JOSEPHSON EFFECT IN HYBRID SUPERCONDUCTOR NORMAL METAL STRUCTURES

A THESIS

SUBMITTED TO THE DEPARTMENT OF PHYSICS

AND THE INSTITUTE OF ENGINEERING AND SCIENCE

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

BY

Özgür Çakır

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Master of Science.

Prof. Igor Kulik (Supervisor)

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Abstract

JOSEPHSON EFFECT IN HYBRID SUPERCONDUCTOR-NORMAL METAL STRUCTURES

Özgür Çakır
M. S. in Physics
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September 2000

The clean SNS junction, with a phase difference χ across the normal metal barrier and without Fermi level mismatch between the components, is modeled by a step like pair potential. The quasiclassical equations are obtained from the Gorkov equations by the elimination of crystal momenta and are valid in the existence of only the off-diagonal potential(pair potential). Then for the SNS junction, the quasiclassical Green functions are obtained. At zero temperature, in the long barrier limit($d \gg \xi_0$), the current is found to have a sawtooth dependence on the phase difference, the amplitude of which is inversely proportional to the thickness of the normal layer. At finite temperatures in the limits $d \gg \xi_T$ and $T_c \gg T$, the current is found to have exponential dependence on the thickness of the normal layer, $T\exp(-d/\xi_T)\sin\chi$, where $\xi_T = v_F/2\pi T$.

An extension of single SNS structures is the periodic SNS structure which may as well exhibit Josephson effect. It is simulated by a periodic step-wise pair potential, where the phase of the pair potential changes by some constant value in subsequent superconducting islands. Bogoliubov equations in the semiclassical

limit are employed, yielding the density of states(DOS). In the DOS, there appears some forbidden energy regions, which point out to a band structure.

Keywords: Superconductivity, Josephson Effect, Hybrid Structures, Weak Links, Proximity Effect, Quasiclassical Green Functions, Bogoliubov-de Gennes Equations

Özet

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Fizik Yüksek Lisans
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11 Eylül 2000

Üstüniletken elektrotlar arasında χ kadar faz farkı olan ve bileşenleri arasında Fermi seviyesi farkı bulunmayan saf SNS eklemi, basamak şeklinde bir çift potansiyel modeli ile incelenecektir. Ancak çapraz olmayan potansiyellerin varlığında geçerli yarı-klasik denklemler, Gorkov denklemlerinden örgü momentumunun elimine edilmesiyle elde edilmekte ve SNS eklemi için çözülmektedir. Sıfır sıcaklıkta, kalın bariyer limitinde Josephson akımının faz farkına bağlılığının doğrusal olduğu ve büyüklüğünün normal tabaka kalınlığıyla ters orantılı degiştiği gösterildi. Düşük sıcaklıklarda, $d \gg \xi_T$ ve $T_c \gg T$ limitlerinde akımın $T\exp(-d/\xi_T)\sin\chi$ şeklinde, normal tabakanın kalınlığına üstel şekilde bir bağlılık gösterdiği bulunmuştur($\xi_T = v_F/2\pi T$).

Peryodik SNS yapısı basamak şeklinde periyodik çift potansiyel modeli ile ele alınacak ve üstüniletken adacıkların fazları her periyotta aynı miktarda artmaktadır. Bu sistem için Bogoliubov-de Gennes denklemleri yarı-klasik limitte çözülmekte ve durum yoğunlugu elde edilmektedir. Enerji spektrumunda yasak enerji bölgelerinin varlığı bant yapısına işaret etmektedir.

Anahtar sözcükler: Josephson etkisi, zayıf bağlantılar, melez yapılar, Bogoliubov-de Gennes deklemleri, yarı-klasik Green fonksiyonları

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Chapter 1

Introduction

In 1962 Josephson predicted the existence of a super-current between two superconducting electrodes separated by a tunnel barrier(thin insulating layer),[1]

$$I_s = I_c \sin \Delta\varphi \tag{1.1}$$

without any voltage drop across the barrier, which is called the *stationary Josephson effect*. I_c is the maximum value of the supercurrent that the barrier can support, which is, at the same time, a measure of coupling between two electrodes. $\Delta\varphi = \varphi_1 - \varphi_2$ is the phase difference of the order parameters of the electrodes across the interface,

$$\psi_{1,2} = \psi_0 e^{i\varphi_{1,2}} \tag{1.2}$$

where ψ is the Cooper pair amplitude. In the presence of a constant voltage difference V across the barrier, the supercurrent oscillates in time,

$$\frac{d\Delta\varphi}{dt} = \frac{2eV}{\hbar} \tag{1.3}$$

and is called the *nonstatinary Josephson* effect.

Josephson effect is a remarkable manifestation of macroscopic coherence of the superconducting state. Macroscopic coherence is maintained by the existence of a macroscopic order parameter $\psi = \psi_0(\mathbf{r})e^{i\varphi(\mathbf{r})}$ characterizing the Cooper-pairs(paired electrons) inside the whole specimen, $|\psi|^2$ being the local value of the Cooper-pair density. ψ varies at distances of the order of coherence length ξ, which is much larger than atomic length scales. T_c is the critical temperature of the superconductor. The macroscopic coherence, first postulated by London,[3] was phenomenologically formulated by Ginzburg and Landau in 1950 and they have showed that the order parameter ψ obeys a Schrodinger like equation,[2] with a corresponding conserved current,

$$\begin{aligned} \frac{1}{2m}\left(-i\hbar\nabla - \frac{e^*\mathbf{A}}{c}\right)^2\psi + \alpha\psi + \beta|\psi|^2\psi = 0 \\ J = \frac{n_s e^*\hbar}{m^*}(\nabla\varphi - \frac{e^*}{\hbar c}\mathbf{A}) = e^*|\psi^2|\mathbf{v}_s \end{aligned} \tag{1.4}$$

$n_S = |\psi|^2$ is the local value of the density of Cooper pairs, $\mathbf{v_s}$ is the local superfluid velocity and $\mathbf{A}$ is the vector potential. e^*, m^* are the charge and mass of the Cooper pair respectively.

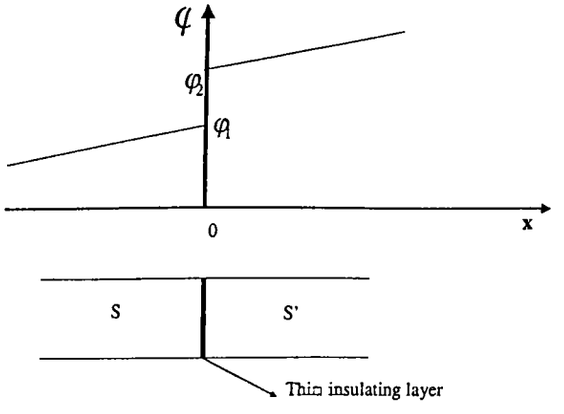

Figure 1.1: Behavior of phase near a tunnel barrier located at x=0

The existence of macroscopic phase coherence manifests itself in a number of macroscopic quantum effects like **(i)**Josephson effect,[1] **(ii)**quantization of magnetic flux in a superconductor,[3] **(iii)** periodic variations of depression in the transition temperature of a hollow superconducting cylinder, with varying magnetic flux(Little-Parks Effect).[4]

Supercurrent through a tunnel junction is accompanied by a jump in the phase at the barrier and is a function of the phase difference $\varphi_1 - \varphi_2$ between the two sides of the interface, which is expected due to gradient term in the current expression(1.5)(Fig-1.1).

The tunnel junctions(SIS) were treated by tunneling Hamiltonian approach, on the grounds that **i**)the thickness of the barrier is much smaller than any other characteristic length(particularly l, ξ_0, mean free path of the electron and coherence length respectively) so that the processes inside the barrier can be ignored **ii**)the transparency of the barrier should be so small that the critical current that the barrier can support is much less than the critical current of the electrodes, so that perturbation with respect to transparency of the barrier can be applied.[5,6,7,8,9]

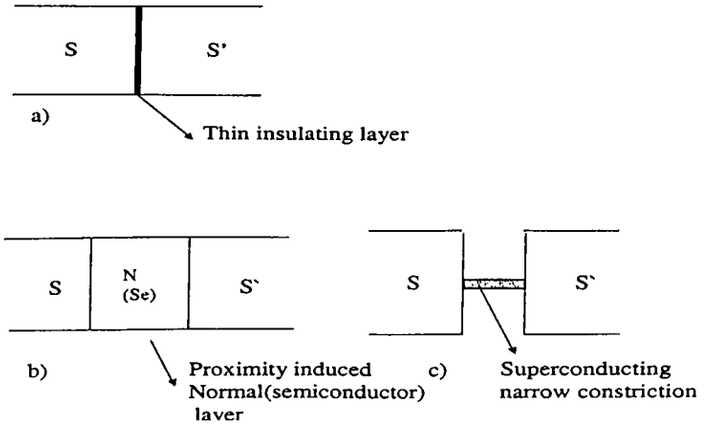

Figure 1.2: (a)SIS (b)SNS (c)ScS type Josephson junctions

However Josephson junctions are not limited by tunnel junctions. Apart from tunnel barrier, Josephson effect can take place in a conducting junction between superconducting electrodes, the critical current through which is much smaller than that of the superconducting electrodes.[32] These junctions are the so called **weak links** which may be achieved using a superconducting connection of small cross-section, or using a substance which is not superconducting, which may be a normal metal or a semiconductor(Fig.1.2). Finite conductivity of the weak link is the main distinguishing property and processes going on inside the junction are taken into account.

Even if the link is normal or semiconducting, it exhibits weak superconducting property due to proximity effect which arises due to leakage of Cooper pairs from the superconducting electrodes into the normal or semiconducting region. The weak links are more advantageous over tunnel junctions, of being low capacitance, of experimental reproducibility, enabling analysis of reduced space dimensionality. Recently there's a growing interest in construction of novel devices using weak links.

Previously, especially during 1970's, although the transport properties of superconducting-normal, semiconductor hybrid structures were extensively studied, because of complications in growing hybrid nano-structures, there hasn't been been any experimental success till 1991.[19,20]

In1958 Bardeen, Cooper and Schrieffer founded the microscopic theory of superconductivity.[12] The ground state of the superconductor is the state in which the electrons with energy in the $|\Delta|$ neighborhood of the Fermi energy, i.e. in the interval $(E-|\Delta|, E+|\Delta|)$ are all condensed into Cooper pairs. Δ is an off-diagonal potential(also called the pair potential) which is proportional to Cooper pair amplitude ψ. At finite temperatures, electron-type and hole-type quasi-particles are excited. The two component wave function $\hat{\Psi}$, with the components ψ_e, ϕ_h, which are electron and hole-type excitations respectively, is governed by Bogoliubov-de Gennes equations(BdG),[10,13] with Δ serving as the off-diagonal potential,

$$(H_e + U)\psi_e(\mathbf{r}) + \Delta(\mathbf{r})\phi_h(\mathbf{r}) = \varepsilon\psi_e(\mathbf{r})$$
$$\Delta^*(\mathbf{r})\psi_e(\mathbf{r}) - (H_e + U)\phi_h(\mathbf{r}) = \varepsilon\phi_h(\mathbf{r}) \tag{1.5}$$

U and Δ are the self consistent potentials,

$$\begin{aligned} U(\mathbf{r}) &= -V\sum_n(|\psi_{en}(\mathbf{r})|^2 f_n + |\phi_{hn}(\mathbf{r})|^2(1-f_n)) \\ \Delta(\mathbf{r}) &= V\sum_n(\phi_{hn}(\mathbf{r})^*\psi_{en}(\mathbf{r})(1-2f_n)) \end{aligned} \tag{1.6}$$

U is the usual Hartree-Fock term, and Δ is the pair potential. Since the BdG equations enable a spatial analysis, it is a powerful tool in handling normal-superconductor interfaces.

In 1965, Andreev discovered a new scattering mechanism which realizes in case of a sharp spatial variation of the off-diagonal potential Δ.[11] In normal region an electron(hole) of energy lower than the gap($|\Delta|$), incident at the superconducting interface reflects back as a hole(electron), accompanied by the creation(annihilation) of a Cooper pair in the superconducting region. Consequently, in a normal metal confined by superconducting specimen, for energies such that $E < |\Delta|$, the energy spectrum becomes discrete as was predicted by Kulik.[15] When the energy of the excitation is larger than the gap parameter, partial Andreev reflection occurs. Andreev reflection is the only scattering mechanism in the mere existence of an off-diagonal potential.

Green's function method is a powerful tool in the study of superconductivity.[30] Single particle temperature Green's functions provide full information about the excitation spectrum and the density of states of quasi-particles. Single particle temperature Green's function is defined as follows

$$G(\mathbf{r}\tau, \mathbf{r}'\tau') = - < T_\tau \Psi_\uparrow(\mathbf{r}\tau)\bar{\Psi}_\uparrow(\mathbf{r}'\tau') > \tag{1.7}$$

where T_τ indicates time ordered product and $< ... >$ is the thermodynamic Gibbs average. $\hat{\Psi}$,$\bar{\hat{\Psi}}$ are the fermion field operators. On the other hand Gorkov Green's function is defined as

$$F^*(\mathbf{r}\tau, \mathbf{r}'\tau') = - < T_\tau \bar{\Psi}_\downarrow(\mathbf{r}\tau)\bar{\Psi}_\uparrow(\mathbf{r}'\tau') > \tag{1.8}$$

One particle Green's function and Gorkov Green's function in the imaginary frequency domain, satisfy the Gorkov equations,

$$\begin{aligned}(i\omega + \frac{\nabla^2}{2m} + \mu)G_\omega(\mathbf{r}, \mathbf{r}') + \Delta(\mathbf{r})F^*_\omega(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \\ (-i\omega + \frac{\nabla^2}{2m} + \mu)\ F^*_\omega(\mathbf{r}, \mathbf{r}') - \Delta^*(\mathbf{r})\ G_\omega(\mathbf{r}, \mathbf{r}') = 0\end{aligned} \tag{1.9}$$

with the self consistency condition,

$$\Delta(\mathbf{r}) \ = \ VF_\omega(\mathbf{r}\tau^+, \mathbf{r}\tau). \tag{1.10}$$

V is the electron-phonon coupling constant. In its present form, Gorkov equations are only applicable to equilibrium situations.

Since the Gorkov's equations are space dependent, they can be employed in studying superconductor-normal hybrid structures.

Zero voltage current through a surface S separating two regions of space, is expressible in terms of one particle Green's functions,

$$\begin{aligned}j \ &= \ 2ieT(\int_{\mathbf{r}\epsilon V'}\int_{\mathbf{r}'\epsilon V} \\ &- \ \int_{\mathbf{r}\epsilon V}\int_{\mathbf{r}'\epsilon V'})d^3\mathbf{r}\ d^3\mathbf{r}'T\sum_\omega \Delta(\mathbf{r})\Delta^*(\mathbf{r}')G^0_\omega(\mathbf{r} - \mathbf{r}')G_{-\omega}(\mathbf{r}', \mathbf{r})\end{aligned} \tag{1.11}$$

which is a quite general, powerful formula derived by Josephson[14] and Kulik[15] independently. $G_\omega(x-x')$ is the full Green's function and $G^0_\omega(x-x')$ is that of the bulk normal metal. V and V' are the volumes on the opposite sides of surface S. This current expression can be employed in order to calculate the zero voltage Josephson current, as long as the Green's functions are known for the particular system.

In the third chapter we will study Josephson effect in SNS structure (Fig.3.1) with vanishing pair potential in N region, and pair potential $\Delta_{1,2} = \Delta_0 e^{i\chi_{1,2}}$ in S, S' banks respectively. The excitation spectrum and density of states, as well as the Green's functions depend on the phase difference between the superconducting banks.

The dc Josephson current in a clean SNS structure was first studied by Kulik[15,16] After determining the quasiparticle wave functions, he constructed Green's functions in terms of the quasi-particle wave functions. Following Kulik, Ishii[23] found the exact Green's functions, first expressing the Green's functions in the form of bulk Green's function plus the bilinear products of bulk Green's functions with the corresponding coefficients. Then he determined the coefficients using the integral equations generated by the Gorkov's equations. After determining the Green's functions both Kulik and Ishii employed the current expression(1.11) in order to determine the Josephson current. On the other hand, Bardeen and Johnson employed the Landau's Galileian transformation principle for excitations in a superfluid in order to obtain the current.[24]

Since the current through SNS, in steady state does not dissipate, the system is in a thermodynamical state. Thus the phase difference of coupled superconductors can be taken as a thermodynamical variable.[27,28,29] Once the phase dependent density of states is known, the thermodynamic potential can be written as

$$\Omega(\chi) = -\frac{1}{\beta}\int_0^\infty dE \ln(1+e^{-\beta E})N(E,\chi). \tag{1.12}$$

Josephson current is nothing but the response of thermodynamic potential to the phase difference χ,

$$J(\chi) = -2e\frac{\partial\Omega}{\partial\chi} \tag{1.13}$$

$$= 2e\frac{1}{\beta}\int_0^\infty dE \ln(1+e^{-\beta E})\frac{\partial N(E,\chi)}{\partial\chi}. \tag{1.14}$$

Due to proximity effect, there is a leakage of Cooper pair condensate from the superconductor to the normal region which makes the normal metal exhibit superconducting properties. In the existence of a phase difference between the superconducting banks, there appears an effective gap in the normal region, which gives an explaination of the superconducting property exhibited by the normal region in SNS.

Since superconductivity is a macroscopic phenomenon, the coherence length($\xi_0 = v_F/\Delta_0$) relevant to superconductivity is much larger than the atomic scales, which gives the idea of filtering out the crystal momentum, being left with the mere information of superconductivity. However elimination of crystal momentum is only possible when the crystal momentum is conserved, i.e., it is required that only the off-diagonal potential Δ exists.

The quasi-classical Green's functions will be found by the elimination of crystal momenta from the Gorkov's equations thus obtaining the equations and the boundary conditions satisfied by them. The differential equations satisfied by quasiclassical are of first order and the boundary conditions are easy to handle. The quasi-classical Green's functions can only be applied when there is only off-diagonal potential Δ, i.e., when there's no diagonal scattering. Then the quasiclassical equations are going to be solved for single SNS structure obtaining the quasi-classical Green's functions for this structure. Then making use of (1.11) the supercurrent will be obtained at $T = 0$ and at finite temperatures.

In the fourth chapter we will be studying periodic SNS structure with cyclic variations of the phase of pair potentials in S islands.(Fig.4.1) In other words, the

phase is increasing in each period by some constant value, where the modulus of the pair potential remains unchanged. For this structure the spectrum and density of states of quasi-particles will be studied. SNS periodic structure with all superconducting islands of the same phase, was first studied by Van Gelder in 1968.[26] The interesting feature about the periodic SNS structure with cyclic phase is the possibility of a Josepson effect. First of all, Bogoliubov-de Gennes equations will be linearized by employing semi-classical approximation,[21] the idea of which is nothing but elimination of crystal momenta from the Bogoliubov-de Gennes equations. Semiclassical equations are valid as long as the crystal momentum is conserved. The semi-classical equations are solved for the SNS periodic structure with cyclic phase thus obtaining the density of states which enables one to write an expression for the Josephson current through this structure.

Chapter 2

Bogoliubov-de Gennes and Gorkov Equations

2.1 Bogoliubov-de Gennes Equations

Electrons with an attractive potential can be described by a method which is a generalization of Hartree-Fock method, in order to describe superconductivity(*Bogoliubov,1959*).[10] The quasi electron and hole excitations can be shown to obey an equation similar to Dirac equation. The Hamiltonian describing the electrons consists of Hamiltonian with an external potential and the two body attractive delta-Dirac potential between electrons. The electronic Hamiltonian without interactions is

$$H_e = \int d^3\mathbf{r} \sum_{\alpha=\uparrow\downarrow} \Psi^\dagger(\mathbf{r}\alpha) \left[\frac{(\mathbf{p} - e\mathbf{A})^2}{2m} + U_0(r) \right] \Psi(\mathbf{r}\alpha) - E_F \tag{2.1}$$

where U_0 is an external potential which is taken to be spin independent(considering nonmagnetic materials), and E_F is the Fermi energy. The field operators satisfy the anticommutation relations

$$\begin{aligned} [\Psi(\mathbf{r}\alpha), \Psi(\mathbf{r}\beta)]_+ &= 0 \\ \left[\Psi^\dagger(\mathbf{r}\alpha), \Psi^\dagger(\mathbf{r}\beta)\right]_+ &= 0 \\ [\Psi(\mathbf{r}\alpha), \Psi\dagger(\mathbf{r}\beta)]_+ &= \delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}'). \end{aligned} \tag{2.2}$$

The interaction Hamiltonian with a two body point interaction $-V\delta(\mathbf{r}-\mathbf{r}')$, can be written in second quantized form as

$$H_1 = -\frac{V}{2}\int d^3\mathbf{r} \sum_{\alpha\beta} \Psi^\dagger(\mathbf{r}\alpha)\Psi^\dagger(\mathbf{r}\beta)\Psi(\mathbf{r}\beta)\Psi(\mathbf{r}\alpha). \tag{2.3}$$

For brevity , removing the summation over spin labels, we can choose the spin indices in the following configuration

$$H_1 = -V\int d^3\mathbf{r}\Psi^\dagger(\mathbf{r}\uparrow)\Psi^\dagger(\mathbf{r}\downarrow)\Psi(\mathbf{r}\downarrow)\Psi(\mathbf{r}\uparrow). \tag{2.4}$$

The rest is to solve the full Hamiltonian

$$H = H_0 + H_1. \tag{2.5}$$

The spin-magnetic field interaction is ignored which is correct for the case $|\Delta| \gg e\hbar H/mc$. However there are quadratic terms which can be decomposed into bilinear forms by making use of mean field approximation. When one is left with the Hamiltonian of bilinear form, there only remains to make a unitary transformation, so as to diagonalize the Hamiltonian. The idea is to approximate the full Hamiltonian to an *effective Hamiltonian* of linear combination of normal ordered bilinear forms,

$$H_{eff} = \int d^3\mathbf{r} \left[\sum_{\alpha} \Psi^{\dagger}(\mathbf{r}\alpha)(H_e + U(\mathbf{r}))\Psi(\mathbf{r}\alpha) \right]$$

$$+\Delta(\mathbf{r})\Psi^{\dagger}(\mathbf{r}\uparrow)\Psi^{\dagger}(\mathbf{r}\downarrow) + \Delta^{*}(\mathbf{r})\Psi(\mathbf{r}\downarrow)\Psi(\mathbf{r}\uparrow) \tag{2.6}$$

with potentials $U(\mathbf{r})$, $\Delta(\mathbf{r})$, which are *self consistently* determined so that this approximation is valid, which will be made clear later. $U(\mathbf{r})$ is the usual Hartree Fock term conserving the number of particles, and $\Delta(\mathbf{r})$ is the pair potential allowing for the condensation of two electrons into a pair and decay of a pair into two electrons with amplitudes $\Delta(\mathbf{r})$, $\Delta^{*}(\mathbf{r})$. In the effective Hamiltonian(2.6), the bilinear form $\Psi(\mathbf{r}\alpha)\Psi(\mathbf{r}\alpha)$ and its Hermitian conjugate are discarded since they identically vanish due to anti-commutation relations(2.2). Assuming that *the media is not magnetic*, the term $\Psi^{\dagger}(\mathbf{r}\uparrow)\Psi(\mathbf{r}\downarrow)$, enabling spin exchange, is also not included. However if needed this term can be preserved within the routine procedure presented here, thus the case for magnetic media can be handled.

By performing Bogoliubov transformations, which are unitary transformations, H_{eff} can be diagonalized,

$$\begin{aligned} \Psi(\mathbf{r}\uparrow) &= \sum_{n} \left(\gamma_{n\uparrow}\psi_n(\mathbf{r}) - \gamma^{\dagger}_{n\downarrow}\phi^{*}_n(\mathbf{r}) \right) \\ \Psi(\mathbf{r}\downarrow) &= \sum_{n} \left(\gamma_{n\downarrow}\psi_n(\mathbf{r}) + \gamma^{\dagger}_{n\uparrow}\phi^{*}_n(\mathbf{r}) \right) . \end{aligned} \tag{2.7}$$

The new annihilation and creation operators γ, $\gamma^{\dagger}$ satisfy anticommutation relations

$$\left[\gamma_{n\alpha}, \gamma_{m\beta}\right]_{+} = 0$$

$$\left[\gamma^{\dagger}_{n\alpha}, \gamma^{\dagger}_{m\beta}\right]_{+} = 0$$

$$\left[\gamma_{n\alpha}, \gamma^{\dagger}_{m\beta}\right]_{+} = \delta_{nm}\delta_{\alpha\beta}. \tag{2.8}$$

This new transformation *preserves the anticommutation relations* which means that this transformation is unitary. This transformation is expected to diagonalize H_{eff}, i.e.,

$$H_{eff} = E_g + \sum_{n,\alpha} \epsilon_n \gamma^{\dagger}_{n\alpha}\gamma_{n\alpha}. \tag{2.9}$$

The requirement for H_{eff} be *diagonal*, following the unitary transformation(2.7) produces the Bogoliubov-de Gennes equations,

$$(H_e + U)\psi(\mathbf{r}) + \Delta\phi(\mathbf{r}) = \varepsilon\psi(\mathbf{r}) \tag{2.10}$$

$$\Delta^*(\mathbf{r})\psi(\mathbf{r}) - (H_e + U)\phi(\mathbf{r}) = \varepsilon\phi(\mathbf{r}) \tag{2.11}$$

which can be rewritten in the form,

$$\mathcal{H}\begin{pmatrix} \psi \\ \phi \end{pmatrix} = \epsilon \begin{pmatrix} \psi \\ \phi \end{pmatrix} \tag{2.12}$$

with $\mathcal{H}$ being hermitian(self adjoint) 2X2 matrix. If $\begin{pmatrix} \dot{\psi} \\ \dot{\phi} \end{pmatrix}$ is a solution to the Bogoliubov-de Gennes equations with eigenvalue ϵ, then $\begin{pmatrix} -\phi^* \\ \psi^* \end{pmatrix}$ is the solution with eigenvalue $-\epsilon$. However, we only keep the solutions of positive energy, since we require a ground state energy.

2.1.1 Self Consistency Condition

In the thermodynamic equilibrium, free energy F_{eff} corresponding to H_{eff} should take its minimum value, as long as the free energy is estimated using the states which diagonalize H_{eff}. In order to assure self consistency, it should be required that free energy corresponding to the full Hamiltonian H calculated using the statistical operator $\rho_{eff} = e^{-\beta H_{eff}}/Tr(e^{-\beta H})$, be stationary i.e. $\delta F = 0$. In other words, the thermal average obtained using the new basis ψ_n, ϕ_n is required to provide a stable free energy for the full Hamiltonian(2.5). Thermal average of an operator $\hat{O}$ is given by,

$$< \hat{O} >= Tr(\rho_{eff}\hat{O}). \tag{2.13}$$

The thermal average of the full Hamiltonian(2.5) is

$$\begin{aligned} < H > &= Tr(\rho_{eff}H) \\ &= \int d^3\mathbf{r} \sum_{\alpha} \left\langle \Psi^{\dagger}(\mathbf{r}\alpha) H_e \Psi(\mathbf{r}\alpha) \right\rangle \\ &- V \int d^3\mathbf{r} \left\langle \Psi^{\dagger}(\mathbf{r}\uparrow)\Psi^{\dagger}(\mathbf{r}\downarrow)\Psi(\mathbf{r}\downarrow)\Psi(\mathbf{r}\uparrow) \right\rangle \end{aligned} \tag{2.14}$$

contains thermal average of quadratic term $< \Psi_1^{\dagger}\Psi_2^{\dagger}\Psi_3\Psi_4 >$ which can be simplified using Wick's theorem,[30,31]

$$\begin{aligned} < \Psi_1^{\dagger}\Psi_2^{\dagger}\Psi_3\Psi_4 > &= < \Psi_1^{\dagger}\Psi_2^{\dagger} >< \Psi_3\Psi_4 > - < \Psi_1^{\dagger}\Psi_3 >< \Psi_2^{\dagger}\Psi_4 > \\ &+ < \Psi_1^{\dagger}\Psi_4 >< \Psi_2^{\dagger}\Psi_3 > \end{aligned} \tag{2.15}$$

However for nonmagnetic media the term $< \Psi^{\dagger}(\mathbf{r} \uparrow \Psi(\mathbf{r} \downarrow) >$ which allows spin exchange identically vanishes and will not be preserved. Free energy corresponding to the full Hamiltonian H is

$$F =< H > -TS \tag{2.16}$$

and the variation in F is given by

$$\delta F = \int d^3\mathbf{r} \sum_{\alpha} \delta[\left\langle \Psi^\dagger(\mathbf{r}\alpha) H_e \Psi(\mathbf{r}\alpha) \right\rangle] - V \sum_{\alpha \neq \beta} \left\langle \Psi^\dagger(\mathbf{r}\alpha)\Psi(\mathbf{r}\alpha) \right\rangle \delta \left[\left\langle \Psi^\dagger(\mathbf{r}\beta)\Psi(\mathbf{r}\beta) \right\rangle \right]$$

$$-V \left(\left\langle \Psi^\dagger(\mathbf{r}\uparrow)\Psi^\dagger(\mathbf{r}\downarrow) \right\rangle \delta \left[\langle \Psi(\mathbf{r}\downarrow)\Psi(\mathbf{r}\uparrow) \rangle \right] + H.C \right). \tag{2.17}$$

We know that H_{eff} is diagonalized by ψ_n, ϕ_n, so free energy estimated using this basis,

$$\begin{aligned} F_{eff} &= < H_{eff} > -TS \\ &= Tr(\rho_{eff} H_{eff}) - TS \end{aligned} \tag{2.18}$$

should take a minimum value in equilibrium, in other words the variation in F_{eff} should vanish,

$$\begin{aligned} \delta F_{eff} &= \delta < H_{eff} > -T\delta S \\ &= \int d^3\mathbf{r} \sum_{\alpha} \delta \left\langle \Psi^\dagger(\mathbf{r}\alpha)(H_e + U(\mathbf{r}))\Psi(\mathbf{r}\alpha) \right\rangle \\ &+ \left(\Delta(\mathbf{r}) \left\langle \Psi^\dagger(\mathbf{r}\uparrow)\Psi^\dagger(\downarrow) \right\rangle + H.C \right) - T\delta S. \end{aligned} \tag{2.20}$$

Comparing (2.17) and (2.19), in order that $\delta F = 0$, the self consistent potentials should be as follows,

$$\begin{aligned} U(\mathbf{r}) &= -V \sum_n \left[|\psi_n(\mathbf{r})|^2 f_n + |\phi_n|^2 (1 - f_n) \right] \\ \Delta(\mathbf{r}) &= V \sum_n \phi_n^*(\mathbf{r}) \psi_n(\mathbf{r}) (1 - 2f_n). \end{aligned} \tag{2.21}$$

The Hartree Fock term $U(\mathbf{r})$ and the pair potential $\Delta(\mathbf{r})$ have fundamentally distinct features. The term $\phi_n^*\psi_n$ has nonvanishing value only in the neighbourhood of Fermi level, so the pair potential strongly depends on temperature. On the other hand, Hartree-Fock term involves summation over all states below the Fermi level, so $U(\mathbf{r})$ has almost no dependence on temperature, so it can be approximated by that of the normal state.

Bogoliubov-de Gennes equations proved to be very useful in studying S-N hybrid structures[15,16,21] BdG equations essentially provide a microscopic description, but can further be extended to macroscopic approach, to account for the Landau-Ginzburg formalism.[13]

2.1.2 Continuity Equation and the Current Expression for Bogoliubov-de Gennes Equations

The Bogoliubov-de Gennes equations satisfy a continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla.\mathbf{j} = 0 \tag{2.22}$$

ρ is the probability density, and $\mathbf{j}$ the probability current density. Time dependent Bogoliubov-de Gennes equations read

$$i\frac{\partial}{\partial t}\begin{pmatrix} \psi \\ \phi \end{pmatrix} = \begin{pmatrix} (H_e + U) & \Delta \\ \Delta^* & -(H_e + U) \end{pmatrix}\begin{pmatrix} \psi \\ \phi \end{pmatrix} . \tag{2.23}$$

The time dependent BdG equations can be put in a more concise form

$$i\partial_t\Psi = \sigma_z(H_e + U)\Psi + \hat{\Delta}\Psi \qquad \Psi = \begin{pmatrix} \psi \\ \phi \end{pmatrix}$$

(2.24)

and the conjugate equations read

$$-i\partial_t \Psi^* = \sigma_z(H_e + U)\Psi^* + \hat{\Delta}^*\Psi^* \tag{2.25}$$

where σ_z is the Pauli spin matrix. $\hat{\Delta}$ is the off diagonal potential. Acting $\bar{\Psi}^*$ from the left hand side on (2.25) and $\bar{\Psi}$ on (2.26), then subtracting the second from the first one, yields

$$i\partial_t |\Psi|^2 = \bar{\Psi}^*\sigma_z(H_e + U)\Psi - \bar{\Psi}\sigma_z(H_e + U)\Psi^* \tag{2.26}$$

$\bar{\Psi} = (\psi, \phi)$, we can make further simplification obtaining the continuity equation with probability and probability current densities as follows

$$\rho = |\psi|^2 + |\phi|^2$$
$$\mathbf{j} = \frac{1}{2mi}(\psi^*\partial_z\psi - \psi\partial_z\psi^*) - \frac{1}{2mi}(\phi^*\partial_z\phi - \phi\partial_z\phi^*) \tag{2.27}$$

2.1.3 Excitation Spectrum of a Superconductor

In this section we will consider the situation when the Cooper pairs have center of mass momentum $2\mathbf{q}$, i.e. if the pair potential is of the form $|\Delta|e^{i2\mathbf{q}.\mathbf{r}}$, the Bogoliubov-de Gennes equations(2.12) yields solutions of the form

$$\psi(\mathbf{r}) = \psi_k e^{i(\mathbf{k}+\mathbf{q}).\mathbf{r}}$$
$$\phi(\mathbf{r}) = \phi_k e^{i(\mathbf{k}-\mathbf{q}).\mathbf{r}} \tag{2.28}$$

$\mathbf{k}$ being the Fermi momentum. In the previous section the current expression in terms of the components of two-component wave function was found. The current arising from the wave functions of quasi-holes and quasi-electrons is

$$\mathbf{J} = e^* \mathbf{v_s} \tag{2.29}$$

where $e^* = 2e$ is the charge of the cooper pair, and $\mathbf{v_s} = \frac{\mathbf{q}}{m}$ is the superfluid velocity.

With the condition that the $k \gg q$ the corresponding energy spectrum becomes

$$\epsilon_k = \epsilon_k^0 + \mathbf{v_S}.\mathbf{k} \tag{2.30}$$

where $\epsilon_k^0 = \sqrt{|\Delta|^2 + \xi_k^2}$, $\xi_k = k^2/2m - E_F$. ϵ_k^0 is the excitation energy in the absence of a current. It is possible to obtain negative values for the product $\mathbf{v_S}.\mathbf{k}$, so the excitation energies in the bulk superconductor of uniform current may well be below $|\Delta|$. However in the case of vanishing superfluid velocity, $|\Delta|$ serves as a gap in the excitation spectrum as in the BCS gap.

2.2 Andreev Reflection

In 1965, Andreev revealed a peculiar scattering mechanism at the normal-superconductor interfaces caused by a sharp spatial variation of gap parameter. An incoming electron of energy below the gap, incident on the superconducting interface reflects back as a hole of opposite spin, as a result of which a Cooper pair is injected into the superconducting region. In the same manner a quasi-hole is reflected as a quasi-electron while absorbing a cooper pair from the superconducting region(Fig.2.1).

This process can be described using Bogoliubov-de Gennes equations.[15,21] Consider a step like gap parameter vanishing in the normal region($x < 0$) and of constant value $\Delta e^{i\Phi}$ in the superconducting region($x > 0$). The problem is

one dimensional, so Bogoliubov-de Gennes equations should be reduced to one dimensional form(see Chapter-3). The incident electron in the N region,

$$x < 0 \rightarrow \quad \Psi_0 = e^{ik_e x} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad k_e = \sqrt{2m(E + \zeta_{q_\perp})} \tag{2.31}$$

$$\zeta_{q_\perp} = \zeta - \frac{q_\perp^2}{2m}$$

is an incoming electron with positive Fermi momentum. The reflected excitations

$$x < 0 \rightarrow \quad \Psi_1 = Ae^{ik_h x} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + Be^{-ik_e x} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad k_h = \sqrt{2m(-E + \zeta_{q_\perp})} \tag{2.32}$$

the first part of which is an outgoing electron with negative Fermi momentum, and the latter is an outgoing hole with positive Fermi momentum. The transmitted excitations for $x > 0$,

$$x > 0 \rightarrow \quad \Psi_2 = Ce^{i\lambda_+ x} \begin{pmatrix} 1 \\ \gamma^+ e^{-i\Phi} \end{pmatrix} + De^{-i\lambda_- x} \begin{pmatrix} 1 \\ \gamma^- e^{-i\Phi} \end{pmatrix} \tag{2.33}$$

$$\lambda_\pm = \sqrt{2m(\zeta_{q_\perp} \pm i\Omega)}, \qquad \gamma_\pm = (E \pm i\Omega)/\Delta_0$$

where the first part of the solution is an electron type excitation and the latter is of hole type, Ω being defined as follows

$$\Omega = \sqrt{E^2 - \Delta_0^2}, \qquad E > \Delta_0$$

$$\Omega = i\sqrt{\Delta_0^2 - E^2}, \qquad E < \Delta_0 \tag{2.34}$$

Matching the wave functions and their derivatives at the boundary $x = 0$ produce the coefficients, and B comes out to be of order of $\Delta_0/\zeta_{q_\perp}$. This means that Fermi momentum is approximately conserved since $\Delta_0/\zeta_{q_\perp}$ is a very small quantity which can safely be ignored. However, this is only valid, when there's no diagonal potential. Charging effects, or defects in the interface or in general, the existence of diagonal potentials produces diagonal scattering as well(i.e. Fermi momentum changes the sign). The conservation of crystal momentum in the lack of diagonal potentials, is the motivation for the *semiclassical approximation* which will be studied in the third chapter. The underlying idea is in studying the case of negative and positive Fermi momenta independently.

For $E < \Delta_0$ the solutions yield $|A| \approx 1$, on the other hand, wave functions attenuate in the superconducting region, as a result the incoming electron reflects back as a hole. This is nothing but the Andreev reflection. This reflection is not a specular one, and is called the retroreflection(Fig-2.1).

If the length of the superconducting islands in SNS structure (S:superconducting, N:Normal) is large enough, the tunneling of Cooper pairs is suppressed and Andreev reflection becomes the prominent transport mechanism at voltages and temperatures below the gap.

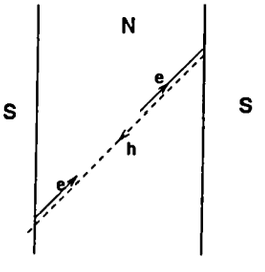

Figure 2.1: Schematic of Andreev reflection:Electron reflects back as a hole, then the hole reflects back as an electron at the S boundary, meanwhile they pick up a phase at the interfaces.

2.3 Tight Binding Approach and Lattice Bogoliubov-de Gennes Equations

The effective Hamiltonian can also be written in the tight binding model,

$$H_{eff} = \sum_i \left\{ (V_j - \mu)\Psi^{\dagger}_{i\sigma}\Psi_{i\sigma} + t_{i,i+1}(\Psi^{\dagger}_{i+1}\Psi_i + H.C.) + (\Delta_i \Psi^{\dagger}_{i\uparrow}\Psi^{\dagger}_{i\downarrow} + H.C) \right\} \quad (2.35)$$

Operators $\Psi_i, (\Psi^{\dagger}_i)$ annihilate(create) electrons on the i'th site. ϵ_i is the onsite energy and it may take different values in the normal and superconducting regions. $t_{i,i+1}$ is the hopping amplitude and there are mainly three values it may take, in the superconducting region, normal region and at the interface. Δ_i is the pairing amplitude. The V term takes account of the normal scattering at the interface. Using the tight binding Hamiltonian a solvable model, for instance, for the N-S interface, we have

$$t_{i,i+1} = \begin{cases} t & i > 0 \\ t'' & i = 0 \\ t' & i < 0 \end{cases}$$

$$\Delta_i = \begin{cases} \Delta & i > 0 \\ 0 & i \leq 0 \end{cases}, \quad V_i = V\delta_{i1} \ . \quad (2.36)$$

The interface is chosen to lie in between $i = 0$ and $i = 1$. Different values of $t_{i,i+1}$ on normal and superconducting regions take account of the Fermi velocity mismatch. Perfect Andreev reflection can be achieved, for instance, by taking $t = t' = t''$. Even for nonzero normal scattering intensity, it is possible to achieve perfect Andreev reflection by fine adjustment of the parameter t''.[22] The tight binding model can be extended to handle the single or periodic SNS structure.

The tight binding Hamiltonian can be diagonalized by performing the Bogoliubov transformations

$$\Psi_{i\sigma} = \sum_{\alpha}(u_{i\alpha}\hat{\gamma}_{\alpha\sigma} - \sigma v^{*}_{i\alpha}\gamma^{\dagger}_{\alpha,-\sigma}) \ . \tag{2.37}$$

By requiring the Hamiltonian be diagonal,

$$H_{eff} = E_G + \sum_{\alpha}\epsilon_{\alpha}\gamma^{\dagger}_{\alpha\sigma}\gamma_{\alpha\sigma} \tag{2.38}$$

the lattice Bogoliubov-de Gennes equations are produced,

$$\begin{aligned} \epsilon_{\alpha}u_{\alpha i} &= t_{i,i-1}u_{\alpha,i-1} + t_{i,i+1}u_{\alpha i+1} + (V_i - \zeta)v_{\alpha i} + \Delta_i v_{\alpha i} \\ \epsilon_{\alpha}v_{\alpha i} &= -t_{i,i-1}v_{\alpha,i-1} - t_{i,i+1}v_{\alpha i+1} + (V_i - \zeta)v_{\alpha i}\Delta^{*}_i u_{\alpha i} \end{aligned} \tag{2.39}$$

the lattice Bdg equations can be solved for the boundaries as in the continuum model and allows fine tuning of parameters s.t. perfect Andreev reflection is received. A thorough discussion of this model and its applications and extensions are presented in [22]

2.4 Green's Functions and Gorkov's Equations

One particle Green's function has full information of the quasiparticle excitation spectrum, density of states and the thermodynamic properties. Especially whenever calculation of the current through hybrid structures is of interest, Green's functions proved to be very useful. Space dependent Green's functions can be employed whenever $S-N$ walls exist. The Green's functions obey Gorkov's

equations, along with the off-diagonal pair potential. 2X2 Green's functions are defined as follows

$$\mathcal{G}_\omega(\mathbf{r}\tau,\mathbf{r}'\tau') = -\left\langle T_\tau \hat{\Psi}(\mathbf{r}\tau)\bar{\hat{\Psi}}(\mathbf{r}'\tau')\right\rangle$$

$$= \begin{pmatrix} G(\mathbf{r}\tau,\mathbf{r}'\tau') & F(\mathbf{r}\tau,\mathbf{r}'\tau') \\ F*(\mathbf{r}\tau,\mathbf{r}'\tau') & -G(\mathbf{r}'\tau',\mathbf{r}\tau) \end{pmatrix} \tag{2.40}$$

where $\hat{\Psi}(\mathbf{r}\tau)$ is a two component field operator

$$\hat{\Psi}(\mathbf{r}\tau) = \begin{pmatrix} \Psi_\uparrow(\mathbf{r}\tau) \\ \bar{\Psi}_\downarrow(\mathbf{r}\tau) \end{pmatrix} . \tag{2.41}$$

The field operators are in Heisenberg picture

$$\Psi_\sigma(\mathbf{r}\tau) = e^{H\tau}\Psi_\sigma(\mathbf{r})e^{-H\tau}$$

$$\bar{\Psi}_\sigma(\mathbf{r}\tau) = e^{H\tau}\Psi^\dagger_\sigma(\mathbf{r})e^{-H\tau} . \tag{2.42}$$

One should note that $\Psi_\sigma(\mathbf{r}\tau)$ is not the adjoint of $\bar{\Psi}_\sigma(\mathbf{r}\tau)$.

Using the $H_{eff}(2.6)$ the equations of motion for the Green's functions can be shown to be as follows

$$(-\frac{\partial}{\partial\tau} - H_e\sigma_z - \hat{\Delta})\hat{\mathcal{G}}_\omega(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')\delta(\tau-\tau') \tag{2.43}$$

$$H_e = -\frac{1}{2m}\frac{\nabla^2}{2m} - \zeta$$

The off-diagonal potential is

$$\hat{\Delta} = \begin{pmatrix} 0 & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & 0 \end{pmatrix} . \tag{2.44}$$

Since the Hamiltonain is time independent, Green's functions are functions of $\tau - \tau'$. Fourier transformation to frequency domain is

$$\begin{aligned} \mathcal{G}_\omega &= T \sum_{\omega=(2n+1)\pi T} e^{-i\omega(\tau-\tau')} \mathcal{G}_\omega(\mathbf{r}, \mathbf{r}') \\ \mathcal{F}^*_\omega &= T \sum_{\omega=(2n+1)\pi T} e^{-i\omega(\tau-\tau')} \mathcal{F}_\omega(\mathbf{r}, \mathbf{r}') \end{aligned} \tag{2.45}$$

Performing the Fourier transformation of the equations of motion, Gorkov's equations are produced

$$(i\omega - H_e\sigma_z - \hat{\Delta})\hat{\mathcal{G}}_\omega(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')\delta(\tau - \tau') \tag{2.46}$$

which must be solved along with self consistency condition

$$\begin{aligned} \Delta(\mathbf{r}) &= -V \langle \Psi(\mathbf{r}\uparrow)\Psi(\mathbf{r}\downarrow) \\ &= VT \sum_\omega e^{-i\omega\eta} \mathcal{F}_\omega(\mathbf{r}, \mathbf{r}) \end{aligned} \tag{2.47}$$

where $\omega = (2n+1)\pi T$ imposes the Fermion statistics and $\eta \to 0$. An elaborate discussion and derivation of the Green's function methods, Gorkov's equations can be found in the references [28,29].

Chapter 3

Josephson Effect in Single SNS Structure

In this chapter, we will employ quasiclassical approximation in order to reduce the order of Gorkov equations, the idea of which is eliminating the Fermi oscillations from the Gorkov equations, thus obtaining the quasiclassical equations for the Green's functions with the corresponding boundary conditions. These quasiclassical differential equations will be solved for the SNS structure exactly, and the quasiclassical Green's functions will be obtained. Then using the current expression(3.29), the Josephson current through the single SNS structure will be obtained.

3.1 Gorkov Equations in One Dimension

The equations satisfied by the thermodynamic Green functions of a pure metal-superconductor system are the second order Gorkov equations differential equations,

$$\begin{aligned}(i\omega + \frac{\nabla^2}{2m} + \mu)G_\omega(\mathbf{r},\mathbf{r}') + \Delta(\mathbf{r})F^*_\omega(\mathbf{r},\mathbf{r}') &= \delta(\mathbf{r}-\mathbf{r}') \\ (-i\omega + \frac{\nabla^2}{2m} + \mu)\ F^*_\omega(\mathbf{r},\mathbf{r}') - \Delta^*(\mathbf{r})\ G_\omega(\mathbf{r},\mathbf{r}') &= 0 \end{aligned} \tag{3.1}$$

together with the self consistency conditions

$$\Delta^*(\mathbf{r}) = V F^*(\mathbf{r}\tau^+, \mathbf{r}\tau) \ . \tag{3.2}$$

Here $\Delta(\mathbf{r})$ is the gap parameter of the superconductor, which becomes zero for normal metal , V is the eletron-phonon coupling constant. The thermodynamic Green's functions appearing in the Gorkov equations are defined as follows

$$G(\mathbf{r}\tau, \mathbf{r}'\tau') = - < T\psi_\sigma(\mathbf{r}\tau)\bar{\psi}_\sigma(\mathbf{r}'\tau') >$$

$$F^*(\mathbf{r}\tau, \mathbf{r}'\tau') = - < T\bar{\psi}_\downarrow(\mathbf{r}\tau)\bar{\psi}_\uparrow(\mathbf{r}'\tau') > \tag{3.3}$$

which can be expressed in the frequency domain as follows

$$G(\mathbf{r}\tau, \mathbf{r}'\tau') = T \sum_{\omega_n=(2n+1)\pi T} e^{-i\omega_n(\tau-\tau')} G_{\omega_n}(\mathbf{r}, \mathbf{r}')$$

$$F^*(\mathbf{r}\tau, \mathbf{r}'\tau') = T \sum_{\omega_n=(2n+1)\pi T} e^{-i\omega_n(\tau-\tau')} F^*_{\omega_n}(\mathbf{r}, \mathbf{r}') \ . \tag{3.4}$$

$F^*_\omega(\mathbf{r}, \mathbf{r}')$ is the Gorkov Green's function and vanishes for normal metal.

The single SNS' structure which extends in the x direction is translationally invariant in the x direction. So we can eliminate transversal degrees of freedom from the Gorkov equations thus obtaining one dimensional Gorkov Equations. In order to achieve this, we perform Fourier transformation to the transverse momenta space as follows

$$G_\omega(\mathbf{r},\mathbf{r}') = \frac{1}{(2\pi)^2}\int d^2\mathbf{k}_\perp e^{i\mathbf{k}_\perp\cdot(\mathbf{r}-\mathbf{r}')}G_{\omega\mathbf{k}_\perp}(x,x')$$
$$F^*_\omega(\mathbf{r},\mathbf{r}') = \frac{1}{(2\pi)^2}\int d^2\mathbf{k}_\perp e^{i\mathbf{k}_\perp\cdot(\mathbf{r}-\mathbf{r}')}F^*_{\omega\mathbf{k}_\perp}(x,x') \tag{3.5}$$

insertion of the Fourier transforms of Green's functions into the Gorkov equations yields the one dimensional Gorkov equations

$$(i\omega - T_x + \mu_{k_\perp})G_{\omega\mathbf{k}_\perp}(x,x') + \Delta(x)F^*_{\omega\mathbf{k}_\perp}(x,x') = \delta(x-x')$$
$$(-i\omega - T_x + \mu_{k_\perp})F^*_{\omega\mathbf{k}_\perp}(x,x') - \Delta^*(x)G_{\omega\mathbf{k}_\perp}(x,x') = 0 \tag{3.6}$$

$$\mu_{k_\perp} = \mu - \frac{k_\perp^2}{2m}\,.$$

3.1.1 Green's Functions for the Bulk Superconductor and Metal

We will find the solution to the one dimensional Gorkov equations(3.6)using the Fourier transform method. Since we are looking for solutions for the bulk metal, we can use the translational invariance of the Green's functions

$$G_{\omega\mathbf{k}_\perp}(x-x') = \int dk_x\ \ G_\omega(\mathbf{k})e^{ik_x(x-x')}$$
$$F_{\omega\mathbf{k}_\perp}(x-x') = \int dk_x\ \ F_\omega(\mathbf{k})e^{ik_x(x-x')} \tag{3.7}$$

where $\mathbf{k}_\perp, k_x$ are transverse and x components respectively. Insertion of the above Fourier transforms into the Gorkov's equations yields

$$G_\omega(\mathbf{k}) = \frac{1}{2\pi}\ \frac{-i\omega + \mu_{k_\perp} - k_x^2/2m}{\omega^2 + \Delta_0^2 + (\mu_{k_\perp} - k_x^2/2m)^2}$$

$$F^*_\omega(\mathbf{k}) = \frac{1}{2\pi} \frac{\Delta^*}{\omega^2 + \Delta_0^2 + (\mu_{k_\perp} - k_x^2/2m)^2} \,. \tag{3.8}$$

Now we can calculate the Green's functions by performing the Fourier integrals

$$\begin{aligned} G_{\omega \mathbf{k}_\perp}(x - x') &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_x \; \frac{-i\omega + \mu_{k_\perp} - k_x^2/2m}{\Omega^2 + (\mu_{k_\perp} - k_x^2/2m)^2} e^{ik_x(x-x')} \\ F^*_{\omega \mathbf{k}_\perp}(x - x') &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_x \; \frac{\Delta^*}{\Omega^2 + (\mu_{k_\perp} - k_x^2/2m)^2} e^{ik_x(x-x')} \end{aligned} \tag{3.9}$$

where $\Omega = \sqrt{\omega^2 + \Delta_0^2}$. Now we are just left with evaluating the integrals using the method of residues. The integrands are peaked at $k_x = \mp p$, where $p = \sqrt{2m\mu_\perp}$, so we can switch to integration variable κ s.t. $k = \kappa \mp p$. We can approximate $\mu_\perp - k^2/2m$ by $\pm v\kappa$ ($v = \sqrt{2\mu_\perp/m}$ is the Fermi velocity in the x direction). After this simplification the Fourier integrals become

$$G_{\omega \mathbf{k}_\perp}(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\kappa \frac{(-i\omega + v\kappa)e^{i(-p+\kappa)(x-x')} + (-i\omega - v\kappa)e^{i(p+\kappa)(x-x')}}{(v\kappa + i\Omega)(v\kappa - i\Omega)}$$

$$F^*_{\omega \mathbf{k}_\perp}(x - x') = \frac{\Delta^*}{2\pi} \int_{-\infty}^{\infty} d\kappa \frac{\cos(p(x - x'))e^{i\kappa(x-x')}}{(v\kappa + i\Omega)(v\kappa - i\Omega)} \,. \tag{3.10}$$

Making use of the method of residues, one can easily evaluate the above integrals and obtain the Green's functions for the bulk superconductor,

$$\begin{aligned} G_{\omega \mathbf{k}_\perp}(x, x') &= \frac{-i}{2v} \left[\left(\frac{\omega}{\Omega} - 1\right) e^{-ip|x-x'|} + \left(\frac{\omega}{\Omega} + 1\right) e^{ip|x-x'|} \right] e^{-\frac{\Omega}{v}|x-x'|} \\ F^*_{\omega \mathbf{k}_\perp}(x, x') &= \frac{\Delta^*}{v\Omega} \cos p|x - x'| e^{-\frac{\Omega}{v}|x-x'|} \end{aligned} \tag{3.11}$$

3.2 Introduction of Quasiclassical Green's Functions

In principle Gorkov equations(3.6) can be solved for any pure superconductor-metal system and the Green's functions can be obtained. However in practice, solving the Gorkov equations even for the single SNS' structure , since the Gorkov equations are of second order, is quite a clumsy job. Luckily there is a way out. It is possible to separate the Fermi oscillations from the Gorkov equations which ends with the Quasiclassical Green's functions satisfying the Quasiclassical equations. We can predict the form of the quasiclassical Green's functions making use of the form of the Green's functions of the bulk normal metal and superconductor with constant gap parameter (3.11). In the Green's functions, the superconductivity manifests itself in the terms which are free from Fermi oscillations. So we expect $e^{\pm ip|x-x'|}$ terms to be unaffected by the phenomena related with superconductivity. This can be rigorously proven by examining the expansion of the Green's functions in terms of the quasiparticle eigenstates of the system,

$$G_\omega(x,x') = \sum_\alpha \left[\frac{\psi_\alpha(x)\psi_\alpha^*(x')}{i\omega - E_\alpha} + \frac{\phi_\alpha^*(x)\phi_\alpha(x')}{i\omega + E_\alpha} \right]$$
$$F_\omega^*(x,x') = \sum_\alpha \left[\frac{\phi_\alpha(x)\psi_\alpha^*(x')}{i\omega - E_\alpha} + \frac{\psi_\alpha^*(x)\phi_\alpha(x')}{i\omega + E_\alpha} \right] \tag{3.12}$$

now we can express the eigenstates in the semiclassical form(see Chap.4)

$$\begin{pmatrix} \psi_\alpha(x) \\ \phi_\alpha(x) \end{pmatrix} = e^{i\sigma px} \begin{pmatrix} u_{\sigma n}(x) \\ v_{\sigma n}(x) \end{pmatrix} \tag{3.13}$$

where $p = \sqrt{2m\mu_{k_\perp}}$ is the Fermi momentum in the x direction, insertion of these eigenstates into (3.12) leads to the form,

$$G_\omega(x,x') = \sum_{n\sigma}\left[\frac{u_{\sigma n}(x)u^*_{\sigma n}(x')}{i\omega - E_{\sigma n}} + \frac{v^*_{-\sigma n}(x)v_{-\sigma n}(x')}{i\omega + E_{-\sigma n}}\right] e^{i\sigma p(x-x')}$$

$$F^*_\omega(x,x') = \sum_{n\sigma}\left[\frac{v_{\sigma n}(x)u^*_{\sigma n}(x')}{i\omega - E_{\sigma n}} - \frac{u^*_{-\sigma n}(x)v_{-\sigma n}(x')}{i\omega + E_{-\sigma n}}\right] e^{i\sigma p(x-x')} \tag{3.14}$$

where $\sigma = \mp 1$. So we can introduce the quasiclassical Green's functions in the following form

$$G_{\omega\mathbf{k}_\perp}(x,x') = \sum_\sigma e^{i\sigma p|x-x'|} g^\sigma_{\omega\mathbf{k}_\perp}(x,x')$$

$$F^*_{\omega\mathbf{k}_\perp}(x,x') = \sum_\sigma e^{i\sigma p|x-x'|} f^\sigma_{\omega\mathbf{k}_\perp}{}^*(x,x') \ . \tag{3.15}$$

$g^\sigma_{\omega\mathbf{k}_\perp}(x,x'), f^\sigma_{\omega\mathbf{k}_\perp}{}^*(x,x')$ are the quasiclassical Green's functions. However, from now on, for simplicity, we will drop the $\mathbf{k}_\perp$. We used $e^{i\sigma p|x-x'|}$ instead of $e^{i\sigma p(x-x')}$ which is a more familiar form from the Green's functions we obtained for the bulk superconductor(3.11). Now we will insert the above quasiclassical forms into the one dimensional Gorkov equations(3.6) and and try to obtain the equations satisfied by the quasiclassical Green's functions. Substitution of quasiclassical forms of Green's functions(3.15) into the Gorkov equations yields

$$\sum_\sigma e^{i\sigma p|x-x'|}\left\{\left(i\omega + i\sigma v\delta(x-x') + i\sigma v\,\mathrm{sgn}(x-x')\partial_x + \frac{\partial_x^2}{2m}\right) g^\sigma_\omega(x,x') + \Delta(x) f^{*\sigma}_\omega(x,x')\right\} = \delta(x-x') \tag{3.16}$$

$$\sum_\sigma e^{i\sigma p|x-x'|}\left\{\left(-i\omega + i\sigma v\delta(x-x') + i\sigma v\,\mathrm{sgn}(x-x')\partial_x + \frac{\partial_x^2}{2m}\right) f^{*\sigma}_\omega(x,x') - \Delta^*(x) g^\sigma_\omega(x,x')\right\} = 0 \ .$$

Now we will simplify the above equations. There are terms with factors $exp\{i\sigma p|x - x'|\}$ which are fast oscillating terms compared with the other functions. We can take the coefficients of these terms to be separately equal to zero for $x \neq x'$ (i.e. when δ Dirac function vanishes). For $x \neq x'$ $\partial_x sgn(x-x') = 2\delta(x - x')$ vanishes. Here a simplification can be made, we can safely take

$$iv\mathrm{sgn}(x - x')\partial_x g^\sigma_\omega \gg \frac{1}{2m}\partial^2_x g^\sigma_\omega$$

$$iv\mathrm{sgn}(x - x')\partial_x f^{*\sigma}_\omega \gg \frac{1}{2m}\partial^2_x f^{*\sigma}_\omega \ . \tag{3.17}$$

So we can just discard the second order derivatives for $x \neq x'$ in the expressions (3.16).

After making the simplifications, for $x \neq x'$, we re left with the **quasiclassical equations**

$$\begin{aligned}(i\omega + i\sigma v\mathrm{sgn}(x - x')\partial_x)g^\sigma_\omega + \Delta(x)f^{*\sigma}_\omega = 0 \\ (-i\omega + i\sigma v\mathrm{sgn}(x - x')\partial_x)f^{*\sigma}_\omega - \Delta^*(x)g^\sigma_\omega = 0 \ .\end{aligned} \tag{3.18}$$

v is the Fermi velocity in the x direction. The above equations should be solved with the appropriate boundary conditions.

Now what remains is to derive the boundary conditions either beginning from the original Gorkov equations, or using the expression (3.16) directly.

In order that the second derivatives in the expression(3.16) exist, the boundary conditions at any point $x = y$ should be as follows

$$\begin{aligned}\sum_\sigma e^{i\sigma p(y-x')} g^\sigma_\omega(x, x') \Big]^{x=y+\epsilon}_{x=y-\epsilon} = 0 \\ \sum_\sigma e^{i\sigma p(y-x')} f^{*\sigma}_\omega(x, x') \Big]^{x=y+\epsilon}_{x=y-\epsilon} = 0\end{aligned} \tag{3.19}$$

where $\epsilon \to 0^+$. Since $e^{i\sigma p(y-x')}$ are fast oscillating terms, the coefficients of $e^{i\sigma p(y-x')}$ can separately be taken to be zero. So these lead to the continuity of quasiclassical Green's functions conditions at any point x,

$$\begin{aligned} g_\omega^+(x+\epsilon,x') - g_\omega^-(x-\epsilon,x') &= 0 \\ f_\omega^{*+}(x+\epsilon,x') - f_\omega^{*-}(x-\epsilon,x') &= 0 \ . \end{aligned} \tag{3.20}$$

The further boundary conditions are obtained by integrating(3.16) in the interval $(y-\epsilon, y+\epsilon)$ for $y \neq x'$, leads to the trivial condition $0=0$. For the case $y=x'$ the integration in $(x'-\epsilon, x'+\epsilon)$ interval yields the normalization

$$\begin{aligned} g_\omega^+(x',x') - g_\omega^-(x',x') &= -i/v \\ f_\omega^+(x',x') - f_\omega^-(x',x') &= 0 \end{aligned} \tag{3.21}$$

We obtained the equations of motion for the quasiclassical Green's functions and the corresponding boundary conditions. The quasiclassical equations are of first order which are easy to handle. Their simplicity will be made clear when the quasiclassical equations are applied to the bulk superconductor with constant gap parameter.

3.2.1 Application of the Quasiclassical Equations to the Bulk Superconductor with Constant Phase

The quasiclassical Green's functions will be obtained for the bulk superconductor by solving the quasiclassical equations with the boundary conditions and compared to the ones obtained directly from the Gorkov equations(3.11). We will solve the quasiclassical equations (3.18)for the bulk superconductor with a constant gap parameter $\Delta = \Delta_0 e^{ix}$, then we will impose the boundary conditions(3.20, 3.21). The solution set for quasiclassical equations (3.18) is given as

$$\mathrm{g}_\omega^+ \propto \begin{pmatrix} 1 \\ \Gamma^\pm e^{-i\chi} \end{pmatrix} e^{\pm\kappa x} \qquad \mathrm{g}_\omega^- \propto \begin{pmatrix} 1 \\ \Gamma^\mp e^{-i\chi} \end{pmatrix} e^{\pm\kappa x} \tag{3.22}$$

$$\Gamma^\pm = -i\frac{\omega \pm \Omega \ \mathrm{sgn}(x-x')}{\Delta_0}$$

$$\kappa = \frac{\Omega}{v} \qquad \mathrm{g}_\omega = \begin{pmatrix} g_\omega^\sigma \\ f_\omega^{*\sigma} \end{pmatrix}$$

χ is the phase of the superconductor.

We can choose for x' any value we wish, due to translational invariance of the system. Suppose we choose $x' = 0$.

For $x < 0$,

$$\mathrm{g}_\omega^1 = A \begin{pmatrix} 1 \\ \gamma^- e^{-i\chi} \end{pmatrix} e^{\kappa x} \qquad \mathrm{g}_\omega^2 = B \begin{pmatrix} 1 \\ \gamma^+ e^{-i\chi} \end{pmatrix} e^{\kappa x} \tag{3.23}$$

for $x > 0$,

$$\mathrm{g}_\omega^+ = C \begin{pmatrix} 1 \\ \gamma^- e^{-i\chi} \end{pmatrix} e^{-\kappa x} \qquad \mathrm{g}_\omega^- = D \begin{pmatrix} 1 \\ \gamma^+ e^{-i\chi} \end{pmatrix} e^{-\kappa x} \tag{3.24}$$

$$\gamma^\pm = -i\frac{\omega \pm \Omega}{\Delta_0}.$$

The above solutions are introduced such that they vanish at infinities. Imposing the boundary conditions at the point $x = 0$ given by (3.20,3.21) we can determine the coefficients,

$$C - A = 0$$

$$B - D = 0$$

$$A - B = -i/v$$
$$A\gamma^- - B\gamma^+ = 0 \tag{3.25}$$

The above equation set can easily be solved and the final solution for the full Green's functions for the bulk superconductor are as follows

$$g_\omega^\sigma = \frac{-i}{2v}(\frac{\omega}{\Omega} + \sigma)e^{-\kappa|x|}$$
$$f_\omega^{*\sigma} = \frac{\Delta^*}{2v\Omega}e^{-\kappa|x-x'|}\,. \tag{3.26}$$

Then the full Green's functions for the bulk superconductor of constant phase, are as follows

$$G_\omega(x) = \frac{-i}{2v}\left[(\frac{\omega}{\Omega}+1)e^{ip|x|} + (\frac{\omega}{\Omega}-1)e^{-ip|x|}\right]e^{-\kappa|x|}$$
$$F_\omega^*(x) = \frac{\Delta^*}{v\Omega}cos(p|x|)e^{-\kappa|x|}. \tag{3.27}$$

So using the quasiclassical equations we were able to obtain the Green's functions for the bulk superconductor correctly(3.11). Just by taking $\Delta_0 \to 0$ we can obtain the Green's functions for the bulk normal metal

$$G_\omega(x) = \frac{-i}{2v}\left[(\mathrm{sgn}\omega+1)e^{ip|x|} + (\mathrm{sgn}\omega-1)e^{-ip|x|}\right]e^{-\kappa|x|}$$
$$F_\omega^*(x) = 0. \tag{3.28}$$

3.3 Calculation of the Josephson Current Through a Single SNS structure

The expression for the current through a single SNS' structure is given by the following expression

$$\begin{aligned} j &= \frac{2ie}{(2\pi)^2} T \int d^2\mathbf{k}_\perp \Big(\int_{x\epsilon S'} \int_{x'\epsilon S} \\ &- \int_{x\epsilon S} \int_{x'\epsilon S'} \Big) dx\, dx' \sum_\omega \Delta(x)\Delta^*(x') G^0_{\omega\mathbf{k}_\perp}(x-x') G_{-\omega\mathbf{k}_\perp}(x',x) \end{aligned} \quad (3.29)$$

$G^0_\omega(x-x')$, $G_\omega(x,x')$ stand for the Green's functions of the bulk normal metal and single SNS' structure, respectively.

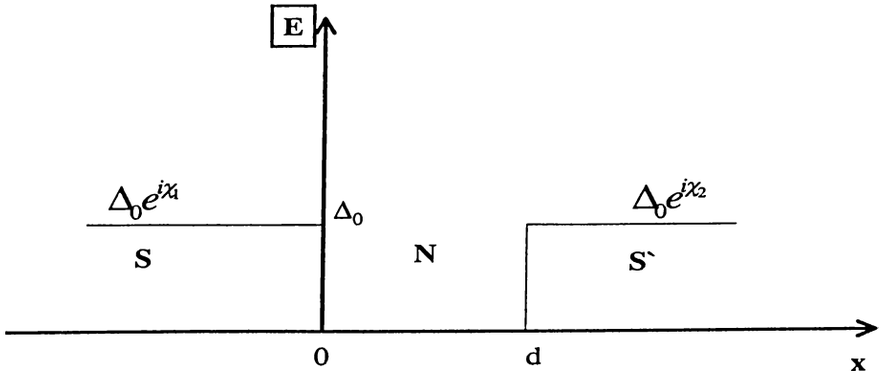

Figure 3.1: Single SNS structure with phase difference χ across the barrier

According to (3.15), the current expression (3.29) can be expressed in terms of quasiclassical Green's functions

$$J = 2ie\frac{\Delta_0^2 T}{(2\pi)^2}\sum_{\omega}\int d^2\mathbf{k}_\perp \left\{\int_d^\infty dx\int_{-\infty}^0 dx' e^{i\chi} - \int_{-\infty}^0 dx\int_d^\infty dx' e^{-i\chi}\right\}$$

$$\{g_{0\omega}^+(x-x')g_{-\omega}^-(x',x) + g_{0\omega}^-(x-x')g_{-\omega}^+(x',x)\} \quad (3.30)$$

where we have discarded the highly oscillating terms, since their contribution is very small.

So, in order to find the Josephson current, we are just left with solving the quasiclassical Green's functions (3.18) for $SNS'(3.1)$. We will take $x' < 0$ (i.e. $x'\epsilon S$) , and write the solution for quasiclassical Green's functions in the four regions and obtain the solution for the case $x'\epsilon S'$ by symmetry considerations. The solution set for the superconducting region with constant phase χ has been presented in (3.22). The solution set for the normal metal is

$$\mathrm{g}_\omega^+ \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-\omega x \mathrm{sgn}(x-x')/v}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\omega x \mathrm{sgn}(x-x')/v}$$

$$\mathrm{g}_\omega^- \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\omega x \mathrm{sgn}(x-x')/v}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\omega x \mathrm{sgn}(x-x')/v} . \quad (3.31)$$

We will be writing the solutions in the four regions, for each region we will pick up the solutions s.t they vanish at infinities, if that region extends to infinity. For $x' < 0$ the solutions in four regions are

$$x < x' \rightarrow \mathrm{g}_\omega^+ = A\begin{pmatrix} 1 \\ \gamma^- e^{-i\chi_1} \end{pmatrix} e^{\kappa x} \qquad \mathrm{g}_\omega^- = B\begin{pmatrix} 1 \\ \gamma^+ e^{-i\chi_1} \end{pmatrix} e^{\kappa x}$$

$$x' < x < 0 \rightarrow \mathbf{g}_\omega^+ = C_1 \begin{pmatrix} 1 \\ \gamma^+ e^{-i\chi_1} \end{pmatrix} e^{\kappa x} + D_1 \begin{pmatrix} 1 \\ \gamma^- e^{-i\chi_1} \end{pmatrix} e^{-\kappa x}$$

$$\mathbf{g}_\omega^- = C_2 \begin{pmatrix} 1 \\ \gamma^- e^{-i\chi_1} \end{pmatrix} e^{\kappa x} + D_2 \begin{pmatrix} 1 \\ \gamma^+ e^{-i\chi_1} \end{pmatrix} e^{-\kappa x}$$

$$0 < x < d \rightarrow \mathbf{g}_\omega^+ = \begin{pmatrix} F_1 e^{-\lambda x} \\ G_1 e^{\lambda x} \end{pmatrix} \qquad \mathbf{g}_\omega^- = \begin{pmatrix} F_2 e^{\lambda x} \\ G_2 e^{-\lambda x} \end{pmatrix}$$

$$d < x \rightarrow \mathbf{g}_\omega^1 = H \begin{pmatrix} 1 \\ \gamma^- e^{-i\chi_2} \end{pmatrix} e^{-\kappa x} \qquad \mathbf{g}_\omega^+ = J \begin{pmatrix} 1 \\ \gamma^+ e^{-i\chi_2} \end{pmatrix} e^{-\kappa x} \tag{3.32}$$

where $\lambda = \omega/v$. Here v is the Fermi velocity in the x direction. Imposing the boundary conditions given by (3.20, 3.21), at $x = x'$ the following boundary conditions are obtained,

$$\begin{aligned}
(A - C_1)e^{\kappa x'} - D_1 e^{-\kappa x'} &= 0 \\
(B - C_2)e^{\kappa x'} - D_2 e^{-\kappa x'} &= 0 \\
(A\gamma^- - C_1\gamma^+)e^{\kappa x'} - D_1\gamma^- e^{-\kappa x'} &= 0 \\
(B\gamma^+ - C_2\gamma^-)e^{\kappa x'} - D_2\gamma^+ e^{-\kappa x'} &= 0 \\
(A - B)e^{\kappa x'} &= -i/v \\
A\gamma^- - B\gamma^+ &= 0 \, .
\end{aligned} \tag{3.33}$$

At $x = 0$, we apply the boundary conditions(3.20) yielding

$$C_1 + D_1 - F_1 = 0$$

$$\begin{aligned} C_2 + D_2 - F_2 &= 0 \\ (C_1\gamma^+ + D_1\gamma^-)e^{-i\chi_1} - G_1 &= 0 \\ (C_2\gamma^- + D_2\gamma^+)e^{-i\chi_1} - G_2 &= 0 \, . \end{aligned} \tag{3.34}$$

At $x = d$, we apply the boundary conditions(3.20),

$$\begin{aligned} F_1 e^{-\lambda d} - He^{-\kappa d} &= 0 \\ F_2 e^{\lambda d} - Je^{-\kappa d} &= 0 \\ G_1 e^{\lambda d} - H\gamma^- e^{-\kappa d - i\chi_2} &= 0 \\ G_2 e^{-\lambda d} - J\gamma^+ e^{-\kappa d - i\chi_2} &= 0 \, . \end{aligned} \tag{3.35}$$

The boundary conditions (3.33,3.34,3.35) can be solved thus obtaining the unknown coefficients. If we examine the current expression(3.30), we only need the quasiclassical Green's functions $g_\omega(x,x')$, $f_\omega(x,x')$ either $x\epsilon S$, $x'\epsilon S'$ or $x\epsilon S'$, $x'\epsilon S$. In our calculations, we have taken $x'\epsilon S$, so for $x\epsilon S'$, $x'\epsilon S$ (3.32) the quasiclassical Green's functions for SNS' are

$$g_\omega^+(x,x') = \frac{i}{v}\frac{Re^{\kappa d}}{e^{-\lambda d+i\chi} - Re^{\lambda d}}e^{-\kappa(x-x')}, \quad g_\omega^-(x,x') = \frac{i}{v}\frac{e^{\kappa d}}{e^{-\lambda d} - Re^{\lambda d+i\chi}}e^{-\kappa(x-x')} \tag{3.36}$$

Considering the symmetry of the problem, we can obtain the solution for $x\epsilon S', x'\epsilon S$ just by the replacement x to $(-x+d)$ and x' to $(-x'+d)$ as well as χ to $-\chi$,

$$g_\omega^+(x,x') = \frac{i}{v}\frac{Re^{\kappa d}}{e^{-\lambda d-i\chi} - Re^{\lambda d}}e^{-\kappa(x'-x)}, \quad g_\omega^-(x,x') = \frac{i}{v}\frac{e^{\kappa d}}{e^{-\lambda d} - Re^{\lambda d-i\chi}}e^{-\kappa(x'-x)} \tag{3.37}$$

Where $R = (\omega + \Omega)/(\omega - \Omega)$ and $\chi = \chi_1 - \chi_2$ is the phase difference between the two sides, $\lambda = \omega/v$. The bulk metal quasiclassical Green's functions are also needed(3.28)

$$g^{+}_{0\omega}(x,x') = \frac{-i}{2v}(1+\mathrm{sgn}\omega)e^{\frac{-\omega}{v}|x-x'|} \qquad g^{-}_{0\omega}(x,x') = \frac{i}{2v}(1-\mathrm{sgn}\omega)e^{\frac{\omega}{v}|x-x'|} \quad (3.38)$$

Now we are left with evaluating the current expression (3.30). Replacing $d^2\mathbf{k}_\perp = 2\pi k_\perp dk_\perp = -m^2 2\pi v dv$, where v is the Fermi velocity in the x direction, and inserting the quasiclassical Green's functions into (3.30) and performing the space integrals yields

$$J = -\frac{4iem^2T}{2\pi}\sum_{\omega>0}\int_0^{v_F} dv\, v\left(\frac{1}{Re^{\lambda d+i\chi}-e^{-\lambda d}} - \frac{1}{Re^{\lambda d-i\chi}-e^{-\lambda d}}\right)e^{-\lambda d} \quad (3.39)$$

where $\omega = (2n+1)\pi T$, $n = 0, 1, 2..$. This current expression can be put in a concise form

$$J = \frac{8em^2\Delta_0^2}{2\pi}T\sum_{\omega>0}\int_0^{v_f} dv\, v\frac{\sin\chi}{(\omega+\Omega)^2e^{2\omega d/v} + 2\Delta_0^2\cos\chi + (\omega-\Omega)^2e^{-2\omega d/v}} \quad (3.40)$$

From (3.39) the integrand can be written in a form suitable for series expansion,

$$J = \frac{4iem^2T}{2\pi}\int_0^{v_F} dv\, v \sum_{\omega} \frac{e^{-2\lambda d}}{R}\left(\frac{e^{-i\chi}}{1-\frac{e^{-2\lambda d-i\chi}}{R}} - \frac{e^{i\chi}}{1-\frac{e^{-2\lambda d+i\chi}}{R}}\right) . \qquad (3.41)$$

Performing the series expansion gives

$$J = -\frac{8em^2T}{2\pi}\int_0^{v_F} dv\, v \sum_{\omega}\sum_{n=1}^{\infty} e^{-2\frac{\omega}{v}nd}\left(\frac{\omega-\Omega}{\omega+\Omega}\right)^n sin(n\chi) . \qquad (3.42)$$

First the $T = 0$ limit will be considered. As $T \to 0$, the replacement $dn = d\omega/2\pi T$ can safely be done, and summation in (3.41) can be converted into an integral over frequency. To the first order in ω/Ω we can make the following expansion

$$\left(\frac{\omega-\Omega}{\omega+\Omega}\right)^n = (-1)^n\{1 - \frac{2n\omega}{\Omega} + o(\omega^2/\Omega^2)\} . \qquad (3.43)$$

Employing this expansion in the previous current expression(3.42) and performing the integrations, we obtain the following expression for the current at T=0

$$J = \frac{-4\Delta_0 e {k_F}^2}{(2\pi)^2} 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\chi)\{\frac{\xi_0}{3d} - \frac{\xi_0^2}{4d^2} + o(\frac{\xi_0^3}{d^3})\} . \qquad (3.44)$$

There's a summation to be evaluated

$$2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(n\chi)}{n} = \chi(mod(2\pi)) \qquad (3.45)$$

which is nothing else but the periodic sawtooth function, and from (3.44) it can be seen that Josephson current becomes independent of Δ_0 and its amplitude is

inversely proportional to d, in the thick barrier limit, i.e. $d \gg \xi$. One can make the correspondence

$$\frac{\varphi}{d} = \nabla\varphi \tag{3.46}$$

which means that in the thick barrier limit at $T = 0$, Josephson current is proportional to superfluid velocity(Fig.3.2).

At finite temperatures, (however still much below the critical temperature), for $d \gg \xi_T$ where $\xi_T = \frac{v_F}{2\pi T}$, we can retain the lowest order term in the summation over ω,n in the expression for the Josephson current(3.42). In this limit, the integral over v

$$\int_0^{v_F} dv v e^{-2\omega d/v} \tag{3.47}$$

can be put into the following form by making the transformation $v = 2\pi T/x$

$$\pi^2 T^2 \int_{x_F}^{\infty} \frac{1}{x^3} e^{-dx}$$

where $x_F = 2\pi T/v_F$. The condition $d \gg \xi_T$ corresponds to $d \gg 1/x_F$, and the integral yields,

$$\frac{e^{-dx_F}}{2x_F^2} \tag{3.48}$$

The current expression, in the limits $d \gg \xi_T$, and $T \ll T_c$, is

$$J = -\frac{2ek_F^2}{\pi} T e^{-d/\xi_T} sin\chi \tag{3.49}$$

So the phase coherence persists at distances of the order of ξ_T and the classical Josephson effect is attained.

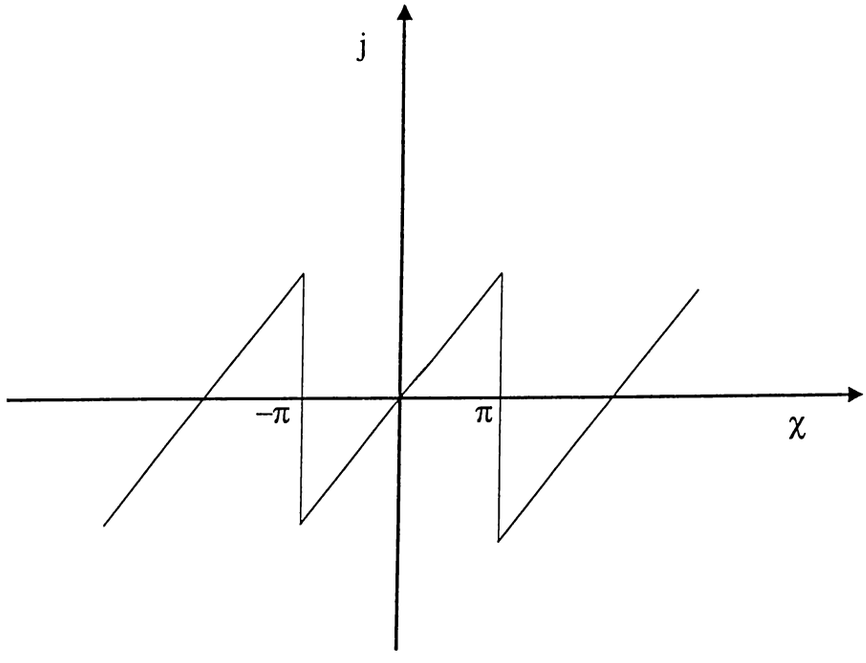

Figure 3.2: Phase dependence of Josephson current at $T = 0$

Chapter 4

Band Structure of Periodic SNS Structures

In this chapter we will consider SNS periodic structure with a periodic step like gap parameter of cyclic phase(Fig.4.1, where $\Delta = \Delta_0 e^{i\Phi}$)). Keeping in mind that gap parameter is an off-diagonal potential in BdG equations, the spectrum of the periodic SNS structure is expected to exhibit *band structure* due to periodicity, with forbidden regions in the energy spectrum. The same structure with all superconducting islands of the same phase was previously studied by Van-Gelder[26] and he identified the formation of band structures(i.e. for $\varphi = 0$). In our problem, the modulus of the gap parameter remains unchanged, however the phase of the superconducting islands is changing in a cyclic manner.

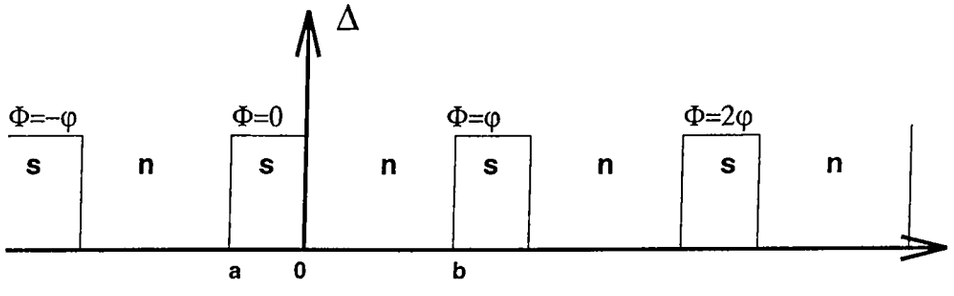

Figure 4.1: Periodic structure where the phase of the gap parameter is changing in a cyclic manner.

Bogoliubov-de Gennes equations(4.1), which has the full information of the quasiparticles, will be employed. First of all, semiclassical approximation of BdG equations will be performed in order to obtain the semiclassical equations, the idea of which is elimination of the fast oscillating terms(i.e., the Fermi oscillatons) thus reducing the order of the equations from the second order to first order. Semiclassical equations are much easier to handle, even in the SNS periodic structure. In our problem, the modulus of the gap parameter is periodic, however the phase of the superconducting islands is changing in a cyclic manner and a condition similar to the Bloch condition should be formulated in order to find the translational properties of the quasiparticle wave functions. Using the translational property of wave functions, the semiclassical equations will be solved thus obtaining the dispersion relation and consequently the DOS for one dimensional SNS structure, the result of which will be extended to the three dimensional case.

Andreev scattering is the key mechanism in this problem. Bogoliubov-de Gennes equations are very well suited to the description of Andreev reflection, since they have the full information of the quasiparticle excitations,[21]

$$\begin{aligned} (H_e + U)\psi + \Delta\phi &= \varepsilon\psi \\ \Delta^*\psi - (H_e + U)\phi &= \varepsilon\phi \ . \end{aligned} \tag{4.1}$$

ε is the energy of the excitation. Here $H_e = -\nabla^2/2m - \zeta$ is the electronic Hamiltonian (ζ is the chemical potential), U and Δ are the self consistent potentials

$$\begin{aligned} U(\mathbf{r}) &= -V\sum_n (|\psi_n(\mathbf{r})|^2 f_n + |\phi_n(\mathbf{r})|^2(1-f_n)) \\ \Delta(\mathbf{r}) &= V\sum_n (\phi_n(\mathbf{r})^*\psi_n(\mathbf{r})(1-2f_n)) \ . \end{aligned} \tag{4.2}$$

The components of the two component wave function (ψ_n, ϕ_n) are wave functions of the electron and hole like excitations. For $\Delta = 0$, i.e. in the normal metal, the components of the two component wave function (ψ_n, ϕ_n) are exactly electron and hole wave functions, respectively. f_n is the Fermi-Dirac distribution function corresponding to the excitation energy ε_n. The self consistent potential $U(\mathbf{r})$ is the usual Hartree-Fock term, which is a sum over all states and as a result of this it has almost no temperature dependence, so it can be approximated by its value for the normal metal and absorbed into the chemical potential. On the other hand, the gap parameter $\Delta(\mathbf{r})$ is strongly dependent on the temperature since $\psi_n^* \phi_n$ term has nonvanishing value only in the neighbourhood of the Fermi energy.

4.1 One Dimensional BdG Equations

The SNS structure(Fig.4.1) of our concern is translationally invariant in the x direction, so the transverse degrees of freedom can be eliminated from the Bogoliubov equations, making use of the Fourier transform into the transverse momenta space,

$$\begin{pmatrix} \psi(\mathbf{r}) \\ \phi(\mathbf{r}) \end{pmatrix} = \frac{1}{(2\pi)^2} \int d^2\mathbf{q}_\perp e^{i\mathbf{q}_\perp \cdot \mathbf{r}} \begin{pmatrix} \psi_{\mathbf{q}_\perp}(x) \\ \phi_{\mathbf{q}_\perp}(x) \end{pmatrix} . \tag{4.3}$$

Insertion of the Fourier transforms into the original Bogoliubov equations produces the one dimensional Bogoliubov equations,

$$(-\frac{1}{2m}\frac{\partial^2}{\partial x^2} - \zeta_{\mathbf{q}_\perp})\psi_{\mathbf{q}_\perp}(x) + \Delta\phi_{\mathbf{q}_\perp}(x) = \varepsilon\psi_{\mathbf{q}_\perp}(x) \tag{4.4}$$

$$\Delta^*\psi_{\mathbf{q}_\perp}(x) - (-\frac{1}{2m}\frac{\partial^2}{\partial_x^2} - \zeta_{\mathbf{q}_\perp})\phi_{\mathbf{q}_\perp}(x) = \varepsilon\phi_{\mathbf{q}_\perp}(x) \tag{4.5}$$

where

$$\zeta_{q_\perp} = \xi - \zeta_\perp^2/2m \ .$$

Owing to the self consistency conditions, in the periodic SNS structure(Fig.4.1), the gap parameter smears out in the neighbourhood of superconducting and normal interfaces. In the neighbourhood of the superconductor-normal metal interface, the gap parameter in the normal region is exponentially small with a decay width proportional to $\xi_0 = v_F/\Delta_0$, and varies over distances of the order of $\xi(T) \propto v_F/\sqrt{T_c(T_c - T)}$ in the superconducting region.[13] The characteristic length in the superconducting region becomes that of the Cooper pair correlation length ξ_0 at $T = 0$. So, the step-like gap model serves to be an acceptable one, if the thickness of the normal and superconducting layers are taken to be much larger than the corresponding coherence lengths. Alternatively, the gap parameter in the normal layers can be taken as zero in the normal region by requiring the coupling constant to be zero for the normal layer.

4.2 Semiclassical Approximation

As shown in the second chapter, an electron incident on an $N - S$ interface, the electron reflects back as a hole and vice versa(Andreev reflection) with amplitudes ($\propto (1 - \Delta_0/E_f)$), thus conserving the crystal momentum as long as the gap parameter is much smaller than the Fermi energy.[15] As long as the chemical energy is the same throughout the structure, and in the absence of digonal scattering crystal momentum is conserved throughout the structure.. Diagonal scattering does not happen, when the electronic Hamiltonian merely consists of kinetic part and the scattering arises only due to off-diagonal potentials.

So the following form can be proposed for the solution of BdG equations, expecting to separate the Fermi oscillations from the BdG equations and its solutions,

$$\begin{pmatrix} \psi \\ \phi \end{pmatrix} = \sum_{\sigma=\pm} \exp(i\sigma q_x x) \begin{pmatrix} u_\sigma(x) \\ v_\sigma(x) \end{pmatrix} . \tag{4.6}$$

The new components u, v are free from Fermi oscillations within the semiclassical approximation i.e. as long as the gap parameter Δ_0 is much smaller than the Fermi energy $q_x^2/2m$. u and v are expected to vary at the order of $\xi_0 = v_x/\Delta_0$. Here the x component of Fermi momentum q_x and Fermi velocity v_x are positive definite. The substitution of the semiclassical form(4.6) into BdG equations yields

$$\begin{aligned} \sum_\sigma e^{i\sigma q_x x} \left[(-i\sigma v_x \partial_x - \partial_x^2/2m - \zeta_{\mathbf{q}_\perp} - \frac{q_x^2}{2m}) u_\sigma + \Delta(x) v_\sigma \right] &= \sum_\sigma e^{i\sigma q_x x} u_\sigma \\ \sum_\sigma e^{i\sigma q_x x} \left[(i\sigma v_x \partial_x + \partial_x^2/2m + \zeta_{\mathbf{q}_\perp} - \frac{q_x^2}{2m}) v_\sigma + \Delta^*(x) u_\sigma \right] &= \sum_\sigma e^{i\sigma q_x x} v_\sigma \end{aligned} \tag{4.7}$$

∂_x denotes ∂/∂_x. $v_x = q_x/m$ is the Fermi velocity in the x direction. u_σ and v_σ vary at distances of the order of $1/\xi_0$, $\partial_x^2 \Psi \propto \Psi/\xi_0^2$, where

$$\Psi = \begin{pmatrix} u \\ v \end{pmatrix} . \tag{4.8}$$

The coherence length $\xi = v_F/\Delta_0$ is much larger than the Fermi wave length $1/q_x$, therefore we can discard the second order derivatives from the BdG equations. There's also the term $\zeta_{\mathbf{q}_\perp}$ canceling $q_x^2/2m$. The semiclassical BdG equations can finally be put into the following form

$$\begin{aligned} -i\sigma v_x \partial_x u_\sigma + \Delta v_\sigma &= \varepsilon u_\sigma \\ i\sigma v_x \partial_x v_\sigma + \Delta^* u_\sigma &= \varepsilon v_\sigma . \end{aligned} \tag{4.9}$$

4.3 Andreev Reflection and Discrete Spectrum of Single SNS Structure[15,11]

In this section, single SNS structure, with a phase difference φ between the superconducting banks, will be studied using semiclassical approximation. As discussed before, an electron in the normal region, with the energy below the gap reflects back as a hole, and vice versa. Owing to Andreev reflection, sub-gap excitations in the normal metal, confined by the superconducting banks exhibit discrete spectrum. Kulik, in a paper dated 1970, revealed the discrete spectrum of single SNS structure.[15]

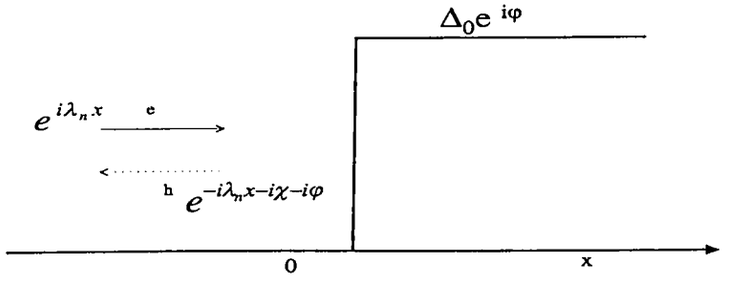

Figure 4.2: Andreev Reflection: An electron incident on a superconducting wall reflects back as a hole, meanwhile picking up a phase with a slight momentum nonconservation of $-2\lambda_n$

The solutions of the semiclassical equations(4.9) for the superconducting region of constant gap parameter $\Delta_0 e^{i\Phi}$, consist of the superpositions of eigenfunctions,

$$\begin{pmatrix} 1 \\ \gamma^{\sigma\sigma'} e^{-i\Phi} \end{pmatrix} e^{\sigma'\lambda_s x}, \quad \sigma' = \mp \tag{4.10}$$

$$\gamma^{\sigma\sigma'} = \frac{\varepsilon + i\sigma\sigma'\sqrt{\Delta_0^2 - \varepsilon^2}}{\Delta_0} \qquad \lambda_s = \frac{\sqrt{\Delta_0^2 - \varepsilon^2}}{v_x}$$

and in the normal region

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\sigma\lambda_n x}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\sigma\lambda_n x} \tag{4.11}$$

$$\lambda_n = \varepsilon / v_x$$

σ refers to sign of the Fermi momentum. The most general solution for a system will be the superposition of these eigenfunctions.

According to semi-classical equations, the excitation spectrum of bulk superconductors with constant gap parameter is as follows

$$\epsilon = \sqrt{\Delta_0^2 + \lambda^2 v^2}\ . \tag{4.12}$$

(4.13)

The excitations of concern take place at small values of λ compred with Fermi momentum. On the other hand, in the normal metal

$$\epsilon = |\lambda v|\ . \tag{4.14}$$

The group velocity of the excitation is given by $d\epsilon/d\lambda$.

4.3.1 Andreev Reflection

An incoming electron in the normal region(Fig.4.2), of energy lower than the gap energy, incident on the superconducting interface has the following wave functions in the normal and the superconducting regions,

$$x < 0 \rightarrow \Psi(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\lambda_n x} + R \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\lambda_n x}$$

$$x > 0 \rightarrow \Psi(x) = T \begin{pmatrix} 1 \\ \gamma^{+-} e^{-i\varphi} \end{pmatrix} e^{-\lambda_s x} \tag{4.15}$$

$$\gamma^{+-} = e^{-i\chi} \qquad \chi = \arccos(E/\Delta_0) \ .$$

The solutions are picked up such that they do not blow up at infinities. Imposition of the continuity conditions yields the reflection coefficient,

$$R = e^{-i\chi - i\varphi} \tag{4.16}$$

Electron reflects back as a hole, and absorbs a phase $\phi = -i\chi - i\varphi$. There's a slight momentum non-conservation amounting to $-2\lambda_n$. The Andreev reflection is made clearly visible from the semiclassical point of view. If the incoming particle were a hole, it would be reflected as an electron, while picking up a phase $\phi = -i\chi + i\varphi$ at the interface.

The incoming electron has positive Fermi momentum, so throughout this process the Fermi momentum is conserved and is always positive. This is a direct consequence of semiclassical approximation which says that the crystal momentum is conserved.

4.3.2 Discrete Spectrum of the Single SNS Structure

For the single SNS structure with pair potentials $\Delta_{1,2} = \Delta_0 e^{\varphi_{1,2}}$ in the S banks, the solution in the three regions are

$$\begin{aligned} x < 0 \rightarrow \Psi(x) &= A \begin{pmatrix} 1 \\ \gamma^{\sigma+} e^{-i\varphi_1} \end{pmatrix} e^{\lambda_s x} \\ 0 < x < d \rightarrow \Psi(x) &= \begin{pmatrix} B e^{i\sigma\lambda_n x} \\ C e^{-i\sigma\lambda_n x} \end{pmatrix} \\ d < x \rightarrow \Psi(x) &= D \begin{pmatrix} 1 \\ \gamma^{\sigma-} e^{-i\varphi_2} \end{pmatrix} e^{-\lambda_s x} . \end{aligned} \tag{4.17}$$

Imposition of the continuity condition produces the equations for the unknown coefficients. For $E < \Delta_0$, this condition leads to the discrete spectrum of energy,[15]

$$E = \frac{v}{d}(\arccos(E/\Delta_0) + \pi n + \sigma\varphi/2) \tag{4.18}$$

where n is an integer(Fig.4.3).

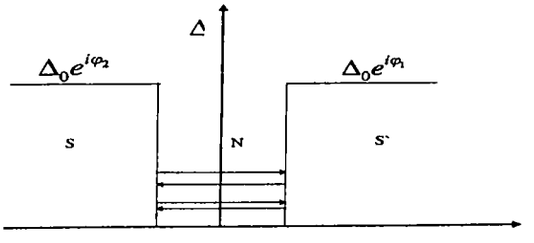

Figure 4.3: Discrete spectrum of quasi-particles in a normal metal confined by superconducting banks.

For $E \ll \Delta_0$, the subgap spectrum reduces to the form,

$$E_n(q_x, \varphi) = \frac{v}{d}((n + \frac{1}{2}\pi + \sigma\varphi/2) \tag{4.19}$$

which can be written in the form,[24,17]

$$E_n(q_x, \varphi) = E_n(q_x, 0) + \sigma v_s q_x \tag{4.20}$$

v_s is interpreted as the superfluid velocity,

$$v_s = \frac{\varphi}{md} \tag{4.21}$$

Then there appears an effective gap

$$\Delta_{eff} = E_0(q_x, 0) = \frac{v_s}{2d} , \tag{4.22}$$

which may be an argument for the superconductive property exhibited by the normal region in single SNS structure when there's a phase difference across the superconducting electrodes.

Discrete spectrum of single SNS' may also be revealed by establishing an analogy with the Bohr-Sommerfeld quantization condition. An incoming electron reflects back as a hole picking up a phase $\phi = -i\chi - i\varphi_2$, then the hole reflects back at the left-hand wall as an electron, picking up a phase $\phi = -i\chi + i\phi_1$, coming back to the initial point. Meanwhile due to the path traversed, electron picks up a phase $2\lambda_n L$. Since the wave functions are standing wave functions, the total phase acquired during one cycle, should be equated to $2\pi n$,

$$2\lambda_n L - 2arccos(E/\Delta_0) - \varphi = 2\pi n$$

which is the quantization condition.

4.4 Symmetries of the Bogoliubov-de Gennes Equations for the Periodic SNS Structure

Now we are going to solve the semiclassical equations for periodic SNS structure with cyclic phase variations(Fig.4.1). First we have to find the requirements of symmetries of the periodic SNS structure on the quasiparticle wave functions (4.9), similar to the Bloch condition for periodic potentials. However one must notice that our system is not exactly periodic, in each period the phase of the gap parameter changes by some certain value φ, while the modulus value Δ_0 remains the same. Let $L = a + b$ be the length of the unit cell. The semiclassical Hamiltonian can be put into the matrix form,

$$\mathcal{H} = \begin{pmatrix} -i\sigma v_x \frac{\partial}{\partial x} & \Delta \\ \Delta^* & i\sigma v_x \frac{\partial}{\partial x} \end{pmatrix} \tag{4.23}$$

and the semiclassical equations read,

$$\mathcal{H}\Psi = \varepsilon\Psi \qquad \Psi = \begin{pmatrix} u \\ v \end{pmatrix} . \tag{4.24}$$

Following from (4.1), the unitary operatorT_φ can be defined as

$$T_\varphi = \begin{pmatrix} exp(-i\varphi/2) & 0 \\ 0 & exp(i\varphi/2) \end{pmatrix} . \tag{4.25}$$

It is a unitary operator which displaces the system(i.e. the Hamiltonian) by one period L,

$$T_\varphi^\dagger \mathcal{H}(x) T_\varphi = \mathcal{H}(x+L) \tag{4.26}$$

$$\tag{4.27}$$

However, this choice of T_φ is not unique. Any diagonal $2X2$ unitary matrix with the ratio of diagonals equal to $e^{-i\varphi}$, will also work. It can be shown that the density of states does not depend on this particular choice.

Now we are going to determine the symmetries of the quasiparticle wave functions under translation. Starting from the eigenvalue equation,

$$\mathcal{H}(x)\Psi(x) = \varepsilon\Psi(x) \tag{4.28}$$

and making use of the unitarity of T_φ, eigenvalue equation can be transformed to,

$$\underbrace{T_\varphi^\dagger \mathcal{H}(x) T_\varphi}_{\mathcal{H}(x+L)} T_\varphi^\dagger \Psi(x) = T_\varphi^\dagger \Psi(x) \tag{4.29}$$

and under displacement of the system by one unit cell, the eigenvalue equation can be put into the form,

$$x \to x + L \to \mathcal{H}(x+L)\Psi(x+L) = \varepsilon\Psi(x+L)\ . \tag{4.30}$$

Here L is the length of the unit cell, $L = a + b$. Comparing the two expressions 4.29 and 4.30, and assuming that there's no degeneracy, we conclude that $\Psi(x+L)$

can differ from $T_\varphi^\dagger \Psi(x)$ only by a phase factor which we can write as e^{ikL},

$$\Psi(x+L) = \exp(ikL) T_\varphi^\dagger \Psi(x) \tag{4.31}$$

Applying the periodic boundary conditions one can easily see that, the value of the wave-vector equals to $k = 2\pi n/\mathcal{P}L$, where $\mathcal{P}$ is the total number number of cells in the specimen and n is an integer. However we should also require that $\mathcal{P}\varphi/2$ is a multiple of 2π, in order to satisfy the periodic boundary condition. As a result, the wave function, under translation by one unit cell, can be related to the original wave functions as

$$\Psi(x+L) = \exp(ikL) \begin{pmatrix} u(x)\exp(i\varphi/2) \\ v(x)exp(-i\varphi/2) \end{pmatrix} \tag{4.32}$$

where the wave vector k is defined in the interval $(-\pi/L, \pi/L)$.

4.5 Solution of the Semiclassical Equations for the Periodic SNS structure

Now we can solve the semiclassical equations for the periodic SNS system(Fig.4.1). Our problem is similar to Kronig-Penney model. We first introduce the solutions in one unit cell and write the solution in the neighbouring cell making use of (4.32), and then match the wave functions at the boundaries provides the wave functions and the energy spectrum.

The solution in the $(-a, b)$ interval will be introduced (Fig.4.1) then the solution in the subsequent cell will be obtained using the condition(4.32),

$$-a < x < 0 \rightarrow \Psi(x) = A_{\sigma\sigma'} \begin{pmatrix} 1 \\ \gamma^{\sigma\sigma'} \end{pmatrix} \exp(\sigma' \lambda_s x)$$

$$0 < x < b \rightarrow \Psi(x) = B_\sigma \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(i\sigma\lambda_n x) + C_\sigma \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp(-i\sigma\lambda_n x)$$

$$b < x < L \rightarrow \Psi(x) = A_{\sigma\sigma'} \begin{pmatrix} e^{i\varphi/2} \\ \gamma^{\sigma\sigma'} e^{-i\varphi/2} \end{pmatrix} \exp(\sigma' \lambda_s (x - L) + ikL) \ . \quad (4.33)$$

We first wrote the solution in the region $-a < x < 0$. Then making use of the Bloch condition we were able to write the solution in the the region $b \leq x \leq (a + b)$ from (4.32) . As a result we have found the solutions in three regions with unknown coefficients. There remains the determination of the unknown coefficients and the condition for the existence of solutions.

Semiclassical equations impose the continuity of the two component wave function Ψ_σ at the boundaries . We impose the continuity of the wave function at x=0 and x=b, receiving four boundary conditions, which suffice to solve for the coefficients $A_{\sigma\sigma'}, B_\sigma, C_\sigma$. For a solution to exist, the determinant of these coefficients should vanish, which leads to the *dispersion relations*

$$\begin{aligned} \cos(kL) = &- \cot\chi \sinh(a \sin\chi/\xi) \sin(b \cos\chi/\xi - \sigma\varphi/2) \\ &+ \cosh(a \sin\chi/\xi)) \cos(b \cos\chi/\xi - \sigma\varphi/2) \end{aligned} \quad (4.34)$$

for $E \leq \Delta_0$, where $E = \Delta_0 \cos\chi$, $\xi = v_x/\Delta_0$ and

$$\begin{aligned} \cos(kL) = &- \coth\chi \sin(a \sinh\chi/\xi) \sin(b \cos\chi/\xi - \sigma\varphi/2) \\ &+ \cos(a \sinh\chi/\xi) \cos(b \cos\chi/\xi - \sigma\varphi/2) \ . \end{aligned} \quad (4.35)$$

for $E \geq \Delta_0$, where $E = \Delta_0 \cosh(\chi)$. For some regions in the energy spectrum, the absolute values of the right-hand sides of dispersion relations 4.34, 4.35 may exceed unity. However this is prohibited since $\cos(kL)$ takes values in the interval $(-1, 1)$. Consequently, there appears forbidden energy regions. Since we restrict the values of the wave vector k be in the zone $(-\pi/L, \pi/L)$, there arises energy branches, in the k versus E graph, which is nothing but a band structure.

The dispersion relations are even functions of the wave vector k, indicating that the dispersion relation $E(k)$ is an even function of k in the zone $k \in [-\frac{\pi}{L}, \frac{\pi}{L}]$. It can be checked that the branches of different bands do not cross each other which means that an allowed energy value corresponds to only one branch.

If the dispersion relations are examined, it can be seen that a sign change in the Fermi velocity is equivalent to a sign change in the phase ϕ, as expected. The dispersion relations suffer a sign change, when $\phi \to \phi + 2\pi$, and are 4π periodic, however after the calculation of the DOS, it will be revealed that the DOS has 2π periodicity, as it should be.

4.6 DOS for the 1D Periodic SNS Structure

Once the dispersion relations are known(4.34, 4.35), the density of states for the SNS periodic structure can be calculated, making use of the well known expression

$$\begin{aligned} N(\epsilon) &= \sum_{i,n} \delta(\epsilon - \epsilon_n(k_i)) \\ &= \sum_n \frac{PL}{2\pi} \int_{-\pi/L}^{\pi/L} dk' \ \ \delta(\epsilon - \epsilon_n(k')) \ . \end{aligned} \tag{4.36}$$

The summation in the first line is over all energy values with quantum numbers indicated by the band number n and the wave vector k_i determined by the periodic boundary conditions. In the second line, the summation over momenta

is converted into an integral over the momentum. Using the properties of the δ-Dirac function we can explicitly write the integrals as follows,

$$\begin{aligned} N(\epsilon) &= \frac{PL}{2\pi}\int_{-\pi/L}^{\pi/L} dk' \left[\frac{\delta(k'-k)}{\frac{d\epsilon}{dk'}|_{k'=k}} + \frac{\delta(k'+k)}{\frac{d\epsilon}{dk'}|_{k'=-k}}\right] \\ &= \frac{PL}{\pi}\frac{dk}{d\epsilon}\,. \end{aligned} \tag{4.37}$$

We shall now find the full expression for the density of states $N(\epsilon)$ for the SNS periodic structure using the dispersion relations (4.34),(4.35).

We can express the density of states as a function of χ, for $E < \Delta_0$,

$$\begin{aligned} N(\chi) &= -\frac{PL}{\pi \sin\chi}\frac{dk}{d\chi} \\ &= \frac{P}{\pi\sqrt{1-f^2}\sin\chi}\frac{df}{d\chi} \end{aligned} \tag{4.38}$$

where $f(\chi)$ is a function

$$\begin{aligned} f = &-\cot\chi \sinh(a\sin\chi/\xi)\sin(b\cos\chi/\xi - \sigma\varphi/2) \\ &+\cosh(a\sin\chi/\xi))\cos(b\cos\chi/\xi - \sigma\varphi/2) \end{aligned}$$

We just made use of the dispersion relation (4.34) in order to derive the density of states. Replacement of χ by $i\chi$ simply produces the DOS for $E > \Delta_0$,

$$N(\chi) = \frac{P}{\pi\sqrt{1-f^2}\sin\chi}\frac{dg}{d\chi} \tag{4.39}$$

where $g(\chi)$ is a function

$$g = -\coth\chi\sin(a\sinh\chi/\xi)\sin(b\cos\chi/\xi - \sigma\varphi/2) + \cos(a\sinh\chi/\xi)\cos(b\cos\chi/\xi - \sigma\varphi/2)$$

The functions f, g are the expressions given by (4.34, 4.35) respectively.

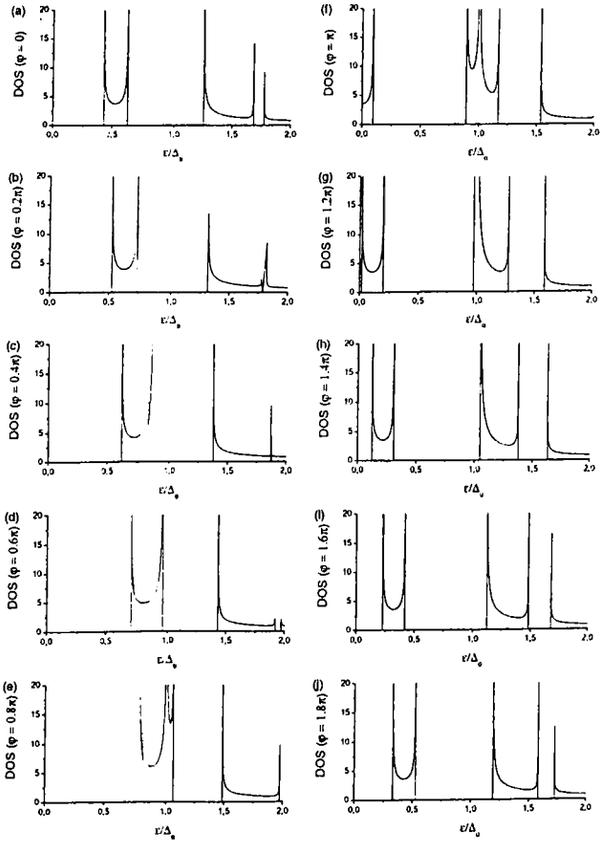

Figure 4.4: DOS for values of φ between 0 and 2π for $\sigma = +$ where $a = 2\xi_0$,$b = 2\xi_0$. Energy is scaled with respect to Δ_0, and DOS is scaled with respect to number of unit cells in the specimen.

In the figure-4.4 the DOS is shown for various values of φ when the length of the normal and superconducting regions are taken to be $2\xi_0$ and $4\xi_0$ in the figure-4.5. The given DOS plots are for states corresponding to positive Fermi momentum($\sigma = +$). When the Fermi momentum changes sign, effectively φ changes sign so it is sufficient to plot for $\sigma = +$ only. As the value of φ increases, the DOS shifts. There appears forbidden energy regions where $N(\epsilon)$ drops to zero.

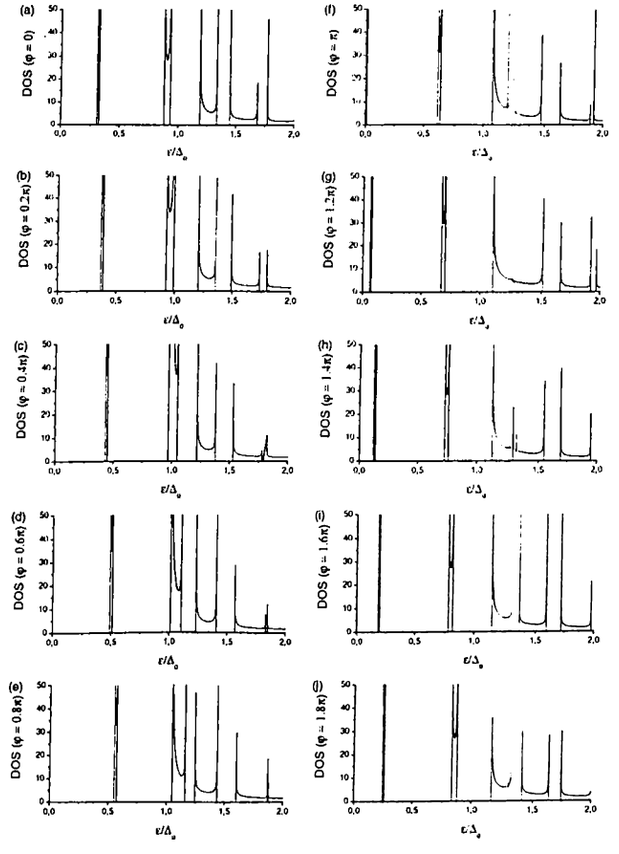

Figure 4.5: DOS for values of φ between 0 and 2π for $\sigma = +$ where $a = 4\xi_0$,$b = 4\xi_0$. Energy is scaled with respect to Δ_0, and DOS is scaled with respect to number of unit cells in the specimen.

The single SNS structure and periodic SNS structure are closely related. Inside the gap there are quantized energy levels, which broadens in the case of SNS periodic structure. DOS, inside the gap, has nonzero values around the corresponding discrete levels for the single SNS structure i.e. single SNS structure with the same length of normal region. The levels broaden due to tunneling. As the length of the superconducting region is increased, we expect the allowed regions inside the gap to become narrower. If the length of the normal region is increased, the number of bound levels for the single SNS structure increases, in relation with this, for the periodic SNS structure, the number of allowed regions inside the gap increases(Fig.??).

4.7 DOS for the Three Dimensional Case

The procedure is as in the one dimensional case, but this time we have to make an integration over the transverse momenta. We will employ an expression similar to the expression 4.36,

$$\begin{aligned} N(\epsilon) &= \sum_{i,n} \delta(\epsilon - \epsilon_n(k_i)) \\ &= \sum_n \frac{\mathcal{P}L}{(2\pi)^2} \int_0^{q_F} d^2\mathbf{q}_\perp \int_{-\pi/L}^{\pi/L} dk' \;\; \delta(\epsilon - \epsilon_n(k', v_x)) \end{aligned} \tag{4.40}$$

There's an extra integration over the transverse Fermi momenta. The expression can be put in a more explicit form by making use of the properties of the δ-Dirac function,

$$N(\epsilon) = \frac{m^2\mathbf{P}L}{(2\pi)^2} \int_0^{v_F} dv_x v_x \int_{-\pi/L}^{\pi/L} dk' \;\; \left[\frac{\delta(k'-k)}{\frac{d\epsilon}{dk'}|_{k'=k}} + \frac{\delta(k'+k)}{\frac{d\epsilon}{dk'}|_{k'=-k}} \right]$$

$$= \frac{m^2 PL}{2\pi^2}\int_0^{v_F} dv_x v_x \frac{dk}{d\epsilon} \tag{4.41}$$

writing the DOS as a function of χ, for $E \leq \Delta_0$

$$\begin{aligned} N(\chi) &= -\frac{m^2 PL}{2\pi^2 \sin\chi}\int_0^{v_F} dv_x v_x \frac{dk}{d\chi} \\ &= \frac{m^2 P}{2\pi^2 \sin\chi}\int_0^{v_F} dv_x v_x \frac{1}{\sqrt{1-f^2}}\frac{df}{d\chi} \end{aligned} \tag{4.42}$$

and for $E > \Delta_0$

$$N(\epsilon) = -\frac{m^2 P}{2\pi^2 \sinh\chi}\int_0^{v_F} dv_x v_x \frac{1}{\sqrt{1-f^2}}\frac{df}{d\chi} \tag{4.43}$$

The functions f, g were given in 4.38, 4.39.

4.8 Josephson Current Through the Periodic SNS Structure

We have obtained the phase dependent density of states for the periodic SNS structure, then the thermodynamic potential can be written as

$$\Omega(\chi) = -\frac{1}{\beta}\int_0^{\infty} dE \ln(1+e^{-\beta E}) N(E,\varphi) \tag{4.44}$$

Josephson current is the response of thermodynamic potential to the phase difference φ,

$$J(\chi) = -2e\frac{\partial\Omega}{\partial\varphi} \tag{4.45}$$

$$= 2e\frac{1}{\beta}\int_0^\infty dE \ln(1+e^{-\beta E})\frac{\partial N(E,\varphi)}{\partial\varphi} \tag{4.46}$$

In principle, the Josephson current through the periodic SNS structure can be obtained just by performing an integration over the energy.

Chapter 5

Conclusion

The quasiclassical equations together with the boundary and normalization conditions, are obtained and are solved for single SNS structure. The quasiclassical equations are valid only when the diagonal potentials are absent, which means that there is no Fermi level mismatch, no impurities, no potential barriers at the interfaces throughout the structure. For single SNS structure, at zero temperature in the thick barrier limit, we have shown that the relationship between Josephson current and the phase difference is a saw-tooth function, and is independent of gap parameter. The critical current of SNS structure is inversely proportional to the length of the normal region. This result dramatically differs from the classical Josephson effect in which the Josephson current exponentially decreases with the barrier thickness. All harmonics of phase contribute to the the current, and as a result, sawtooth phase dependence of current emerges. At zero temperature the phase coherence between superconducting electrodes persists even in a very long normal barrier of length exceeding the coherence length ξ_0. At low temperature, Andreev reflection becomes the dominant transport mechanism,[11] and the current carrying states are the subgap discrete responsible for this behavior.[15]

In SNS structures, there's a close relationship between the characteristic lengths of proximity effect and the supercurrent. At $T = 0$, and in the case of an $S - N$ interface, the Gorkov Green function decays on a power law in the normal

metal due to the proximity effect, $F \propto 1/r$,[18] while the super-current through the SNS system also decays on a power law of d, $J \propto 1/d$.

At finite temperatures, such that $d \gg \xi_T, T_c \gg T$ where $\xi_T = v_F/2\pi T$, the supercurrent through SNS structure is found to decay as $\exp(-d/\xi_T)$. This length scale ξ_T also coincides with the characteristic decay length of Cooper pair amplitude F in the normal region in case of the $S-N$ proximity.[18]

ξ_T is the thermal coherence length which is also defined to be the length at which electron and hole phase difference amounts to π under Andreev reflection. Thus the proximity effect, Andreev reflection and Josephson current through the SNS structure are fundamentally interrelated.

Density of states for single SNS structure is phase dependent, consequently the thermodynamic potential of the system becomes phase dependent. In the case of a steady current, the energy attains its minimum at a nonzero phase difference. There's a similarity between interpretation of persistent currents in normal metal loops and supercurrent through SNS structure, in this respect.

In the last chapter, the SNS periodic structure was studied. The symmetries of the quasiparticle wave functions are obtained, and the density of states is calculated for 1d and 3d cases and it is shown that DOS is phase dependent. There comes out forbidden energy regions which indicate a band structure.

There's a close relationship between single and periodic SNS structures. In the periodic case, the discrete levels are broadened due to tunneling of quasiparticles into the energy bands of finite width. Also the symmetries of the system impose further restrictions on the energy spectrum.

SNS periodic structure is also a candidate for the Josephson effect. Taking the phase difference as a thermodynamical variable, an expression for the Josephson current can be obtained. Carrying out the integrations over the energy will yield the Josephson current as a derivative of the thermodynamic potential with respect to phase difference of the neighbouring superconducting layers.

The pair potentials are taken to be step-like in single SNS structure and in the periodic SNS structure. In the normal region, the pair potential vanishes since the electron-phonon interaction is zero or repulsive. However, there's always a

gap parameter depression in the superconducting region, at characteristic length $\xi(T) \propto \xi_0/\sqrt{1-T/T_c}$ which diverges at T_c and reduces to zero temperature coherence length ξ_0 at low temperature. So, a step-like pair potential is acceptable only at low temperature and in the thick barrier limit.

Since there's a sharp change of phase in the normal region, a possible small phase gradient in the superconducting regions and the depression of pair potential at the interfaces can be ignored. An important measure of this is the comparison of the critical Josephson current to the bulk critical current. Since the bulk critical current[27] is of the order of $Nev_F\Delta_0/E_F$(N is the DOS at the Fermi level), the ratio of zero temperature critical Josephson current for a single SNS structure to that of bulk superconductors is of the order ξ_0/d, and at finite temperature this ratio becomes $(\xi_0/d)\exp(-d/\xi_T)$. So in the thick barrier limit, the self consistency condition for the negation of gradient and depression of pair potential in the electrodes is acceptable.

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