WEINGARTEN SURFACES ARISING FROM SOLITON THEORY

A THESIS

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS

AND THE INSTITUTE OF ENGINEERING AND SCIENCES

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

BY

ÖZGÜR CEYHAN

AUGUST, 1998

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Metin Gürses(Principal Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Alexander Klyachko

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ABSTRACT

WEINGARTEN SURFACES ARISING FROM SOLITON THEORY

Özgür Ceyhan
M. S. in Mathematics
Advisor: Prof. Dr. Metin Gürses
August, 1999

In this work we presented a method for constructing surfaces in $\mathbb{R}^3$ associated with the symmetries of Gauss-Mainardi-Codazzi equations. We show that among these surfaces the sphere has a unique role. Under constant gauge transformations all integrable equations are mapped to a sphere. Furthermore we prove that all compact surfaces generated by symmetries of the sine-Gordon equation are homeomorphic to sphere. We also construct some Weingarten surfaces arising from the deformations of sine-Gordon, sinh-Gordon, nonlinear Schrödinger and modified Korteweg-de Vries equations.

Keywords and Phrases: Solitons, integrable surfaces, Weingarten surfaces.

ÖZET

SOLİTON TEORİSİNDEN TÜRETİLEN WEINGARTEN YÜZEYLERİ

Özgür Ceyhan
Matematik Bölümü Yuksek Lisans
Danışman: Prof. Dr. Metin Gürses
Ağustos, 1999

Bu çalışmada Gauss-Mainardi-Codazzi denklemlerinin simetrileri ile bağıntılı $\mathbb{R}^3$ teki yüzeyleri türetmek için bir yöntem verildi. Bu yüzeyler arasında kürenin özel bir yeri olduğu belirlendi. Tüm entegre edilebilir denklemlerin sabit ayar dönüşümlerinden elde edilen yüzeylerin küre olduğu kanıtlandı. Ayrıca sine-Gordon denkleminin simetrileri kullanılarak türetilen tüm kompakt yüzeylerin küreye homeomorfik olduğu gösterildi. Sine-Gordon, sinh-Gordon, doğrusal olmayan Schrödinger ve değişik Korteweg-de Vries denklemlerinin simetrileri ile bağıntılı bazi Weingarten yüzeyleri verildi.

Anahtar Kelimeler ve İfadeler: Solitonlar, entegre edilebilir yüzeyler, Weingarten yüzeyleri.

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ACKNOWLEDGMENTS

I would like to express my sincere gratitude to Prof. Dr. Metin Gürses for his excellent supervision. His patience and expert guidence bring my research up to this point.

I am greatful to Burak Gürel who read large portions of manuscript at an early stage, and his detailed comments enhanced both the content and the style of the final product.

I owe special thanks to my family, in particular Metin Eşmen whose encouregement have supported my graduate study from its begining.

I am indebted to all my friends for all they have done for me.

As on several previous occasions, the most thanks goes to my best friend who offered encouragement, understanding, patience, and help in many different ways. Thanks, Nur.

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Chapter 1

Introduction

The latter period of the nineteenth century and the early part of this century saw a great deal of activity in the study of special classes of surfaces in three dimensional Euclidean space (see, e.g. [27]-[31]). Typical examples include minimal surfaces, surfaces of constant mean curvature and surfaces of constant Gaussian curvature. Gauss equations that describe surfaces in three dimensional space have been studied in detail from various points of view. One of the classical problems of differential geometry was the study of the connections between geometry of submanifolds and nonlinear partial differential equations (PDEs). Probably sine-Gordon and Liouville equations are the best known examples. They describe minimal and pseudospherical surfaces respectively. They arise as the compatibility condition of the Gauss-Weingarten equations of a surface under a suitable parametrization. At that time many features of integrability of the sine-Gordon, Liouville and some other integrable equations were discovered.

On the other hand, the works of Kruskal-Zabursky, Lax, AKNS, Zakharov-Shabat,... introduced a technique (inverse spectral transform) for solving nonlinear PDEs, in the 1960's (see, e.g. [21]-[25]). This method allows one to solve a number of nonlinear PDEs. Nonlinear PDEs integrable by the inverse spectral transformation possess some remarkable properties such as soliton solutions, an infinite number of conservation laws, infinite symmetry groups, special Hamiltonian structures,...

A key element of the inverse spectral transformation method is the representation (Lax representation) of the nonlinear PDE

$$U_{1,2} - U_{2,1} + [U_1, U_2] = 0$$

as a compatibility condition of certain system of linear equations

$$\Phi_{,k} = U_k \Phi, \quad k = 1, 2.$$

Lax representation has a transparent geometrical interpretation. We may identify these equations with Gauss-Mainardi-Codazzi (GMC) equations represented as the compatibility condition of linear equations for the moving frame (Gauss-Weingarten equations). Due to the analogy between GMC and Lax equations, for a long time, surface theory was used as a source of integrable equations (see e.g [8]-[20]). In the last decade, the attitude is to use soliton theory in understanding some local and global properties of surfaces, (e.g. [1]-[14]).

In this work we investigate the relationship between the generalized symmetries and the associated surfaces in $\mathbb{R}^3$. In chapter 2, readers are reminded of the basic notions and equations of differential geometry of surfaces. Sym's formulation of soliton surfaces and its recent generalization given by Fokas and Gelfand are presented in the first section of chapter 3. In the next section, a general method of constructing immersion functions by using the symmetries of GMC equations is discussed. In following sections of chapter 3, some local and global properties of particular surfaces are described. We show that surface associated with constant gauge transformation is a sphere and investigate the symmetry surfaces of the sine-Gordon equation. We show that compact, connected, oriented sine-Gordon surfaces are homeomorphic to sphere. In last chapter, we constructed several Weingarten surfaces arising from the symmetries of the sine-Gordon, sinh-Gordon, nonlinear Schröndinger and modified Korteweg-de Vries equations. In Appendix, Gauss-Mainardi-Codazzi-Ricci equations are given for higher dimensional embedded or immersed manifolds of arbitrary codimensions.

Chapter 2

Surfaces in $\mathbb{R}^3$

In this chapter we shall give a brief survey of two dimensional surfaces immersed in $\mathbb{R}^3$. For the Gauss-Mainardi-Codazzi equations, we use the corresponding equations given in Appendix D for dimension $m = 2$. For further details of two dimensional surfaces see [33, 34].

2.1 Elements of the Theory of Surfaces

Our interest is now directed toward some elementary concepts of surfaces immersed in $\mathbb{R}^3$.

Definition 2.1 *Let $M \subset \mathbb{R}^3$ be a surface, with the inclusion map $F : U \subset \mathbb{R}^2 \to \mathbb{R}^3$. Then the first fundamental form (or equivalently induced metric g) is $F^* < .,. >$ where $< .,. >$ is the usual inner product on $\mathbb{R}^3$.*

We write the first fundamental form tensor on M as

$$(ds_I)^2 = g_{11}dx^1 \otimes dx^1 + 2g_{12}dx^1 \otimes dx^2 + g_{22}dx^2 \otimes dx^2,$$

where we define functions $g_{ij},\ i,j = 1,2$ directly by using the local coordinates (x^1, x^2) of $U \subset \mathbb{R}^2$ with $F : U \to \mathbb{R}^3$

$$\begin{aligned} g_{ij} &= < \tfrac{\partial F}{\partial x^i}, \tfrac{\partial F}{\partial x^j} > \\ &= < F_{,i}, F_{,j} >, \quad i,j = 1,2. \end{aligned}$$

In the sequel we shall use lower indices ",i" for the differentiation with respect to the coordinate x^i.

We next deal with the properties of the Gauss map $\nu : M \to S^2$, namely a unit normal differentiable vector field, which can at least be defined in a neighborhood of each point $p \in M$. (M is assumed to be orientable)

Definition 2.2 *In terms of ν, the second fundamental form Ω on M is defined as*

$$\begin{aligned} \Omega(p)(X,Y) &= \prec -d\nu(X), Y >_p \\ &= < -\nu_*(X), Y >_p . \end{aligned}$$

In particular, by considering an immersion $F : U \to \mathbb{R}^3$, for $U \subset \mathbb{R}^2$, the second fundamental form can be defined directly on U in terms of local coordinates by

$$b_{ij} =< -\nu_{,i}, F_{,j} >=< \nu, F_{,ij} > .$$

Definition 2.3 *Let the matrix S with the coefficients $b^i_j = g^{ik}b_{kj}$ represent the "shape operator". Eigenvalues k_1 and k_2 of S, are defined as the principle curvatures and then Gauss curvature K and mean curvature H are defined as*

$$\begin{aligned} K &= det(S) = k_1 k_2, \\ H &= tr(S) = k_1 + k_2. \end{aligned}$$

Definition 2.4 *If there exists a function f such that $f(K,H) = 0$ (or equivalently $\tilde{f}(k_1,k_2) = 0$) then the corresponding surface is called a Weingarten surface.*

2.2 Gauss and Mainardi-Codazzi Equations

Let $F : U \subset \mathbb{R}^2 \to M \subset \mathbb{R}^3$ be a local parametrization . Then it is possible to assign a trihedron to every point $p \in M$ given by the vectors $F_{,1}, F_{,2}$ and ν.

We may express motion of this frame along M by the Gauss and Weingarten equations

$$F_{,ij} = \Gamma^k_{ij} F_{,k} + b_{ij}, \nu \tag{2.1}$$
$$\nu_{,i} = g^{hj} b_{ij} F_{,h} \quad , \quad h,i,j,k = 1,2, \tag{2.2}$$

where summation over repeated indices is assumed.

The above set of partial differential equations are integrable if certain compatibility conditions are satisfied. Setting $F_{,ijk} = F_{,ikj}$ and assuming the linear independence of $F_{,1}, F_{,2}$ and ν, we get:

Lemma 2.5 *Integrability conditions of (2.1) and (2.2) reduce to set of equations:*

$$\Gamma^l_{ik,j} - \Gamma^l_{ij,k} + \Gamma^h_{ik}\Gamma^l_{hj} - \Gamma^h_{ij}\Gamma^l_{hk} = b_{ik}b^l_j - b_{ij}b^l_k, \tag{2.3}$$
$$b_{ik,j} - b_{ij,k} + \Gamma^h_{ik} b_{hj} - \Gamma^h_{ij} b_{hk} = 0. \tag{2.4}$$

Proof: m=2 case of theorem (A.3), (A.4) and lemma (A.6). □

Right hand side of the equation (2.3) is the Riemann curvature tensor. Then

$$R_{1212} = b_{11}b_{22} - b_{12}b_{12},$$

is the Gauss equation for surfaces in $\mathbb{R}^3$. Hence that intrinsically defined Gauss curvature K is given by (see [32, 33]):

$$K = \frac{< R(F_{,1}, F_{,2})F_{,2}, F_{,1} >}{< F_{,1}, F_{,1} >< F_{,2}, F_{,2} > - < F_{,1}, F_{,2} >^2} = \frac{b_{11}b_{22} - b_{12}b_{12}}{g_{11}g_{22} - g_{12}g_{12}},$$

and if we take a look at equation (2.4), it reduces to the following set of equations

$$\begin{aligned} b_{12,1} - b_{11,2} + \Gamma^h_{12} b_{h1} - \Gamma^h_{11} b_{h2} &= 0, \\ b_{22,1} - b_{21,2} + \Gamma^h_{22} b_{h1} - \Gamma^h_{21} b_2 &= 0, \end{aligned} \tag{2.5}$$

which are called Mainardi-Codazzi equations. The integrability conditions $\nu_{,ij} = \nu_{,ji}$ are satisfied automatically by the Mainardi-Codazzi equations.

Example 1: (Surfaces of Revolution) Let $M \subset \mathbb{R}^3$ be the set obtained by rotating a regular plane curve C about an axis in the plane which does not meet the curve; let the xz plane be the plane of curve the C and the z axis be

the rotation axis. Let C be parametrized by $\alpha(x) = (\phi(x), 0, \psi(x))$. We can compute the Christoffel symbols for a surface of revolution parametrized by:

$$F(x^1, x^2) = (\phi(x^2)\cos x^1, \phi(x^2)\sin x^1, \psi(x^2)) \ , \ \phi(x^2) \neq 0.$$

Since

$$g_{11} = \phi(x^2)^2 \ , \ g_{12} = 0 \ , \ g_{22} = \phi(x^2)^2 + \psi(x^2)^2,$$

we obtain the Christoffel symbols to be:

$$\begin{aligned}
\Gamma^1_{11} &= 0 \ , \ \Gamma^2_{11} = -\frac{\phi\phi'}{(\phi')^2 + (\psi')^2}, \\
\Gamma^1_{12} &= \frac{\phi\phi'}{\phi^2} \ , \ \Gamma^2_{12} = 0, \\
\Gamma^1_{22} &= 0 \ , \ \Gamma^2_{22} = \frac{\phi'\phi'' + \psi'\psi''}{(\phi')^2 + (\psi')^2}.
\end{aligned}$$

If we let x^2 be the arclength parameter of the curve (i.e. $(\phi')^2 + (\psi')^2 = 1$), then the Gauss and mean curvatures are given by

$$K = -\frac{\phi''}{\phi} \ , \ H = \frac{1}{2}\frac{-\psi' + \phi(\psi'\phi'' - \psi''\phi')}{\phi}$$

and the Gauss equation (2.3) reduces to

$$\phi'' + K\phi = 0.$$

Mainardi-Codazzi equations (2.5) are identically satisfied.

Chapter 3

Soliton Surfaces

It is well a known fact that existence of Lax pair for a differential equation entails existence of infinitely many symmetries. The symmetry group of system of PDEs is the group of transformations that map solutions of the system to other solutions. Here in this chapter we shall present an explicit formulation of the immersion functions that associated with each symmetry of a given soliton equation.

3.1 Surfaces Immersed in $\mathbb{R}^3$ as Surfaces in Lie Algebras

Our goal in this section is to reformulate the classical theory of surfaces in a form familiar to the soliton theory, which makes an application of the analytical methods of this theory to integrable cases possible.

Formulas for the moving frames associated with integrable equations can be integrated. This issue was first suggested by A. Sym [8]-[14] and generalized by Fokas and Gel'fand [2], Fokas, Gel'fand, Finkel, Liu [3] and Cieśliński [6]. This approaches were applied to several soliton equations [1]-[14].

In section (3.1.1), the moving frame for a general surface is described in terms of $su(2)$ algebra. In the sections (3.1.2) and (3.1.3), Sym, Fokas-Gel'fand, and Cieśliński approaches are presented.

3.1.1 Immersions in $\mathbb{R}^3$

Let $F : U \subset \mathbb{R}^2 \to M \subset \mathbb{R}^3$ be an immersion and $\nu(x^1, x^2)$ be the unit normal field along M. Then $F_{,1}, F_{,2}$ and ν define a basis in $\mathbb{R}^3$. As we have seen in previous chapter, the motion of this basis on M is characterized by Gauss-Weingarten equations (2.1), (2.2). Now let us consider following orthonormal basis

$$e_1 = \frac{F_{,1}}{\sqrt{g_{11}}} \quad , \quad e_2 = \frac{g_{11}F_{,2} - g_{12}F_{,1}}{\sqrt{g_{11}det(g)}}.$$

Let us consider this moving frame on M in 3×3 matrix form $E^\top = (e_1, e_2, \nu)$. Then the Gauss-Weingarten equations for the frame E become

$$E_{,k} = \Lambda_k E \quad , \quad k = 1, 2, \tag{3.1}$$

and Gauss-Mainardi-Codazzi equations are

$$\Lambda_{1,2} - \Lambda_{2,1} + [\Lambda_1, \Lambda_2] = 0, \tag{3.2}$$

where the matrices E and Λ_k, $k = 1, 2$ have value in $SO(3)$ and $so(3)$ respectively. It is convenient to use the isomorphism $so(3) \simeq su(2)$ to rewrite equation (3.1) in terms 2×2 complex matrices. Let $\Phi(x^1, x^2)$ be an $SU(2)$ valued function, then we can write these matrices explicitly as follows

$$\Phi_{,k} \;=\; U_k \, \Phi \quad , \quad k = 1, 2, \tag{3.3}$$

where

$$U_k = -\frac{i}{2}\begin{pmatrix} \alpha_k & \beta_k \\ \overline{\beta_k} & -\alpha_k \end{pmatrix} \quad , \quad k = 1, 2,$$

and

$$\alpha_k = \frac{\Gamma^2_{1k}\sqrt{det(g)}}{g_{11}} \quad , \quad \beta_k = \frac{g_{11}b_{2k} - g_{12}b_{1k}}{\sqrt{g_{11}det(g)}} - \frac{ib_{1k}}{\sqrt{g_{11}}}.$$

We rewrite the compatibility conditions given by (3.2) as

$$U_{1,2} - U_{2,1} - [U_1, U_2] = 0. \tag{3.4}$$

Finally we summarize the result given in above arguments with following theorem.

Theorem 3.1 *[2] Let $U_k = U_k(x^1, x^2) \in su(2), k = 1, 2$ be differentiable functions of x^1, x^2 in some neighborhood of $\mathbb{R}^2$. Assume that each U_k satisfy (3.4) then equations (3.3) define a 2-dimensional surface $\Phi \in SU(2)$.*

Remark 3.2 *In the context of integrable systems equation (3.3) is known as the Lax equation and equation (3.4) as the zero curvature condition. However, in order to apply inverse spectral transform one needs to insert a spectral parameter in (3.3), In the following sections we shall consider such cases.*

3.1.2 Soliton Surfaces Approach

An interesting connection between classical geometry of surfaces and the symmetries of soliton equations is first given by Sym in [8]-[14].

Theorem 3.3 *[8] Let $U_k = U_k(x^1, x^2, \lambda) \in su(2), k = 1, 2$ be differentiable functions of x^1, x^2 and λ which satisfy (3.3) and (3.4). Assume that Gauss-Mainardi-Codazzi equations (3.4) are independent of λ. Then*

$$F_{,k} = \Phi^{-1}\frac{\partial U_k}{\partial \lambda}\Phi, \quad k = 1, 2 \tag{3.5}$$

define a tangent space and

$$F = \Phi^{-1}\frac{\partial \Phi}{\partial \lambda} + C \tag{3.6}$$

defines an explicit immersion function of the surface associated with the λ translation symmetry of equation the (3.4) where C is constant $su(2)$ matrix.

Proof: We define an $SU(2)$ valued function by $\tilde{F}(x, \lambda) = \Phi^{-1}(x, \lambda_0)\Phi(x, \lambda)$ which is known as the Pohlmeyer transformation. The equation (3.3) yields

$$\tilde{F}_{,k} = \Phi^{-1}(x, \lambda_0)[U_{k,\lambda}(x, \lambda_o)(\lambda - \lambda_0) + \cdots]\Phi(x, \lambda_0)\tilde{F} \tag{3.7}$$

whose integrability conditions of equation (3.7) (i.e. $\tilde{F}_{,kl} = \tilde{F}_{,lk}$) are

$$(\Phi^{-1}U_{k,\lambda}\Phi)_{,l} = (\Phi^{-1}U_{l,\lambda}\Phi)_{,k}. \tag{3.8}$$

The equation (3.8) implies, the existence of an $su(2)$ valued function $F = F(x, \lambda)$ such that

$$F_{,k} = \Phi^{-1}U_{k,\lambda}\Phi \ , \ k = 1, 2.$$

The equation (3.5) can be integrated to get

$$F = \Phi^{-1}\Phi_{,\lambda} + C,$$

where C is a constant $su(2)$ matrix.Adding term C is equivalent to a rigid motion. Hence we may take $C = 0$. The equation (3.6) is interpreted as a coordinate representation of the λ family of the surfaces in $su(2)$. The Gauss-Weingarten equations are equivalent to

$$F_{,kl} = \Phi^{-1}(U_{k,l\lambda} + [U_{k,\lambda}, U_l])\Phi$$

and the Gauss-Mainardi-Codazzi equations ($F_{,jkl} = F_{,jlk}$)

$$(U_{1,2} - U_{2,1} + [U_1, U_2])_{,\lambda} = 0$$

are identically satisfied by virtue of (3.4). □

By using the scalar product on $su(2)$

$$< A, B >= -\frac{1}{2} trace(AB)\ ,\ |A| = \sqrt{< A, A >}, \tag{3.9}$$

induced metric $g_{ij} =< F_{,i}, F_{,j} >=< U_{i,\lambda}, U_{j,\lambda} >$, $i, j = 1, 2$ on the surface is defined. And the frame on the surface $(F_{,1}, F_{,2}, \nu)$ is determined by the normal vector

$$\nu = \frac{[F_{,1}, F_{,2}]}{\sqrt{det(g)}}.$$

3.1.3 The Fokas-Gelfand Approach

Now we will give the generalization of Sym's formula orginally formulated in [2].

Theorem 3.4 *[2] Let $U_k = U_k(x^1, x^2, \lambda) \in su(2), k = 1, 2$ be differentiable functions of x^1, x^2 and λ which satisfy (3.3) and (3.4). Assume that the equation (3.4) is independent of λ. Consider the function $F \in su(2)$ implicitly given by*

$$F_{,k} = \Phi^{-1} A_k \Phi, \quad k = 1, 2 \tag{3.10}$$

where $A_k \in su(2)$, $k = 1, 2$. Then F defines a surfaces in $su(2)$ iff the equations (3.10) are compatible i.e.

$$A_{1,2} - A_{2,1} + [A_1, U_2] + [U_1, A_2] = 0 \tag{3.11}$$

is satisfied.

Corollary 3.5 *Let us define a frame on the surface which satisfies the conditions of theorem (3.4), i.e.*

$$F_{,1} = \Phi^{-1} A_1 \Phi \ , \ F_{,2} = \Phi^{-1} A_2 \Phi \ , \ \nu = \Phi^{-1} A_3 \Phi$$

where

$$A_3 = \frac{[A_1, A_2]}{|[A_1, A_2]|}.$$

Then the first and second fundamental forms can be expressed explicitly as

$$\begin{aligned} (ds_I)^2 = & < A_1, A_1 > (dx^1)^2 + 2 < A_1, A_2 > dx^1 dx^2 \\ + & < A_2, A_2 > (dx^2)^2, \\ (ds_{II})^2 = & < A_{1,1} + [A_1, U_1], A_3 > (dx^1)^2 \\ + & 2 < A_{1,2} + [A_1, U_2], A_3 > dx^1 dx^2 \\ + & < A_{2,2} + [A_2, U_2], A_3 > (dx^2)^2. \end{aligned} \tag{3.12}$$

The theorem (3.4) characterizes surfaces in terms of an arbitrary parametrization. The classical formulations of well known geometrical parametrizations can be obtain as particular cases of this theorem. An example of this theorem is given below.

Theorem 3.6 *[2] Let $\sigma_j, j = 1, 2, 3$ denote the Pauli spin matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.13}$$

Consider an arbitrary immersion function F in $\mathbb{R}^3$ implicitly defined by

$$F_{,1} = -i\Phi^{-1} a \sigma_1 \Phi \ , \ F_{,2} = -i\Phi^{-1}(b_1\sigma_1 + b_2\sigma_2)\Phi, \ a \neq 0, \ b_2 \neq 0,$$

where

$$\Phi_{,k} = U_k(x^1, x^2, \lambda)\Phi, \ k = 1, 2$$

are compatible (i.e. satisfy equation (3.3)). Then the functions

$$U_k(x^1, x^2, \lambda) = -\frac{i}{2} \sum_{\alpha=1}^{3} U_k^{\alpha}(x^1, x^2, \lambda)\sigma_{\alpha} \ , \ k = 1, 2 \tag{3.14}$$

are defined by

$$\begin{aligned} U_1^3 &= \frac{1}{b_2}(\frac{\partial a}{\partial x^2} - \frac{\partial b_1}{\partial x^1}), \\ U_2^2 &= \frac{1}{a}(b_1 U_1^2 - b_2 U_1^1), \\ U_2^3 &= \frac{1}{ab_2}(b_1\frac{\partial a}{\partial x^2} - b_1\frac{\partial b_1}{\partial x^1} + b_2\frac{\partial b_2}{\partial x^1}). \end{aligned} \tag{3.15}$$

The first and second fundamental forms of the surface are

$$\begin{aligned} (ds_I)^2 &= a^2(dx^1)^2 + 2ab_1 dx^1 dx^2 + ((b_1)^2 + (b_2)^2)(dx^2)^2, \\ (ds_{II})^2 &= aU_1^2(dx^1)^2 + 2aU_2^2 dx^1 dx^2 + (b_1 U_2^2 - b_2 U_2^1)(dx^2)^2. \end{aligned} \tag{3.16}$$

This surface is unique up to a rigid motion in space. The Gauss and mean curvatures are

$$K = -(\frac{U_1^1}{a}) + \frac{U_1^2}{a}(\frac{b_1 U_1^1 - aU_2^1}{ab_2}) \ , \ H = \frac{U_1^2}{a} + \frac{b_1 U_1^1 - aU_2^1}{ab_2}. \tag{3.17}$$

A frame on this surface is given by $F_{,1}, F_{,2}$ and $\nu = -i\Phi^{-1}\sigma_3\Phi$.

Proof: These results follow from theorem (3.4) with the choices $A_1 = -ia\sigma_1, A_2 = -i(b_1\sigma_1 + b_2\sigma_2)$, and then the equation (3.11) become

$$aU_2^2 + b_2 U_1^1 - b_1 U_1^2 = 0 \ , \ b_{1,1} + b_2 U_1^3 - a_{,2} = 0 \ , \ b_{2,1} + aU_2^3 - b_1 U_1^3 = 0.$$

Solving these for U_1^3, U_2^2 and U_2^3 we obtain equation (3.15). Using the results of corollary (3.5) we find equations (3.16) and (3.17). □

Example 1:(Parametric Lines of Curvature) [2] Letting $b_1 = U_2^2 = 0$, and introducing the notations $b = b_2, f = \frac{U_1^2}{a}, h = -\frac{U_2^1}{b}$, the Gauss-Mainardi-Codazzi equations (3.4) become

$$\frac{\partial}{\partial x^2}(\frac{1}{b}\frac{\partial a}{\partial x^2}) + \frac{\partial}{\partial x^1}(\frac{1}{a}\frac{\partial b}{\partial x^1}) + abfh = 0,$$

$$\frac{\partial}{\partial x^1}(bh) - f\frac{\partial b}{\partial x^1} = 0,$$

$$\frac{\partial}{\partial x^2}(af) - h\frac{\partial a}{\partial x^2} = 0.$$

The first and second fundamental forms are

$$(ds_I)^2 = a^2(dx^1)^2 + b^2(dx^2)^2 \ , \ (ds_{II})^2 = a^2 f(dx^1)^2 + b^2 h(dx^2)^2.$$

A frame on this surface is determined by

$$F_{,1} = -i\Phi^{-1}a\sigma_1\Phi \ , \ F_{,2} = -i\Phi^{-2}b\sigma_2\Phi \ , \ \nu = -i\Phi^{-1}\sigma_3\Phi.$$

The Gauss and mean curvatures of this surface are

$$K = fh \quad , \quad H = f + h.$$

3.2 A Generalized Immersion Function

An important step in applying the outlined method in section (3.1.3) is to solve the following problem:

For a given differential equation in the form (3.4) with the Lax pair (3.3) find a class of functions A_1, A_2 for which one can construct explicitly the immersion function F and hence an associated surface in $\mathbb{R}^3$.

One of the solutions of this problem is given in section (3.1.2) via Sym (or Sym-Tafel) formula (2.3.5) and (2.3.6). A generalization of Sym's formula was given by Fokas and Gelfand in [2]. A further generalization of this formula can be found in [3] and [6].

Proposition 3.7 *Suppose that Φ is a $SU(2)$ valued solution of Lax equations (3.3) for a given differential equation (3.4). Let δ be an operator representing the infinitesimal transformations. Then the equations (3.10) with*

$$A_k = \delta U_k + ([\partial_{x^k}, \delta]\Phi)\Phi^{-1}, \quad k = 1, 2 \tag{3.18}$$

are compatible and F is given explicity by

$$F = \Phi^{-1}\delta\Phi. \tag{3.19}$$

Proof: The compatibility conditions can be easily verified taking into account that $\Phi_{,k} = U_k\Phi$, $(\Phi^{-1})_{,k} = -\Phi^{-1}U_k$ and $U_{k,l} - U_{l,k} + [U_k, U_l] = 0$ for $k, l = 1, 2$. Differentiating F we obtain the above expression for A_1, A_2. □

All known symmetries of an integrable equation can be considered as particular cases of the δ. For instance $\delta = \partial_{x^1}$ is the infinitesimal generator of

the symmetry corresponding to translation along x^1 direction. A nontrivial example is $\delta = R$ where R is the recursion operator for (3.4) (if it exists).

Let us reformulate the proposition in a detailed way for the case in which the equation (3.4) reduce to a single partial differential equation.

Definition 3.8 *The Frèchet derivative of the differential function $U[\theta]$ in the direction of ϕ, denoted by $U'(\phi)$, is*

$$U'(\phi) = \frac{\partial}{\partial \epsilon} U[\theta + \epsilon\phi]\Big|_{\epsilon=0}$$

where ϵ is a real parameter.

Theorem 3.9 *[3] Let $U_k, k = 1,2$ be parametrized by λ and the scalar function $\theta(u,v)$, where the compatibility equation (3.4) reduces to a single PDE of $\theta(u,v)$ independent of λ. Define $A_k = A_k(x^1, x^2, \lambda) \in su(2)$ by*

$$A_k = \alpha \frac{\partial U_k}{\partial \lambda} + \frac{\partial M}{\partial x^k} + [M, U_k] + U_k'(\phi) \ , \ k = 1,2, \tag{3.20}$$

where $\alpha(\lambda)$ is an arbitrary scalar function of λ, $M(x^1, x^2, \lambda) \in su(2)$ is an arbitrary function of x^1, x^2, λ, the scalar $\phi(u,v)$ is a symmetry of the equation (3.4) and prime denotes Frèchet differentiation . Then there exists a family of surfaces with immersions $F(x^1, x^2, \lambda) \in su(2)$ in terms of A_1, A_2 and Φ. Furthermore, F is given up to an additive constant $C(\lambda) \in su(2)$ by

$$F = \Phi^{-1}(\alpha \frac{\partial \Phi}{\partial \lambda} + M\Phi + \Phi'(\phi)). \tag{3.21}$$

Proof: Theorem (3.9) is a special case of lemma (3.7) where $[\partial_k, \delta] = 0$. It can be verified directly if $U_k, k = 1,2$ satisfy the equation (3.3) and if ϕ is a symmetry of an integrable nonlinear PDE satisfied by θ, then the functions $A_k, k = 1,2$ are defined by the equations (3.20). This implies the existence of the immersion function F.

It is possible to establish this result avoiding most of the computations. Extending the definition of a symmetry from scalar functions to functions in Lie algebras, it follows that the pair A_1, A_2 is a symmetry of the pair U_1, U_2. Indeed replacing U_k by $U_k + \epsilon A_k$ for $k = 1,2$, the $O(\epsilon)$ term of the resulting

equation is (3.10). Thus finding a pair $A_k, k = 1,2$ is equivalent to finding a symmetry for the pair $U_k, k = 1,2$. The pair A_1, A_2 defined by equation (3.20) corresponds to three different types of symmetries:

(i) The integrable equation (3.4) is independent of λ, thus λ translation is a trivial symmetry for this equation. This yields ,

$$A_k = \alpha \frac{\partial U_k}{\partial \lambda} \ , \ k = 1,2.$$

where $\alpha = \alpha(\lambda)$ is an arbitrary function of λ.

(ii) Equation (3.4) is invariant under the gauge transformation $\Phi \rightarrow S\Phi$ and

$$U_k \rightarrow SU_kS^{-1} + \frac{\partial S}{\partial x^k}S^{-1} \ , \ k = 1,2. \tag{3.22}$$

Letting $S = I + \epsilon M$, where I denotes the identity matrix, the expression in (3.22) become,

$$U_k \rightarrow U_k + \epsilon(\frac{\partial M}{\partial x^k} + [M, U_k]) + O(\epsilon^2) \ , \ k = 1,2.$$

Thus

$$A_k = \frac{\partial M}{\partial x^k} + [M, U_k] \ , \ k = 1,2.$$

(iii) Let ϕ be a symmetry of the equation of (3.4). Then Frèchet differentiation gives

$$U_k \rightarrow U_k(\theta + \epsilon\phi) = U_k + \epsilon U_k'(\phi) + O(\epsilon^2), \quad k = 1,2$$

which implies:

$$A_k = U_k'(\phi), \quad k = 1,2.$$

Linear combination of the above three symmetries gives rise to equations (3.20).□

Example 2: (Theorem (1.2) in [2]) Let

$$M = f_1(x^1, x^2)U_1 + f_2(x^1, x^2)U_2 + M_0$$

where $M_0 \in su(2)$ is a constant matrix and $\alpha(\lambda), f_1(x^1, x^2), f_2(x^1, x^2)$ are scalar functions with the arguments indicated. Then equations (2.3.20) become

$$A_1 \ = \ \alpha(\lambda)\frac{\partial U_1}{\partial \lambda} + \frac{\partial f_1}{\partial x^1}U_1 + f_1\frac{\partial U_1}{\partial x^1} + \frac{\partial f_2}{\partial x^1}U_2$$

$$
\begin{aligned}
& + f_2 \frac{\partial U_1}{\partial x^2} + f_3 [M_0, U_1] + U_1'(\phi), \\
A_2 \;=\; & \alpha(\lambda) \frac{\partial U_2}{\partial \lambda} + \frac{\partial f_1}{\partial x^2} U_1 + f_1 \frac{\partial U_2}{\partial x^1} + \frac{\partial f_2}{\partial x^2} U_2 + \\
& f_2 \frac{\partial U_2}{\partial x^2} + f_3 [M_0, U_2] + U_2'(\phi),
\end{aligned}
$$

and immersion function takes the form

$$
F = \Phi^{-1} \left(\alpha \frac{\partial \Phi}{\partial \lambda} + f_1 \, \partial_{x^1} \Phi + f_2 \partial_{x^2} \Phi + M_0 \Phi + \Phi'(\phi) \right).
$$

Example 3: (Parallel Surfaces) if $\tilde{F}$ in $su(2)$ is parallel to F then $F - \tilde{F} = a\nu = a\Phi^{-1} A_3 \Phi$, where ν is the unit normal vector (i.e. $< A_3, A_3 >= 1$) to the surface F (also to $\tilde{F}$) and a is a constant (distance between surfaces). One can easily observe that parallel surfaces can be given by virtue of the generalized immersion function. It is enough to set M and $\phi = 0$ and

$$
\tilde{F} = \Phi^{-1} \delta \Phi \quad , \quad a\nu = a\Phi^{-1} A_3 \Phi = \Phi^{-1} M_0 \Phi
$$

Remark 3.10 *In sections (3.1) and (3.2) we have considered surfaces in* $\mathbb{R}^3$ *as surfaces in* $su(2)$ *algebra. The whole approach for immersions of dimension* $\dim M > 2$ *becomes considerably more difficult. But the notion given in proposition (3.7) can be extended to immersions into a lie algebra* $\mathfrak{g}$ *(let* $\dim \mathfrak{g} = m$*) of higher codimension. Let*

$$
\Phi_{,k} = U_k \Phi \ , \ k = 1, \cdots, n < m \tag{3.23}
$$

denote the system of equations where $U_k(x, \lambda)$ *are smooth functions of* λ *and coordinates* $x = (x^1, \cdots, x^n)$*. The functions* Φ *take values in a semisimple matrix group* G *and* $U_k \in \mathfrak{g}$*, the lie algebra of* G*. The integrability conditions of this overdetermined system of equations require that*

$$
U_{k,l} - U_{l,k} + [U_k, U_l] = 0, \ k < l = 1, \cdots, n \tag{3.24}
$$

Equations (3.23) can be interpreted as defining a G-valued connection (G representation of Gauss-Weingarten equations (A.4) and (A.5)) with equation (3.24) (Gauss-Mainardi-Codazzi-Ricci equations in this representation).

We now need a prescription for constructing an immersion F *associted with the symmetry of (3.24) . Introduce an arbitrary variation* δ*. The from (3.23) we get*

$$
(\delta \Phi)_{,k} = \delta U_k \Phi + U_k \delta \Phi
$$

and consequently

$$(\Phi^{-1}\delta\Phi)_{,k} = \Phi^{-1}(\delta U_k + [\partial_k, \delta])\Phi$$

so that there exists a function $F : \mathbb{R}^n \to \mathfrak{g}$ *such that*

$$F_{,k} = \Phi^{-1}\delta U_k \Phi$$

and

$$F = \Phi^{-1}\delta\Phi + C, \quad C \in \mathfrak{g}.$$

Notice that $C = C(\lambda)$*is arbitrary in this last equation. For this matrix group* G *we calculate the geometrical quantities by using nondegenerate invariant bilinear form* $< .,. >: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$

$$g_{kl} =< F_{,k}, F_{,l} >=< \delta U_k, \delta U_l >, \quad k, l = 1, \cdots, n$$

and by introducing normal vector fields ν_r, $r = n+1, \cdots, m = dim\ G$

$$\beta^k_{lr} =< -U_l \delta U_k + \delta U_{k,l} + \delta U_l U_k, \nu_r >$$

This formulation gives whole algorithm for constructing the soliton immersions:

(i) Find a soliton system with Lax represenntation (3.23) for which $n <$ *dim* $\mathfrak{g}$.

(ii) Construct an orthonormal basis for g.

(iii) Construct a function $F : \mathbb{R}^n \to \mathfrak{g}$ *from a variation* $\delta : G \to TG$ *which defines a canonical map* $G \to \mathfrak{g}$ *under left translation.*

This approach is similar to the one developed by Sym (Sym considered only the immersed submanifold with dimension 2), and the recent work of Dodd gives this construction for arbitrary dimension and codimension, [5].

3.3 Immersions Associated With The Symmetries of The Integrable Gauss-Mainardi-Codazzi Equations

The theorem (3.9) provides an algorithmic approach to construct the surface by starting from a suitable Lax pair. We shall apply this technique to construct

surfaces associated with the constant gauge transformation for arbitrary Lax equations and associated with symmetries of the Sine-Gordon equation

3.3.1 Immersions Associated With the Constant Gauge Transformations

Now let us work out the surfaces generated by the constant $SU(2)$ rotations of Φ, i.e. by a constant $su(2)$ matrix M_0

Theorem 3.11 *[1] Let $A_k = [M_0, U_k], k = 1,2$, where $M_0 \in su(2)$ is a constant matrix. Then $K = \frac{1}{|M_0|^2}$ and $H = \frac{-2\varepsilon}{|M_0|}$, where $\varepsilon = \pm 1$ and $|M_0| = \sqrt{< M_0, M_0 >}$. Hence all such deformed surfaces are spheres with radii $|M_0|$ where the immersion function is*

$$F = \Phi^{-1} M_0 \Phi.$$

Proof: Let $U_k = \frac{i}{2}\sum_{\alpha=1}^{3} U_k^\alpha \sigma_\alpha$ for $k = 1,2$ be any Lax pair and $M_0 = \frac{i}{2}\sum_{\alpha=1}^{3} m^\alpha \sigma_\alpha$ be a constant $su(2)$ matrix, where σ_α, $j = 1,2,3$ denotes Pauli spin matrices. Since $[\sigma_\alpha, \sigma_\beta] = 2i\epsilon_{\alpha\beta\gamma}\sigma_\gamma$, we have

$$A_k = [M_0, U_k] = -\frac{i}{2}\epsilon_{\alpha\beta\gamma}\, m^\alpha\, U_k^\beta\, \sigma_\gamma.$$

To calculate the normal vector field $\nu = \Phi^{-1}A_3\Phi$, we need $[A_1, A_2]$

$$\begin{aligned}
[A_1, A_2] &= -\frac{i}{2}\,\epsilon_{\alpha\beta\gamma}\,\epsilon_{\alpha\delta\zeta}\,\epsilon_{\eta\xi\beta}\, m^\delta\, U_1^\zeta\, m^\eta\, U_2^\xi\, \sigma_\gamma \\
&= -\frac{i}{2}\,(\delta_{\beta\delta}\delta_{\gamma\zeta} - \delta_{\beta\zeta}\,\delta_{\gamma\delta})\,\epsilon_{\eta\xi\beta}\, m^\delta\, U_1^\zeta\, m^\eta\, U_2^\xi\, \sigma_\gamma \\
&= \tau M_0
\end{aligned}$$

since

$$\epsilon_{\eta\xi\delta}\, m^\delta\, U_1^\zeta\, m^\eta\, U_2^\xi\, \sigma_\zeta = (< U_2, [M_0, M_0] >)\, U_1 = 0$$

and

$$-\frac{i}{2}\epsilon_{\eta\xi\zeta}\, m^\delta\, U_1^\zeta\, m^\eta\, U_2^\xi \sigma_\delta = -4(< M_0, [U_2, U_1] >)\, M_0. \tag{3.25}$$

Letting $\varepsilon = \frac{\tau}{|\tau|}$, we find

$$A_3 = \frac{\varepsilon}{|M_0|}\, M_0, \tag{3.26}$$

hence

$$< A_{1,1}, A_3 >=< A_{1,2}, A_3 >=< A_{2,2}, A_3 >= 0. \tag{3.27}$$

Using these equations it follows that

$$d_{ij} = -\frac{\varepsilon}{|M_0|}\, g_{ij}. \tag{3.28}$$

Hence $S = \frac{-\varepsilon}{|M_0|}\, I$, where I is the identity matrix. Thus

$$K = det(S) = \frac{1}{|M_0|^2}, \tag{3.29}$$

$$H = tr(S) = -\frac{2\,\varepsilon}{|M_0|}, \tag{3.30}$$

claim follows. □

This result is quite interesting. Lax pair is arbitrary so that under the rigid $SU(2)$ rotations all integrable equations are mapped into a sphere.

3.3.2 Immersions Associated With The Sine-Gordon Equation

Both in the classical differential geometry and integrable nonlinear partial differential equations, sine-Gordon equation for smooth function $\theta(x^1, x^2)$

$$\frac{\partial^2 \theta}{\partial x^1 \partial x^2} = \sin\theta, \tag{3.31}$$

is of special interest. The Gauss-Mainardi-Codazzi system of any pseudospherical surface endowed with the so-called asymptotic coordinates reduces to the sine-Gordon equation.

The Lax pair for the sine-Gordon equation is given by (3.3) with

$$U_1 = \frac{i}{2}\left(-\theta_{,1}\,\sigma_1 + \lambda\,\sigma_3\right)\ ,\ U_2 = \frac{i}{2\lambda}(\sin\theta\sigma_2 - \cos\theta\sigma_3), \tag{3.32}$$

where $\theta(x^1, x^2) \in \mathbb{R}$ and λ is an arbitrary constant. Let φ be a symmetry of equation (3.31), i.e. let φ be a solution of

$$\frac{\partial^2 \varphi}{\partial x^1 \partial x^2} = \varphi\,\cos\theta. \tag{3.33}$$

There exists infinitely many explicit solutions of equation (3.33) in terms of θ and its higher derivatives. The first few are (see [26])

$$\theta_{,1}\ ,\ \theta_{,2}\ ,\ \theta_{,111}+\frac{\theta_{,1}^3}{2}\ ,\ \theta_{,222}+\frac{\theta_{,2}^3}{2}\ ,\ \cdots \tag{3.34}$$

starting from the third one all such solutions are called the generalized symmetries of (3.31). Then for each φ theorem (3.9) (with $\alpha = 0$, $M = 0$) implies a surface constructed by

$$A_1 = -\frac{i}{2}\frac{\partial\varphi}{\partial x^1}\sigma_1\ ,\ A_2 = \frac{i}{2\lambda}\varphi(\cos\theta\,\sigma_2 + \sin\theta\,\sigma_3). \tag{3.35}$$

We now study the surfaces corresponding to these generalized symmetries.

Lemma 3.12 *[3] Let M be the surface generated by a generalized symmetry of the sine-Gordon equation. That is, let M be the surface generated by U_k and $A_k, k = 1,2$ defined by equations (3.32) and (3.35) respectively. The first and second fundamental forms, the Gaussian and the mean curvatures of this surface*

$$F \ = \ \Phi^{-1}\,\Phi'(\varphi) \tag{3.36}$$

are given by

$$\begin{aligned} ds_I^2 &= \tfrac{1}{4}(\varphi_{,1}^2\,(dx^1)^2 + \tfrac{1}{\lambda^2}\,\varphi^2\,(dx^2)^2), \\ ds_{II}^2 &= \tfrac{1}{2}(\lambda\varphi_{,1}\sin\theta(dx^1)^2 + \tfrac{1}{\lambda}\varphi\theta_{,2}(dx^2)^2), \end{aligned} \tag{3.37}$$

$$K = \frac{4\lambda^2\theta_{,2}\sin\theta}{\varphi\varphi_{,1}}\ ,\ H = \frac{2\lambda(\varphi_{,1}\theta_{,2} + \varphi\sin\theta)}{\varphi\varphi_{,1}}. \tag{3.38}$$

Proof: Applying corollary (3.5) to the frame defined by (3.35) we get

$$\begin{aligned} g_{11} &= < A_1, A_1 > = \frac{\varphi_{,1}^2}{4}, \\ g_{12} &= < A_1, A_2 > = 0, \\ g_{22} &= < A_2, A_2 > = \frac{\varphi^2}{4\lambda^2}, \end{aligned}$$

and

$$\begin{aligned} b_{11} &= < A_{1,1} + [A_1, U_1], A_3 > = \frac{\lambda\varphi_{,1}\sin\theta}{2} \\ b_{12} &= < A_{1,2} + [A_1, U_2], A_3 > = 0 \\ b_{22} &= < A_{2,2} + [A_2, U_2], A_3 > = \frac{\varphi\theta_{,2}}{2\lambda} \end{aligned}$$

where

$$A_3 \;=\; -i\,(\sin\theta\sigma_2 - \cos\theta\sigma_3)$$

Using the equation (3.37) the Gauss and mean curvatures (3.38) are obtained directly. □

An immediate corollary of the above lemma is:

Corollary 3.13 *[1] Let M be the particular surface defined in the above lemma corresponding to $\varphi = \theta_{,2}$. Then this surface is the sphere with*

$$\begin{aligned} ds_I^2 &= \frac{1}{4}(\sin^2\theta\,(dx^1)^2 + \frac{\theta_{,2}^2}{\lambda^2}\,(dx^2)^2), \\ ds_{II}^2 &= \frac{\lambda}{2}\,(\sin^2\theta\,(dx^1)^2 + \frac{\theta_{,2}^2}{\lambda^2}\,(dx^2)^2), \\ K &= 4\lambda^2 \quad , \quad H = 4\lambda. \end{aligned} \tag{3.39}$$

We now present a global result regarding the above surfaces.

Theorem 3.14 *[1] Let M be the surface defined in lemma (3.12) in terms of a generalized symmetry of the sine-Gordon equation. If M is a compact, connected and oriented surface then it is homeomorphic to a sphere.*

Proof: All compact, connected and oriented surfaces with the same Euler-Poincare characteristics are homeomorphic, [34]. For compact surfaces the Euler-Poincare characteristics χ is given by Gauss-Bonnet theorem

$$\chi = \frac{1}{2\pi}\int\int_S \sqrt{det(g)}\,K\,dx^1\,dx^2. \tag{3.40}$$

Since $det(g) = \frac{\varphi^2\varphi_{,1}^2}{16\lambda^2}$, then the integrand $\sqrt{det(g)}\,K$ simply becomes

$$\sqrt{det(g)}\,K \;=\; \lambda\,\theta_{,2}\sin\theta. \tag{3.41}$$

Hence χ is independent of symmetry φ

$$\chi = \frac{\lambda}{2\pi}\int\int_S \theta_{,2}\,\sin\theta\,dx^1\,dx^2. \tag{3.42}$$

This proves that χ has the same value for all generalized symmetries and hence for all sine-Gordon deformed surfaces. Thus in order to calculate χ it is enough to choose the simplest case. According to the Corollary (3.13) the choice $\varphi = \theta_{,2}$ leads to a sphere with radius $\frac{1}{2\lambda}$ where $\chi = 2$. In this example since curvature density have the same form with eqn. (3.41), then $\chi > 0$ for all compact sine-Gordon symmetry surfaces. Hence (with orientation) all deformed surfaces have the Euler-Poincare characteristics $\chi = 2$. Therefore they are all homeomorphic to a sphere. This completes the proof of the theorem.□

Remark 3.15 *Consider the case, immersion (3.36) is smooth, in the preceding theorem. Since continuous and smooth categories are same for two dimensional manifolds, then compact, connected and oriented surfaces associated with the symmetries of sine-Gordon equation are diffeomorphic to sphere. If there are any such surfaces other than the sphere with $K > 0$ then they must be ovaloids.*

Solitonic solutions of the sine-Gordon equation satisfy the rapidly decaying conditions , $\theta(\pm\infty) = 0$, $\theta_{,1}(\pm\infty) = 0$, $\theta_{,2}(\pm\infty) = 0$,.... Then for such a case we have the following lemma

Lemma 3.16 *[1] Let M be the surface defined in Lemma (3.12). Suppose that this surface is non-compact. If the associated solution $\theta(x^1, x^2)$ of the sine-Gordon equation satisfies the conditions that $\theta, \theta_{,1}, \theta_{,2}, \ldots$ tend to zero as $x^1 \to \pm\infty$ then*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{det(g)}\, K\, dx^1 dx^2 = 0. \tag{3.43}$$

Proof: $(\theta_{,2})^2|_{c_1}^{c_2}$ tends to zero as $c_1, c_2 \to \pm\infty$ so (3.43) does.□

We now consider a different class of immersed surfaces which are also constructed from solutions of the sine-Gordon equation such as

$$F \;=\; \mu\, \Phi^{-1}\, \frac{\partial \Phi}{\partial \lambda}.$$

Lemma 3.17 *[1] Let M be the surface constructed by $U_k, k = 1, 2$ which are given by the equation (3.32) and by $A_k = \mu \frac{\partial U_k}{\partial \lambda}, k = 1, 2$ where μ depends on λ. Then M is a surface of constant negative curvature.*

Proof: Frame on M is defined by

$$\begin{aligned} A_1 &= \frac{i}{2}\mu\sigma_3, \\ A_2 &= -\frac{i}{2}\frac{\mu}{\lambda^2}(\sin\theta\sigma_2 - \cos\theta\sigma_2), \\ A_3 &= \pm i\,\sigma_1. \end{aligned} \tag{3.44}$$

Corollary (3.5) allows us to calculate the geometrical quantities given above. This surface has the following fundamental forms and curvatures

$$\begin{aligned} ds_I^2 &= \tfrac{\mu^2}{4}((dx^1)^2 + \tfrac{2}{\lambda^2}\cos\theta\, dx^1\, dx^2 + \tfrac{1}{\lambda^4}(dx^2)^2), \\ ds_{II}^2 &= \pm\tfrac{\mu}{\lambda}\sin\theta\, dx^1\, dx^2, \end{aligned}$$

$$K = -\frac{4\lambda^2}{\mu^2}\ ,\ H = \pm\frac{4\lambda}{\mu}\cot(\theta).$$

□

Corollary 3.18 *[1] Let θ be a rapidly decaying solution of the sine-Gordon equation and M be the surface defined in lemma (3.17). Then*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{det(g)}\,K\,dx^1\,dx^2 = 0.$$

Proof: This is a consequence of

$$\sqrt{det(g)}\,K = -\sin\theta = -\theta_{,12}.$$

□

We now consider yet different class of immersions associated with solutions of the sine-Gordon equation, in the form

$$F = \Phi^{-1}\,(\mu\,\partial_\lambda + \frac{ip}{2}\,\sigma_1)\,\Phi$$

Lemma 3.19 *[1] Let M be the surface constructed by $U_k, k = 1,2$ which are defined by equation (3.32) and, by $A_k = \mu\frac{\partial U_k}{\partial\lambda} + \frac{ip}{2}[\sigma_1, U_k], k = 1,2$ with $\mu = \lambda p$. Then*

$$\begin{aligned} ds_I^2 &= \frac{p^2}{2}(\lambda^2(dx^1)^2 - 2\sin\theta\, dx^1\, dx^2 + \frac{1}{\lambda^2}(dx^2)^2), \\ ds_{II}^2 &= \frac{p}{2}[\lambda^2(dx^1)^2 - 2(\sin\theta + \cos\theta)\, dx^1\, dx^2 + \frac{1}{\lambda^2}(dx^2)^2], \end{aligned}$$

$$K = -\frac{2}{p^2}\tan\theta \quad , \quad H = \frac{2}{p} - \frac{2}{p}\tan\theta.$$

The curvature density $\sqrt{det(g)}\,K$ *has a form similar to the one in corollary (3.18). Thus* $\sqrt{det(g)}\,K = -\sin\theta = -\theta_{,12}$.

Proof: Frame on M defined by

$$\begin{aligned} A_1 &= \frac{i}{2}\,\lambda\, p\,(\sigma_2 + \sigma_3), \\ A_2 &= \frac{i}{2}\,\frac{p}{\lambda}\,((\cos\theta + \sin\theta)\,\sigma_2 + (\cos\theta - \sin\theta)\sigma_3), \\ A_3 &= i\sigma_1. \end{aligned}$$

Then claim follows by using corollary (3.5).□

The following corollary of the Lemma (3.19) is for the solitonic solutions of the sine-Gordon equation

Corollary 3.20 *[1] Let* θ *be a rapidly decaying solution of the sine-Gordon equation and* M *be the surface defined in lemma (3.19). Then*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{det(g)}\,K\,dx^1 dx^2 = 0. \tag{3.45}$$

Chapter 4

Weingarten Surfaces

In this chapter, making use of the generalized immersion function established in chapter 3, we shall construct Weingarten surfaces arising from some other nonlinear partial differential equations. The classical description of Weingarten surfaces is studied in [35].

4.1 Linear Weingarten Surfaces

In this section we will study equations on which the surfaces associated with their symmetries hold a relation

$$f(K, H) = \alpha\, K + \beta\, H + \gamma = 0.$$

4.1.1 The Sine-Gordon Equation

Now start from the Lax representation of sine-Gordon equation given in equation (3.32):

$$U_1 = \frac{i}{2}\left(-\theta_{,1}\,\sigma_1 + \lambda\,\sigma_3\right)\ ,\ U_2 = \frac{i}{2\lambda}(\sin\theta\sigma_2 - \cos\theta\sigma_3).$$

Lemma 4.1 *[1] Let M be the surface constructed from U_1 and U_2 defined by equations (3.32) and from $A_k = \mu\frac{\partial U_k}{\partial\lambda} + \frac{ip}{2}[\sigma_1, U_k], k = 1, 2$. This surface whose immersion function is given as*

$$F \;=\; \Phi^{-1}\,\left(\mu\;\partial_\lambda + \frac{ip}{2}\;\sigma_1\right)\,\Phi,$$

satisfies the following Weingarten relation

$$(\mu^2 + \lambda^2 p^2) K + 2p\lambda^2 H + 4\lambda^2 = 0. \tag{4.1}$$

and it is parallel to a space of negative constant curvature. The distance between these surfaces is $\frac{p}{4}$.

Proof: Weingarten relation (4.1) can be directly verified with a tedious calculation.

Let K_0 and H_0 be the Gaussian and mean curvatures of a surface M_0 with constant curvature K_0 and let M be parallel to M_0 then

$$K_0 = \frac{K}{1 - 2aH + a^2 K}\ ,\ H_0 = \frac{H - aK}{1 - 2aH + a^2 K} \tag{4.2}$$

where a is a constant [34]. Hence comparing the first equation above and (4.1) we find that

$$a = \frac{p}{4}\ ,\ K_0 = -\frac{16\lambda^2}{3p^2 + 4\mu^2}.$$

Thus M is parallel to a surface M_0 with negative constant curvature and $\frac{p}{4}$ is the distance between the surfaces.□

We have the following corollary to the lemma (4.1).

Corollary 4.2 *The surfaces equidistant to pseudospherical surfaces are linear Weingaten surfaces and according to lemma (4.1) in a certain coordinate system all such surfaces can be characterized by the sine-Gordon equation.*

4.1.2 The Sinh-Gordon Equation

The sinh-Gordon equation defined by

$$\theta_{,11} + \theta_{,22} + \frac{1}{4}(H_0^2 e^{2\theta} - e^{-2\theta}) = 0 \tag{4.3}$$

where $\theta(x^1, x^2) \in \mathbb{R}$ and $H_0 \neq 0$ is real constant. This equation usually associated with surfaces of the constant mean curvature H_0. In what follows we will show that this equation can also be used to construct several other classes of interesting surfaces.

Lemma 4.3 *Let $\theta(x^1, x^2) \in \mathbb{R}$ be a solution of the sinh-Gordon equation (4.3), where $H_0 \neq 0$ is a real constant. Define $U_k, A_k \in su(2)$, $k = 1, 2$, by*

$$\begin{aligned} U_1 &= \tfrac{i}{4}[\cos\lambda(H_0 e^{\theta} + e^{-\theta})\sigma_1 - \sin\lambda(H_0 e^{\theta} - e^{-\theta})\sigma_2 + 2\theta_{,2}\sigma_3], \\ U_2 &= -\tfrac{i}{4}[\sin\lambda(H_0 e^{\theta} + e^{-\theta})\sigma_1 + \cos\lambda(H_0 e^{\theta} - e^{-\theta})\sigma_2 + 2\theta_{,1}\sigma_3], \end{aligned} \tag{4.4}$$

$$A_k = 2\mu\frac{\partial U_k}{\partial\lambda} + \frac{ip}{2}[\sigma_3, U_k] \ , \ k = 1, 2 \tag{4.5}$$

where μ and p are real constants. Then the associated surface M with the immersion function

$$F \ = \ \Phi^{-1} \ (2\mu \ \partial_\lambda + \frac{ip}{2} \ \sigma_3) \ \Phi$$

satisfies the following Weingarten relation

$$(p^2 - 4\mu^2)\, K + 2p\, H + 4 = 0. \tag{4.6}$$

There are some particular limiting cases. If $p = \pm 2\mu$, S is a surface of constant mean curvature

$$\begin{aligned} p &= 2\mu \ , \ H = -\tfrac{1}{\mu} \ , \ K = \tfrac{e^{4\theta} H_0^2 - 1}{4\mu^2 H_0^2 e^{4\theta}}, \\ p &= -2\mu \ , \ H = \tfrac{1}{\mu} \ , \ K = -\tfrac{e^{4\theta} H_0^2 - 1}{4\mu^2}. \end{aligned}$$

If $p = 0$, S is a surface of constant positive Gaussian curvature,

$$\begin{aligned} K &= \tfrac{1}{\mu^2}, \\ H &= -(\tfrac{2}{\mu})\tfrac{H_0^2 e^{4\theta} + 1}{H_0^2 e^{4\theta} - 1} \end{aligned}$$

If $\mu = 0$, S is sphere.

Proof: Direct application of the theorem (3.9) and corollary (3.5) gives the stated result. The surface M associated with the symmetry given in (4.5) has the following fundamental forms and curvatures

$$\begin{aligned} g_{11} &= \frac{1}{16e^{2\theta}}\left([e^{2\theta}H_0^2(2\mu + p) + (p - 2\mu)]^2 + \right. \\ &\quad \left. 4\, H_0\, (4\mu^2 - p^2)\, \sin^2\lambda\, e^{2\theta}\right), \\ g_{12} &= \frac{H_0\,(4\mu^2 - p^2)\,\sin 2\lambda}{8}, \\ g_{22} &= \frac{1}{16e^{2\theta}}\left([e^{2\theta}H_0^2(2\mu + p) - (p - 2\mu)]^2 - \right. \\ &\quad \left. 4\, H_0\, (4\mu^2 - p^2)\, \sin^2\lambda\, e^{2\theta}\right) \end{aligned}$$

$$
\begin{aligned}
b_{11} &= \frac{-H_0^2\, e^{4\theta}\,(p+2\mu) - p + 2\mu - 2p\, H_0 \cos 2\lambda\, e^{2\theta}}{8\, e^{2\theta}}, \\
b_{12} &= \frac{p\, H_0 \sin 2\lambda}{4} \\
b_{22} &= \frac{-H_0^2\, e^{4\theta}\,(p+2\mu) - p + 2\mu + 2p\, H_0 \cos 2\lambda\, e^{2\theta}}{8\, e^{2\theta}},
\end{aligned}
$$

$$
\begin{aligned}
K &= 4\frac{e^{4\theta} H_0^2 - 1}{e^{4\theta} H_0^2 (2\mu+p)^2 - (2\mu-p)^2}, \\
H &= -4\frac{e^{4\theta} H_0^2 (2\mu+p) + (2\mu-p)}{e^{4\theta} H_0^2 (2\mu+p)^2 - (2\mu-p)^2}.
\end{aligned}
$$

and satisfies the following Weingarten relation given in (4.6). By arguments similar to the ones used in proving lemma (4.3), it can be shown that this surface is parallel to surface whose curvature is

$$K_0 = \frac{16}{16\mu^2 - 3p^2}$$

constant. Distance between surfaces are $a = \frac{p}{4}$.□

Corollary 4.4 *The surfaces equidistant to constant curvature surfaces are linear Weingaten surfaces and according to lemma (4.3), in a certain coordinate system all such surfaces can be characterized by sinh-Gordon equation.*

In the case of $H_0 = 0$ the sinh-Gordon equation has some particular geometric interpretation. The sinh-Gordon equation reduces to the Liouville equation

$$\theta_{,11} + \theta_{,22} - \frac{1}{4} e^{-2\theta} = 0. \tag{4.7}$$

We have the following lemma:

Lemma 4.5 *Let $\theta(x^1, x^2) \in \mathbb{R}$ be a solution of the Liouville equation (4.7). Define $U_k, A_k, k = 1, 2$ by*

$$
\begin{aligned}
U_1 &= \frac{i}{4}(e^{-\theta} \cos\lambda\, \sigma_1 + e^{-\theta} \sin\lambda\, \sigma_2 + 2\theta_{,2}\, \sigma_3), \\
U_2 &= -\frac{i}{4}(e^{-\theta} \sin\lambda\, \sigma_1 - e^{-\theta} \cos\lambda\, \sigma_2 + 2\theta_{,1}\, \sigma_3),
\end{aligned}
$$

where $A_k, k = 1, 2$ are given in (4.5) with $p \neq \pm 2\mu$. The associated surface S has the following fundamental forms and curvatures

$$
\begin{aligned}
K &= \frac{4}{(2\mu - p)^2}, \\
H &= -\frac{4}{2\mu - p}.
\end{aligned}
$$

Thus for any μ, p with $p \neq 2\mu$, S is a sphere.

Proof: Direct result of theorems (3.9),(3.4) and corollary (3.5) with symmetry given in (4.5)).□

4.2 Nonlinear Weingarten Surfaces

4.2.1 The Nonlinear Schrödinger Equation

The nonlinear Schrödinger equation is an equation for a complex function $\psi(x^1, x^2)$.

$$i\psi_{,2} = \psi_{,11} + 2|\psi|^2\psi.$$

Letting $\psi(u,v) = r(u,v) + is(u,v)$, the real valued functions r and s satisfy

$$\begin{aligned} r_{,2} &= s_{,11} + 2\,s\,(r^2 + s^2), \\ s_{,2} &= -r_{,11} - 2\,r\,(r^2 + s^2). \end{aligned} \tag{4.8}$$

The associated matrices $U_k, k = 1,2$ defining NLS's Lax pair are given by

$$\begin{aligned} U_1 &= \tfrac{i}{2}\begin{pmatrix} -2\lambda & 2(s-ir) \\ 2(s+ir) & 2\lambda \end{pmatrix}, \\ U_2 &= -\tfrac{i}{2}\begin{pmatrix} -4\lambda^2 + 2(r^2+s^2) & v_1 - iv_2 \\ v_1 + iv_2 & 4\lambda^2 - 2(r^2+s^2) \end{pmatrix}, \end{aligned} \tag{4.9}$$

where

$$v_1 = 2r_{,1} + 4\lambda s \quad , \quad v_2 = -2s_{,1} + 4\lambda r. \tag{4.10}$$

Lemma 4.6 *Let $U_k, k = 1,2$ be defined by equations (4.9), where r and s satisfy the integrable nonlinear equations defined by (4.8), and v_1, v_2 be defined by (4.10). Let A_k be defined by $A_k = \mu\,\frac{\partial U_k}{\partial \lambda}, k = 1,2$, where μ is a real constant, i.e. let*

$$A_1 = \frac{i}{2}\begin{pmatrix} -2\mu & 0 \\ 0 & 2\mu \end{pmatrix} \quad , A_2 = -\frac{i}{2}\begin{pmatrix} -8\lambda\mu & 4\mu(s-ir) \\ 4\mu(s+ir) & 8\lambda\mu \end{pmatrix}., \tag{4.11}$$

Then geometrical quantities of the surface M with the immersion function

$$F = \mu\Phi^{-1}\frac{\partial \Phi}{\partial \lambda}$$

associated with the $U_k, A_k, k = 1, 2$ *are*

$$\begin{aligned} ds_I^2 &= \mu^2 [(dx^1 - 4\lambda\, dx^2)^2 + 4\, q^2\, (dx^2)^2], \\ ds_{II}^2 &= -2\mu\, q[dx^1 - (-\phi_{,1} + 2\lambda)dx^2]^2 + 2\mu\, q_{,11}(dx^2)^2, \end{aligned}$$

$$\begin{aligned} K &= -\frac{q_{,11}}{\mu^2\, q}, \\ H &= \frac{q_{,11} - q\,(\phi_{,1} + 2\lambda^2) - 4\, q^3}{2\mu q^2}. \end{aligned}$$

which can be expressed in terms of the new variables

$$r = q\cos\phi\ ,\ s = q\sin\phi,$$

In terms of these variables the NLS (4.8) become

$$\begin{aligned} q\,\phi_{,2} &= -q_{,11} - 2q^3 + q\phi_{,1}^2, \\ q_{,2} &= q\phi_{,11} + 2q_{,1}\phi_{,1}. \end{aligned} \tag{4.12}$$

Proof: Use frame defined by (4.11) and corollary (3.5).□

In particular if $\phi = \nu x^2$, $q = q(x^1)$, where ν is a real constant, then $q(x^1)$ satisfies

$$q'' = -2q^3 - \nu q. \tag{4.13}$$

Lemma 4.7 *Let* $U_k, A_k, k = 1, 2$ *be defined by the equations (4.9), and (4.11) where* $r = q(x^1)\sin(\nu x^2)$, $s = q(x^1)\cos(\nu x^2)$, λ, ν, μ *are constants and* $q(x^1)$ *satisfies (4.13). Then the associated surface* S *is a Weingarten surface which satisfies the relation*

$$2\mu^2 H^2\,(\mu^2\, K - \nu) = (3\mu^2\, K + 4\,\lambda^2 - 2\,\nu)^2.$$

If $\nu = -4\lambda^2$ *the above Weingarten relation becomes quadratic,*

$$K - \frac{2}{9}\, H^2 + \frac{4\lambda^2}{9\mu^2} = 0.$$

4.2.2 The mKdV Equation

Let $\rho(x^1, x^2)$ satisfy the so called modified Korteweg-de Vries equation

$$\rho_{,2} = \rho_{,111} + \frac{3}{2}\,\rho^2\,\rho_{,1}. \tag{4.14}$$

The associated matrices$U_k, k = 1,2$ which define its Lax pair are given by

$$\begin{aligned} U_1 &= \tfrac{i}{2}\begin{pmatrix} \lambda & -\rho \\ -\rho & -\lambda \end{pmatrix}, \\ U_2 &= -\tfrac{i}{2}\begin{pmatrix} -\frac{\lambda\rho^2}{2} + \lambda^3 & v_1 - iv_2 \\ v_1 + iv_2 & \frac{\lambda\rho^2}{2} - \lambda^3 \end{pmatrix}, \end{aligned} \tag{4.15}$$

where

$$v_1 = \rho_{,11} + \frac{\rho^3}{2} - \lambda^2\rho \ \ , \ v_2 = -\lambda\rho_{,1}. \tag{4.16}$$

Lemma 4.8 *Let $U_k, k = 1,2$ be defined by the equations (4.15), where $\rho(x^1, x^2) \in \mathbb{R}$ satisfies the mKdV equation (4.14) and v_1, v_2 be defined by the equation (4.16). Let $A_k, k = 1,2$ be defined by $A_k = \mu \frac{\partial U_k}{\partial \lambda}, k = 1,2$, where μ is a real constant, i.e. let*

$$\begin{aligned} A_1 &= \tfrac{i}{2}\begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}, \\ A_2 &= \tfrac{i}{2}\begin{pmatrix} \frac{-\mu\rho^2}{2} + 3\mu\lambda^2 & -2\mu\lambda\rho + i\mu\rho_{,1} \\ -2\mu\lambda\rho - i\mu\rho_{,1} & \frac{\mu\rho^2}{2} - 3\mu\lambda^2 \end{pmatrix}. \end{aligned} \tag{4.17}$$

The geometrical quantities of the surface M associated with the $U_k, A_k, k = 1,2$,

$$F = \mu\, \Phi^{-1}\, \frac{\partial \Phi}{\partial \lambda}$$

are given by

$$\begin{aligned} K &= \tfrac{4\lambda^2}{\mu^2(\rho_{,1}^2 + 4\lambda^2\rho^2)^2}[4\rho^3\rho_{,1111} - 4\rho^2\rho_{,1}\rho_{,111} - 4\rho^2(\rho_{,11})^2 \\ &\quad +4\rho\rho_{,1}^2\rho_{,11} - 4\lambda^2\rho^3\rho_{,11} + 4\rho^5\rho_{,11} - \rho_{,1}^4 + 8\rho^4\rho_{,1}^2], \\ H &= \tfrac{4\lambda}{\mu(\rho_{,1}^2 + 4\lambda^2\rho^2)^{3/2}}[-\rho\rho_{,1111} + \rho_{,1}\rho_{,111} - 3\lambda^2\rho\rho_{,11} \\ &\quad -\rho^3\rho_{,11} + 2\lambda^2\rho_{,1}^2 - 3\rho^2\rho_{,1}^2 - 4\lambda^4\rho^2 - 4\lambda^2\rho^4], \end{aligned} \tag{4.18}$$

$$\begin{aligned} ds_I^2 &= \tfrac{\mu^2}{4}[(dx^1 + \tfrac{1}{2}(\rho^2 - 6\lambda^2)\,dx^2)^2 + (\rho_{,1}^2 + 4\lambda^2\rho^2)(dx^2)^2], \\ ds_{II}^2 &= \tfrac{\lambda\mu}{(\rho_{,1}^2 + 4\lambda^2\rho^2)^{1/2}}[-\rho^2(dx^1)^2 + (-2\rho\rho_{,11} + \rho_{,1}^2 + 2\lambda^2\rho^2 - \rho^4)\,dx^1dx^2 \\ &\quad +\tfrac{1}{4}(-4\rho\rho_{,1111} + 4\rho_{,1}\rho_{,1111} + 12\lambda^2\rho\rho_{,11} - 8\rho^3\rho_{,11} - \\ &\quad 4\lambda^2\rho_{,1}^2 - 6\rho^2\rho_{,1}^2 - 4\lambda^4\rho^2 + 4\lambda^2\rho^4 - \rho^6)(dx^2)^2 \end{aligned} \tag{4.19}$$

Proof: Direct result of equation (4.17) and corollary (3.4). □

A particular reduction of the above surface M is a Weingarten surface with a complicated Weingarten relation.

Corollary 4.9 *Let $U_k, k = 1, 2$ be defined by the equations (4.15), where λ, μ, α are constants and suppose as a particular case, that $\rho(x^1, x^2) = \rho(x^1 + \alpha x^2)$ satisfies*

$$\rho'' = \alpha\,\rho - \frac{\rho^3}{2}. \tag{4.20}$$

Then the associated surface M is a Weingarten surface satisfying the relation

$$\begin{aligned}\mu^2\,H^2\,\rho^2\,[4(\alpha+4\lambda^2)-\rho^2]^3 = 16\lambda^2\,[\rho^4 - \\ 6\rho^2\,(\alpha+4\lambda^2) - 8\lambda^2\,(\alpha+4\lambda^2)]^2,\end{aligned} \tag{4.21}$$

where

$$\rho^2 = 4(\alpha+4\lambda^2) + \frac{16\,\lambda^2}{\mu}\sqrt{\frac{\alpha+4\lambda^2}{K+4\lambda^2/\mu^2}}.$$

It is interesting that using a different Lax pair for equation (4.20) it is possible to obtain a Weingarten surface simpler than the above one in (4.21)

Lemma 4.10 *Let $U_k, k = 1, 2$ defined by*

$$\begin{aligned} U_1 &= \tfrac{i}{2}\begin{pmatrix}\lambda & -\rho\\ -\rho & -\lambda\end{pmatrix},\\ U_2 &= -\tfrac{i}{2}\begin{pmatrix}\frac{\rho^2}{2} - (\alpha+\alpha\lambda+\lambda^2) & (\alpha+\lambda)\,\rho - i\rho_{,1}\\ (\alpha+\lambda)\,\rho + i\rho_{,1} & -\frac{\rho^2}{2} + (\alpha+\alpha\lambda+\lambda^2)\end{pmatrix}\end{aligned} \tag{4.22}$$

where λ, α are constants and ρ satisfy the equation (4.20). Let $A_k, k = 1, 2$ be defined by $A_k = \mu\frac{\partial U_k}{\partial\lambda}$, i.e. let

$$\begin{aligned} A_1 &= \tfrac{i}{2}\begin{pmatrix}\mu & 0\\ 0 & -\mu\end{pmatrix},\\ A_2 &= -\tfrac{i}{2}\begin{pmatrix}-(\alpha\mu+2\mu\,\lambda) & \mu\,\rho\\ \mu\,\rho & \alpha\mu+2\mu\,\lambda\end{pmatrix}.\end{aligned} \tag{4.23}$$

This surface M with the immersion function

$$F = \mu\,\Phi^{-1}\,\frac{\partial\Phi}{\partial\lambda} \tag{4.24}$$

is a Weingarten surface satisfying the relation

$$2\mu^2\,H^2\,(\mu^2\,K + 4\alpha) = [3\mu^2\,K + 4\lambda^2 + 8\alpha]^2. \tag{4.25}$$

In the special case $\alpha = \lambda^2$ the relation becomes

$$2\mu^2\,H^2 = 9\,[\mu^2\,K + 4\lambda^2]. \tag{4.26}$$

Proof: The geometrical quantities of the surface M associated with the $U_k, A_k, k = 1, 2$ are given by

$$K = \frac{2}{\mu^2}\left[\rho^2 - 2\alpha\right] \ , \ H = \frac{1}{\mu\,\rho}\left[3\rho^2 + 2(\lambda^2 - \alpha)\right],$$

$$\begin{aligned} ds_I^2 &= \tfrac{\mu^2}{4}\left[(dx^1 + (\alpha + 2\lambda)\,dx^2)^2 + \rho^2\,(dx^2)^2\right], \\ ds_{II}^2 &= \tfrac{\mu\rho}{2}\left[dx^1 + (\alpha + \lambda)\,dx^2\right]^2 + \tfrac{\mu\rho}{4}\,(\rho^2 - 2\alpha)\,(dx^2)^2. \end{aligned}$$

by using corollary (3.5).□

Chapter 5

Conclusion

In this work we presented a procedure of the construction of surfaces in $\mathbb{R}^3$ associated with the symmetries of integrable nonlinear partial differential equations within the framework of surfaces on Lie groups and on Lie algebras. We applied this method to some well-known integrable equations and obtained several symmetry surfaces. In particular we investigated some global properties of surfaces arisen from constant gauge transformation and symmetries of sine-Gordon equation. We showed that under rigid $SU(2)$ rotations all integrable equations are mapped to sphere. In the case of sine-Gordon equation, we proved that all compact sine-Gordon symmetry surfaces are homeomorphic to sphere. Besides we have constructed several Weingarten surfaces associated with symmetries of some soliton equations. We found some explicit linear and nonlinear Weingarten surfaces generated by the symmetries of sine-Gordon, sinh-Gordon, mKdV and nonlinear Schrödinger equations. Some characterization results are given for linear Weingarten surfaces.

However, many questions remain open and deserve further investigation. The logical continuation seems to consider the equations whose Lax pair is given on other Lie algebras e.g. $su(1,1), sl_2(\mathbb{R})$. Another interesting problem is the characterization of nonlinear Weingarten surfaces by the symmetries of soliton equations. These questions are being pursued further.

Appendix A

Fundamental Equations for Submanifolds

In this appendix we begin by considering higher dimensional embedded or immersed manifolds, of higher codimensions. For interested readers we refer to [32, 33].

A.1 Fundamental Equations for Submanifolds

Let $F : M^m \to N^n$ be an immersion of an m-dimensional Riemannian manifold (M, F^*g^N) into an n-dimensional Riemannian manifold (N, g^N). For every $p \in M$, we have $T_pN = T_pM \oplus T_pM^\perp$, and we use this decomposition to define two projections, $\top : T_pN \to T_pM$ and $\perp : T_pN \to T_pM^\perp$. $\Gamma(TM)$ and $\Gamma(TM^\perp)$ are the sets of tangent and normal vector fields respectively. For vector fields $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(TM^\perp)$ we write

$$\begin{aligned} \nabla^N_X Y &= \top(\nabla^N_X Y) + \perp(\nabla^N_X Y), \\ \nabla^N_X \xi &= \top(\nabla^N_X \xi) + \perp(\nabla^N_X \xi), \end{aligned} \tag{A.1}$$

where ∇^N_X denotes the connection in N. $\top\nabla^N_X$ and $\perp\nabla^N_X$ induces connections on TM and on $TM^\perp$, denoted as ∇^M_X and D_X respectively. And we will denote $A_\xi(X) = -\top(\nabla^N_X \xi)$.

Definition A.1 *The second fundamental form tensor of M is $s(X,Y) = \perp(\nabla^N_X Y)$. If we choose $\nu_{m+1}, ..., \nu_n \in \Gamma(TM^\perp)$ such that $< \nu_r, \nu_s >= \epsilon_{rs}$ where $\epsilon_{rs} = \pm\delta_{rs}$ defined in a neighborhood of a point $p \in U \subset M$, we define $n - m$ real valued second fundamental forms Ω^r by*

$$\Omega(X,Y)^r =< \nabla^N_X Y, \nu_r >=< s(X,Y), \nu_r > . \tag{A.2}$$

Definition A.2 *Connection D defined above is called the normal connection. So we introduce the normal fundamental forms β_r^s, by*

$$\beta_r^s(X) =< \nabla_X^N \nu_r, \nu_s >=< D_X \nu_r, \nu_s > . \tag{A.3}$$

With notation that we have just introduced , we may rewrite decompositions given in equations (A.1)

$$\nabla_X^N Y = \top(\nabla_X^N Y) + s(X,Y), \tag{A.4}$$
$$\nabla_X^N \xi = A_\xi(X) + D_X \xi, \tag{A.5}$$

which are called the Gauss formula and the Weingarten equations respectively.

Theorem A.3 *Let M^m be a submanifold of the Riemannian manifold N^n, for X, Y, Z and W are tangent fields along M, we have the Gauss equation*

$$\begin{aligned} &< R^N(X,Y)Z, W > - < R^M(X,Y)Z, W > \\ &=< s(X,Z), s(Y,W) > - < s(X,W), s(Y,Z) > . \end{aligned} \tag{A.6}$$

Proof: We have

$$\nabla_X^N \nabla_Y^N Z = \nabla_X^M \nabla_Y^M Z + s(X, \nabla_Y^M Z) + \nabla_X^N (s(Y,Z)),$$

similarly

$$\nabla_Y^N \nabla_X^N Z = \nabla_Y^M \nabla_X^M Z + s(Y, \nabla_X^M Z) + \nabla_Y^N (s(X,Z)),$$

as well as

$$\nabla_{[X,Y]}^N Z = \nabla_{[X,Y]}^M Z + s([X,Y], Z).$$

Substituting last three equations and noting that W is orthogonal to any term $s(.,.)$, we obtain

$$\begin{aligned} &< R^N(X,Y)Z, W >=< R^M(X,Y)Z, W > \\ &+ < \nabla_X^N(s(Y,Z)) - \nabla_Y^N(s(X,Z)), W > . \end{aligned}$$

On the other hand, since $< s(Y,Z), W >= 0$ we have

$$0 = X < s(Y,Z), W >=< \nabla_X^N s(Y,Z), W > + < s(Y,Z), s(X,W) > .$$

Desired result is obtained by substituting the last equation and the similar expression with X and Y interchanged. □

Theorem A.4 *Let M be a submanifold of a Riemannian manifold N. Let $\nu_r \in \Gamma(TM^{\perp})$ where $r = m+1,..,n$ with corresponding Ω^r and β^r_s. Then for all tangent fields X, Y, Z along M, we have Mainardi-Codazzi equations*

$$\begin{aligned} < R^N(X,Y)Z, \ \nu_r \ > &= (\nabla^M_X \Omega^r)(Y,Z) - (\nabla^M_Y \Omega^r)(X,Z) \\ &+ \ \epsilon_{rt}(\Omega^s(Y,Z)\beta^t_s(X) - \Omega^s(X,Z)\beta^t_s(W)). \end{aligned} \tag{A.7}$$

Proof: Since D is the connection on the normal bundle of M we get

$$D_X(s(Y,Z) = X(\Omega^s(Y,Z))\nu_s + \Omega^s D_X \nu_s,$$

moreover

$$s(\nabla^M_X Y, Z) + s(Y, \nabla^M_X Z) = \Omega^s(\nabla^M_X Y, Z)\nu_s + \Omega^s(Y\nabla^M_X, Z)\nu_s.$$

Then these two equation give

$$\begin{aligned} D_X(s(Y,Z) \ &- \ s(\nabla^M_X Y, Z) - s(Y, \nabla^M_X Z) \\ &= \ (\nabla^M_X \Omega^s)(Y,Z) + \Omega^s(Y,Z) D_X \nu_s, \end{aligned}$$

and hence

$$\begin{aligned} < D_X(s(Y,Z) \ &- \ s(\nabla^M_X Y, Z) - s(Y, \nabla^M_X Z), \nu_r > \\ &= \ (\nabla^M_X \Omega^r)(Y,Z) + \epsilon_{rt}(\Omega^s(Y,Z)\beta^t_s(X). \end{aligned}$$

Finally there is a similar equation by interchanging X and Y. After the substitution $\perp R^N(X,Y)Z = D_X(s(Y,Z) - s(\nabla^M_X Y, Z) - s(Y, \nabla^M_X Z) - (D_Y(s(X,Z) - s(\nabla^M_Y X, Z) - s(X, \nabla^M_Y Z))$, we obtain the Mainardi-Codazzi equations. □

Before introducing the Ricci equations, we introduce one more operation. Given tangent fields X and Y along M and a basis $U_1,..,U_m$ such that $< U_i, U_j >= g^{ij} = \pm\delta_{ij}$, we set

$$\Omega^r * \Omega^s(X,Y) = g^{ij}\Omega^r(X,U_i)\Omega^s(Y,U_j).$$

Theorem A.5 *Let M be a submanifold of N. If $\nu_r \in \Gamma(TM^{\perp})$ where $r = m+1,..,n$ with corresponding Ω^r and β^s_r, then for all $X, Y \in \Gamma(TM)$, we have the Ricci equations*

$$\begin{aligned} < R^N(X,Y)\nu_r, \nu_s > &= \Omega^r * \Omega^s(X,Y) - \Omega^r * \Omega^s(Y,X) \\ +(\nabla^M_X \beta^r_s)(Y) - (\nabla^M_Y \beta^r_s)(X) &+ \epsilon_{th}(\beta^r_t(X)\beta^h_s(Y) - \beta^r_t(Y)\beta^s_t(X)). \end{aligned} \tag{A.8}$$

Proof: By using the Gauss formula and Weingarten equations

$$\nabla^N_X \nabla^N_Y \xi = -\nabla^M_X A_\xi(Y) - s(X, A_\xi(Y)) - A_{D_Y\xi}(X) + D_X D_Y \xi,$$

we get the normal components

$$\perp(\nabla^N_X \nabla^N_Y \xi) = -s(X, A_\xi(Y)) + D_X D_Y \xi,$$
$$\perp(\nabla^N_Y \nabla^N_X \xi) = -s(Y, A_\xi(X)) + D_Y D_X \xi,$$

and

$$\perp(\nabla^N_{[X,Y]})\xi = D_{[X,Y]}\xi.$$

Thus we obtain

$$\perp R^N(X,Y)\xi = R_D(X,Y)\xi + s(A_\xi(X), Y) - s(A_\xi(Y), X),$$

where R_D denotes the curvature of the normal connection. Now if $U_1, .., U_m$ is a given basis of T_pM

$$A_{\nu_r}(X) = \Omega(X, U_i)U_i,$$

and

$$< s(A_{\nu_r}(X), Y), \nu_s >= \Omega^r * \Omega^s(X,Y),$$

we also have

$$< D_X D_Y \nu_r, \nu_s >= X(\beta^s_r(Y) - \epsilon_{th}\beta^t_r(X)\beta^s_h(Y),$$
$$< D_{[X,Y]}\nu_r, \nu_s >= \beta^s_r(\nabla^M_X Y) - \beta^s_r(\nabla^M_Y X).$$

Using the above equations we obtain the Ricci equations. □

We have seen that the Gauss and Mainardi-Codazzi equations are precisely the integrability conditions of the Gauss formula.The integrability conditions of the Weingarten equations lead to two sets of equations. One set reduces to the Mainardi-Codazzi equations, the other set is the Ricci equations. Therefore three fundamental equations give the complete set of equations for smooth immersions.

A.2 Gauss, Mainardi-Codazzi and Ricci Equations in Local Coordinates

We shall rewrite Gauss, Mainardi-Codazzi and Ricci equations in terms of a basis [32] :

Let M be an immersed submanifold in N. We consider a coordinate system $y^1, .., y^n$ on a neighborhood of $U \subset N$, with the metric

$$< ., . >^N = g^N_{\alpha\beta} dy^\alpha \otimes dy^\beta,$$

and let $x^1, .., x^m$ be the coordinates on M, with

$$< ., . >^M = g^M_{ij} dx^i \otimes dx^j,$$

When we consider the local parametrization of M as $y^\alpha = y^\alpha(x^1, .., x^m)$. ∂_i for $i = 1, .., m$ is the corresponding basis of T_pM.

$$g^N_{\alpha\beta} y^\alpha_{:i} y^\beta_{:j} = g^M_{ij},$$

where $y^\alpha_{:i}$ denotes the covariant differentiation of y^α with respect to x^i.

To write the Gauss, Mainardi-Codazzi and Ricci equations, all we need is to write Ω^r and β^s_r in terms of the local coordinates. For the local parametrization $y^\alpha = y^\alpha(x^1, .., x^m)$ of M, we can write the basis of T_pM in terms of coordinates of N as $\partial_i = y^\alpha_{:i} \partial_\alpha$. Then we have

$$\nabla^N_{\partial_i} \partial_j = (y^\alpha_{:ij} + \Gamma^\alpha_{\rho\sigma} y^\rho_{:i} y^\sigma_{:j}) \partial_\alpha,$$

and definition of Ω^r gives

$$\begin{aligned} \Omega^r_{ij} &= \Omega^r(\partial_i, \partial_j) =< \nabla^N_{\partial_i} \partial_j, \nu_r > \\ &= g^N_{\alpha\beta} \nu^\beta_r (y^\alpha_{:ij} + \Gamma^\alpha_{\rho\sigma} y^\rho_{:i} y^\sigma_{:j}) \partial_\alpha. \end{aligned} \tag{A.9}$$

Now we have

$$\nabla^N_{\partial_i} \nu_r = (\nu^\alpha_{r:i} + \Gamma^\alpha_{\rho\sigma} \nu^\rho_r y^\sigma_{:i}) \partial_\alpha.$$

Similarly normal fundamental forms can be written in terms of local coordinates

$$\begin{aligned} \beta^s_{ri} &= \beta^s_r(\partial_i) =< \nabla^N_{\partial_i} \nu_r, \nu_s > \\ &= g^N_{\alpha\beta} \nu^\beta_s (\nu^\alpha_{r:i} + \Gamma^\alpha_{\rho\sigma} \nu^\rho_r y^\sigma_{:i}) \partial_\alpha. \end{aligned} \tag{A.10}$$

Now we can write all fundamental equations of immersed manifolds Let $X = \partial_i = y^\alpha_{:i} \partial_\alpha$, $Y = \partial_j = y^\beta_{:j} \partial_\beta$, $Z = \partial_k = y^\gamma_{:k} \partial_\gamma$, and $W = \partial_l = y^\delta_{:l} \partial_\delta$, then we have the following lemma

Lemma A.6 *In local coordinates the Gauss-Mainardi-Codazzi-Ricci equations are respectively given by*

$$R^N_{\alpha\beta\gamma\delta}y^\alpha_{:i}y^\beta_{:j}y^\gamma_{:k}y^\delta_{:l} = R^M_{ijkl} + \epsilon_{rs}(\Omega^r_{ik}\Omega^s_{jl} - \Omega^r_{il}\Omega^s_{jk}), \tag{A.11}$$

and if $\nu_r = \nu^\gamma_r\partial_\gamma$ *and* $\nu_s = \nu^\delta_s\partial_\delta$, *we get Mainardi-Codazzi and Ricci equations as follows*

$$R^N_{\alpha\beta\gamma\delta}y^\alpha_{:i}y^\beta_{:j}y^\gamma_{:k}\nu^\delta_r = \Omega^r_{jk:i} - \Omega^r_{ik:j} + \epsilon_{rs}(\Omega^s_{jk}\beta^r_{si} - \Omega^s_{ik}\beta^r_{sj}), \tag{A.12}$$

$$\begin{aligned} R^N_{\alpha\beta\gamma\delta}y^\alpha_{:i}y^\beta_{:j}\nu^\gamma_r\nu^\delta_s = g^{ij}(\Omega^r_{ik}\Omega^s_{jl} - \Omega^r_{jk}\Omega^s_{il}) + \\ \beta^s_{rj:i} - \beta^s_{ri:j} + \epsilon_{th}(\beta^s_{ti}\beta^h_{rj} - \beta^s_{tj}\beta^h_{ri}). \end{aligned} \tag{A.13}$$

A.3 Immersions into Constant Curvature Spaces

It is useful to examine the form which our fundamental equations take when the ambient space N has constant curvature K_0. For the case $K_0 > 0$ the manifold N is just the n-sphere of radius $\frac{1}{\sqrt{K_0}}$. For $K_0 < 0$, we obtain an analogous submanifold $n+1$ dimensional Euclidean space E^{n+1} by considering a pseudo-Riemannian metric on E^{n+1}. $K_0 = 0$ is trivially E^n.

In each case mentioned above, curvature tensor R^N of N satisfies

$$< R^N(\partial_\alpha, \partial_\beta)\partial_\gamma, \partial_\delta >= K_0(g^N_{\alpha\delta}g^N_{\beta\gamma} - g^N_{\alpha\gamma}g^N_{\beta\delta}),$$

where $\partial_\alpha, \partial_\beta, \partial_\gamma, \partial_\delta$ are the basis element of T_pN

Now Gauss, Mainardi-Codazzi and Ricci equations can be simplified by using this restriction on R^N. They take the form

the Gauss equations

$$K_0(g^N_{\alpha\delta}g^N_{\beta\gamma} - g^N_{\alpha\gamma}g^N_{\beta\delta})y^\alpha_{:i}y^\beta_{:j}y^\gamma_{:k}y^\delta_{:l} = R^M_{ijkl} + \epsilon_{sr}(\Omega^r_{ik}\Omega^s_{jl} - \Omega^r_{il}\Omega^s_{jk}).$$

the Mainardi-Codazzi equations

$$K_0(g^N_{\alpha\delta}g^N_{\beta\gamma} - g^N_{\alpha\gamma}g^N_{\beta\delta})y^\alpha_{:i}y^\beta_{:j}y^\gamma_{:k}\nu^\delta_r = \Omega^r_{jk:i} - \Omega^r_{ik:j} + \epsilon_{rs}(\Omega^s_{jk}\beta^r_{si} - \Omega^s_{ik}\beta^r_{sj}).$$

the Ricci equations

$$\begin{aligned} K_0(g^N_{\alpha\delta}g^N_{\beta\gamma} - g^N_{\alpha\gamma}g^N_{\beta\delta})y^\alpha_{:i}y^\beta_{:j}\nu^\gamma_r\nu^\delta_s = g^{ij}(\Omega^r_{ik}\Omega^s_{jl} - \Omega^r_{jk}\Omega^s_{il}) + \\ \beta^s_{rj:i} - \beta^s_{ri:j} + \epsilon_{th}(\beta^s_{ti}\beta^h_{rj} - \beta^s_{tj}\beta^h_{ri}). \end{aligned}$$

respectively.

As a final remark we can write fundamental equations of immersion of M into $N = E^n$ i.e. curvature tensor $R^N = 0$, take the form

the Gauss equations

$$R^M_{ijkl} + \epsilon_{sr}(\Omega^r_{ik}\Omega^s_{jl} - \Omega^r_{il}\Omega^s_{jk}) = 0.$$

the Mainardi-Codazzi equations

$$\Omega^r_{jk:i} - \Omega^r_{ik:j} + \epsilon_{rs}(\Omega^s_{jk}\beta^r_{si} - \Omega^s_{ik}\beta^r_{sj}) = 0.$$

the Ricci equations

$$g^{ij}(\Omega^r_{ik}\Omega^s_{jl} - \Omega^r_{jk}\Omega^s_{il}) + \beta^s_{rj:i} - \beta^s_{ri:j} + \epsilon_{th}(\beta^s_{ti}\beta^h_{rj} - \beta^s_{tj}\beta^h_{ri}) = 0.$$

respectively.

A.4 Immersions of Hypersurfaces

Let us consider a more specific situation where M^m is a hypesurface in N^{m+1}, that is, a submanifold of codimension 1. In the case of hypersurfaces we can locally choose a unit normal vector field ν on a neighborhood of $p \in U \subset M$. Then Weingarten equations reduce to

$$< \nabla^N_X \nu, Y >= - < \nu, \nabla^N_X Y >=< \nu, s(X,Y) >,$$

and since $< \nu, Y >= 0$ and $< \nu, \nu >= 1$ along M,we have

$$\begin{aligned} 0 &= X < \nu, Y >=< \nabla^N_X \nu, Y > + < \nu, \nabla^N_X Y >, \\ 0 &= X < \nu, \nu >= 2 < \nabla^N_X \nu, \nu > . \end{aligned}$$

Hence

$$\begin{aligned} D_X \nu &= 0, \\ s(X,Y) &= \Omega(X,Y)\nu, \end{aligned}$$

or equivalently normal fundamental form β^{n+1}_{n+1} vanishes. We can compute

the Gauss equations

$$R^N_{\alpha\beta\gamma\delta} y^\alpha_{:i} y^\beta_{:j} y^\gamma_{:k} y^\delta_{:l} = R^M_{ijkl} + \epsilon_{rs}(\Omega^r_{ik}\Omega^s_{jl} - \Omega^r_{il}\Omega^s_{jk}).$$

and the Mainardi-Codazzi equations

$$R^N_{\alpha\beta\gamma\delta} y^\alpha_{:i} y^\beta_{:j} y^\gamma_{:k} \nu^\delta_r = \Omega^r_{jk:i} - \Omega^r_{ik:j}.$$

The Ricci equations are trivial if M is a hypersurface. And finally if we let N have zero constant curvature, then equations simply become

the Gauss equation

$$R^M_{ijkl} + \epsilon_{rs}(\Omega^r_{ik}\Omega^s_{jl} - \Omega^r_{il}\Omega^s_{jk}) = 0.$$

and the Mainardi-Codazzi equations

$$\Omega^r_{jk:i} = \Omega^r_{ik:j}.$$

respectively.

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