STRONGLY INTERACTING ONE-DIMENSIONAL BOSE CONDENSATES

A THESIS
SUBMITTED TO THE DEPARTMENT OF PHYSICS
AND THE INSTITUTE OF ENGINEERING AND SCIENCE
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
KAMİL ERKAN

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By
Kamil Erkan
September 2000

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Master of Science.

Prof. Bilal Tanatar (Supervisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Master of Science.

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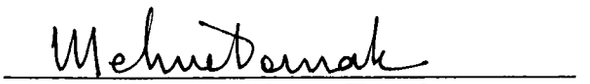

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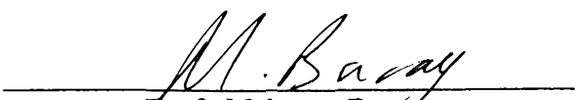

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Abstract

STRONGLY INTERACTING ONE-DIMENSIONAL BOSE CONDENSATES

Kamil Erkan
M. S. in Physics
Supervisor: Prof. Bilal Tanatar
September 2000

Recent observation of Bose-Einstein condensation in dilute alkali gases led to a great interest in this area both experimentally and theoretically. The most important characteristics of a Bose-Einstein condensate is that it consists of a large number of atoms occupying a single quantum state. This kind of a feature seen in photons led to the production of widely-used photon lasers. Coherent state of atoms may lead to the production of atom lasers in near future.

The well-known Bogoliubov model to explain the nature of Bose-Einstein condensates of trapped dilute gases is valid when the interaction between particles is weak. However, as the number of atoms is increased, the interaction effects lead to a significant contribution in the system. Several attempts were made to improve the Bogoliubov model and to explain strongly interacting systems but these treatments are accurate up to a finite strength of the coupling .

One-dimensional Bose systems is important because exact solution of the homogenous problem exists. Also it is a good testing ground to study interaction effects since only two-body interactions play role in these systems. Furthermore,

experimental realization of one-dimensional systems are attracting a great deal of interest into the present problem.

We investigate a somewhat different method to study the properties of strongly coupled Bose condensates in one-dimensional space. It uses the so-called Kohn-Sham theory to solve the problem by considering the exact solution of the homogenous one-dimensional Bose gas. The new approach reveals that interactions are expressed by a ψ^5 term in the strongly coupled regime in contrast to a ψ^3 term in weak coupling regime. The model is applied to several types of trap potentials by performing a numerical minimization. We also improve the model for the case of a finite temperature. We observe that the system has a non-zero critical temperature which suggests a real phase transition in one-dimensional space. In the last part, we work on the stability of a two-component condensate in a harmonic trap potential. We find that for a wide range of system parameters either a coexisting or a phase-segregated mixture can be obtained.

Keywords:

Bose-Einstein condensation, Gross-Pitaevskii equation, mean-field theory, Kohn-Sham equation, Thomas-Fermi approximation, two-gas model, one-dimensional Bose gas, Bose gas, two-component, strong interaction

Özet

KUVVETLİ ETKİLEŞİMLİ TEK BOYUTLU BOSE YOĞUNLUKLARI

Kamil Erkan
Fizik Yüksek Lisans
Tez Yöneticisi: Prof. Bilal Tanatar
Eylül 2000

Bose-Einstein yoğunlaşmasının, yakın geçmişte alkali gazlarda gözlenmesi bu konuya deneysel ve teorik alanda büyük bir ilginin uyanmasına sebep olmuştur. Bose-Einstein yoğunluğunun en önemli özelliği, tek bir kuvantum durumunda olan çok sayıda atomdan oluşmuş olmasıdır. Fotonlarda böyle bir özelliğin görülmesi, günümüzde birçok alanda kullanılan foton lazerlerinin üretilmesini beraberinde getirmiştir. Benzer bir durumun atomlarda gözlenmesi, gelecekte atom lazerlerinin üretilmesini beraberinde getirebilir.

Seyreltik atomlardan oluşan Bose-Einstein yoğunluğunun açıklanmasında sıkça kullanılan Bogoliubov modeli, atomlar arasında zayıf etkileşim olduğunda doğru sonuçlar vermektedir. Fakat, sistemdeki atom sayısı arttığında atomlar arası etkileşimin de önemi artmaktadır. Kuvvetli etkileşimli sistemleri açıklamak amacıyla Bogoliubov modeli daha da geliştirilmiştir. Fakat bu uygulamalar ancak belli bir etkileşim seviyesine kadar iyi sonuçlar vermektedir.

Tek boyutlu Bose sistemleri, homojen problemin tam sonucu olmasından dolayı önemlidir. Bunun yanında bir boyutlu sistemler, atomlar arası etkileşimleri incelemek için uygun bir zemindir. Çünkü sistemde sadece iki-madde etkileşimleri

rol oynamaktadır. Bunun ötesinde, tek boyutlu sistemlerin deneysel olarak oluşturulabilmesi birçok bilim adamının bu konuya ilgi duymasına sebep olmuştur.

Biz, yüksek etkileşimli tek boyutlu Bose sistemlerini incelemek amacıyla biraz farklı bir yöntem kullanıyoruz. Yeni yöntem, homojen ve tek-boyutlu Bose sistemlerinin kesin çözümünü kullanarak, Kohn-Sham teorisini probleme uygulamaktadır. Sistemin kuvantum durumunu ψ ile ifade edersek, yeni model, yüksek etkileşimi ψ^5 in bir fonksiyonu olarak tanımlamaktadır. Zayıf etkileşim durumunda ise bu bağıntı ψ^3 şeklindedir. Yeni modeli, sıfır sıcaklığı sınırında, nümerik minimizasyon yoluyla, çeşitli sıkıştırma potansiyelleri üzerine uyguluyoruz. Bunun yanında, modelimizi sıfırdan yüksek sıcaklıklar için geliştiriyoruz. Nümerik çüzüm yoluyla, sistemin sıfırdan yüksek değerde bir kritik sıcaklığa sahip olduğunu görüyoruz. Bu sonuç, bir boyutlu uzayda gerçek bir faz geçişinin oldugunu göstermektedir. Son olarak harmonik potansiyel altinda, iki bileşenli yoğunlukların kararlılığını inceliyoruz. Sistem parametrelerinin çok geniş bir aralığı için, üst-üste veya ayrık-fazlı karışımlar elde edilebileceğini görüyoruz.

Anahtar sözcükler: Bose-Einstein Yoğunlaşması, Gross-Pitaevskii denklemi, ortalama-alan teorisi, Kohn-Sham denklemi, Thomas-Fermi yaklaşımı, iki-gaz modeli, tek-boyutlu Bose gazı, Bose gazı, iki-bileşenli, kuvvetli etkileşim

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Chapter 1

Introduction

Bose-Einstein condensation (BEC) has been an important area of low temperature physics for many years because of its key-role in the fundamental phenomena such as superfluidity and superconductivity. The recent observation of BEC in dilute alkali gases attracted the interest of many physicists into the present problem.[1]

Evidence for BEC in superfluid helium emerged from the analysis of the momentum distribution of the atoms measured in neutron scattering experiments.[2] In recent years, BEC has been also investigated in the gas of paraexitons in semiconductors.[3] However, an unambiguous signature for BEC was difficult to find in such systems. A milestone in the history of the BEC was the experiment performed by Cornell and collaborators at JILA group of the University of Colorado.[4] Vapors of rubidium were confined in magnetic harmonic traps and cooled down to temperatures of the order of nanokelvin. Such low temperatures were achieved by slowing down each atom using laser light.[5] In the same years, first evidence of BEC in Li were shown by Bradley *et al.*[6] and in Na were shown by Davis *et al.*.[7] Recently, more than 20 groups in the world reported their achievements of BEC in alkali atoms.[8]

The first evidence for the condensation emerged from time-of-flight measurements. The atoms were left to expand by switching off the confining trap and then imaged by optical methods. A sharp peak in the velocity distribution was

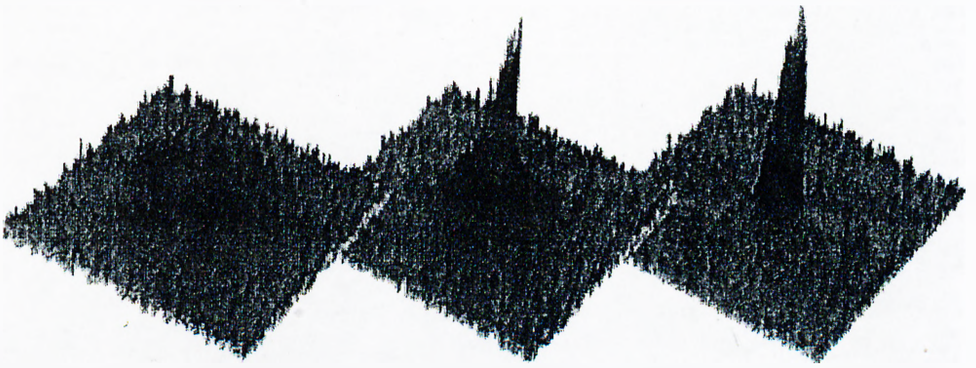

Figure 1.1: Velocity distribution of the particles after release. A large population of particles near the zero velocity region is the evidence of Bose-Einstein condensation.

then observed below a certain temperature, providing a clear signature for BEC (Fig.1.1). More reliable evidence was seen in anisotropic trap potentials. The occurrence of anisotropy in the momentum distributions has been interpreted as an important signature of BEC. Because of this, if the particles, instead of being in the ground state, were thermally distributed among many eigenstates at higher energy, their distribution function would be isotropic in the momentum space according to the equipartition principle. In Fig.1.2, momentum distribution of a condensate in a cylindrically symmetric trap is seen. Difference in the horizontal and vertical direction is a clear evidence of BEC. That is why cylindrically symmetrical type potentials are chosen in most of BEC experiments.

The most striking feature of BEC is that wave-like behavior of matter predicted by quantum mechanics is exhibited on a macroscopic scale due to the condensation of, typically, millions of identical atoms into the same state. This is counter-intuitive to our daily experience of the world, where objects are distinguishable and behave like particles that follows classical trajectories described by Newton's law of motion. In BEC experiments, this classical description of the system is initially valid when the atomic sample is loaded

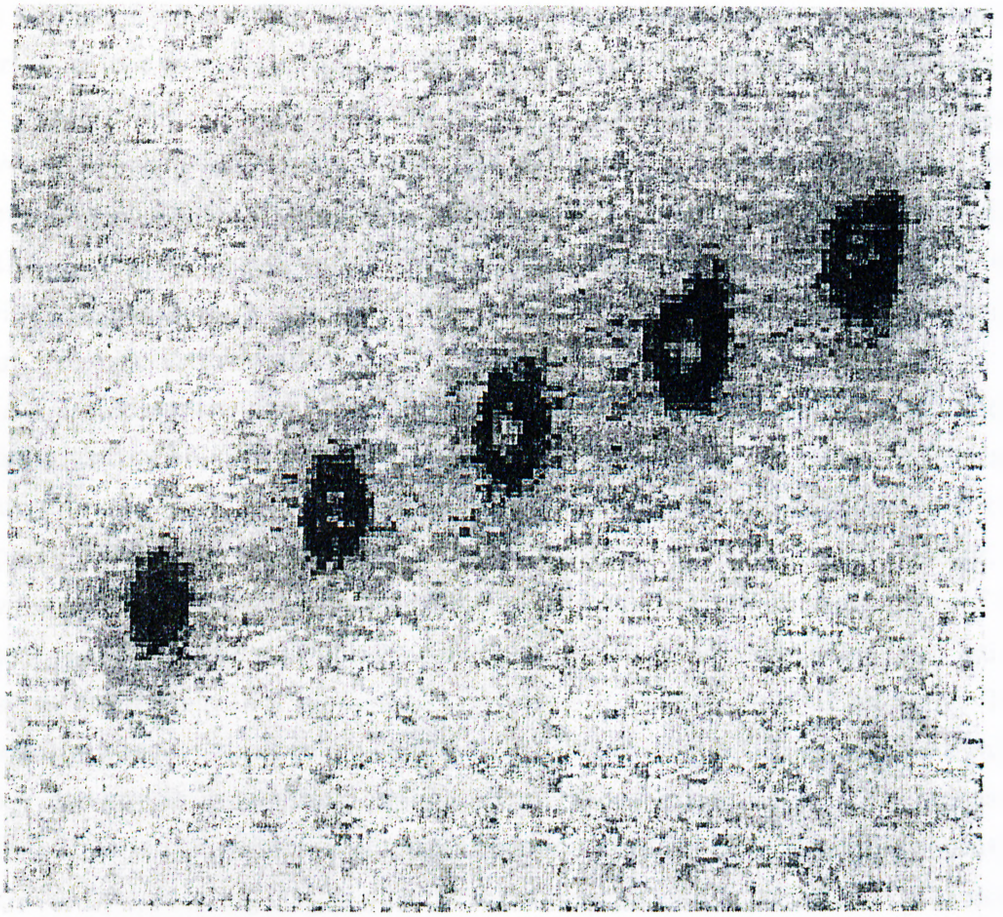

Figure 1.2: Top view of the velocity distribution in a cylindrically symmetric system. Egg-shaped distributions are another manifestation of the condensation.

into a vacuum cell and laser cooling begins, followed by evaporative cooling in a second stage of the experiment. As the temperature of the gas approaches some critical value T_c ($\approx 10^{-7}$K) the wavelike nature of the particles become important to consider. As the gas is cooled down and wave nature of the particles become distinguishable. Each wave-packet has a characteristic wavelength $\lambda = (2\pi^2\hbar^2/mk_BT)^{1/2}$ called the thermal de Broglie wavelength, where m is the mass of the atom and T is the temperature. As the temperature is lowered close to T_c, the de Broglie wavelength becomes comparable to the mean-free-path and wave-packets begin to overlap. At this point it becomes impossible to distinguish

one particle from another. Due to this indistinguishability, the particles become correlated in a particular way, depending on whether they are bosons or fermions. For bosons, this correlation causes all of the particles "condense" into a single wave-packet with the longest wavelength permitted by the size of the trap. Since photons are bosons, they too exhibit this phase transition, which takes the form of the well-known laser, which produces a phase-coherent beam of light. One obvious application of the BEC, then, would be to produce a phase-coherent beam of atoms namely atom laser.[9]

Alkali atoms are well suited to laser-based methods because their optical transitions can be excited by available lasers and because they have a favorable internal energy-level structure for cooling to very low temperatures. Once they trapped, their temperature can be lowered further by evaporative cooling. It is worth noticing that in these conditions, the equilibrium configuration of the system would be the solid phase. Thus, in order to observe BEC, one has to preserve the system in a metastable gas phase for a sufficiently long time. This is possible because three body collisions are rare events in dilute and cold gases, whose lifetime is hence long enough to carry out experiments. The measure of diluteness is the ratio a/r of the characteristic range of interaction potential, expressed by scattering length a to the mean interparticle separation r. For ratios near 1, as in the case of liquid ^{4}He, the simple picture of BEC fails completely. For alkali gases of current interest a/r is about 0.01. However, at $T = 0$, alkali atoms show 99% condensation whereas for liquid helium the fraction is only on the order of 10%.

Despite the very dilute nature of these gases, the combination of BEC and harmonic trapping greatly enhances the effects of the atom-atom interactions on important measurable quantities. For instance, the central density of the interacting gas at very low temperature can be easily one or two orders of magnitude smaller than the density predicted for an ideal gas in the same trap. Despite the inhomogeneity of these systems, which makes the solution of the many body problem nontrivial, the dilute nature of the gas allows one to describe the effects of the interaction in a rather fundamental way. More clearly, interaction

between particles in a dilute atomic gas have a weak effect, so that the wave-like condensate dominate the system and collisions can be treated perturbatively. In this case, the interaction of all particles on a single particle can be summed to give an averaged effect, as a first approximation. This approach is called *mean field theory* and with suitable approximations leads to the *Gross-Pitaevskii equation* which describes the time-evolution of the condensate. The effect of interactions gives rise to a density dependent effective potential that makes the dynamics of the condensate nonlinear. Gross-Pitaevskii theory is reliable to explain various properties of a condensate such as density distribution, collective excitations, finite temperature excitations and stability of two-component condensates. We give a more detailed explanation of the theory in chapter 2 starting with the fundamental expressions of the many-body problem. The discussion in chapter 2 includes effects of non-condensed particles and Gross-Pitaevskii equation is presented as the zero temperature limit of the model. The aim of such study is to give a clear insight of the picture to the reader from a mathematical point of view. As a second motivation, the analytical tools used in this part are also applied to the problem which discussed in chapter 3. In the last part of chapter 2, we give a brief discussion on stability of two-component condensates using Gross-Pitaveskii formalism.

Since the system is dilute one can apply ideal gas approximation by neglecting atom-atom interactions. In this limit, almost all predictions are analytical and relatively simple since solution of the many-body problem is not more difficult than the one-particle case. The ground state wave function of N particles for a homogenous gas is a plane wave having energy $\varepsilon = p^2/2m$. Distribution of particles in the phase space is calculated by taking the Fourier transform of the ground state wave function which yields a Dirac delta solution. Thermal particles obey Bose-Einstein distribution.

However, applying an external potential changes the picture drastically. By using magnetic traps, a confinement potential can be safely approximated with a quadratic form

$$V_{ext} = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) \tag{1.1}$$

where ω_x, ω_y and ω_z are trapping frequencies in corresponding directions. For simplicity we can take $\omega_x = \omega_y = \omega_z = \omega$ which enables us to treat the problem in a spherically symmetric space. Then the many-body hamiltonian is the sum of each single particle hamiltonian whose eigenvalues have the form

$$\varepsilon_n = \left(n + \frac{1}{2}\right)\hbar\omega \tag{1.2}$$

where n is an integer. The ground state is found by taking $n = 0$ which is given by

$$\psi(r) = \left(\frac{m\omega}{\hbar}\right)^{\frac{3}{4}} \exp\left(-\frac{m\omega}{2\hbar}r^2\right). \tag{1.3}$$

Eq.(1.3) describes the distribution of particles in the condensed phase which have a Gaussian profile. The size of the cloud is found by calculating the half-width of the Gaussian which is

$$a_{ho} = \sqrt{\frac{\hbar}{m\omega}}. \tag{1.4}$$

This quantity is important since it gives a characteristic length scale for treating the problem. In available experiments it is typically in the order of $a_{ho} \simeq 1\mu m$. All length scales involved in the condensation phenomenon are some multiples of this quantity.

Phase space distribution is obtained by taking the Fourier transform of Eq.(1.3) which again yields again a Gaussian. The appearance of the condensate as a narrow peak in both coordinate and phase space is the most peculiar feature of trapped Bose gases having important consequences. This is different from the case of uniform gas where the particles condense into a zero momentum state but any condensation in coordinate space can not be revealed.

At temperature T the total number of particles in the grand canonical ensemble

$$N = N_0 + \int_0^\infty \frac{\rho(\varepsilon)}{e^{\frac{\varepsilon-\mu}{k_B T}} - 1} d\varepsilon \tag{1.5}$$

where N_0 is the number of particles in the ground state and k_B is the Boltzman constant. $\rho(\varepsilon)$ can be evaluated by calculating the differential volume in the

spherically symmetric phase space which gives $\rho(\varepsilon) = \frac{1}{2}\hbar\omega^{-3}\varepsilon^2$. At this point we should clarify that a discrete distribution of energy ε can be approximated to a continuous distribution. It is valid as long as number of trapped atoms is large and $k_BT \gg \hbar\omega$. In available experiments with N ranging from a few thousand to several millions[4] the transition temperature is 20-200 times larger than $\hbar\omega$. This means that our semi-classical approximation is reliable as far as experimental conditions are concerned. By proper substitutions the integral in Eq.(1.5) is taken to give

$$N - N_0 = \xi(3)\left(\frac{k_BT}{\hbar\omega}\right)^3 \tag{1.6}$$

where $\xi(n)$ is the Riemann-zeta function. By imposing $N_0 \to 0$ to Eq.(1.5) one can find the critical temperature at which condensation starts as

$$T_c = \hbar\omega\left(\frac{N}{\xi(3)}\right)^{\frac{1}{3}} \tag{1.7}$$

in ideal gas limit. Inserting above equation into Eq.(1.6) we have a simpler form as

$$\frac{N}{N_0} = 1 - \left(\frac{T}{T_0}\right)^3. \tag{1.8}$$

These results can be compared with the well-known theory of uniform Bose gas.[30] In this case the density of states is given by $\rho(\varepsilon) = \frac{V}{2\pi}(2m\hbar^2)^{3/2}\varepsilon^{\frac{1}{2}}$ to give the equation for the critical temperature as

$$\frac{N}{N_0} = 1 - \left(\frac{T}{T_0}\right)^{3/2}. \tag{1.9}$$

The non-interacting harmonic oscillator model has guided experimentalists to proper value of critical temperature. In fact measured transition temperatures were found to be very close to ideal values.

The importance of trapping is seen clearly when the dimensionality is being considered. Density of states of a uniform Bose gas is $\rho(\varepsilon) \propto \varepsilon^{d-1}$. One can easily verify that the integral Eq.(1.5) gives a convergent solution only if $d = 3$ which

predicts that BEC does not occur in one and two dimensional space. However for trapped gases density of states $\rho(\varepsilon) \propto \varepsilon^{\frac{d-2}{2}}$. This reveals that a trapping potential enables realization of BEC in two dimensional system. BEC in one-dimensional space seem to be prohibited even with the addition of trapping potential. However it should be remembered that discrete level structure is approximated by a continuous density of states in Eq.(1.5) under the assumption that the level spacing is negligible compared to the temperature as mentioned before. Secondly above calculations is performed in the thermodynamic limit i.e. $N \to \infty$. However recent BEC experiments on atomic gases were performed with number of particles N ranging from a few thousand to a few million which is quite small compared to the thermodynamic limit. It was shown that[24]in finite systems lowering the dimension increases critical temperature as $T_c \approx N^{1/d}$ and is therefore favorable for BEC. This is in contrast to the standard result obtained in the thermodynamic limit which states that BEC is not possible in one-dimensional systems as explained before.[4] Van Druten and Ketterle[10] investigated properties of one-dimensional ideal bosons in a finite sized system. They found that BEC occurs if the external parameters are properly chosen. They called it the "two-step condensation " because the condensation in the third direction occurs after some critical point where the condensation in other directions formed before that. So, this kind of transition could be expressed as a quasi-one-dimensional transition rather than a real phase-transition. It is finally worth pointing that the above discussion concerns the behavior of the ideal Bose gas. Effects of two-body interactions are expected to modify the nature of the phase transition in a deep way in reduced dimensionality.

In above discussion the possibility of BEC in harmonic trap potentials was considered because of its conventional use and simplicity in mathematics. It is equally experimentally practical to use a power-law type potential like $V(x) \propto |x|^{\eta}$ for $\eta < 2$ besides its complexity in mathematics compared to harmonic type potentials. Bagnato and Kleppner showed that BEC of one-dimensional ideal atoms does occur in a power-law type potentials even in the thermodynamic limit.[11] They showed that the system has a finite T_c between $\eta = 0$ and $\eta = 2$.

Recently the realization of BEC under power-law potentials for interacting one-dimensional bosons in finite sized systems was also shown by using Monte Carlo simulation method[12] and the semiclassical two-gas model.[13]

It was argued that the standard Bogoliubov model (yielding Gross-Pitaevskii equation), which assumes that the interactions between bosons are weak, does not give a satisfactory explanation when the number of atoms in the condensate is large .[14,15] In this thesis, we aim to study on one-dimensional trapped bosons in strongly interacting regime. For this purpose we use a model explored by Tanatar.[16] The one-dimensional system is a good testing ground to investigate effects of interactions in a Bose-Einstein condensate. Firstly, we should point out that the exact solution of one-dimensional interacting bosons exist.[17] Secondly, because of its physical nature, we only deal with two-body collisions. As a last point, we should remark that experimental realization of quasi one-dimensional systems is performed by using highly asymmetrical potentials. Therefore the theory can be tested directly by experiments.[18] Detailed explanation of the new theory and some applications of it are given in chapter 3. The applications include solving the eigenvalue equation describing the current problem in various trap potentials, determining critical temperature and, lastly, stability of the system when there are two different types of atoms forming a two-component structure.[19]

Chapter 2

Gross-Pitaevskii Theory

The basic idea for a mean field description of a dilute Bose gas was formulated by Bogoliubov.[20] Gross[21]and Pitaevskii[22]developed the theory of the condensate of weakly interacting Bose gas afterwards. Although the theories are not realistic to study the physical properties of ^{4}He at low temperature, they are well suited for condensed low density alkali gas.

The problem of N interacting bosons is very complicated in general. The dynamical evolution of the systems is described by an N-body wave function. In the case of a weakly interacting Bose gas, however, strong simplifications occur in the description of the system under the occurrence of the BEC phenomenon. In the formalism of second quantization a system of N interacting bosons in the grand canonical ensemble interacting through a potential U_{int} and confined by an external potential U_{ext}, is described by the following Hamiltonian:

$$\begin{aligned} H &= \frac{\hbar^2}{2m}\int \nabla\hat{\psi}^\dagger\nabla\hat{\psi} + \int (U_{ext}(\mathbf{r}) - \mu)\hat{\psi}^\dagger\hat{\psi} \\ &\quad + \frac{1}{2}\int \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')U_{int}(\mathbf{r},\mathbf{r}')\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}')d\mathbf{r}'d\mathbf{r} \end{aligned} \tag{2.1}$$

where $\hat{\psi}^\dagger$, $\hat{\psi}$ are field operators satisfying Bose commutation relations,

$$[\hat{\psi}_i^\dagger(\mathbf{r}), \hat{\psi}_j(\mathbf{r}')] = \delta_{ij}(\mathbf{r}-\mathbf{r}'), \quad [\hat{\psi}(\mathbf{r}), \hat{\psi}(\mathbf{r}')] = 0, \quad [\hat{\psi}^\dagger(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}')] = 0. \tag{2.2}$$

The dynamics follows from the Heisenberg equation of motion for the time dependent operators $\hat{\psi}^{\dagger}(\mathbf{r},t)$ and $\hat{\psi}(\mathbf{r},t)$

$$i\hbar\frac{\partial\hat{\psi}(\mathbf{r},t)}{\partial t}=\left[-\frac{\hbar^2}{2m}\nabla^2+U_{ext}(\mathbf{r})-\mu+\int\hat{\psi}^{\dagger}(\mathbf{r}',t)U_{int}\hat{\psi}(\mathbf{r}',t)dr'\right]\hat{\psi}(\mathbf{r},t). \quad (2.3)$$

We assume that it is sufficient to describe interactions between atoms by a two-body potential $U_{int}=U(|\mathbf{r}_i-\mathbf{r}_j|)$ although in reality interactions between two particles are modified by the presence of a third particle due to their finite extent, such effect lead to three body combinations, for example. However, in the dilute limit, these effects should be negligible.[20] Because we are only interested in very low temperatures at which BEC occurs, the kinetic energy in a collision is very low, so that we need keep only the s-wave term in a partial wave expansion. Furthermore, a generic form for the interatomic potential can be used that is independent of the details of the two body potential, but depends only on the s-wave scattering length represented by a. A pseudo-potential, given by

$$U_{int}(|\mathbf{r}-\mathbf{r}'|)=g\delta^3(\mathbf{r}-\mathbf{r}') \quad (2.4)$$

can be used, which reproduces the correct s-wave scattering length for a two-body collision.[30] The parameter g is given as

$$g=\frac{4\pi\hbar^2a}{m} \quad (2.5)$$

which is derived from the solution of the homogenous problem. Hence for an interaction denoted by Eq.(2.4) the Heisenberg equation of motion is converted into

$$i\hbar\frac{\partial}{\partial t}\hat{\psi}(\mathbf{r},t)=\left(-\frac{\hbar^2\nabla^2}{2m}+U_{ext}(\mathbf{r})-\mu\right)\hat{\psi}(\mathbf{r},t)+g\hat{\psi}^{\dagger}(\mathbf{r},t)\hat{\psi}(\mathbf{r},t)\hat{\psi}(\mathbf{r},t). \quad (2.6)$$

To simplify the calculations we consider a large box of volume V with periodic boundary conditions. If the creation and annihilation operators $a_0^{\dagger}$ and a_0 for the

zero momentum mode are applied to the ground state they satisfy the following equations:[25]

$$\begin{aligned} a_0|\psi_0(N)\rangle &= N^{1/2}|\psi_0(N-1)\rangle \\ a_0^\dagger|\psi_0(N)\rangle &= (N+1)^{1/2}|\psi_0(N+1)\rangle \end{aligned} \tag{2.7}$$

where $\psi_0(N) = |N, 0, 0, \ldots\rangle$ is the ground state of N bosons.

Operators a_0 and $a_0^\dagger$ for a Bose system multiply the ground state by $N^{1/2}$ and $(N+1)^{1/2}$ which is evidently large. Since it is generally preferable to deal with intensive variables, we shall introduce the operators

$$\xi_0 \equiv V^{-1/2}a_0, \qquad \xi_0^\dagger \equiv V^{-1/2}a_0^\dagger \tag{2.8}$$

with the following properties:

$$[\xi_0, \xi_0^\dagger] = V^{-1} \tag{2.9}$$

$$\xi_0|\psi_0(N)\rangle = \left(\frac{N}{V}\right)^{1/2}|\psi_0(N-1)\rangle \tag{2.10}$$

$$\xi_0^\dagger|\psi_0(N)\rangle = \left(\frac{N+1}{V}\right)^{1/2}|\psi_0(N+1)\rangle. \tag{2.11}$$

Although ξ_0 and $\xi_0^\dagger$ each multiply $|\psi_0\rangle$ by a finite factor, their commutator vanishes in the thermodynamic limit ($N \to \infty$, $V \to \infty$, $N/V \to$ const.). Hence it is permissible to treat the operators ξ_0 and $\xi_0^\dagger$ as c-numbers (Bogoliubov approximation), as long as we consider only states where a finite fraction of the particles occupies the $k = 0$ mode. This approximate procedure clearly neglects fluctuations in the occupation number of the condensate. In an interacting system, the interparticle potential energy reduces the occupation of the preferred mode, so that the ground state expectation value

$$\langle\psi_0|\xi_0^\dagger\xi_0|\psi_0\rangle = N_0V^{-1} = n_0 \tag{2.12}$$

is less than the total density $n = N/V$. Nevertheless, the Bogoliubov replacement at ξ_0 and $\xi_0^\dagger$ by c-numbers correctly describes the interacting ground state in the thermodynamic limit whenever the number of particles in the zero momentum state remains a finite function of N. We are therefore led to write the boson field operator as

$$\hat{\psi}(\mathbf{r}) = \xi_0 + \sum_k{}' V^{-1/2} \mathrm{e}^{ikx} a_k = \xi_0 + \tilde{\psi}(\mathbf{r}) = n_0^{1/2} + \tilde{\psi}(\mathbf{r}) \tag{2.13}$$

where the prime means to omit the term $k = 0$. The operator $\hat{\psi}(\mathbf{r})$ has no zero-momentum components, and ξ_0 is a constant c-number. Therefore, $\tilde{\psi}(\mathbf{r})$ represents the non-condensed states.

Conservation of momentum implies that

$$\langle \psi_0 | \tilde{\psi} | \psi_0 \rangle = 0. \tag{2.14}$$

Then the quantity ξ_0 could be interpreted as the ground state expectation value of the field operator

$$\langle \psi_0 | \hat{\psi}(\mathbf{r}) | \psi_0 \rangle = \xi_0. \tag{2.15}$$

After this point, it is convenient to replace ξ_0 by a suitable function of $\mathbf{r}$ representing the ground state. Then we have

$$\langle \hat{\psi}(\mathbf{r}, t) \rangle = \xi_0 \equiv \Phi(\mathbf{r}) \tag{2.16}$$

where a short-hand notation is used for the expectation value of $\hat{\psi}$. Consequently time dependent Bose field operator takes the form

$$\hat{\psi}(\mathbf{r}, t) = \Phi(\mathbf{r}) + \tilde{\psi}(\mathbf{r}, t). \tag{2.17}$$

Using the explicit form of the Bose field operator given in Eq.(2.17), the interaction term in Eq.(2.6) can be rewritten in the following form:

$$\begin{aligned}\hat{\psi}^\dagger(\mathbf{r},t)\hat{\psi}(\mathbf{r},t)\hat{\psi}(\mathbf{r},t) &= (\Phi^*(\mathbf{r})+\tilde{\psi}^\dagger(\mathbf{r},t))(\Phi(\mathbf{r})+\tilde{\psi}(\mathbf{r},t))(\Phi(\mathbf{r})+\tilde{\psi}(\mathbf{r},t)) \\ &= |\Phi(\mathbf{r})|^2\Phi(\mathbf{r}) + 2|\Phi(\mathbf{r})|^2\tilde{\psi}(\mathbf{r},t) + \Phi^2(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r},t) \\ &\quad +\Phi^*(\mathbf{r})\tilde{\psi}(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t) + 2\Phi(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t) \\ &\quad +\tilde{\psi}^\dagger(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t). \qquad (2.18)\end{aligned}$$

We treat the last term in Eq(2.18) in the self-consistent mean-field approximation, namely

$$\tilde{\psi}^\dagger(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t) \simeq 2\langle\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle\tilde{\psi}(\mathbf{r},t) + \langle\tilde{\psi}(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle\tilde{\psi}^\dagger(\mathbf{r},t) \qquad (2.19)$$

and then Eq.(2.18) reduces to

$$\begin{aligned}\hat{\psi}^\dagger(\mathbf{r},t)\hat{\psi}(\mathbf{r},t)\hat{\psi}(\mathbf{r},t) &= |\Phi(\mathbf{r})|^2\Phi(\mathbf{r}) \\ &+ 2\left[|\Phi(\mathbf{r})|^2 + \langle\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle\right]\tilde{\psi}(\mathbf{r},t) + \left[\Phi^2(\mathbf{r}) + \langle\tilde{\psi}(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle\right]\tilde{\psi}^\dagger(\mathbf{r},t) \\ &+ 2\Phi(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t) + \Phi^*(\mathbf{r})\tilde{\psi}(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t). \qquad (2.20)\end{aligned}$$

The time independent, spatially-inhomogeneous Bose order parameter $\Phi(\mathbf{r})$ is given directly by taking an average over Eq(2.6)

$$\left(-\frac{\hbar^2\nabla^2}{2m} + U_{ext}(\mathbf{r}) - \mu\right)\Phi(\mathbf{r}) + g\langle\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r})\rangle = 0. \qquad (2.21)$$

On the other hand, after taking the anomalous average of Eq.(2.20), the linear terms in $\tilde{\psi}(\mathbf{r},t)$ vanish since $\langle\tilde{\psi}(\mathbf{r},t)\rangle = \langle\tilde{\psi}(\mathbf{r})\rangle = 0$ and we are left with

$$\langle\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r})\rangle = |\Phi(\mathbf{r})|^2\Phi(\mathbf{r}) + 2\Phi(\mathbf{r})\langle\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle + \Phi^*(\mathbf{r})\langle\tilde{\psi}(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle. \qquad (2.22)$$

Using this in Eq(2.21), we get

$$\left(-\frac{\nabla^2}{2m} + U_{ext}(\mathbf{r}) - \mu\right)\Phi(\mathbf{r}) + g[\Phi^2(\mathbf{r}) + 2\langle\tilde{\psi}^\dagger\tilde{\psi}\rangle]\Phi(\mathbf{r}) + g\langle\tilde{\psi}\tilde{\psi}\rangle\Phi^*(\mathbf{r}) = 0. \qquad (2.23)$$

When we introduce the local densities

$$n_0(\mathbf{r}) \equiv |\Phi(\mathbf{r})|^2, \quad n_1(\mathbf{r}) \equiv \langle\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle, \quad n_2(\mathbf{r}) \equiv \langle\tilde{\psi}(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle, \qquad (2.24)$$

Eq.(2.23) could be rewritten as

$$\left(-\frac{\nabla^2}{2m} + U_{ext}(\mathbf{r}) - \mu\right)\Phi(\mathbf{r}) + g[n_0(\mathbf{r}) + 2n_1(\mathbf{r})]\Phi(\mathbf{r}) + gn_2(\mathbf{r})\Phi^*(\mathbf{r}) = 0. \quad (2.25)$$

Here the terms n_1 and n_2 comes from the finite temperature excitations whereas n_0 represents the density of the condensed atoms. Eq.(2.24) involving the excitations in a cold Bose gas has not a simple solution although it is evaluated in the dilute limit.

At temperatures $T \approx 0$, most of the atoms are in the condensate ($N_0 \approx N$). If we set both $n_1(\mathbf{r})$ and $n_2(\mathbf{r})$ to zero Eq.(2.25) reduces to the well known Gross-Pitaevskii equation:[21,22]

$$\left(-\frac{\hbar^2\nabla^2}{2m} + U_{ext}(\mathbf{r}) - \mu + g|\Phi(\mathbf{r})|^2\right)\Phi(\mathbf{r}) = 0 \quad (2.26)$$

which means that $\Phi(\mathbf{r}) = n_0(\mathbf{r})$ is described by a closed non-linear Schrödinger equation.

Another way of approximating Eq.(2.25) would be to keep $n_1(\mathbf{r})$ finite but neglect the anomalous density $n_2(\mathbf{r})$ which is a small quantity compared to $n_0(\mathbf{r})$ and $n_1(\mathbf{r})$.[49] In this case we get

$$\left(-\frac{\hbar^2\nabla^2}{2m} + U_{ext}(\mathbf{r}) - \mu + g[|\Phi(\mathbf{r})|^2 + 2n_1(\mathbf{r})]\right)\Phi(\mathbf{r}) = 0. \quad (2.27)$$

This approximation was used by Popov[23] in a homogenous gas to discuss the finite temperature region closed to BEC transition. This kind of approximation seems to give a reasonable first approximation for the excitation spectrum in the dilute Bose gas at *all* temperatures in addition to simplicity in the solution. Because of these reasons, we follow the Popov approximation to investigate finite temperature properties of one-dimensional bosons.

2.1 Thomas-Fermi Limit

At this point, it could be useful to consider the limiting case of many atoms in the condensate ($N \to \infty$) which is known as Thomas-Fermi limit. In this

case the kinetic energy term in Eq.(2.26) becomes negligible. Upon setting the kinetic energy term to zero and solving for Φ, the ground state solution in the Thomas-Fermi limit becomes

$$\Phi_{TF}(\mathbf{r}) = \sqrt{\frac{\mu_{TF} - U_{ext}}{g}}\ \theta(\mu_{TF} - U_{ext}) \tag{2.28}$$

It could be useful to consider the specific case of the condensate in an isotropic harmonic potential $U_{ext}(\mathbf{r}) = \frac{1}{2}m\omega_0^2 r^2$ where ω_0 represents the trap frequency. The Thomas-Fermi radius is determined by solving the expression $U_{ext}(\mathbf{r}) = \mu_{TF}$, which gives

$$r_{TF} = \sqrt{\frac{2\mu_{TF}}{m\omega_0^2}}. \tag{2.29}$$

The chemical potential is determined for the normalization condition

$$4\pi \int_0^{r_{TF}} r^2 dr |\Phi_{TF}|^2 = N. \tag{2.30}$$

After evaluating the integral, we find that the chemical potential in the Thomas-Fermi limit is

$$\mu_{TF} = \frac{\hbar\omega}{2}\left(\frac{15N_0 a}{a_\perp}\right)^{\frac{2}{5}} \tag{2.31}$$

where $a_\perp = \sqrt{\frac{\hbar}{m\omega}}$. Thomas-Fermi approximation has a very wide use since it gives an analytical solution of the problem. Also, it allows us to check the consistency of results found by numerical calculations.

More generally the trapping potential is not isotropic but has the form

$$U_{ext} = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2). \tag{2.32}$$

The system then could be modeled as a system of one-dimensional bosons in each direction.[24] Another possibility is to have $\omega_x = \omega_y = \omega_\perp$ and $\omega_z = \omega_\parallel$ which produce a cylindrically symmetric cigar shaped structure. The symmetry axis is then the z-axis. if we define

$$\lambda = \frac{\omega_\parallel}{\omega_\perp}, \tag{2.33}$$

which is generally called as *asymmetry parameter*, the external potential can be written as

$$U_{ext} = \frac{1}{2}m\omega_{\perp}^2(x^2 + y^2 + \lambda^2 z^2). \tag{2.34}$$

After performing the substitutions $\lambda z \rightarrow z'$ and $r' \rightarrow x^2 + y^2 + z'^2$ we obtain again an isotropic form

$$U_{ext} = \frac{1}{2}m\omega_{\perp}^2 r'^2 \tag{2.35}$$

with the addition of the asymmetry parameter λ.

2.2 Two-Component Structures

By combining different kind of alkali atoms (Rb-Na, Li-Rb, Na-Li etc.) or different hyperfine states of the same atom ($|1,1\rangle$ and $|2,2\rangle$ spin states of Rb) into the same space one can have coexisting or overlapping condensates.[26–28] Two species are distinguished by their mass or scattering lengths. Total energy of the system is given by

$$\begin{aligned} K &= E(\psi_1, \psi_2) - \mu_1 N_1 - \mu_2 N_2 && (2.36) \\ &= \sum_{i=1,2} \int \left[\frac{\hbar^2}{2m_i} |\nabla \psi_i|^2 + U_i |\psi_i|^2 \right] d\mathbf{r} \\ &+ \sum_{i,j=1,2} \int \left[g_{ii} |\psi_i|^2 + 2 g_{ij} |\psi_j|^2 \right] d\mathbf{r} - \mu_1 N_1 - \mu_2 N_2 && (2.37) \\ &= \int F(\psi_1, \psi_2, \mathbf{r}) d\mathbf{r} && (2.38) \end{aligned}$$

where $\psi_i (i = 1, 2)$ represent wave functions of the first and second components and $U_i (i = 1, 2)$ are the external potentials of each component which, in principle, could differ from each other. The parameters g_{ii} $(i = 1, 2)$ represent mean field interactions in same component whereas g_{ij} $(i, j = 1, 2)$ represent the mean field interaction between different components. The Lagrange multipliers, the chemical potentials μ_i $(i = 1, 2)$, are constrained by the normalization condition $\int d\mathbf{r} |\psi_i(\mathbf{r})|^2 = N_i$ $(i = 1, 2)$ with N_i being the number of particles of the ith

condensate. To understand the ground state properties of the system one can apply a functional minimization ($\delta K = 0$) to Eq.(2.38) which means that the $F(\psi_1, \psi_2, x)$ satisfy the Euler-Lagrange equation[43]

$$\frac{d}{dt}\left[\frac{\partial F}{\partial \dot{\psi}_i}\right] - \frac{\partial F}{\partial \psi_i} = 0 \tag{2.39}$$

for each component. This leads to the coupled Gross-Pitaevskii Equations for ψ_1 and ψ_2 as

$$\left[-\frac{\hbar^2\nabla^2}{2m_1} + U_1(\mathbf{r}) + g_{11}|\psi_1|^2 + g_{12}|\psi_2|^2\right]\psi_1 = \mu_1\psi_1 \tag{2.40}$$

and

$$\left[-\frac{\hbar^2\nabla^2}{2m_2} + U_2(\mathbf{r}) + g_{22}|\psi_2|^2 + g_{21}|\psi_1|^2\right]\psi_2 = \mu_2\psi_2. \tag{2.41}$$

It can be inferred from above equations that existence of a second component is seen as an additional effective potential for the first component. To have a stable minimum point in the two dimensional $F(\psi_1, \psi_2, \mathbf{r})$ space one must have further $(\partial^2 F/\partial\psi_1^2)(\partial^2 F/\partial\psi_2^2) - (\partial^2 F/\partial\psi_1\partial\psi_2)^2 > 0$. In the absence of an external potential and neglecting the kinetic energy terms in Eq.(2.40) and Eq.(2.41) we conclude that

$$g_{12} > \sqrt{g_{11}g_{22}}. \tag{2.42}$$

Stability of a mixture of two condensates under an external potential can be analyzed by considering a square well potential with periodic boundary conditions. This kind of approach gives us a lot of physical insight about the problem.

With this kind of simplified potential coupled nonlinear Gross-Pitaevskii equation have an obvious homogeneous solution: Condensate densities can be defined as $\rho_i = |\psi_i|^2 = \frac{N_i}{V}$ where V is the volume of the square well potential trap. By neglecting the kinetic energy terms the chemical potentials are extracted from Eq.(2.40) and Eq(2.41) to give $\mu_1 = g_{11}\rho_1 + g_{12}\rho_2$ and $\mu_2 = g_{22}\rho_2 + g_{12}\rho_1$. The corresponding total energy of the system is

$$E_{ho} = \frac{1}{2}\left[g_{11}\frac{N_1^2}{V} + g_{22}\frac{N_2^2}{V} + 2g_{12}\frac{N_1N_2}{V}\right]. \tag{2.43}$$

Eq.(2.43) reveals that for a small g_{12} total energy is increased with increasing interaction parameters g_{11} and g_{22}, implying that the excitations are stable. Therefore in this parameter regime the homogeneous state is the ground state and condensates sit on top of each other.

Let us consider the case of the inhomogeneous state in which the two condensates mutually exclude each other. For the moment we ignore the thickness of the interface and the corresponding extra energy. In this way we again temporarily ignore the derivative terms in Eq.(2.40) and Eq.(2.41) in determining the effective condensate wave functions.[29] Let V_i be the volume inside the trap occupied by the condensate i. We have $|\psi_i|^2 = \rho_{i0} = N_i/V_i$ and the total energy of the inhomogeneous state $E_{in} = \frac{1}{2}\sum_{i=1,2} g_{ii}\frac{N_i^2}{V_i}$. Minimizing E_{in} with respect to V_1 or V_2 under the constraint $V_1 + V_2 = V$, we obtain the spatial volume occupied by each condensate:

$$V_1 = \frac{1}{1+\sqrt{\frac{g_{22}}{g_{11}}\frac{N_2}{N_1}}}V, \qquad V_2 = \frac{1}{1+\sqrt{\frac{g_{11}}{g_{22}}\frac{N_1}{N_2}}}V \tag{2.44}$$

The corresponding condensate densities are

$$\rho_{10} = \left(1+\sqrt{\frac{g_{22}}{g_{11}}\frac{N_2}{N_1}}\right)\frac{N_1}{V}, \qquad \rho_{20} = \sqrt{\frac{g_{11}}{g_{22}}}\,\rho_{10}\ , \tag{2.45}$$

and the chemical potentials $\mu_i = g_{ii}\,\rho_{i0}$. The total energy for this inhomogeneous state is

$$E_{in} = \frac{1}{2}\left[g_{11}\frac{N_1^2}{V} + g_{22}\frac{N_2^2}{V} + 2\sqrt{g_{11}g_{22}}\frac{N_1N_2}{V}\right]. \tag{2.46}$$

The energy difference from the homogeneous state is then

$$\Delta E = E_{in} - E_{ho} = -\left(g_{12} - \sqrt{g_{11}g_{22}}\right)\frac{N_1N_2}{V}\ . \tag{2.47}$$

This equation reveals that for a large enough mutual repulsive interaction, that is, if

$$g_{12} > \sqrt{g_{11}g_{22}}\ , \tag{2.48}$$

then the inhomogeneous state has a lower total energy. Hence, for sufficiently large values of g_{12} phase separated condensates are favorable in a trapped system.

Chapter 3

Strong Coupling in One-dimension

Although absence of any condensation of ideal[30]and interacting[31] uniform bosons in the thermodynamic limit in the one-dimensional space, finite size effects and an external trapping potential may modify the picture.[32–34] It was shown by making an exact diagonalization of the many-body hamiltonian of $N \approx 40$ particles, a real phase transition occurs in trapped finite sized systems.[35] Also it was shown that Gross-Pitaevskii theory gives accurate results in these systems in the low density limit.

Exact solution of interacting homogenous one-dimensional bosons exists.[17] This allows one to compare the accuracy of other approximate theories. It was shown by numerical analysis of the exact solution, Gross-Pitaeskii Theory (Bogoliubov model) gives correct results only when the density is small (weak interaction limit).[17,36] Bogoliubov model is seen to be valid up to $\gamma \simeq 2$ where $\gamma = \frac{g}{\rho}$. Here g is the coupling strength as defined before and ρ is the density of atoms. It was shown that the first order perturbation theory (Bogoliubov model) have a large discrepancies from the second order perturbation theory when the interaction is strong.[16] Recently, Bogoliubov model was improved with the addition of local-field corrections by using[37] the method so called STLS approach[38] and using[39] so called VS scheme.[40] However, these models give reliable

results up to $\gamma \leq 10$. In fact, both STLS and VS approaches are based on the perturbation theory so they are not expected to yield reliable results in the strong coupling regime.

A theory to investigate the properties of strongly interacting ($\gamma \to \infty$) bosons could be derived[16]by using exact solution of homogenous bosons within the spirit of Kohn-Sham theory.[41] In the following section we briefly mention about the original Kohn-Sham theory. Afterwards, the equations governing strongly interacting bosons are derived using Kohn-Sham approach.

3.1 Kohn-Sham Theory

The functional representing the total energy of an interacting inhomogeneous Bose gas in one-dimension could be written as

$$E = N\int dx \left[\frac{\hbar^2}{2m}\psi\nabla^2\psi + V_{ext}|\psi|^2\right] + E_{int}(\rho) \tag{3.1}$$

where E_{int} is the term coming from interaction between bosons and $\rho(x) = N|\psi(x)|^2$ represents the density of atoms at point x. For an appropriate value of $\psi(x)$ the total energy is minimum. For an arbitrary system one can give no simple solution for $E_{int}(\rho)$. However if $\rho(x)$ is sufficiently slowly varying, because of diluteness, one can show[42] that

$$E_{int}(\rho) = \int \rho\epsilon(\rho)dx \tag{3.2}$$

where $\epsilon(\rho)$ represents the interparticle energy per atom of a uniform Bose gas with density $\rho(x)$. We can regard $\epsilon(\rho)$ as known from theories of the homogenous Bose gas.[17,36]

After the approximation of the interaction term, Eq.(3.1) takes the form,

$$E = \int dx \left[\frac{\hbar^2}{2m}\sqrt{\rho(x)}\nabla^2\sqrt{\rho(x)} + V_{ext}(x)\rho(x) + \epsilon(\rho(x))\rho(x)\right]. \tag{3.3}$$

The equation representing ground state is obtained by minimizing the energy functional Eq.(3.3) with respect to ρ. If we perform a functional minimization,

$\frac{\delta E}{\delta \psi^*}$, to Eq.(3.3) to satisfy the Euler-Lagrange equation,[43]we are left with the following eigenvalue equation:

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_{ext} + \frac{d(\epsilon(\rho)\rho)}{d\rho}\right]\psi = \mu\psi \tag{3.4}$$

where the chemical potential μ comes into the equation as a Lagrange multiplier in the Euler-Lagrange equation. Eq.(3.4) is in the form of *Kohn – Sham equation* which were firstly applied to interacting electron gas.[41] However, it can be conveniently used for interacting bosons. By using the Kohn-Sham approach one can also develop an alternative theory to Gross-Pitaevskii theory for an inhomogeneous Bose gas. Gross-Pitaevskii theory gives closer results to Kohn-Sham theory as long as higher order terms in the perturbation expansion is concerned.[44] This means that Kohn-Sham theory gives more appropriate results than the well-known Gross-Pitaevskii theory in dilute systems.

3.2 Strongly Coupled Bosons

It was shown that BEC is not realized in interacting homogenous one and two dimensional systems in the thermodynamic limit.[31] The problem of homogenous interacting one-dimensional bosons with a repulsive delta-function potential (Bogoliubov model) were calculated exactly.[17] Exact solution by Lieb and Liniger includes collection of three coupled equations so numerical analysis is inescapable for such exact solution. However when the interaction between particles are large, certain approximation can be used to find an analytical expression. As a result, the ground state energy of homogenous system of interactions bosons in the large g limit is

$$E(\rho)/N = \epsilon(\rho) = \frac{\hbar^2}{2m}\frac{\pi^2}{3}\left(\frac{\gamma}{\gamma+2}\right)\rho^2. \tag{3.5}$$

If the interaction is very strong i.e. $\gamma \to \infty$, Eq.(3.5) becomes

$$E(\rho)/N = \epsilon(\rho) = \frac{\hbar^2}{2m}\frac{\pi^2}{3}\rho^2. \tag{3.6}$$

Then this exact result can be used to evaluate the system described by Eq.(3.4). For strongly interacting bosons Eq.(3.4) takes the form

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_{ext}(x) + \frac{\hbar^2}{2m}\pi^2\rho^2(x)\right]\psi(x) = \mu\psi(x) \tag{3.7}$$

or, replacing ρ^2 by ψ^4, we obtain

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_{ext}(x) + \frac{\hbar^2}{2m}\pi^2\psi^4\right]\psi(x) = \mu\psi(x). \tag{3.8}$$

It is numerically and visually better to make all parameters dimensionless. The best and most conventional way is to use harmonic oscillator units which are deduced from the half-width of the Gaussian coming from the solutions of non-interacting bosons. It is calculated to be $a_{ho} = \sqrt{\frac{\hbar}{m\omega}}$. Length scale x can be converted into dimensionless form by $x \rightarrow a_{ho}x$. In this case radius of the condensate becomes some multiples of the harmonic oscillator length. Furthermore, we can convert energy terms E and the wave function ψ into dimensionless form by $E \longrightarrow \hbar\omega E$ and $\psi \longrightarrow \sqrt{\frac{N}{a_{HO}}}\psi$ respectively. Here N represents the total number of particles in the condensate. After these manipulations we are left with

$$\left[-\frac{1}{2}\frac{d^2}{dx^2} + V_{ext} + \frac{\pi^2}{2}N^2\psi^4\right]\psi = \mu\psi. \tag{3.9}$$

The interesting point about the last expression is that in very strong coupling regime non-linear Schrödinger equation involves the fifth power of ψ. Recently, Kolomeisky *et al.*[45] obtained the same ψ^5 dependence using renormalization group analysis and the concept of equivalence between bosons and spinless fermions in strong coupling regime. As in the normal case it is tempting to apply Thomas-Fermi approximation which means neglecting the kinetic energy term in Eq.(3.9). Condensate density is then found to be

$$\rho(x) = \psi^2(x) = \frac{1}{N}\sqrt{\frac{2}{\pi^2}(\mu - V_{ext}(x))}. \tag{3.10}$$

The chemical potential is calculated from the normalization of the wave function

$$\int |\psi_{TF}(\mu, x)|^2 dx = 1. \tag{3.11}$$

The external potential could be taken very generally as

$$V_{ext} = \frac{1}{2}x^{\eta} \tag{3.12}$$

in dimensionless form. Here η controls the strength of the confinement. For the special case of a harmonic trap potential, $\eta = 2$.

We solved both Eq.(3.9) and Eq.(3.10) using numerical methods. The numerical solution to find the exact results (Eq.(3.9)) were performed by *steepest − descent method.*[46] It consists of projecting onto the minimum of the total energy functional an initial trial state by propagating it in imaginary time. At this point it could be useful to say something about this method. The method has been improved for more complicated problems but for our case the explanation given below is sufficient.

A time-dependent wave function $\psi(x,t)$, where t is fictitious time variable, is evaluated at different time steps, starting from an arbitrary trial function and converging to the exact solution $\psi(x,\infty) \equiv \psi(x)$. The time evolution can be formulated in terms of the equation

$$\frac{\partial}{\partial t}\psi(x,t) = -\frac{\tilde{\delta}E/N}{\tilde{\delta}\psi(x,t)} \tag{3.13}$$

where $\tilde{\delta}$ indicates the constrained functional derivative that preserve the normalization condition along the time evaluation. This equation defines a trajectory in the wave function space in which at each step one moves a little bit down the gradient $-\frac{\tilde{\delta}E/N}{\tilde{\delta}\psi(x,t)}$. The constrained functional derivative is obtained by adding the normalization condition to the functional derivative

$$-\frac{\tilde{\delta}E/N}{\tilde{\delta}\psi(x,t)} = H\psi(x,t) \tag{3.14}$$

where H represents the hamiltonian. The end product is the minimization of energy, which corresponds to $(\partial\psi/\partial t) = 0$. As far as a numerical solution is concerned, one chooses an arbitrary time step Δt and iterates the equation

$$\psi(x,t+\Delta t) \simeq \psi(x,t) - \Delta t H\psi(x,t) \tag{3.15}$$

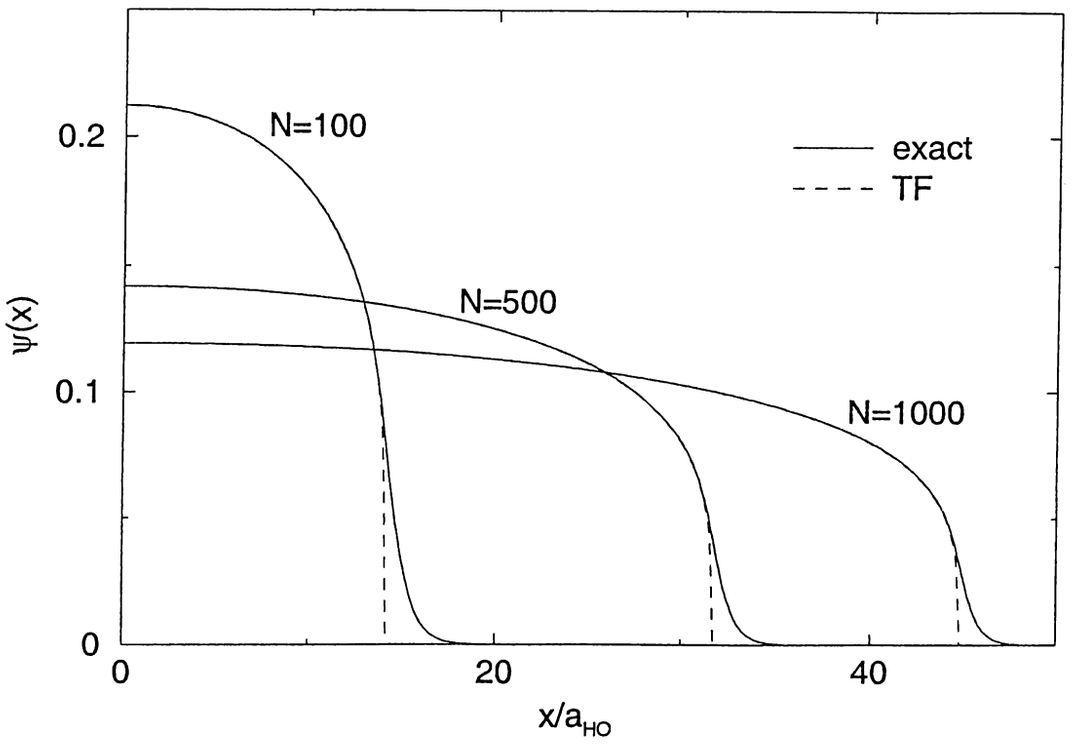

Figure 3.1: Ground State wave function of strongly coupled one-dimensional bosons in harmonic potential. Dashed lines represent the results of Thomas-Fermi approximation.

with normalizing ψ at each iteration. The time step controls the rate of convergence. The number of iterations in imaginary time depends on the degree of the convergence which is controlled by Δt. Since the internal energy is a local functional each iteration is very fast. For 1000 grid points of ψ and for $\Delta t = 10^{-4}$, functional minimization takes no more than 3-6 min. of CPU on a Pentium-Pro 200 processor.

Comparison of TF approximation and exact solution for $\eta = 2$ is shown in Fig.(3.1). We see that as the number of particles increases, TF approximation gives more accurate results compared to the exact solution. Only in the boundary of the condensate kinetic energy term makes a significant contribution. Nevertheless, Thomas-Fermi approximation is still a reasonable approximation in our case. The radius of the condensate can be calculated[48] by considering the

equivalence of the interaction term $\frac{\pi^2}{2}(\frac{N}{R})^2$ and potential energy term $\frac{1}{2}R^2$ near the boundary. From this point of view, it is found to be $R = (\pi N)^{\frac{1}{2}}$ which is in agreement with the numerical solution shown in Fig.(3.1).

Above results clearly indicate condensation of the one-dimensional bosons in the very strong coupling regime. However, above consideration is valid in the limit of zero temperature. It would be more realistic to include the effect of temperature. For this purpose, a finite temperature formalism is developed in the following section.

3.3 Finite Temperature Formalism

The model which was developed in chapter 2 can also be applied in the present case. The starting point is again the Heisenberg equation of motion presented before but with a different hamiltonian:

$$\begin{aligned} i\frac{\partial}{\partial t}\hat{\psi}(x,t) &= \left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}x^\eta - \mu\right)\hat{\psi}(x,t) \\ &+ \frac{\pi^2}{2}\hat{\psi}^\dagger(x,t)\hat{\psi}(x,t)\hat{\psi}^\dagger(x,t)\hat{\psi}(x,t)\hat{\psi}(x,t). \end{aligned} \tag{3.16}$$

Let

$$\hat{\psi}(x,t) = \langle\hat{\psi}(x,t)\rangle + \tilde{\psi}(x,t) = \Phi(x) + \tilde{\psi}(x,t) \tag{3.17}$$

where $\tilde{\psi}(x,t)$ represents the finite temperature excitations as explained before. The interaction term in Eq.(3.16) can be treated using explicit representation of $\hat{\psi}$ as

$$\hat{\psi}^\dagger\hat{\psi}\hat{\psi}^\dagger\hat{\psi}\hat{\psi} = (\Phi^* + \tilde{\psi}^\dagger)(\Phi + \tilde{\psi})(\Phi^* + \tilde{\psi}^\dagger)(\Phi + \tilde{\psi})(\Phi + \tilde{\psi}) \tag{3.18}$$

which leads to 32 terms after the expansion. By taking the anomalous average of Eq.(3.16), terms linear in $\tilde{\psi}$ and $\tilde{\psi}^\dagger$ vanishes because $\langle\tilde{\psi}\rangle = \langle\tilde{\psi}^\dagger\rangle = 0$. Also the cubic and fifth order terms in $\tilde{\psi}$ can be converted into a linear form within the self-consistent mean field approximation[49] and they also give zero contribution. We then remain with

$$\begin{aligned}\langle\hat{\psi}^\dagger\hat{\psi}\hat{\psi}^\dagger\hat{\psi}\hat{\psi}\rangle &= \Phi|\Phi|^4 + 5\Phi|\Phi|^2\langle\tilde{\psi}^\dagger\tilde{\psi}\rangle + \Phi|\Phi|^2\langle\tilde{\psi}\tilde{\psi}^\dagger\rangle \\ &+ 2\Phi\langle\tilde{\psi}^\dagger\tilde{\psi}^\dagger\tilde{\psi}\tilde{\psi}\rangle + 2\Phi\langle\tilde{\psi}^\dagger\tilde{\psi}\tilde{\psi}^\dagger\tilde{\psi}\rangle. \qquad (3.19)\end{aligned}$$

Last term involves only off-diagonal terms which represent anomalous fluctuations and has a small contribution as explained before (Popov approximation). For the forth term one can write the following equality by the self-consistent mean field approximation as

$$\begin{aligned}\langle\tilde{\psi}^\dagger\tilde{\psi}^\dagger\tilde{\psi}\tilde{\psi}\rangle &\simeq 2\langle\tilde{\psi}^\dagger\tilde{\psi}\rangle\langle\tilde{\psi}^\dagger\tilde{\psi}\rangle + \langle\langle\tilde{\psi}^\dagger\tilde{\psi}\rangle + \langle\tilde{\psi}^\dagger\tilde{\psi}\rangle\langle\tilde{\psi}^\dagger\tilde{\psi}\rangle + \langle\tilde{\psi}^\dagger\tilde{\psi}^\dagger\rangle\rangle \\ &\simeq 2\langle\tilde{\psi}^\dagger\tilde{\psi}\rangle\langle\tilde{\psi}^\dagger\tilde{\psi}\rangle. \qquad (3.20)\end{aligned}$$

Consequently Eq.(3.19) becomes

$$\langle\hat{\psi}^\dagger\hat{\psi}\hat{\psi}^\dagger\hat{\psi}\hat{\psi}\rangle = \Phi|\Phi|^4 + 6\Phi|\Phi|^2\langle\tilde{\psi}^\dagger\tilde{\psi}\rangle + 2\langle\tilde{\psi}^\dagger\tilde{\psi}\rangle\langle\tilde{\psi}^\dagger\tilde{\psi}\rangle. \qquad (3.21)$$

A more rigorous discussion about the above treatment can be found in Ref.(25) and Ref.(49).

On the other hand time-independent equation of motion is obtained by taking the average of Eq.(3.16) which is

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^\eta - \mu\right)\Phi(x) + \frac{\pi^2}{2}\langle\hat{\psi}^\dagger\hat{\psi}\hat{\psi}^\dagger\hat{\psi}\hat{\psi}\rangle = 0. \qquad (3.22)$$

Putting Eq.(3.21) into above equation we conclude that

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^\eta + \frac{\pi^2}{2}|\Phi|^4 + 3\pi^2|\Phi|^2 n_1(x) + \pi^2 n_1^2(x)\right)\Phi = \mu\Phi. \qquad (3.23)$$

Eq.(3.23) includes second power of the term representing thermal particles whereas the corresponding equation (Eq.(2.26)) in the weak coupling case includes only the first power of it. Thermal particles can be treated as ideal bosons subject to an effective potential[50]

$$V_{eff} = \frac{1}{2}x^\eta + \pi^2|\Phi|^4 + 3\pi^2|\Phi|^2 n_1(x) + \pi^2 n_1^2(x). \qquad (3.24)$$

The term involving the effects of condensed particles only is multiplied by 2 since all condensed particles are distinguishable from non-condensed particles.[50] For a well defined chemical potential μ, the thermal average of non-condensed particles can be written as an integration of Bose-distribution function over all momentum space as

$$\begin{aligned} n_1(x) &= \int dp \frac{1}{\exp(\frac{p^2}{2m} + V_{eff} - \mu)/T - 1} \\ &= (T)^{\frac{1}{2}} \sum_{j \geq 1} \frac{1}{j^{\frac{1}{2}}} \exp[-j(V_{eff} - \mu)/T] \end{aligned} \tag{3.25}$$

where we used dimensionless parameters as defined before. Since the total number of particles is conserved one can write

$$N = \int \Phi^2(x)dx + \int n_1(x)dx. \tag{3.26}$$

When kinetic energy term is neglected in Eq.(3.23), $\Phi(x)$ can be extracted using the formula for the roots of a quadratic polynomial and one remains with the following analytical expression:

$$\Phi^2(x) = \frac{-3\pi n_1(x) + \sqrt{9\pi^2 n_1^2(x) - 2(\pi^2 n_1^2(x) + x^\eta/2 - \mu)}}{\pi} \tag{3.27}$$

for $x < x_0$ and $\Phi(x) = 0$ for $x > x_0$. The turning point x_0 is obtained from $\Phi^2(x_0) = 0$ which gives

$$\mu = \pi^2 n_1^2(x_0) + \frac{x_0^\eta}{2}. \tag{3.28}$$

A self-consistent solution can be proposed for Eq.(3.23)-Eq.(3.28). Since equations are coupled, an iterative solution is necessary. We can find a critical temperature for a given particle number (N) and trapping strength (η) by increasing temperature (T) until there are no particles in the condensed state ($N_0 = 0$). The numerical results giving the distribution of condensed and non-condensed particles are given in Fig.(3.2) for $\eta = 1$ and 2. Temperature scale is in dimensionless form such that $T \to \frac{k_B}{\hbar\omega} T$. We see from the figure that non-condensed particles are concentrated at the boundary of the condensate to

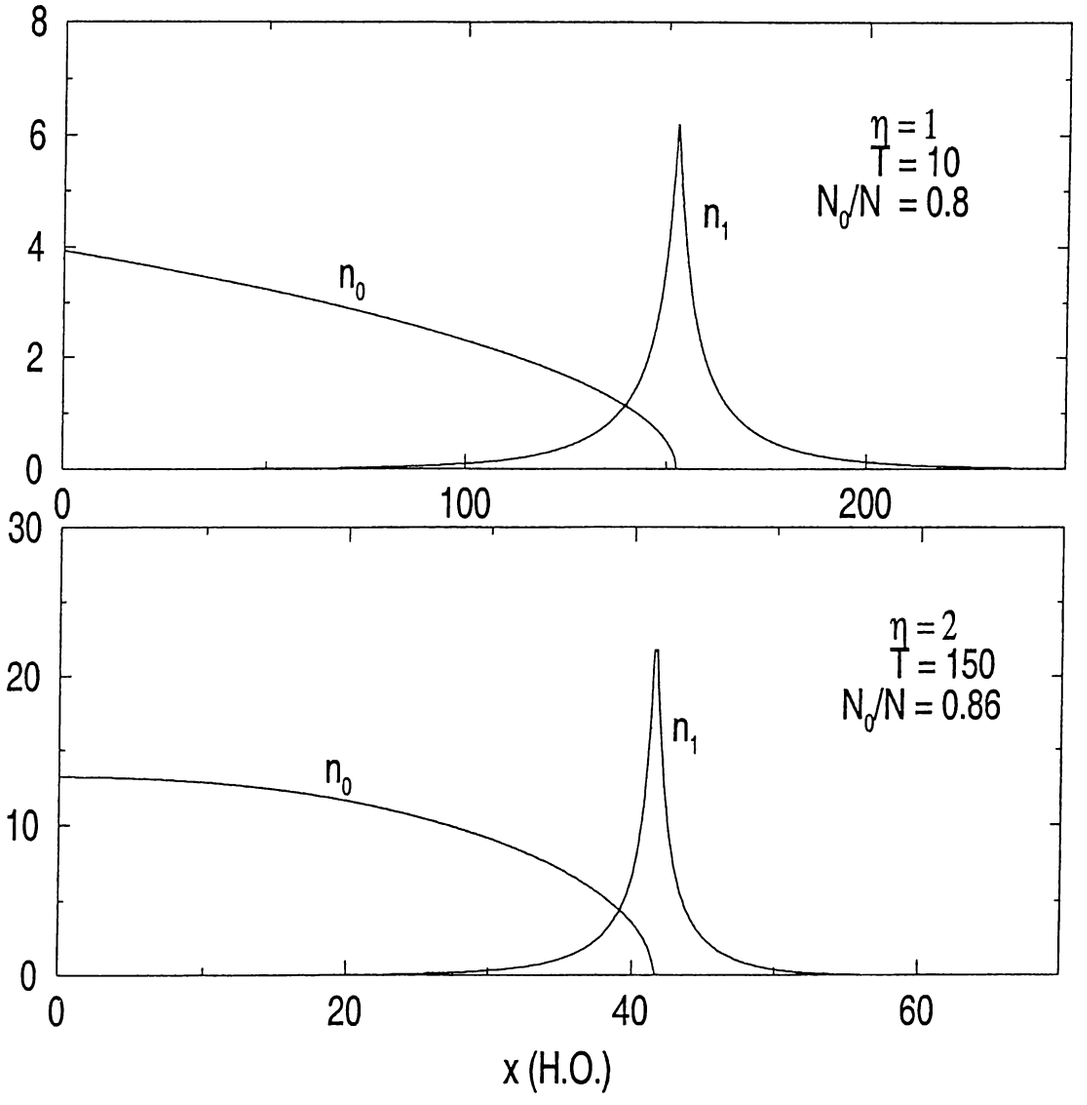

Figure 3.2: Distribution of condensed and non-condensed atoms for $\eta = 1$ and $\eta = 2$.

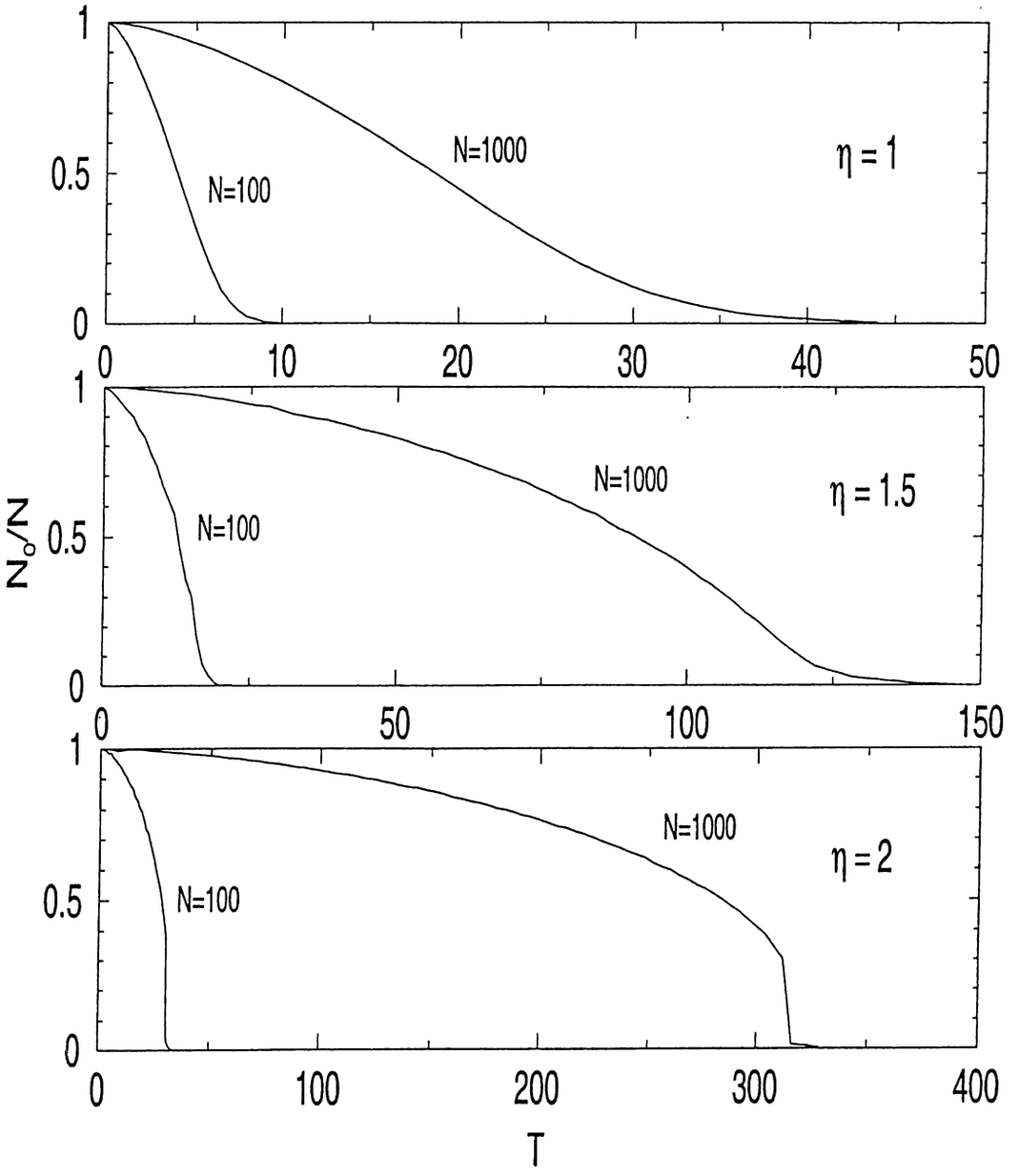

Figure 3.3: Ratio of number of condensed atoms as a function of temperature for $\eta = 1, 1.5$ and 2.

minimize the total energy. The picture also shows the advantage of evaporative cooling which based on isolating thermal particles from the condensate by reducing the external potential. By this way, firstly the thermal particles escape from the trap. Sharp edges in distribution functions are because of neglecting kinetic energies of the condensates which are effective near the boundaries. The most important outcome of this calculations is the indication of a true phase transition at $T > 0$. We also calculate the critical temperature for different values of N and η. In various panels of Fig.(3.3) ratio of condensed atoms are shown as a function of temperature. The critical temperature shows increasing behavior when η is increased. Also, we see from Fig.(3.3) that T_c increases considerably with increasing number of total particles.

3.4 Two-component Condensates

In this section, we consider the behavior of two-component condensates trapped by a harmonic potential. To use the same type of atoms with different hyperfine states is not a proper way for our case since the only distinguishing factor, coupling constant (g), disappears in the very strong coupling limit. Instead, we can use different type of atoms with different masses. Total energy in Kohn-Sham form is given by

$$\begin{aligned} E(\psi_1,\psi_2) &= \int dx[\frac{\hbar^2}{2m_1}\left|\frac{d\psi_1}{dx}\right|^2+\frac{\hbar^2}{2m_2}\left|\frac{d\psi_2}{dx}\right|^2+\frac{1}{2}m_1\omega^2x^2|\psi_1|^2+\frac{1}{2}m_2\omega^2x^2|\psi_2|^2 \\ &+ \epsilon(\rho_1)\rho_1+\epsilon(\rho_2)\rho_2+\epsilon(\rho_1)\rho_2+\epsilon(\rho_2)\rho_1], \end{aligned} \tag{3.29}$$

where $\psi_1=\sqrt{\rho_1}$ and $\psi_2=\sqrt{\rho_2}$ represent the wave function of the first and second condensate having masses m_1 and m_2 respectively. The normalization condition is such that $\int dx\psi_i^2(x)=N_i(i=1,2)$. In a more general case trap frequencies can be considered as different.

The homogenous solution of $\epsilon(\rho_1)$ and $\epsilon(\rho_2)$ for very strongly coupled bosons were discussed previously. Inserting the results into Eq.(3.29) we remain with

$$E(\psi_1,\psi_2) = \int dx[\frac{\hbar^2}{2m_1}\left|\frac{d\psi_1}{dx}\right|^2+\frac{\hbar^2}{2m_2}\left|\frac{d\psi_2}{dx}\right|^2+\frac{1}{2}m_1\omega^2x^2|\psi_1|^2+\frac{1}{2}m_2\omega^2x^2|\psi_2|^2$$

$$+ \quad \frac{\hbar^2\pi^2}{6m_1}\psi_1^6 + \frac{\hbar^2\pi^2}{6m_2}\psi_2^6 + \frac{\hbar^2\pi^2}{6m_1}\psi_1^4\psi_2^2 + \frac{\hbar^2\pi^2}{6m_2}\psi_1^2\psi_2^4]. \tag{3.30}$$

When we can again rescale all length terms as $x \longrightarrow a_{ho}x$ ($a_{ho} = \sqrt{\frac{\hbar}{m_1\omega}}$), energy terms as $E \longrightarrow \hbar\omega E$ and $\psi_i \longrightarrow \sqrt{\frac{N_i}{a_{ho}}}\psi_i$ (i=1,2). As a result Eq.(3.30) becomes

$$\begin{aligned} E(\psi_1,\psi_2) &= \int dx[\frac{1}{2}N_1\left|\frac{d\psi_1}{dx}\right|^2 + \frac{1}{2}\delta N_2\left|\frac{d\psi_2}{dx}\right|^2 + \frac{1}{2}N_1x^2|\psi_1|^2 + \frac{1}{2}\frac{N_2}{\delta}x^2|\psi_2|^2 \\ &+ \frac{\pi^2}{6}N_1^3\psi_1^6 + \frac{\pi^2}{6}\delta N_2^3\psi_2^6 + \frac{\pi^2}{6}N_1^2N_2\psi_1^4\psi_2^2 + \frac{\pi^2}{6}N_1N_2^2\psi_1^2\psi_2^4] \end{aligned} \tag{3.31}$$

where $\delta = \frac{m_1}{m_2}$ and wave functions, ψ_1 and ψ_2, are normalized to unity in this case. By applying a functional minimization to $E(\psi_1,\psi_2)$, one obtains two coupled nonlinear equations for the each condensate:

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 + \frac{\pi^2}{2}N_1^2\psi_1^4 + \frac{\pi^2}{3}N_1N_2\psi_1^2\psi_2^2 + \delta\frac{\pi^2}{6}N_2^2\psi_2^4\right)\psi_1 = \mu_1\psi_1 \tag{3.32}$$

$$\left(-\frac{1}{2}\delta\frac{d^2}{dx^2} + \frac{1}{2\delta}x^2 + \frac{\pi^2}{2}\delta N_2^2\psi_2^4 + \frac{\pi^2}{3}\delta N_1N_2\psi_1^2\psi_2^2 + \frac{\pi^2}{6}N_1^2\psi_1^4\right)\psi_2 = \mu_2\psi_2 \tag{3.33}$$

where μ_1 and μ_2 are the chemical potential of each condensate coming from the solution of the time dependent equation of motion. There is a clear asymmetry between Eq.(3.32) and Eq.(3.33) because of the mass difference of the atoms. Spatial distribution of the condensates can be calculated by solving both Eq.(3.32) and Eq.(3.33) in a self-consistent way. Iterations are carried out until minimum of Eq.(3.31) is achieved. Different configurations can be obtained by changing either the number of atoms in the system (N_1, N_2) or the mass ratio (δ).

In Fig.(3.4) and Fig(3.5) some examples for coexisting and phase segregated condensates are shown respectively. In the absence of an external potential the condensates are totally separated by repulsive interaction and a stable coexisting condensate mixture never be achieved. However, under an external trap, two condensates can form a two-component structure. In this case the actual stability criterion must be a density dependent expression since the reason of the stability

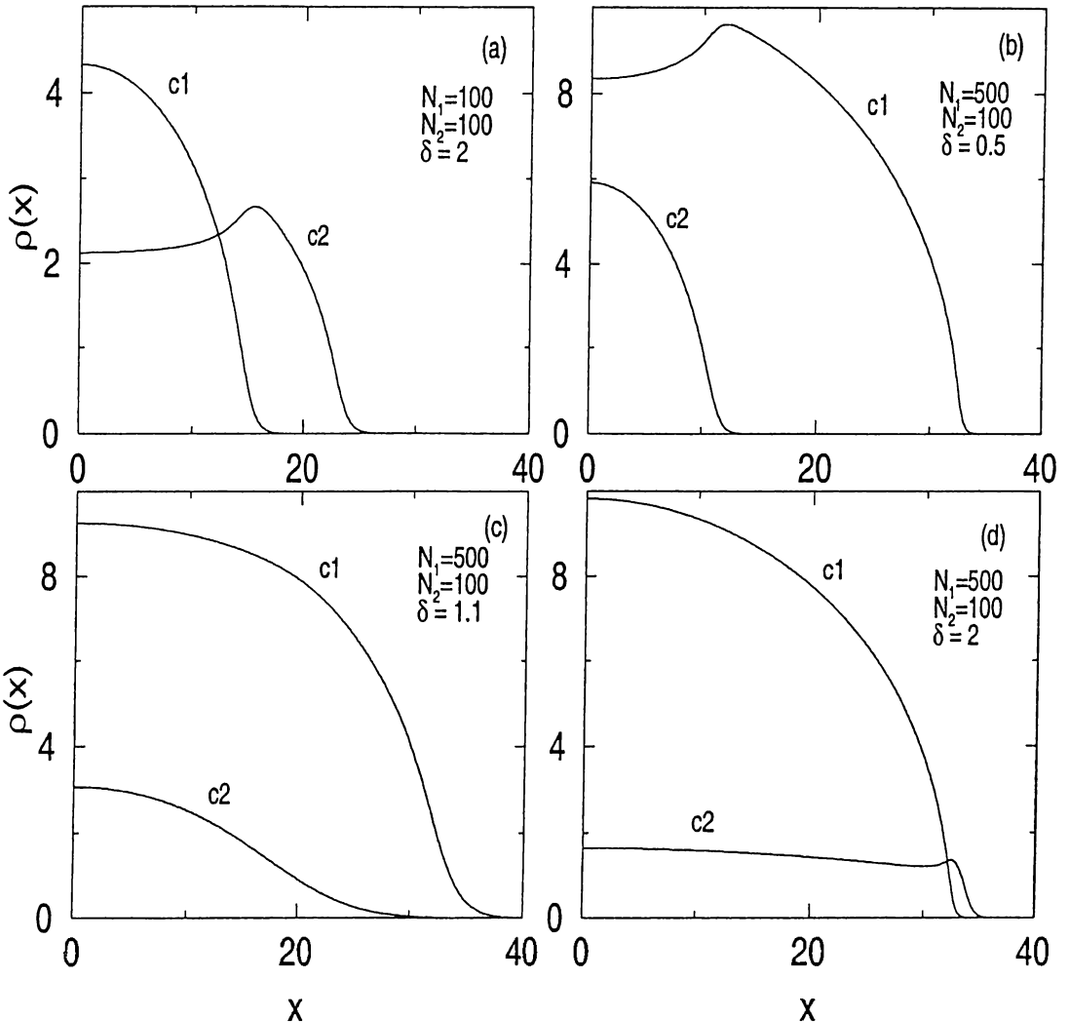

Figure 3.4: Two-component mixture showing a coexisting distribution.

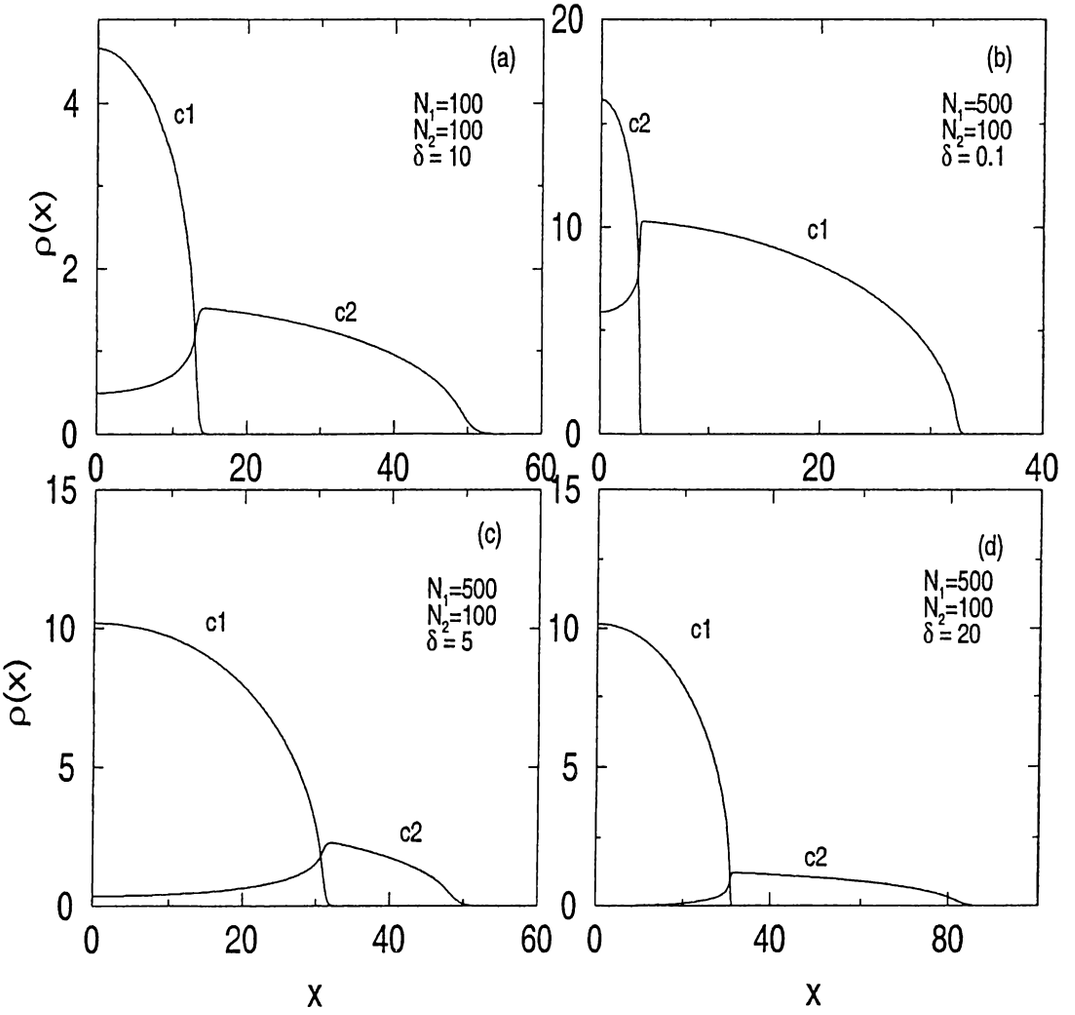

Figure 3.5: Two-component mixture showing a phase segregated distribution.

is the interaction between species. To examine the stability of the mixture let us look at the system when $N_1 >> N_2$. When the kinetic energy term is neglected in Eq.(3.32) we obtain :

$$\psi_1^4(x) = \frac{2}{\pi^2 N_1^2}\left(\mu_1 - \frac{x^2}{2}\right). \tag{3.34}$$

This form of ψ_1 can be inserted into Eq.(3.33) to obtain

$$\left[-\frac{1}{2}\delta\frac{d^2}{dx^2} + \frac{\mu_1}{3} + \frac{x^2}{2}\left(\frac{1}{\delta} - \frac{1}{3}\right)\right]\psi_2 = \mu_2\psi_2. \tag{3.35}$$

$\delta = 3$ is a critical point for the second condensate since the second term in the effective potential $\frac{\mu_1}{3} + \frac{x^2}{2}(\frac{1}{\delta} - \frac{1}{3})$ changes sign. For $\delta < 3$ minimum of the effective potential is obtained when $x_0 = 0$ (x_0 represents location of the peak of the second condensate). In this case both condensates are centered at the same point so that we obtain coexisting condensates. On the other hand for $\delta > 3$ the minimum is obtained at $x_0 > 0$. In this case second condensate sits apart from the first one so that overlapping condensates are formed. We calculated numerically the location of the peaks of each condensate for different values of δ. Result for $N_1 = 1000$ and $N_2 = 10$ is shown in Fig.(3.6). It is clearly seen that $\delta = 3$ is the critical point and confirms our theoretical predictions. Another conclusion revealed from the figure is that even for large values of δ we may obtain a stable two-component mixture. We solved the problem for $\delta = 30$ and observed an overlapping two-component mixture. This shows that for a large range of system parameters a stable mixture can be constructed in the strong coupling regime.

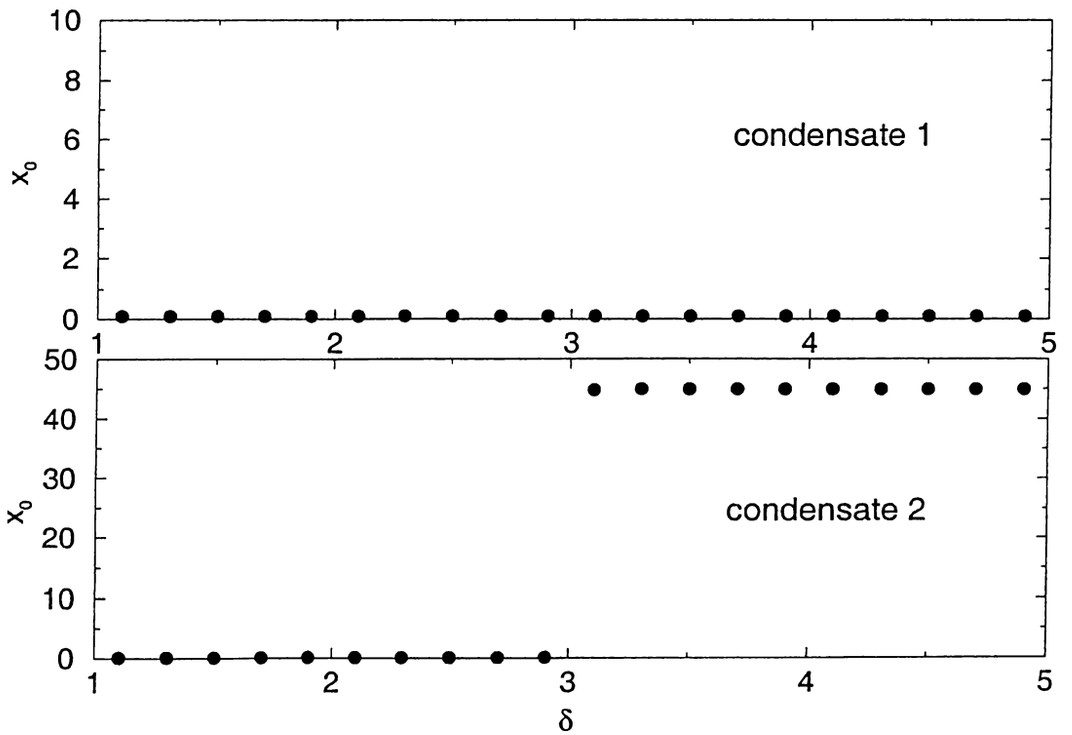

Figure 3.6: Location of the peak values of each component for different mass ratios.

Chapter 4

Conclusions

We studied properties of strongly coupled one-dimensional Bose-Einstein condensates in several type of traps. Since well-known Gross-Pitaevskii theory is not satisfactory to explain the properties of strongly coupled bosons we used a different approach which was firstly derived by Tanatar.[16] The main result of this approach is that interaction energy is a function of ψ^5, where ψ^2 represents the density of the condensate, in strong coupling regime in contrast to ψ^3 dependence in weak coupling regime. We assumed a harmonic trapping potential to plot the distribution of condensate in the zero temperature limit. We saw that even for small number of particles the density is reduced in the center of the cloud compared to the weak interaction case. By neglecting kinetic energy terms we compared the analytical result found from Thomas-Fermi approximation and solution of the exact solution. We observed that except the boundaries two solutions gave very close results. As a next step, it could be interesting to look at vortex state solutions and the effects of quantum fluctuations in zero temperature limit. One other possible direction could be to look at the response of the system against a periodic potential.

An important outcome of this thesis is to explore a finite temperature formalism to the current problem to see whether there is a non-zero critical temperature . In order to do this, we applied finite temperature self-consistent mean field theory for bosons. The resulting equations were solved numerically

and revealed that there is a finite critical temperature for strongly interacting one-dimensional bosons. It proves that a Bose-Einstein condensation like behavior occurs in trapped systems even in the strong interaction limit. By changing various parameters and calculating the critical temperature we observed that T_c is highly dependent on system parameters N and η. We also verified the prediction $T_c \propto N^{1/d}$ for harmonic trapping potentials[24] (figure 3.3). Finite temperature properties can also be studied using Monte-Carlo simulation method which is a more reliable way to investigate the properties of many-body systems.

In the last part, we looked at the stability of two-component strongly coupled condensates in harmonic trap potentials. Because of vanishing the coupling constant in strong-coupling limit, components are distinguished by their masses only. Our calculations revealed that for a wide spectrum of mass ratio(δ), a stable mixture can be obtained. After certain value of the mass ratio ($\delta \approx 3$), we saw that coexisting condensates separate each other and one remains with a phase-segregated mixture. A possible future direction of this analysis could be to apply the equations to interacting boson-fermion mixtures.

Quantum transport properties of a condensate is an important issue for the future production of atom lasers. It requires the solution of the time dependent equation of motion. It was shown that the interaction strength plays an important role in the transport phenomenon.[51] The present model can be used to predict the quantum motion of strongly interacting condensed bosons.

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