathcal{D}_{xy}$ and the associated possible cactus decompositios of Example 5.4.

Clearly, $\bar{L}_o = 2$ for this system as also revealed by the decomposition of Figure 5.6:(c).

Another upper bound for $\bar{L}_o$ can be obtained using the results of the previous section:

Fact 5.3

$$\bar{L}_o \leq \min\{\gamma(\mathcal{P}_n^*)\}$$

where the minimization is over all possible choices of a set of n particular paths $\mathcal{P}_1^*, \cdots, \mathcal{P}_n^*$ as stated in Theorem 5.1.

Proof: Obvious. □

We finish this section by the following two examples which illustrate the result of Fact 5.3.

Example 5.5 For the system of Example 5.2, we know from Fact 5.2 that $\bar{L}_o \leq 2$. However, by the first choice of the set of particular paths of Example 5.2, Fact 5.3 also gives the same bound, and indeed $\bar{L}_o = 2$.

Example 5.6 Consider the digraph of Figure 5.7.

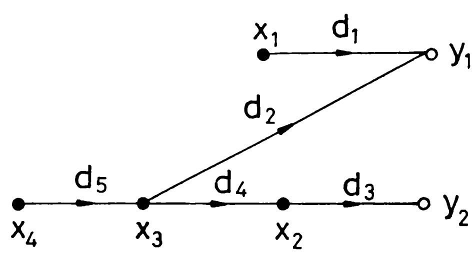


Figure 5.7. $\mathcal{D}_{xy}$ of Example 5.6.

Again, Fact 2 gives $\bar{L}_o \leq 3$, whereas the existence of the choice of the set of particular paths as

$$\begin{aligned}
\mathcal{P}_1^* &= \overline{(x_1, y_1)} = \{d_1\} \\
\mathcal{P}_2^* &= \overline{(x_2, y_2)} = \{d_3\} \\
\mathcal{P}_3^* &= \overline{(x_4, y_1)} = \{d_5, d_2\} \\
\mathcal{P}_4^* &= \overline{(x_3, y_2)} = \{d_4, d_3\}
\end{aligned}$$

reveals that $\bar{L}_o \leq 2$ (clearly $\bar{L}_o = 2$, as $\bar{L}_o \geq 2$ by Fact 5.1).

Chapter 6

AN ALGEBRAIC STUDY ON GENERICITY OF SEVERAL RESULTS ON POLE ASSIGNABILITY AND STABILIZABILITY

In the previous chapters we studied generic pole assignability and stabilizability problems using a graph-theoretic approach. The results we have obtained were essentially algebraic ones, which were stated in a graphical framework. Although the graphical approach provides extreme simplicity in testing certain sufficient conditions for pole assignability and stabilizability, it has a serious limitation: No similarity transformation, which changes the system structure or destroys algebraic independence of nonzero parameters, is allowed in a graphical analysis. Therefore a more general algebraic approach would be preferable for those systems which are not already in a canonical form that allows for the use of graphical procedures without any transformation. In this chapter, we consider genericity of some well-known results on pole assignability and stabilizability of certain classes of systems following an algebraic approach.

6.1 Pole-Assignability By Dynamic Output Feedback

The first problem we consider is pole assignment in single input/multi output, structurally controllable and observable systems, single input/single output version of which was considered in Section 3.3.2.

Consider a system represented as

$$\mathcal{S}: \quad \begin{aligned} \dot{x} &= Ax + bu \\ y &= Cx, \end{aligned} \tag{6.1}$$

and a dynamic output feedback controller

$$\hat{\mathcal{S}}: \quad \begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}y \\ u &= \hat{c}^T\hat{x} + \hat{f}^T y, \end{aligned} \tag{6.2}$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}$, $y \in \mathcal{R}^r$ and $\hat{x} \in \mathcal{R}^{\hat{n}}$. As has already been discussed in Section 3.3.2, the pole assignment problem for the pair $(\mathcal{S}, \hat{\mathcal{S}})$ is equivalent to the pole assignment problem for the augmented pair $(\mathcal{S}_a, \mathcal{F}_a)$, where $\mathcal{S}_a$ is described by

$$\mathcal{S}_a: \quad \begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} b & \\ & I \end{bmatrix} \begin{bmatrix} u \\ \hat{u} \end{bmatrix} \\ \begin{bmatrix} y \\ \hat{y} \end{bmatrix} &= \begin{bmatrix} C & \\ & I \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}, \end{aligned} \tag{6.3}$$

and

$$\mathcal{F}_a: \quad \begin{bmatrix} u \\ \hat{u} \end{bmatrix} = \begin{bmatrix} \hat{f}^T & \hat{c}^T \\ \hat{B} & \hat{A} \end{bmatrix} \begin{bmatrix} y \\ \hat{y} \end{bmatrix} \tag{6.4}$$

represents an equivalent constant output feedback law for $\mathcal{S}_a$. Again, as in Section 3.3.2, we assume that the pair $(\hat{A}, \hat{c}^T)$ is in observable canonical form of (3.21), which corresponds to the case where some of the feedback variables have already been fixed at either zero or one.

Let us first consider the case when $\hat{n} = 0$, that is, when $\hat{\mathcal{S}}$ of (6.1) reduces to constant output feedback

$$\mathcal{F}: \quad u = \hat{f}^T y, \tag{6.5}$$

so that $\mathcal{S}_a = \mathcal{S}$ and $\mathcal{F}_a = \mathcal{F}$. Rewriting (6.5) as

$$\mathcal{F}: \quad u = f^T x, \quad f^T = \hat{f}^T C, \tag{6.6}$$

and using the result of Section 3.3.1, we observe that the coefficients of the closed-loop characteristic polynomial can be expressed as

$$p = \alpha + Ef = \alpha + EC^T \hat{f}, \tag{6.7}$$

where $\alpha = \alpha(d)$ and $E = E(d)$ are as in the proof of Theorem 3.1. Let us partition the matrix $\bar{C}^T = EC^T$ into its rows as

$$\bar{C}^T = EC^T = \begin{bmatrix} \bar{c}_1^T \\ \bar{c}_2^T \\ \vdots \\ \bar{c}_n^T \end{bmatrix}, \tag{6.8}$$

and rewrite (6.7) explicitly as

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} \alpha_1(d) \\ \alpha_2(d) \\ \vdots \\ \alpha_n(d) \end{bmatrix} + \begin{bmatrix} \bar{c}_1^T(d) \\ \bar{c}_2^T(d) \\ \vdots \\ \bar{c}_n^T(d) \end{bmatrix} \hat{f}. \tag{6.9}$$

Note that $\alpha_k(d)$ in 6.9 are due to d-cycle families of width k, and $\bar{c}_k^T(d)\hat{f}$ are due to simple $\hat{f}$-cycle families of the same width in the closed-loop digraph $\mathcal{D}_a(\mathcal{F}_a) = \mathcal{D}(\hat{f})$.

Next consider the case when $\hat{n} = 1$, for which the closed-loop digraph $\mathcal{D}_a(\mathcal{F}_a)$ is shown in Figure 6.1. An inspection of Figure 6.1 reveals the following facts about the cycle families in $\mathcal{D}_a(\mathcal{F}_a)$:

(i) Any d-cycle family of width k in $\mathcal{D}(\hat{f})$ also appears in $\mathcal{D}_a(\mathcal{F}_a)$; and in addition, forms an f-cycle family of width k+1 together with the cycle $\{\overline{(\hat{u}_1, \hat{y}_1)}, (\hat{y}_1, \hat{u}_1)\}$,

(ii) The same is true for the $\hat{f}$-cycle families of $\mathcal{D}(\hat{f})$,

(iii) The cycle $\{\overline{(\hat{u}_1, \hat{y}_1)}, (\hat{y}_1, \hat{u}_1)\}$ is a single f-cycle family of width 1 by itself,

$\mathcal{D}_a(\mathcal{F}_a)$:

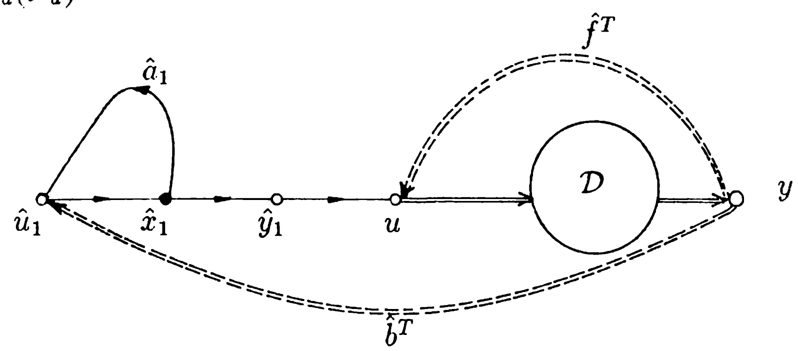


Figure 6.1. Illustration of $\mathcal{D}_a(\mathcal{F}_a)$ for $\hat{n} = 1$

(iv) Every f-cycle family of width k+1 formed by the f-cycle in (iii) and an $\hat{f}$-cycle family of width k in $\mathcal{D}(\hat{f})$ is accompanied by a simple f-cycle family of width $k+1$ which contains an f-edge from y to $\hat{u}_1$.

Based on these facts it is easy to see that the coefficients of the closed-loop characteristic polynomial are given as

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ p_{n+1} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & \bar{c}_1^T & 0 \\ \alpha_1 & \bar{c}_2^T & \bar{c}_1^T \\ \vdots & \vdots & \vdots \\ \alpha_{n-1} & \bar{c}_n^T & \bar{c}_{n-1}^T \\ \alpha_n & 0 & \bar{c}_n^T \end{bmatrix} \begin{bmatrix} -\hat{a}_1 \\ \hat{f} \\ \hat{b} - \hat{a}_1\hat{f} \end{bmatrix} \tag{6.10}$$

Similarly, in the most general case, we have

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ p_{n+1} \\ \vdots \\ p_{n+\hat{n}} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \left[\begin{array}{cc|cccc} 1 & & \bar{c}_1^T & 0 & & \\ \alpha_1 & & \bar{c}_2^T & \bar{c}_1^T & & \\ \vdots & 1 & \vdots & \vdots & & 0 \\ \alpha_{n-1} & \alpha_1 & \bar{c}_n^T & \bar{c}_{n-1}^T & & \bar{c}_1^T \\ \alpha_n & \vdots & 0 & \bar{c}_n^T & & \vdots \\ & \alpha_{n-1} & & & \ddots & \bar{c}_{n-1}^T \\ & \alpha_n & & & & \bar{c}_n^T \end{array}\right] \begin{bmatrix} -\hat{a}_1 \\ \vdots \\ -\hat{a}_{\hat{n}} \\ \hat{f} \\ \hat{b}_1 - \hat{a}_1\hat{f} \\ \vdots \\ \hat{b}_{\hat{n}} - \hat{a}_{\hat{n}}\hat{f} \end{bmatrix} \tag{6.11}$$

where $\hat{B} = [\hat{b}_1 \ \ \hat{b}_2 \ \ \cdots \ \ \hat{b}_{\hat{n}}]$. Rewriting 6.11 in compact form as

$$p = \alpha(d) + \Omega_{\hat{n}}(d)\tilde{f}, \tag{6.12}$$

where $\tilde{f}$ are the auxiliary variables in Corollary 3.1, we observe that $\mathcal{S}(\hat{\mathcal{S}})$ is generically pole assignable if and only if

$$\bar{\rho}(\Omega_{\hat{n}}) = n + \hat{n} \tag{6.13}$$

A sufficient condition for (6.13) to hold is given below:

Lemma 6.1 *If $\bar{L}_0$ is the generic observability index defined in Section 5.3, then (6.13) is satisfied for $\hat{n} = \bar{L}_0 - 1$.*

The proof of the lemma is based on the following fact.

Fact 6.1 *In $\mathcal{D}_{ux}$ there exist a spanning cactus $\mathcal{D}_c$ and an ordering of the state vertices $x_1, x_2, \cdots, x_n$, such that any x_j which is adjacent to x_1 in $\mathcal{D}_{ux}$, occurs in the same bunch of $\mathcal{D}_c$ as x_1.*

Proof: Let $\mathcal{D}_c$ be an arbitrary cactus spanning $\mathcal{D}_{ux}$, the vertices of which are ordered according to the enumeration scheme of Lemma 3.4. If $\mathcal{D}_c$ consists of a single bunch, then there is nothing to prove. Otherwise, order the bunches of $\mathcal{D}_c$ as $\mathcal{B}_1, \mathcal{B}_2, \cdots, \mathcal{B}_t$, where $x_1 \in \mathcal{B}_1$ and $\mathcal{B}_t$ is the terminal bunch. If any x_j which occurs in some $\mathcal{B}_l, l > 1$, is adjacent to x_1 in $\mathcal{D}_{ux}$, then modify $\mathcal{D}_c$ by deleting the edge which connects $\mathcal{B}_l$ to u and adding the edge (x_1, x_j) to $\mathcal{D}_c$. This way $\mathcal{B}_l$ is combined with $\mathcal{B}_1$ to form a single bunch. Reorder the vertices of the modified cactus, and repeat the same process, until either $\mathcal{D}_c$ consists of a single bunch, or else, no vertex of any $\mathcal{B}_l, l > 1$, is adjacent to x_1 in $\mathcal{D}_{ux}$.

Proof of Lemma 6.1: Fact 6.1 implies that, after a scaling of the weight of the edge (u, x_1) in $\mathcal{D}_{ux}$, the matrices A and b in (6.1) can be assumed to have the forms

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ * \\ \vdots \\ * \end{bmatrix}, \tag{6.14}$$

where $*$ denotes any zero or nonzero element. In (6.14), $a_{1j} \neq 0$ implies that there exists an edge (x_j, x_1) in $\mathcal{D}_{ux}$, which, by Fact 6.1, takes part in a cycle covering x_1 and x_j.

Now, let $\bar{R}_L(d)$ denote the Rosenbrock's L-step observability matrix defined in (5.3), with each block column postmultiplied by E^T, that is,

$$\bar{R}_L = \begin{bmatrix} \bar{C} & & & \\ E^T & -\bar{A} & & \\ & E^T & & \\ & & & \\ & & -\bar{A} & \\ & & \bar{C} & \\ & & E^T & -\bar{A} \\ & & & \bar{C} \end{bmatrix} \tag{6.15}$$

where $\bar{A} = AE^T$, and $\bar{C}$ is as defined in (6.8).

Perform the following column operations on $\bar{R}_L$: Starting with the first block column, add i-th column of block k to the $(i-1)$st column of block $k+1$, $i = 2, \cdots, n, k = 1, 2, \cdots, L-1$. The resulting matrix $\tilde{R}_L$ has the structure illustrated below for $L = 3$.

$$\tilde{R}_3 = \left[\begin{array}{c|c|cccc|ccccc} \bar{c}_1 & \bar{c}_2 \cdots \bar{c}_n & \bar{c}_2 & \cdots & \bar{c}_n & 0 & \bar{c}_3 & \cdots & \bar{c}_n & 0 & 0 \\ \hline 1 & * \cdots * & \tilde{a}_{11} & \cdots & \tilde{a}_{1n-1} & \tilde{a}_{1n_1} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n-1} & \tilde{a}_{1n} & 0 \\ \cline{1-2} * & & * & \cdots & * & * & * & \cdots & * & * & 0 \\ \vdots & E_1^T & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & 0 \\ * & & * & \cdots & * & * & * & \cdots & * & * & 0 \\ \hline & & \bar{c}_1 & \cdots & \bar{c}_{n-1} & \bar{c}_n & \bar{c}_2 & \cdots & \bar{c}_{n-1} & \bar{c}_n & 0 \\ \cline{3-11} & & 1 & \cdots & * & * & \tilde{a}_{11} & \cdots & \tilde{a}_{1n-2} & \tilde{a}_{1n-1} & \tilde{a}_{1n} \\ \cline{3-6} & & * & & & & * & \cdots & * & * & * \\ & & \vdots & & E_1^T & & \vdots & & \vdots & \vdots & \vdots \\ & & * & & & & * & \cdots & * & * & * \\ \cline{3-11} & & & & & & \bar{c}_1 & \cdots & \bar{c}_{n-2} & \bar{c}_{n-1} & \bar{c}_n \end{array}\right] \tag{6.16}$$

By the proof of Theorem 3.3, E_1^T has full generic rank $n-1$, so that

$$\bar{\rho}[\bar{R}_L(d)] = \bar{\rho}[\tilde{R}_L(d)] = (L-1)(n-1) + \bar{\rho}[R_L^*(d)] \tag{6.17}$$

where

$$R_L^*(d) = \begin{bmatrix} \bar{c}_1 & \bar{c}_2 & \cdots & \bar{c}_n & & & & & \\ 1 & \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} & & & & \\ & \bar{c}_1 & \bar{c}_2 & \cdots & \bar{c}_n & & & & \\ & 1 & \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} & & & \\ & & \ddots & & & & & & \\ & & & 1 & \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ & & & & \bar{c}_1 & \bar{c}_2 & & \bar{c}_n \end{bmatrix} \tag{6.18}$$

with $2L-1$ block rows, is the matrix obtained from $\tilde{R}(d)$ by deleting the rows and columns that correspond to rows and columns of E_1^T's.

We now claim that

$$\bar{\rho}[R_L^*(d)] = \bar{\rho}[\Omega_{L-1}(d)], \tag{6.19}$$

where $\Omega_{\hat{n}}$ is defined in (6.11). To prove the claim, first note that the first row elements of $\bar{A}$ in (6.15) are related to the first row elements of A in (6.14) as

$$\begin{bmatrix} \bar{a}_{11} \\ \bar{a}_{12} \\ \vdots \\ \bar{a}_{1n} \end{bmatrix} = E \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{bmatrix}. \tag{6.20}$$

If in $\mathcal{D}_{ux}$, the edges (x_j, x_1) that correspond to nonzero a_{1j}'s were replaced by hypothetical f-edges (x_j, u), then the left-hand side of (6.20) would represent the coefficients of the characteristic polynomial of the resulting hypothetical closed-loop system, as the weight of the edge (u, x_1) is normalized to unity. Therefore, each $\bar{a}_{1j}$ in (6.20) is nothing but the sum of the weights of all cycle families of width j in $\mathcal{D}_{ux}$ which contains x_1. Next we note that in

$$E^T = \begin{bmatrix} 1 & e_{12} & \cdots & e_{1n} \\ * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \end{bmatrix},$$

each nonzero $e_{1j}, j \geq 2$, is the sum of the weights the cycle families of width $j-1$ not containing x_1. As a result of these two observations we conclude that each $\tilde{a}_{1j} = \bar{a}_{1j} + e_{1,j+1}$ in (6.16) represents the sum of the weights of all cycle families of width j in $\mathcal{D}_{ux}$, so that $\tilde{a}_{1j} = \alpha_j$, where α_j are as in (6.9). Hence, R_L^* is nothing but the transpose of Ω_{L-1} with columns rearranged, so that (6.19) is true.

Finally, the definition of the generic observability index $\bar{L}_0$, together with (6.17) and (6.19) implies that

$$\bar{\rho}[\Omega_{L_o-1}(d)] = \bar{\rho}[R_{L_o}(d)] = n + \bar{L}_o - 1,$$

completing the proof of Lemma 6.1.

Combining the result of Lemma 6.1 with (6.11), we reach the following conclusion about stabilizability of $\mathcal{S}(\hat{\mathcal{S}})$, the proof of which is obvious.

Theorem 6.1 *The single input, structurally controllable and structurally observable system of (6.1) is generically stabilizable by an $(\bar{L}_0 - 1)$st order dynamic output feedback controller of (6.2), where $\bar{L}_0$ is the generic observability index of $\mathcal{S}$.*

Before closing the section, we finally note that the dual result applies to single-output systems with $\bar{L}_0$ replaced with the generic controllability index $\bar{L}_c$.

6.2 Stabilization Of A Class Of Interconnected Systems Using Decentralized State Feedback

The next problem we consider is the stabilization of the interconnected system consisting of controllable subsystems as described in (4.5) using decentralized constant feedback of the form (4.6). We assume, as in Section 4.3, that the interconnections satisfy the matching conditions in (4.8); however, we do not require the subsystems to be in any specific form.

We assume, without loss of generality, that the input vectors are of the form

$$b_i = \begin{bmatrix} b_{i1} & b_{i2} & \cdots & b_{iq_i} & 0 & \cdots & 0 \end{bmatrix}^T, \quad 1 \le i \le N, \tag{6.21}$$

where $1 \le q_i \le n_i$ and $b_{ik} \ne 0, 1 \le k \le q_i$. We also express the matching conditions as

$$A_{ij} = b_i h_{ij}^T, \quad i,j = 1,2,\cdots,N, \tag{6.22}$$

where

$$h_{ij}^T = \begin{bmatrix} h_1^{ij} & h_2^{ij} & \cdots & h_{n_j}^{ij} \end{bmatrix}. \tag{6.23}$$

The overall system then has the representation

$$\mathcal{S}(\mathcal{F}): \quad \dot{x} = (A_D + B_D F_D + B_D H)x, \tag{6.24}$$

where

$$A_D = diag\ \{A_1, A_2, \cdots, A_N\},$$

B_D and F_D are defined similarly, and $H = (h_{ij}^T)_{N \times N}$.

Keeping in mind that the $k-th$ coefficient p_k of the closed-loop characteristic polynomial consists of product terms, each of which corresponds to a nonzero term in the determinantal expansion of some $k \times k$ principal minor of $A_D + B_D F_D + B_D H$, we can write

$$p_k = p_k^o + p_k^f + p_k^h, \quad 1 \le k \le n, \tag{6.25}$$

where p_k^h contains all product terms which include one or more h-parameters; p_k^f contains those which include one or more f-parameters but no h-parameters; and p_k^0 is a constant due to parameters of A_D. Obviously, there are, in general, more than one product terms in p_k^h which contain exactly the same h- and f-parameters; and similarly, more than one product terms in p_k^f which contain the same f-parameters. However, some of such terms cancel each other algebraically; and the remaining terms which differ only in a- or b-parameters can be grouped together to form a single product term. We can, therefore, assume that no two product terms in p_k^h contains exactly the same h- and f-parameters, and no two terms in p_k^f contains exactly the same f-parameters.

Now using the matching conditions (6.22) and simple matrix manipulations, it is not too difficult to see that

(i) if a (grouped) product term in p_k^h contains h_p^{ij}, $1 \leq p \leq n_j$; $1 \leq i, j \leq N$, then it contains no $f_{iq}, 1 \leq q \leq n_i$; and vice versa; and

(ii) to every (grouped) product term in p_k^h there corresponds a (grouped) product term in p_k^f, which contains more f-parameters than the former.

These observations guarantee that choosing high feedback gains for the decoupled subsystems as in the proof of Theorem 4.2, the terms in p_k^f can be made to dominate over p_k^0 and p_k^h in (6.25), so that the poles of the overall closed-loop interconnected system approach to those of the closed-loop decoupled subsystems. Moreover, genericity of pole-assignability of the decoupled subsystems, which was proved in Theorem 3.2, implies genericity of stabilizability of $\mathcal{S}(F)$ of (6.24). We state this result as a theorem.

Theorem 6.2 *The interconnected system described in (4.5), in which the subsystems are structurally controllable, and the interconnections satisfy the matching conditions in (4.8), is generically stabilizable using decentralized constant state feedback.*

6.3 Stabilization of a Class of Interconnected Systems Using Decentralized Dynamic Output Feedback

The final problem we study is the generic stabilizability of a class of interconnected systems using decentralized dynamic output feedback. The interconnected system we consider consists of structurally controllable and structurally observable single input/single output subsystems described as

$$\mathcal{S}_i: \quad \begin{aligned} \dot{x}_i &= A_i x_i + \textstyle\sum_{j=1}^{N} A_{ij} x_j + b_i u_i, \\ y_i &= c_i^T x_i, \quad i = 1, 2, \cdots, N, \end{aligned} \tag{6.26}$$

where $x_i \in \mathcal{R}^{n_i}$ and $u_i, y_i \in \mathcal{R}$. To each decoupled subsystem

$$\mathcal{S}_i^D: \quad \begin{aligned} \dot{x}_i &= A_i x_i + b_i u_i, \\ y_i &= c_i^T x_i, \quad i = 1, 2, \cdots, N, \end{aligned} \tag{6.27}$$

obtained from (6.26) by setting $A_{ij} = 0$, we apply local dynamic output feedback

$$\hat{\mathcal{S}}_i: \quad \begin{aligned} \dot{\hat{x}}_i &= \hat{A}_i \hat{x}_i + \hat{b}_i y_i, \\ u_i &= \hat{c}_i^T \hat{x}_i + \hat{f} y_i, \end{aligned} \tag{6.28}$$

where $\hat{x}_i \in \mathcal{R}^{n_i - 1}$.

As in Sections 3.3.2 and 6.1, we interpret the dynamic output feedback $\hat{\mathcal{S}}_i$ in (6.28) applied to $\mathcal{S}_i^D$ of (6.27) as a constant output feedback $\mathcal{F}_{ai}$ applied to an augmented subsystem $\mathcal{S}_{ai}^D$, where $\mathcal{S}_{ai}^D$ and $\mathcal{F}_{ai}$ are as in (6.3) and (6.4). Also, we choose $(\hat{A}_i, \hat{c}i^T)$ to be in observable canonical form of (3.21), which corresponds to fixing all but $2n_i - 1$ elements of $\mathcal{F}_{ai}$ at zero or one.

The class of interconnected systems we consider is characterized by the following two assumptions:

(i) The decoupled subsystems $\mathcal{S}_i^D$ of (6.27) generically have no transmission zeros,

(ii) There exists a subset $\mathcal{M}$ of the set $\mathcal{N} = \{1, 2, \cdots, N\}$ such that the interconnection matrices A_{ij} in (6.26) satisfy

$$A_{ij} = b_i h_{ij}^T \quad , \quad i \in \mathcal{M}, \; j \in \mathcal{N} \tag{6.29}$$

$$A_{ji} = g_{ji} c_i^T \quad , \quad i \in \mathcal{N} - \mathcal{M}, \; j \in \mathcal{N} \tag{6.30}$$

where $h_{ij} = [h_1^{ij} \; h_2^{ij} \; \cdots \; h_{n_j}^{ij}]^T$ and $g_{ji} = [g_1^{ji} \; g_2^{ji} \; \cdots \; g_{n_j}^{ji}]^T$. In other words, we assume that for each $\mathcal{S}_i$, either the interaction from any other subsystem has the same effect on $\mathcal{S}_i$ as the control input u_i (when $i \in \mathcal{M}$) or the interaction from $\mathcal{S}_i$ to any other subsystem is a reproduction of the measured output y_i (when $i \in \mathcal{N} - \mathcal{M}$).

With this set-up, we now state our main result as follows.

Theorem 6.3 *Under the assumptions (i) and (ii) above, the interconnected system consisting of the subsystems $\mathcal{S}_i$ of (6.26) is generically stabilizable by decentralized output feedback controllers $\hat{\mathcal{S}}_i$ of (6.27).*

The proof of Theorem 6.3 depends on the following characterization of transmission zeros of a single input/single output system by Reinschke[52].

Lemma 6.2 *Let $\mathcal{D}(e_f)$ denote the closed-loop digraph of a single input/single output system described by (6.27) with a feedback edge e_f of weight $w(e_f) = -1$. Then the coefficient β_k of the numerator polynomial $\beta(s) = \beta_1 s^{n-1} + \beta_2 s^{n-2} + \cdots + \beta_n$ of the transfer function of the open-loop system are given by*

$$\beta_k = \sum_{\omega(\mathcal{CF})=k} (-1)^{\gamma(\mathcal{CF})} \omega(\mathcal{CF}), \quad k = 1, 2, \cdots, n, \tag{6.31}$$

where the summation is carried out for all cycle families of width k which include e_f.

Proof of Theorem 6.3: The closed-loop digraph of the equivalent augmented system is of the form $\mathcal{D}_a(\mathcal{F}_a) = (\mathcal{V}_a, \mathcal{E}_a \cup \mathcal{E}_f \cup \mathcal{E}_c)$, where $\mathcal{D}_a^D(\mathcal{F}_a) = (\mathcal{V}_a, \mathcal{E}_a \cup \mathcal{E}_f)$ is a collection of decoupled closed-loop digraphs $\mathcal{D}_{ai}^D(\mathcal{F}_{ai}) = (\mathcal{V}_{ai}, \mathcal{E}_{ai} \cup \mathcal{E}_{fi})$ associated with the decoupled augmented subsystems $\mathcal{S}_{ai}^D(\mathcal{F}_{ai})$, and $\mathcal{E}_c$ is the set of coupling edges due to nonzero parameters of A_{ij}, $i, j \in \mathcal{N}$.

Each closed-loop decoupled subsystem graph $\mathcal{D}_{ai}^D(\mathcal{F}_{ai})$ has the structure shown in Figure 6.2

Assumption (i) concerning the decoupled subsystem implies that the coefficients β_k^i of the numerator polynomial of the transfer function of $\mathcal{S}_i^D$ are all zero except $\beta_{n_i}^i$. Graphically, this means that in $\mathcal{D}_i^D(\hat{f}_i)$ there exists only one cycle that includes $\hat{f}_i$, and this particular cycle is of width n. Equivalently, in $\mathcal{D}_i^D$ there is a unique input/output path which covers all the state vertices

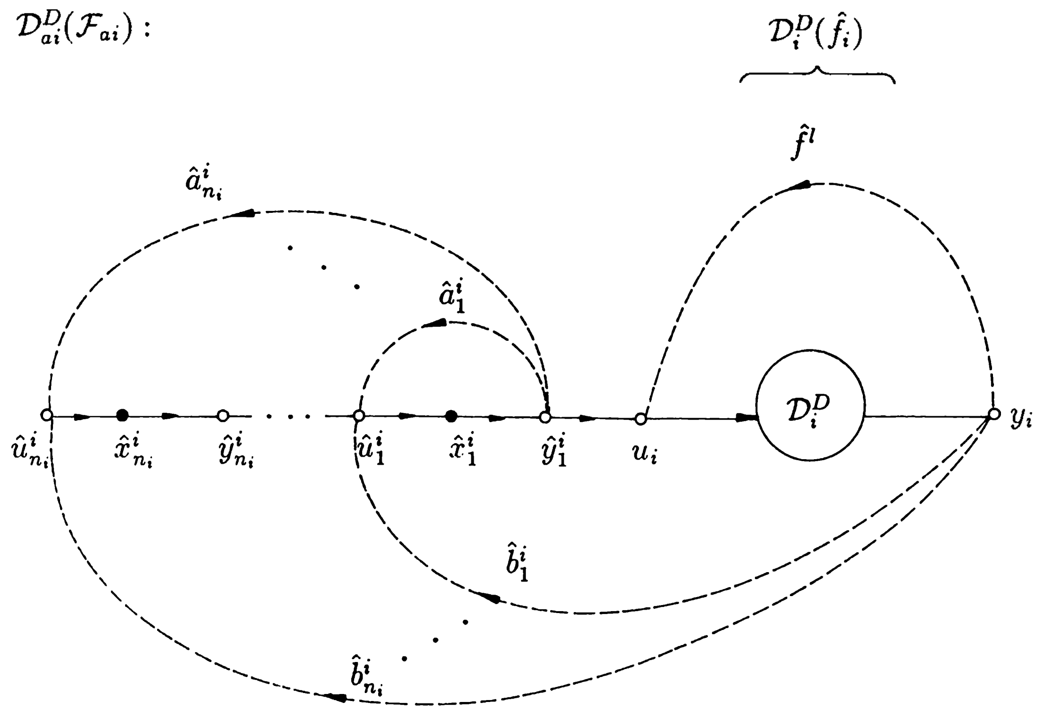


Figure 6.2. Closed-loop decoupled subsystem graph $\mathcal{D}^D_{ai}(\mathcal{F}_{ai})$

$x^i_k, 1 \leq k \leq n_i, 1 \leq i \leq N$. Translated into the structure of the matrices A_i, b_i and c_i^T, assumption (i) means

$$
\begin{aligned}
A_i &= \begin{bmatrix} * & * & * & \cdots & * & * & * \\ * & * & * & \cdots & * & * & * \\ 0 & * & * & \cdots & * & * & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & * & * & * \\ 0 & 0 & 0 & \cdots & * & * & * \\ 0 & 0 & 0 & \cdots & 0 & * & * \end{bmatrix}, \quad b_i = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\
c_i^T &= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} .
\end{aligned}
\tag{6.32}
$$

Since each $\mathcal{S}^D_i$ is structurally controllable and observable, by Theorem 6.1, $\mathcal{S}^D_{ai}(\mathcal{F}_{ai})$ are generically pole assignable. In particular, the variable feedback gains $\hat{a}^i_l, \hat{f}^i$ and $\hat{b}^i_l, 1 \leq l \leq n_i - 1, 1 \leq i \leq N$ can be computed from (6.11), which, due to (6.32), takes the form

$$\begin{bmatrix} p_1^i \\ p_2^i \\ \vdots \\ p_{n_i}^i \\ p_{n_{i+1}}^i \\ \vdots \\ p_{2n_i-1}^i \end{bmatrix} = \left[\begin{array}{cc|ccc} 1 & & & & \\ \alpha_1^i & & & & \\ \vdots & 1 & & & \\ \alpha_{n_i-1}^i & \alpha_1^i & 1 & & \\ \alpha_{n_i}^i & \vdots & & 1 & \\ & \alpha_{n_i-1}^i & & & \\ & \alpha_{n_i}^i & & & 1 \end{array}\right] \begin{bmatrix} \hat{a}_1^i \\ \vdots \\ \hat{a}_{n_i}^i \\ \hat{f}^i \\ \hat{b}_1^i + \hat{f}^i \hat{a}_1^i \\ \vdots \\ \hat{b}_{n_i-1}^i + \hat{f}^i \hat{a}_{n_i-1}^i \end{bmatrix} \tag{6.33}$$

We choose the feedback gains so as to place the poles of the closed-loop decoupled subsystems $\mathcal{S}_{ai}^D(\mathcal{F}_{ai})$ at $-\rho\sigma_l^i$, where $\sigma_l^i > 0, 1 \le l \le 2n_i - 1, 1 \le i \le N$, and $\rho > 0$ is an arbitrarily large parameter. Then, $p_l^i \in \mathcal{O}(\rho^l)$, and (6.33) implies that

$$\begin{bmatrix} \hat{a}_1^i \\ \vdots \\ \hat{a}_{n_i-1}^i \\ \hat{f}^i \\ \hat{b}_1^i \\ \vdots \\ \hat{b}_{n_i-1}^i \end{bmatrix} = \begin{bmatrix} \psi_1^i(d,p) \\ \vdots \\ \psi_{n_i-1}^i(d,p) \\ \psi_{n_i}^i(d,p) \\ \psi_{n_i+1}^i(d,p) \\ \vdots \\ \psi_{2n_i-1}^i(d,p) \end{bmatrix}, \tag{6.34}$$

where $\psi_l^i(d,\rho)$ are polynomials in ρ of degree l, with coefficients being polynomials in d. That is, $deg\psi_l^i(d,\cdot) = l$, $1 \le l \le 2n_i - 1$.

The structure of $\mathcal{D}_{ai}^D(\mathcal{F}_{ai})$ shown in Figure 6.2, together with assumption (ii) also implies that no cycle in $\mathcal{D}_a(\mathcal{F}_a)$, which includes a coupling edge due to a nonzero parameter of some h_{ij}^T or g_{ij}, can include a feedback edge. To see this, consider such a cycle $\mathcal{C}_c$ which pass through the state vertices of $\mathcal{D}_l^D, l \in \mathcal{L} \subset \mathcal{N}$. Let $\mathcal{L}_M = \mathcal{L} \cap \mathcal{M}$ and $\mathcal{L}_{N-M} = \mathcal{L} - \mathcal{L}_M = \mathcal{L} \cap (\mathcal{N} - \mathcal{M})$. Then assumption (ii) implies that $\mathcal{C}_c$ should cover the state vertices x_1^l for $l \in \mathcal{L}_M$ and $x_{n_l}^l$ for $l \in \mathcal{L}_{N-M}$. Therefore, $\mathcal{C}_c$ cannot contain any $\hat{f}_l$ or $\hat{b}_{q_l}^l$, $1 \le q_l \le n_l - 1, l \in \mathcal{L}$. Obviously, $\mathcal{C}_c$ cannot contain any $\hat{f}_m$ or $\hat{b}_{q_m}^m, m \in \mathcal{N} - \mathcal{L}$ either, for then it would have cover x_{nm}^m, contradicting definition of the set $\mathcal{L}$. Finally, that $\mathcal{C}_c$ cannot contain any $\hat{a}$-type feedback edge is clear from the structure of $\mathcal{D}_{ai}^D(\mathcal{F}_{ai})$.

Now consider an arbitrary cycle family $\mathcal{CF}_c$ of an arbitrary width in $\mathcal{D}_a(\mathcal{F}_a)$, which includes a coupling edge. Let $\mathcal{CF}_c = \mathcal{C}_{c1} \cup \mathcal{CF}_{\hat{a}1} \cup \cdots \cup \mathcal{C}_{cK} \cup \mathcal{CF}_{\hat{a}K} \cup \mathcal{CF}_d \cup \mathcal{CF}_f$, where

- $\mathcal{C}_{ck}$, $\mathcal{C}_{\hat{a}k}$, $1 \leq k \leq K$, and the cycles in the cycle families $\mathcal{CF}_d$ and $\mathcal{CF}_f$ are all disjoint,

- each $\mathcal{C}_{ck}$ is a simple d-cycle which includes a coupling edge, and covers some (or all) state vertices of $\mathcal{D}_l^D, l \in \mathcal{L}_k$, where $\mathcal{L}_k \subset \mathcal{N}$ are disjoint, $1 \leq k \leq K$,

- for each $1 \leq k \leq K$, $\mathcal{CF}_{\hat{a}k}$ is an $\hat{a}$-cycle family, which consists of simple $\hat{a}$-cycles $\mathcal{C}_{\hat{a}l}^k$ formed by the feedback edges $\hat{a}_{q_l}^l$, $1 \leq q_l \leq n_l - 1$, $l \in \mathcal{L}_k$, where not all cycles need to exist (in fact, $\mathcal{CF}_{\hat{a}k}$ may be empty),

- $\mathcal{CF}_d$ and $\mathcal{CF}_f$ are families of simple d- or f-cycles in $\mathcal{D}_{ai}^D(\mathcal{F}_{ai})$, $i \in \mathcal{N} - \bigcup \mathcal{L}_k$.

Let each coupling cycle $\mathcal{C}_{ck}$, $1 \leq k \leq K$, cover $\bar{n}_l$ state vertices of $\mathcal{D}_l^D, l \in \mathcal{L}_k$. We now construct another cycle family $\bar{\mathcal{CF}}$ of the same width as $\mathcal{CF}_c$, which includes no coupling edges as follows:

$$\overline{\mathcal{CF}} = \overline{\mathcal{CF}}_1 \cup \cdots \cup \overline{\mathcal{CF}}_K \cup \mathcal{CF}_d \cup \mathcal{CF}_f,$$

where each $\overline{\mathcal{CF}}_k$ consists of simple cycles $\overline{\mathcal{C}}_l^k$, $l \in \mathcal{L}_k$, which include

$$\begin{cases} \hat{a}_{\bar{n}_l+q_l}^l & , \ if\ \bar{n}_l + q_l < n_l \\ \hat{f}^l & , \ if\ \bar{n}_l + q_l = n_l \\ \hat{b}_{\bar{n}+q_l-n_l} & , \ if\ \bar{n}_l + q_l > n_l \end{cases} \tag{6.35}$$

Note that, in each case $\gamma(\overline{\mathcal{C}}_l^k) = \bar{n}_l + q_l$, so that $\gamma(\overline{\mathcal{CF}}_k) = \sum_{l \in \mathcal{L}} \bar{n} + \sum_{l \in \mathcal{L}} q_l = \gamma(\mathcal{C}_{ck}) + \gamma(\mathcal{CF}_{\hat{a}k})$, and therefore, $\gamma(\overline{\mathcal{CF}}) = \gamma(\mathcal{CF}_c)$.

We now compare weights of $\mathcal{CF}_c$ and $\overline{\mathcal{CF}}$. By the choice of the feedback gains as in (6.34), $\omega(\mathcal{CF}_c = \psi_c(d, \rho)$ and $\omega(\overline{\mathcal{CF}}) = \overline{\psi}(d, \rho)$ are both polynomials in ρ. By definition of $\mathcal{CF}_c$ we have

$$\omega(\mathcal{CF}_c) = \prod_{k=1}^{K} \omega(\mathcal{C}_{ck}) \cdot \omega(\mathcal{CF}_d) \cdot \omega(\mathcal{CF}_f) \cdot \prod_{k=1}^{K} \prod_{l \in \mathcal{L}_k} \omega(\mathcal{C}_{\hat{a}l}^k).$$

so that

$$deg\ \psi_c(d,\cdot) = deg\ \psi_f(d,\cdot) + \sum_{k=1}^{K}\sum_{l\in\mathcal{L}_k} deg\ \psi^k_{\hat{a}l}(d,\cdot) \tag{6.36}$$

$$= deg\ \psi_f(d,\cdot) + \sum_{k=1}^{K}\sum_{l\in\mathcal{L}_k} q_l. \tag{6.37}$$

On the other hand, by construction of $\overline{\mathcal{CF}}$, we have

$$deg\ \overline{\psi}(d,\cdot) = deg\ \psi_f(d,\cdot) + \sum_{k=1}^{K}\sum_{l\in\mathcal{L}_k} deg\ \overline{\psi}^k_l(d,\cdot) \tag{6.38}$$

$$= deg\ \psi_f(d,\cdot) + \sum_{k=1}^{K}\sum_{l\in\mathcal{L}_k} \bar{n}_l + q_l, \tag{6.39}$$

where the last equality follows from (6.34) and (6.35).

As a result, associated with every cycle family $\mathcal{CF}_c$ which includes a coupling edge, we have another cycle family $\overline{\mathcal{CF}}$, which includes no coupling edges such that $\gamma(\overline{\mathcal{CF}}) = \gamma(\mathcal{CF}_c)$, and $deg\ \overline{\psi}(d,\cdot) > \psi_c(d,\cdot)$. The proof then follows the same lines as the proof of Theorem 4.2 on letting $\rho \to \infty$.

Chapter 7

CONCLUSION AND SUGGESTIONS FOR FURTHER RESEARCH

This thesis concerns a qualitative analysis of certain (potential) system properties, namely, pole assignability, stabilizability and observability. In the following, while we summarize and comment on the results of each chapter, we also give suggestions for points which need further studying.

We know that the well-known result of Brasch and Pearson [49] which states that all the poles of a controllable and observable system can be assigned arbitrarily using a dynamic feedback compensator of order $L = min\{L_c, L_o\} - 1$, where L_c and L_o are the controllability and observability indices of the system, is overly sufficient. This can be explained by the fact that their algebraic criterion does not take into account the structure of the system, which actually plays the most important role in the solvability of the problem. In Chapter 3, we investigate arbitrary pole assignability as a structural property of the system by means of digraphs and prove two main theorems, namely, Theorem 3.1 and Theorem 3.2, that provide graphical sufficient conditions for structural pole assignability. Indeed, our results show that, in some systems for which we would normally attempt to use dynamic output feedback in order to place all the poles at desired locations, it is sufficient to use constant output feedback or at least a compensator of smaller order. Furthermore, the conditions, being in terms of the digraph, turn out to be sufficient for generic pole assignability, too. We prove that certain classes of systems which are known to be generically arbitrarily pole assignable satisfy the conditions of one or the other of these two theorems, also demonstrating the nontriviality of the theorems.

Note that Theorem 3.2, which is a slightly generalized version of Theorem 3.1, represents a special case of Corollary 3.1, stated in graph-theoretic terms. It corresponds to the case when ψ_k of (3.11) contains a single quadratic term and it seems possible to obtain more general results by considering modifications of this theorem to cover other forms of ψ_k. For example, in showing the pole assignability of structurally controllable and observable systems with dynamic output feedback, via Theorem 3.2, we had to limit ourselves to a class of systems with a certain structure. However, as verified by Example 3.9 of Section 3.3.2, modifying Theorem 3.2 somehow to include the case when ψ_k contains linear terms in addition to a single quadratic term might solve this problem. On the other hand, Corollary 3.1 is still a special case of some other result, namely, Lemma 3.2, which possibly has hints for characterizing a broader class of pole assignable structures.

In Chapter 4, we extend the approach used in the preceding chapter, to investigate structural stabilizability. Assuming high gain feedback, we state and prove three results each characterizing a class of structurally and at the same time generically stabilizable systems. Similar to the situation in Chapter 3, the first two problems in this chapter describe special cases of the algebraic result of Lemma 4.1. This lemma demands a very limiting structure for the coefficients of the closed-loop characteristic polynomial which might probably be relaxed, hence allowing for more general graphical results on stabilizability.

In Chapter 5, we present a graphical interpretation of the observability matrix and provide a new graphical criterion necessary and sufficient for the structural observability of a system. Generic observability index is defined and lower and upper bounds for it are obtained. We actually aimed at but were unsuccessful in developing a graphical interpretation of the generic observability index which would have possibly led to a graphical method to compute the index. This requires checking the generic linear independence of rows with elements which are not necessarily algebraically independent and hence it is extremely difficult to progress in this way. An alternative way of attacking this problem might be through consideration of Rosenbrock's extended observability matrix instead of the regular observability matrix.

Chapter 6 considers genericity of some well-known results on pole assignability and stabilizability of classes of systems with certain structures, using an algebraic approach in conjunction with some purely graph-theoretic results. In addition to the cases studied here, it might be worthwhile to consider the problem of stabilizability using dynamic output feedback for other classes of systems, for example for interconnected systems whose subsystems have stable zeros.

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