CONSISTENCY AND POPULATION MONOTONICITY IN SOCIAL AND ECONOMIC NETWORKS

A Master's Thesis

by

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I certify that I have read this thesis and in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

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ABSTRACT

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In this study, we analyze consistency and population monotonicity principles focusing on the pairwise stability solution in social and economic networks. First, it is examined which allocation rules and value functions lead to the consistent pairwise stable graphs. Second, population monotonic allocation rules with respect to the pairwise stability solution are analyzed.

Keywords: Consistency, population monotonicity, allocation rule, value function, pairwise stability.

ÖZET

SOSYAL VE EKONOMİK AĞLARDA TUTARLILIK VE NÜFUS TEKDÜZENLİĞİ

Özgür Yılmaz

İktisat Bölümü

Tez Yöneticisi: Asst. Prof. Tarık Kara

Agustos 1999

Bu çalışmada tutarlılık ve nüfus tekdüzenliliğini inceledik. Burada odak noktası olarak sosyal ve ekonomik ağlardaki ikili (zayıf) denge çözümü ele alındı. İlk olarak, ikili dengeyi tutarlı kılan paylaştırma kuralları ve değer fonksiyonları üzerinde duruldu. İkinci olarak ise, ikili dengeye göre nüfus tekdüzenliliği arzeden paylaştırma kuralları analiz edildi.

Anahtar Kelimeler: Tutarlılık, nüfus tekdüzenliliği, paylaştırma kuralı, değer fonksiyonu, ikili denge.

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Contents

Chapter 1

Introduction

1.1 Introduction

A *network structure* or *graph* describes the interaction between agents. Here, the nodes represent the agents and an arc exists between two nodes if the corresponding agents interact bilaterally (by a graph we mean an undirected graph, hence unilateral interactions will not be considered in this work). The setting is that multiple links between any two agents are not allowed and an agent cannot have one or more links onto himself.

In the context of graph structures, the need arises to give predictions concerning which networks are likely to form (this leads to the *stability* notion). Self-interested agents can choose to form new links or severe existing links(costs associated with these are assumed to be zero). The specification is that a *value function* gives the value of each graph or network, while an *allocation rule* gives the distribution of value to the individual players forming the network. (Jackson and Wolinsky,1996) With this specification, the question arises at the stability notion. The definition of a stable graph embodies the idea that the agents have the right to form or severe links. The formation of a link requires the consent of both agents involved, but severance can be done unilaterally.

Many social networks such as information transmission, roads between towns, computer networks, internal organization of firms, etc. can be modelled with graph structures having the above setting.

We will analyze two axioms, consistency and population monotonicity, in the domain of graph structures.

In most of social choice theory, game theory, and economic theory the number of agents is assumed to be a fixed number and this number is not allowed to vary. The other alternative is to define solutions for problems involving groups of different number of agents. When a solution is defined in this way, the problem arises at the relation between its components of different groups of some(different) number of agents, and this is the question what *consistency* deals with: how should the components of solutions be linked across cardinalities?

An informal description of the *consistency* principle is the following: a solution is *consistent* if for any admissible problem, whenever it recommends some outcome x as its solution outcome, then it recommends the restriction of x to any subgroup as the solution outcome of the reduced problem faced by this subgroup: this is the problem obtained from the original one by attributing to the members of the complementary subgroup their components of x.(Thomson,1996)

The concept of the reduced problem is crucial in understanding consistency principle. Given a problem $D \in \mathcal{D}^N$ ($\mathcal{D}^N$ represents the class of problems that the members of N could face), and an alternative x in the feasible set of D, the *reduced problem of D relative to the subgroup N' of N and x* is the problem comprising all the alternatives of D at which the members of the complementary subgroup $N \setminus N'$ receive their components of x.(Thomson,1996) So, once the problem $D \in \mathcal{D}^N$ has been solved at x by applying the solution φ, the problem appears to the members of the subgroup N' in such a way that the members in the complementary subgroup $N \setminus N'$ receive the payoffs $x_{N \setminus N'}$ assigned to them by φ. The interpretation is that the agents are promised specific welfare levels and certain payoffs are paid to them when leaving.

Another property is population monotonicity. The question is whether the agents are affected in the same direction as their circumstances change (Thomson,1994), e.g. the case of newcomers. The number of agents is allowed to vary and solutions are investigated for admissible cardinalities. The axiom of *population monotonicity* is meant to help us relate the recommendations made by solutions as the number of agents varies. (Thomson, 1994)

In this work we investigate the consistency and population monotonicity principles where the agents are endowed with graph structures. The solution concept we use here is the *pairwise stability* which is a relatively weak notion. In the next chapter, we present the allocation rules and value functions under which the pairwise stability is consistent. In the third chapter our concern will be the population monotonicity. We conclude in chapter four.

Chapter 2

Consistency of Pairwise Stability Solution in Social and Economic Networks

2.1 Basic Notations and Definitons

2.1.1 Graphs, Value Function, Allocation Rule, and Stability

Let N be the finite set of players. The network relations among these players are formally represented by graphs whose nodes are identified with the players and whose arcs capture the pairwise relations.

The complete graph, denoted g^N, is the set of all subsets of N of size 2. The set of all possible graphs on N is then $G^N = \{g | g \subset g^N\}$. Let ij denote the subset of N containing i and j and is referred to as the *link* ij. The interpretation is that if $ij \in g$, then nodes i and j are directly connected, while if $ij \notin g$, then nodes i and j are not directly connected[1].

Let $g+ij$ denote the graph obtained by adding the link ij to the existing graph g and $g-ij$ denote the graph obtained by deleting the link ij from the existing graph g (i.e. $g+ij = g \cup \{ij\}$ and $g - ij = g \setminus \{ij\}$).

Let $g\mid_S = \{ij \in g | i \in S \text{ and } j \in S\}$ for $S \subseteq N$ denote the restriction of the graph g to S.

Let $N(g) = \{i | \exists j \text{ s.t. } ij \in g\}$ denote the set of agents linked to some other agent(s) and $n(g)$ be the cardinality of $N(g)$; $D_S(g) = \{i \in S | \nexists j \in N \setminus \{i\} \text{ s.t. } ij \in g\}$ and $d_S(g)$ be the cardinality of $D_S(g)$. Note that $D_N(g)$ represents the disconnected agents in g and hence $N(g) = N \setminus D_N(g)$.

A *path* is a non-empty graph of the form $p = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\}$ where $N' = \{x_0, \ldots, x_k\}$ is a subset of N.

The graph $g' \subset g$ is a *component of* g, if for all $i \in N(g')$ and $j \in N(g')$, $i \neq j$, there exists a path in g' connecting i and j, and for any $i \in N(g')$ and $j \in N(g)$, $ij \in g$ implies that $ij \in g'$.

The *value function of a graph* is represented by $v : \{g | g \subset g^N\} \rightarrow \mathbb{R}$. The set of all such functions is V.

An *allocation rule* $Y : \{g|g \subset g^N\} \times V \rightarrow \mathbb{R}^N$ describes how the value associated with each network is distributed to the individual players.(The set of all such allocation rules is $\mathcal{Y}$.) $Y_i(g, v)$ is the payoff to player i from graph g under the value function v.

[1]The graphs analyzed here are non-directed. That is, it is not possible for one individual to link to another, without having the second individual also linked to the first, and furthermore, multiple links between any two agents and an agent having link(s) onto himself are not allowed as stated in the previous section.

Now we need to define a notion which captures the stability of a network. The definition of a stable graph embodies the idea that the agents have the right to form or severe links. The formation of a link requires the consent of both agents involved, but severance can be done unilaterally. Here, the cost of formation and severance is assumed to be zero as stated in the introduction.

Definition 1 A graph g is *pairwise stable* respect to v and Y if
(i) for all $ij \in g$, $Y_i(g,v) \geq Y_i(g-ij,v)$ and $Y_j(g,v) \geq Y_j(g-ij,v)$ and
(ii) for all $ij \notin g$, if $Y_i(g,v) < Y_i(g+ij,v)$ then $Y_j(g,v) > Y_j(g+ij,v)$.

We shall say that g is *defeated* by g' if $g' = g - ij$ and (i) is violated for ij, or if $g' = g + ij$ and (ii) is violated for ij.

Now, we will define some restrictions on Y and v.

Definition 2 Given a permutation $\pi : N \to N$, let $g^\pi = \{ij | i = \pi(k), j = \pi(l), kl \in g\}$. Let v^π be defined by $v^\pi(g^\pi) = v(g)$.

Definition 3 The allocation rule Y is *anonymous* if, for any permutation π, $Y_{\pi(i)}(g^\pi, v^\pi) = Y_i(g,v)$.

Anonymity states that if all that has changed is the names of the agents (and not anything concerning their relative positions or production values), then the allocations they receive should not change. In other words, the anonymity of Y requires that the information used to decide on allocations be obtained from the function v and the structure of g, and not from the label of the individual.

Definition 4 An allocation rule is *balanced* if $\sum_i Y_i(g,v) = v(g)$ for all v and g.

A stronger notion of balance, component balance, requires Y to allocate resources generated by any component to that component. Let $C(g)$ denote the set of components of g.

Definition 5 A value function v is *component additive* if $v(g) = \sum_{h \in C(g)} v(h)$.

Definition 6 The rule Y is *component balanced* if $\sum_{i \in N(h)} Y_i(g, v) = v(h)$ for every g and $h \in C(g)$ and component additive v.

Note that the definition of component balance only applies when v is component additive. Otherwise balancedness will be contradicted.

Definition 7 The value function v is *anonymous* if $v(g^{\pi}) = v(g)$ for all permutations π and graphs g.

Definition 8 The value function v satisfies *monotonicity* if for all $g \in G$, for all $i, j \in N$, $v(g + ij) \geq v(g)$, and satisfies *strict monotonicity* if for all $g \in G$, for all $i, j \in N$, $v(g + ij) > v(g)$.

There are two basic allocation rules: equal split rule and the Shapley value. The equal split rule is the simplest one and allocates the value of a graph (the value of the component if v is component additive) equally among the agents involved in that graph (component):

Definition 9 The equal split rule, $\bar{Y}$, is defined as follows:

$$\bar{Y} = \begin{cases} \frac{v(h)}{n(h)} \text{ where } i \in N(h) \text{ and } h \in C(g) & \text{if } v \text{ is component additive} \\ \frac{v(g)}{n(g)} \text{ for all } i \in N(g) & \text{otherwise} \end{cases}$$

The Shapley value allocates the value considering each agent's marginal contribution:

Definition 10 The TU-game U in characteristic function form is defined as follows: given (v, g), $U_{v,g}(S) = \sum_{h \in C(g|_S)} v(h)$ for each $S \subseteq N$. The Shapley value of a game U is $SV_i(U) = \sum_{S \subset N \setminus i}(U(S \cup \{i\}) - U(S))\frac{s!(n-s-1)!}{n!}$.

2.1.2 Consistency

The consistency principle applies to social networks as follows:

Definition 11 Given a value function v, and an allocation rule Y, the solution $\varphi : \mathcal{Y} \times V \to G^N$ [2] is complement consistent with respect to v and Y if for every $g \in \varphi(Y, v)$ and for every $N' \subset N$, we have $g\,|_{N'} \in \varphi(Y, \bar{v})$ where

$$\bar{v}(g') = v(g' \cup (g \setminus g\,|_{N'})) - \sum_{i \in N \setminus N'} Y_i(g, v) \quad \forall g' \in G^{N'} \tag{2.1}$$

Definition 12 Given a value function v, and an allocation rule Y, the solution $\varphi : \mathcal{Y} \times V \to G^N$ is max consistent with respect to v and Y if for every $g \in \varphi(Y, v)$ and for every $N' \subset N$, we have $g\,|_{N'} \in \varphi(Y, \bar{v})$ where

$$\bar{v}(g') = \max\{v(g' \cup g'') - \sum_{i \in N \setminus N'} Y_i(g, v) | \quad g''\,|_{N'} = \varnothing\} \quad \forall g' \in G^{N'} \tag{2.2}$$

The max consistency can be defined in various ways: the reduced value function (2.2) can be modified as:

$$\bar{v}(g') = \max\{v(g' \cup g'') - \sum_{i \in N(g'') \setminus N'} Y_i(g, v) | \quad g''\,|_{N'} = \varnothing\} \quad \forall g' \in G^{N'} \tag{2.3}$$

[2] In this analysis we simply take φ as the pairwise stability solution

2.2 Results

We will assume that the disconnected agents receive zero-payoff and the value of a graph is zero if and only if it is empty, i.e. $Y_i(g,v) = 0$ for all $i \in D_N(g)$ and $v(g) = 0$ if and only if $g = \varnothing$.

Lemma 1 If g is a pairwise stable graph, and allocation rule is equal split, then $d_N(g) < 2$.

Proof: Assume that g is a pairwise stable graph and $d_N(g) \geq 2$. Now take any $i, j \in D_N(g)$. Then, $Y_i(g,v) = Y_j(g,v) = 0$ due to the above assumption. If v is not component additive, then $Y_i(g+ij,v) = Y_j(g+ij,v) = \frac{v(g+ij)}{n(g)+2} > 0$, otherwise we have $Y_i(g+ij,v) = Y_j(g+ij,v) = \frac{v(\{ij\})}{2} > 0$. In either case, $g+ij$ defeats g. Contradiction.

Proposition 1 Let $\varphi : \mathcal{Y} \times V \to G^N$ be the pairwise stability solution. If v is not component additive and Y is the equal split rule, then the solution φ is complement consistent with respect to v and Y.

Proof: Let $g \in \varphi(Y,v)$ and take any $N' \subset N$. Now, $\bar{v}(g\mid_{N'}) = v(g) - \sum_{i \in N \setminus N'} \bar{Y}_i(g,v) = (n\prime - d_{N'}(g))\bar{Y}_i(g,v)$ which implies $\bar{Y}_i(g\mid_{N'},\bar{v}) = \bar{Y}_i(g,v)$ for all $i \in N'$. Now take any $ij \in g\mid_{N'}$. First assume that $n_i(g), n_j(g) \geq 2$. In this case, we have $\bar{v}(g\mid_{N'} - ij) = v(g-ij) - \sum_{i \in N \setminus N'} \bar{Y}_i(g,v) = (n - d_N(g))\bar{Y}_i(g-ij,v) - (n - n' - d_{N\setminus N'}(g))\bar{Y}_i(g,v)$. Since $i \in D_{N'}(g\mid_{N'})$ does not imply that $\bar{Y}_i(g\mid_{N'},\bar{v}) = 0$ whenever $i \notin D_{N'}(g)$, the last equality implies that $\bar{Y}_i(g\mid_{N'} - ij,\bar{v}) = \bar{Y}_j(g\mid_{N'} - ij,\bar{v}) = \frac{n-d_N(g)}{n'-d_{N'}(g)}(\bar{Y}_i(g-ij,v) - \bar{Y}_i(g,v)) + \bar{Y}_i(g,v) \leq \bar{Y}_i(g,v) = \bar{Y}_i(g\mid_{N'},\bar{v})$. The inequality holds because $\bar{Y}_i(g-ij,v) - \bar{Y}_i(g,v) \leq 0$ due to the pairwise stability of g. Now without loss of generality assume that $n_j(g) > n_i(g) = 1$. Then, $\bar{Y}_i(g\mid_{N'} - ij,\bar{v}) = 0 <$

$\bar{Y}_i(g\mid_{N'},\bar{v})$ and $\bar{Y}_j(g\mid_{N'} -ij,\bar{v}) = \frac{n-d_N(g)-1}{n'-d_{N'}(g)-1}(\bar{Y}_j(g-ij,v) - \bar{Y}_j(g,v)) + \bar{Y}_j(g,v) \leq \bar{Y}_j(g,v) = \bar{Y}_j(g\mid_{N'},\bar{v})$. Therefore, $g\mid_{N'}$ is not defeated by $g' = g\mid_{N'} -ij$ for any $ij \in g\mid_{N'}$. Now take any $i,j \in N'$ such that $ij \notin g\mid_{ij}$. First assume that $i,j \in N(g)$. In this case, we have $\bar{v}(g\mid_{N'} +ij) = v(g+ij) - \sum_{i\in N\setminus N'}\bar{Y}_i(g,v) = (n - d\mid_N(g))\bar{Y}_i(g+ij,v) - (n-n'-d_{N\setminus N'}(g))\bar{Y}_i(g,v)$. Since there are $n'-d_{N'}(g)$ agents in N' to share this value, we have $\bar{Y}_i(g\mid_{N'} +ij,\bar{v}) = \bar{Y}_j(g\mid_{N'} +ij,\bar{v}) = \frac{n-d_N(g)}{n'-d_{N'}(g)}(\bar{Y}_i(g+ij,v) - \bar{Y}_i(g,v)) + \bar{Y}_i(g,v) \leq \bar{Y}_i(g,v) = \bar{Y}_i(g\mid_{N'},\bar{v})$. Now without loss of generality assume that $D_N = \{i\}$. (Note that there can be at most one agent which is disconnected in g by Lemma 1, and since g is a pairwise stable graph, and $\bar{Y}_i(g+ij,v) > \bar{Y}_i(g,v) = 0$ we have that $\bar{Y}_j(g+ij,v) < \bar{Y}_j(g,v)$).Then, $\bar{v}(g\mid_{N'} +ij) = v(g+ij) - \sum_{i\in N\setminus N'}\bar{Y}_i(g,v) = n\bar{Y}_j(g-ij,v) - (n-n')\bar{Y}_j(g,v)$ implying that $\bar{Y}_j(g\mid_{N'} +ij,\bar{v}) = \frac{n}{n'}(\bar{Y}_j(g+ij,v) - \bar{Y}_j(g,v)) + \bar{Y}_j(g,v) < \bar{Y}_j(g,v) = \bar{Y}_j(g\mid_{N'},\bar{v})$. Hence, $g\mid_{N'}$ is not defeated by $g' = g\mid_{N'} +ij$ for any $i,j \in N'$ such that $ij \notin g\mid_{N'}$. This completes the proof.

Proposition 2 There exists a component additive value function v such that the pairwise stability solution is not complement consistent with respect to v and Y when Y is the equal split rule.

Proof: Let $N = \{i,j,k,v,w\}$ and consider the *component additive* v such that $v(\{ij,jk\}) = 3, v(\{ij,jk,ik\}) = 2, v(\{ij\}) = 1, v(\{vw\}) = 10, v(\{ij,jk,vw,vi\}) = v(\{ij,jk,vw,vj\}) = v(\{ij,jk,vw,vk\}) = v(\{ij,jk,vw,wi\}) = v(\{ij,jk,vw,wj\}) = v(\{ij,jk,vw,wk\}) = 22$. Assume that $Y = \bar{Y}$. Then the graph $g = \{ij,jk,vw\}$ is pairwise stable. Now, let $N' = \{j,k,v,w\}$. Then, $g\mid_{N'} = \{jk,vw\}$, and $\bar{Y}_l(g,v) = \bar{Y}_l(g\mid_{N'},\bar{v})$ for every $l \in N'$ since the payoff of agent l in the restricted graph $g\mid_{N'}$ depends on $\bar{v}(h)$ where $l \in N(h)$ and $h \in C(g)$. Consider vj and the payoffs of j and v: we have $\bar{v}(g\mid_{N'} +vj) = v(g+jv) - \bar{Y}_i(g,v) = 21$ implying that $\bar{Y}_j(g\mid_{N'}$

$+vj, \bar{v}) = 5\frac{1}{4} > 1 = \bar{Y}_j(g \mid_{N'}, \bar{v})$ and $\bar{Y}_v(g \mid_{N'} +vj, \bar{v}) = 5\frac{1}{4} > 5 = \bar{Y}_v(g \mid_{N'}, \bar{v})$. Hence, $g \mid_{N'}$ is defeated by $g' = g \mid_{N'} +vj$ and $g \mid_{N'} \notin \varphi(\bar{Y}, \bar{v})$. Therefore, pairwise stability is not complement consistent.

Lemma 2 If v satisfies strict monotonicity, and the allocation rule is the Shapley value of the game U, then the unique pairwise stable graph is the complete graph.

Proof: Let v be a strict monotonic value function and $Y_i(g,v) = SV_i(U_{v,g})$. Assume there exists a pairwise stable graph g such that $g \subsetneq g^N$. Take any $ij \notin g$. For every $S \subset N \setminus i$ such that $j \in S$, we have $U_{v,g+ij}(S \cup \{i\}) > U_{v,g}(S \cup \{i\})$ and $U_{v,g+ij}(S) = U_{v,g}(S)$. Since for all other subsets $T \subset N \setminus i$ with $j \notin T$, $(U(T \cup \{i\}) - U(T))\frac{t!(n-t-1)!}{n!}$ remains the same when g is changed to $g + ij$, we have that $SV_i(U_{v,g+ij}) > SV_i(U_{v,g})$ and the same holds for agent j. Hence, g is defeated by $g' = g + ij$ and $g \notin \varphi(Y, v)$.

Proposition 3 If v satisfies strict monotonicity, and the allocation rule is the Shapley value of the game U, then pairwise stability solution satisfies complement consistency with respect to v and Y.

Proof: Let v be a strict monotonic value function and $Y_i(g,v) = SV_i(U_{v,g})$. Then unique pairwise stable graph is g^N by the above lemma. Clearly, for every $N' \subset N$, we have $g^N \mid_{N'} = g^{N'}$. Take any $ij \in g^{N'}$. By the same argument used in the lemma above, $g^{N'}$ defeats $g = g^{N'} - ij$. Therefore, $g^{N'} \in \varphi(Y, \bar{v})$.

Lemma 3 If v is not component additive and satisfies strict monotonicity, and the allocation rule is equal split, then $g^N \in \varphi(\bar{Y}, v)$, i.e. complete graph is pairwise stable and the only other candidate g for pairwise stability is $g^{N \setminus i}$ with $D_N(g) = \{i\}$ for some $i \in N$.

Proof: Assume that v is not component additive and satisfies strict monotonicity, and the allocation rule is equal split. It is trivial that complete graph is pairwise stable. (Consider g^N and take any $i, j \in N$. Then, $\bar{Y}_i(g^N, v) = \bar{Y}_j(g^N, v) = \frac{v(g^N)}{n} > \frac{v(g^N - ij)}{n} = \bar{Y}_j(g^N - ij, v)$.) Now, consider the graph $g = g^{N\setminus i}$ for some i. Since agent i is disconnected, his payoff is zero. Take any agent $j \in N \setminus \{i\}$. By forming a link with agent i, agent j may loose since the number of agents to share the value of the graph $g + ij$ (although $v(g + ij) > v(g)$) increases by one. If this is true for all the agents in $N \setminus i$, then the graph g will be pairwise stable. (Note that g defeats also $g - kl$ for any $k, l \in N \setminus i$.) By using the same argument above, we conclude that no other graph can be pairwise stable. This completes the proof.

Proposition 4 Assume $\bar{v}$ is defined as (2.2). If v is not component additive and satisfies strict monotonicity, and the allocation rule is equal split, then pairwise stability solution is max consistent with respect to v and Y.

Proof: Assume that v is not component additive and satisfies strict monotonicity, and the allocation rule is equal split. We know from the above lemma that complete graph is pairwise stable. Take any $N' \subset N$. Since the term $\sum_{i \in N \setminus N'} Y_i(g, v)$ in the equation of the reduced value function is constant and does not depend on g'', $g = \{kl | k \in N' \text{ and } l \in N \setminus N'\}$ maximizes $\bar{v}(g')$ for every $g' \in G^{N'}$. This reduces max consistency to the complement consistency for the complete graph, and we also know from Proposition 1 that pairwise stability solution is complement consistent when v is not component additive. (The same argument applies when $g = g^{N\setminus i}$ with $D_N(g) = \{i\}$ for some $i \in N$ is pairwise stable and $i \in N'$) Now assume $g = g^{N\setminus i}$ with $D_N(g) = \{i\}$ for some $i \in N$ is pairwise stable and take any N' such that $i \in N \setminus N'$. Then, $g\,|_{N'} = g^{N'}$ and $\bar{v}(g^{N'}) > \bar{v}(g^{N'} - kl)$ for any $k, l \in N'$ with the number of agents remaining the same in both graphs. Therefore, $g\mid_{N'}$ is not defeated. This completes the proof.

Proposition 5 Assume $\bar{v}$ is defined as (2.3). There exists a strictly monotonic value function v, which is not component additive, such that the pairwise stability solution is not max consistent with respect to v and Y when the allocation rule is equal split.

Proof: Let $N = 8$ and consider the following value function which is not component additive and satisfies strict monotonicity: $v(g^{N\setminus i}) = 77$ with $\bar{Y}_i(g^{N\setminus i}, v) = 0$ i.e. i is disconnected, and $\bar{Y}_j(g^{N\setminus i}, v) = 11$ for every $j \in N \setminus i$; $v(g^{N\setminus i} + ij) = 80$ for every $j \in N \setminus i$ such that $\bar{Y}_k(g^{N\setminus i} + ij, v) = 10$ for all $k \in N$. Hence, $g^{N\setminus i} \in \varphi(\bar{Y}, v)$. Now, let $N \setminus N' = \{k, l, m\}$ and $\bar{v}(g^{N\setminus i}\mid_{N'}) = v(g^{N\setminus\{i,l,m\}}) - \bar{Y}_k(g^{N\setminus i}, v) = 56 - 11 = 45$. (Note that $\bar{v}(g^{N\setminus i}\mid_{N'})$ is maximized through $g'' = \{kj \mid j \in N' \setminus i\}$ and $N(g'') \setminus N' = \{k\}$.) Then we have $\bar{Y}_j(g^{N\setminus i}\mid_{N'}, \bar{v}) = 11\frac{1}{4}$. Now let $\bar{v}(g^{N\setminus i}\mid_{N'} + ij) = v(g^{N\setminus\{i,l,m\}} + ij) - \bar{Y}_k(g^{N\setminus i}, v) = 71 - 11 = 60$. Then, $\bar{Y}_j(g^{N\setminus i}\mid_{N'} + ij, \bar{v}) = 12 > 11\frac{1}{4} = \bar{Y}_j(g^{N\setminus i}\mid_{N'}, \bar{v})$. Hence, $g^{N\setminus i}\mid_{N'} \notin \varphi(\bar{Y}, \bar{v})$.

Chapter 3

Population Monotonic Allocation Rules with respect to the Pairwise Stability Solution in Social and Economic Networks

3.1 Basic Notations and Definitions

Consider the case that there are new agent(s) joining the set of agents N. An allocation rule is *population monotonic* with respect to a solution concept if all agents in the initial group N gain or all of them loose at the new solution outcome. In this section we will use the following notation: $v_N : \{g|g \subset g^N\} \to \mathbb{R}$, i.e. v is defined for all set of players. Now, we can define *population monotonicity* formally:

Definition 13 Given a value function v, the allocation rule Y satisfies population momotonicity with respect to the solution $\varphi : \mathcal{Y} \times V \to G$ if there exists

$g \in \varphi(Y, v_N)$ and $g' \in \varphi(Y, v_{N'})$ with $N \subset N'$ such that for any $i \in N$, $Y_i(g, v_N) \leq Y_i(g', v_{N'})[Y_i(g, v_N) \geq Y_i(g', v_{N'})]$ implies $Y_j(g, v_N) \leq Y_j(g', v_{N'})[Y_j(g, v_N) \geq Y_j(g', v_{N'})]$ for all $j \in N \setminus i$.

3.2 Results

Proposition 6 There exists an anonymous value function v which is not component additive such that equal split rule does not satisfy *population monotonicity* with respect to the pairwise stability solution.

Proof: Let $N = \{i, j\}$, $N' = \{i, j, k\}$, and $N'' = \{i, j, k, l\}$. Now consider the following (anonymous and not component additive) value function: $v_N(\{ij\}) = v_{N'}(\{ij\}) = 2$, $v_{N'}(g^{N'}) = 1$ and $v_N(\{ij, jk\}) = 2$. Then given N', the unique pairwise stable graph is $g = \{ij\}$. Now it is easily constructed such that, given N'', the unique pairwise stable graph is $g^{N''}$ with $v_{N''}(g^{N''}) = 3$. Here, one agent makes profit from joining agent l to N' (he receives $\frac{3}{4}$ instead of 0) but the other two agents loose in this case (they receive $\frac{3}{4}$ instead of 1). Therefore, equal split does not satisfy *population monotonicity* with respect to the pairwise stability solution.

Proposition 7 The equal split rule satisfies *population monotonicity* with respect to the pairwise stability solution when v is not component additive and $v \in V^* = \{v \in V | v_N(g) = 0 \text{ if and only if } D_N(g) \neq \varnothing\}$.

Proof: Assume v is not component additive and $v \in V^* = \{v \in V |\, v_N(g) = 0 \text{ if and only if } D_N(g) \neq \varnothing\}$. In this case, any graph with $d_N(g) > 0$ cannot be pairwise stable. Since also v is not component additive, every agent receives the same payoff in a pairwise stable graph. Hence, equal split rule satisfies population monotonicity with respect to the pairwise stability solution.

Proposition 8 The equal split rule satisfies *population monotonicity* with respect to the pairwise stability solution when v is strictly monotonic.

Proof: Assume that v is strictly monotonic. Since complete graph is pairwise stable for every set of agents, when $Y = \bar{Y}$ and v is strictly monotonic, by Lemma 3, clearly, equal split rule satisfies population monotonicity with respect to the pairwise stability solution.

Proposition 9 There exists an anonymous value function v such that the Shapley allocation does not satisfy population monotonicity with respect to the pairwise stability solution.

Proof: Let v be anonymous and defined as follows: $v(\{ij\}) = v(\{ij, jk, ik\}) = v(\{ij, jk, , kl, il\}) = 1$, $v(\{ij, ik\}) = v(g^{N|_{n=4}}) = v(\{ij, jk, kl\}) = v(\{ij, kl\}) = 2$, $v(\{ij, jk, kl, il, ik\}) = v(\{ij, ik, il\}) = v(\{ij, ik, kl, il\}) = 3$. Then we have that $\varphi(SV, v_{N=\{i,j\}}) = \{ij\}$, and $\varphi(SV, v_{N'=\{i,j,k\}}) = \{ij, ik\}$, and $\varphi(SV, v_{N''=\{i,j,k,l\}}) = \{ij, jk, kl, il, ik\}$. But then

$$SV_i(\varphi(SV, v_{N'=\{i,j,k\}}), v_{N'=\{i,j,k\}}) = 1 > SV_i(\varphi(SV, v_{N''=\{i,j,k,l\}}), v_{N'=\{i,j,k,l\}}) = \frac{2}{3}$$

whereas

$$SV_j(\varphi(SV, v_{N'=\{i,j,k\}}), v_{N'=\{i,j,k\}}) = \frac{1}{2} < SV_j(\varphi(SV, v_{N''=\{i,j,k,l\}}), v_{N'=\{i,j,k,l\}}) = \frac{5}{6}$$

. Hence, Shapley allocation does not satisfy population monotonicity with respect to the pairwise stability solution. (Note that $g = \{ij, jk\}$ is also pairwise stable since v is anonymous. But the result does not change, this time the payoff of j decreases whereas the payoff of i increases.)

Chapter 4

Conclusion

4.1 Conclusion

In this study we analyzed consistency and population monotonicity axioms in the context of graph structures. One of the main results is that pairwise stability solution appears to be complement consistent when the allocation rule is equal split and the value function is not component additive. But in case of component additivity of v, complement consistency is not attained. For max consistency, we found that monotonicity and again non-component additivity is crucial.

As far as the population monotonicity is concerned, we need more restrictions on the value functions. The equal split rule, for example, does not satisfy population monotonicity even if the value function is anonymous and not component additive. The Shapley value also does not satisfy population monotonicity when the value function is anonymous.

There are several directions in which this analysis can be extended. First, a stronger notion of stability appears to be central for further research. Secondly, the allocation rule and value function can be characterized for the principles analyzed in this study.

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