A TIME-BASED CONTROL POLICY FOR A PERISHABLE INVENTORY SYSTEM WITH LOST SALES

A THESIS
SUBMITTED TO THE DEPARTMENT OF INDUSTRIAL ENGINEERING
AND THE INSTITUTE OF ENGINEERING AND SCIENCE
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
Eylem Tekin
July 1998

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Master of Science.

Assoc. Prof. Ülkü Gürler (Supervisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Master of Science.

Assist. Prof. Emre Berk

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Master of Science.

Assoc. Prof. İhsan Sabuncuoğlu

Approved for the Institute of Engineering and Science:

Prof. Mehmet Baray,
Director of Institute of Engineering and Science

Abstract

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Eylem Tekin
M. S. in Industrial Engineering
Supervisor: Assoc. Prof. Ülkü Gürler
03 07 1998

In this study, we propose a new time-based policy for continuous review inventory systems where the products have fixed life times and unmet demands are lost. We derive the exact expressions of the key operating characteristics of the model. Based on these performance measures, we optimize the relevant costs subject to a service level criterion, namely the average fraction of time out of stock. A numerical analysis is provided to validate and compare our model with conventional policies. We also investigate some special cases of the time-based policy which are applicable to the products with infinite life times.

Keywords: Inventory, perishable, lost sales.

Özet

RAF ÖMRÜ OLAN ÜRÜNLER İÇİN ZAMANA DAYALI BİR ENVANTER POLİTİKASI

Eylem Tekin
Endüstri Mühenlisliği Yüksek Lisans
Tez Yöneticisi: Doç. Ülkü Gürler
03 07 1998

Bu çalışmada stok dışı talebin kaybedildiği ve stoktaki malların sabit bir ömrü olduğu sistemler için zamana dayalı bir envanter politikası geliştirilmiştir. Sözü edilen envanter politikası için düşünülen sistemde, iki talep arasındaki zamanın üstel dağıldığı ve sabit bir bekleme süresinin olduğu varsayılmaktadır. Problemin analitik çözümü için rassal süreçler teorisinden yararlanılmıştır. Sürekli gözden geçirilen envanter sistemleri için servis kısıtı altında uzun vadede ortalama maliyet ifade edilmiştir.

Anahtar sözcükler: Envanter, raf ömrü, talebin kaybedildiği ortamlar.

To Yeşim and to my parents

Acknowledgement

I would like to express my deep gratitute to Assoc. Prof. Ülkü Gürler and Assist. Prof. Emre Berk for their supervision in this thesis work. I learned a lot from them. They always supported me throughout my studies.

I am indebted to Assoc. Prof. İhsan Sabuncuoğlu for showing keen interest in the subject matter and accepting to read and review this thesis.

I am grateful to Ersin Keçecioğlu for his helps in typing this thesis and invaluable encouragements during my studies. My special thanks are for Yeşim Tekin for her never-ending moral support.

I would like to thank my officemates A. Gürhan Kök and Bahar Deler who were always helpful and understanding. We shared a lot in the last two years.

I also wish to thank Alper Kuş, Ayşin Oktay, Ebru Voyvoda, Erdem Ofli, Evrim Tekeşin, and Murat Temizsoy who were always with me during the six years of my life at Bilkent University. I am forever grateful for their keen friendship.

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Chapter 1

INTRODUCTION

Inventory management is a fundamental problem which arises in all areas of business administration. Mathematical models form the basis of most of the inventory control systems today, which are designed to answer two major questions: When should a replenishment order be placed and how much should the order quantity be.

The method of analysis and the applicability of any model depend on the assumptions about the underlying physical system. There are a number of key types of assumptions regarding the structure of an inventory model. Nahmias [20] classifies these assumptions as follows.

1. Continuous review vs. periodic review
2. Deterministic vs. random vs. unknown demand
3. Stationary vs. nonstationary models
4. Single period vs. finite horizon vs. infinite horizon
5. Backorder vs. lost sales
6. Average vs. discounted cost
7. Instant delivery vs. positive lead time

8. Infinite lifetime vs. perishability

9. Single vs. multiple products

10. Single installation vs. multi-echolon

Beginning with Harris's [13] *EOQ* formula , a considerable amount of literature has been devoted to inventory control problems in order to determine the optimal ordering policies. Uncertainty in the demand is the most significant issue that is handled in many inventory control studies. The traditional approach has been to minimize the expected costs with respect to the decision variables regarding the order quantity and reorder point. In this study, we will focus on the single-item, single-location inventory control problems.

Most of the research in this area is based on the assumption that the products in the inventories have infinite lifetimes and the inventory system operates under the (s, S) policy (or the continuous review version of the (s, S) policy which is the (Q, r) policy). The exact analysis of the (s, S) policy is available in the literature for the full backlogging case. Optimality of (s, S) policies for this case is also proven. Therefore, with full backlogging assumption, there is a vast literature on the algorithms and approximations for computing optimal and near optimal solutions of (s,S) policies.

On the other hand, for the case where unsatisfied demands are lost, the problem becomes very complex and the optimal ordering policy cannot be computed by analytical means. What makes lost sales analysis more complex is that unlike the backorders case, when the system is out of stock, the amount on hand plus on order does not change by a demand arrival. Therefore, it is not possible to consider the changes in the amount on hand independent of the amount on hand plus on order, so the procedure which is used to compute the distribution of on hand inventory from inventory position does not apply. Therefore, for nonperishable products there is not much done in the literature on the inventory replenishment problems with lost sales. The analysis of this case is restricted to Poisson demands and one order outstanding assumption.

Although one of the basic implicit assumptions of most inventory control models has been the infinite lifetime of products, there are also many types of products with limited shelf lifes, which are referred as perishable goods. If the shelf life of an item in the inventory is long or if the rate of deterioration is low and negligible, the perishability can be ignored in some cases. However, in many situations the existence of a shelf life plays a major role and its impact should be considered explicitly. Foodstuffs, blood inventories, drugs, volatile liquids which are used in industry are some common examples of such perishable inventories. Since the conventional ordering policies may not be appropriate when applied to perishable inventories, mathematical modeling of such systems has been an interesting research topic in inventory theory. A considerable literature is devoted to the inventories where products may have a fixed lifetime or a random lifetime. The existing studies in this area consider that the perishable inventories operate under the (s, S) policy and even for this policy, the means of determining optimal ordering quantity and reorder point is not available in the literature with reasonable general assumptions. Much of the reported literature assumes Poisson demands and instantenaous lead time or imposes a restriction to the policy itself (e.g. $(S-1, S)$ or $(0, S)$). When replenishment lead times are positive, the analysis becomes difficult. The difficulty is that aging can only be applied to units on hand not on order. The state variable would have to include all orders that were placed and the elapsed time since their placement. Unlike the models for nonperishable products, for the cases when there is a shelf life, the problem is harder if backorders are allowed. Schmidt and Nahmias (1985) states that it is unlikely that anyone would be able to find or to use an optimal policy.

In this study, we consider a continuous review inventory system where the products have fixed lifetimes and unmet demands are lost. We propose a new time-based policy to determine the optimal ordering quantity and reorder point. We derive the exact expressions of the key operating characteristics for Poisson demands. Based on these performance measures, we optimize the relevant costs subject to the constraint on the long run average fraction of lost sales. A numerical analysis is provided to validate and compare our model with conventional policies. The model is unique in that a different approach rather than the classical (s, S) type policies is addressed for controlling inventories. We also investigate some special cases of the time-based policy which are applicable to the products with infinite lifetimes.

This thesis work covers the following chapters. In Chapter 2, we present the literature on single-item, single-location inventory control models for perishable goods and random demands. For completeness, we present the major studies that consider infinite lifetimes for the products.

In Chapter 3, we explain the time-based policy and derive the key operating characteristics of the model. We state the optimization problem that we consider explicitly. Some special cases of the model are also examined in this chapter.

In Chapter 4, we present our numerical results on a wide range of parameter settings in comparison with the classical (Q, r) model. In the literature, the computational analysis of proposed models is neither exhaustive nor comprehensive. Hence, this part of the study can be considered as the most exhaustive computational analysis done in the context of perishable inventories.

In Chapter 5, we investigate a special case of our model for items with infinite lifetimes. The model is simple and interesting in that it facilitates a quick and efficient approximate solution procedure for the conventional (Q, r) model where demands follow an arbitrary distribution.

We conclude the thesis work by summarizing our findings and possible future research directions in Chapter 5.

Chapter 2

LITERATURE REVIEW

Inventory management is an area in which operations research has had a significant impact. Although the history of inventory management goes back to the beginning of 20^{th} century, the seminal papers of Arrow, Harris and Marshak [3] and Dvoretsky, Kiefer and Wolfowitz [8,9] are considered as the benchmarks of the modern inventory theory, after which a huge literature on inventory control models has been built on. The book by Hadley and Whitin [12] have a comprehensive discussion on optimal ordering policies and their approximations. Nahmias [20] classifies the inventory control models according to their underlying assumptions and highlights the major techniques and results of the inventory theory literature. Lee and Nahmias [17] give a comprehensive survey on the mathematical models for controlling the inventory of a single item.

Most of the literature on single-item, single-location models considers the (s, S) policies with full backlogging assumption. Scarf [35] establishes the optimality of (s, S) policies for a multi-period dynamic model under full backlogging. Beckman [4] investigates (s, S) policies for continuous review inventory systems and extends the proof for optimality of (s, S) policies for continuous review case. Veinott [41] provides an alternative proof under slightly different conditions.

Sivazlian [38] studies a continuous review inventory system where the interarrival times between unit demands are independently and identically distributed with an arbitrary distribution. Assuming that the reorder point is a nonnegative integer, it is shown that the limiting distribution of the inventory position is uniform and is independent of the distribution of the interarrival times. Optimal decision rules for instantaneous deliveries are given.

Archibald and Silver [2] consider (s, S) policies for a continuous review system with discrete compound Poisson demand, convex holding-shortage cost, fixed ordering cost, and positive lead time. They develop a recursive formula to compute the cost for a given (s, S) pair. In order to determine the optimal (s, S), relations among $s, S, S - s$ and the cost rate are determined.

Sahin [31] assumes a compound renewal time process under the (s, S) policy. He develops expressions for both time dependent and stationary distributions of the net inventory and the inventory position using a renewal theoretic structure. He presents the operating characteristics for both continuous review and periodic review inventory systems. Later, Sahin [32] proves the necessary and sufficient conditions for pseudo convexity of the cost rate function and computes the optimal stationary policy by a one-dimensional search routine.

A considerable effort is given for efficient computation of optimal (s, S) policies. Federgruen and Zipkin [11] present an algorithm to compute an optimal (s, S) policy under stationary data, well-behaved one period cost, discrete demand, and full backlogging assumptions. Porteus [25] also considers a periodic review inventory system with stationary independent demands and infinite planning horizon. He introduces three methods to obtain approximately optimal policies with little computational effort. The paper also provides a detailed survey of other methods for computing (s, S) policies and compares them on a broad range of problem settings. On the other hand, Federgruen and Zheng [10] propose an efficient algorithm for computing an optimal (Q, r) policy in a continuous review inventory system. The computational complexity of the algorithm is linear in Q^*.

Besides the efforts for computing optimal (s, S) policies, some heuristics and approximations are also developed as the computational difficulties make the exact models unattractive in practice. Sahin and Sinha [34] propose an approximation which is derived by using asymptotic results from renewal theory and examine the distribution-free approximation for the order quantity using a wide range of demand distributions and parameter settings. Chen and Zheng [6] develop a heuristic (s, S) policy by providing a closed form formula for $S - s$ and show that the heuristic is within %6 of the optimal cost value.

The study by Schultz [36] considers a special case of (s, S) policy which is one-for-one $(S - 1, S)$ inventory policy. He investigates the conditions under which it is not economical to batch the orders.

The foregoing discussion presents a non-exhaustive review of some basic literature on (s, S) policies with full backlogging assumption. Below we present a review of the literature on inventory control problems with lost sales. As mentioned earlier, the literature on the inventory control problems with lost sales is not as rich as the one for full backlogging case.

One of the earlier works in this area belong to Hadley and Whitin [12]. They consider a continuous review inventory system operating under the (Q, r) policy and Poisson demands. They compute the long run average cost rate function for the case where there is single outstanding order at a time. They also analyze the case $Q = 1$ and lead time is an exponential random variable.

Pressman [26] addresses the periodic review inventory system with lost sales. He introduces a fixed lag (lead time) between the placement and delivery of each order. At the end of each scheduling period, enough stock is ordered so that the stock on hand and on order is raised to a preassigned level. Demands are assumed to be distributed uniformly and demand sizes are discrete with a maximum possible value. Average cost is expressed as a function of the on hand inventory. Nahmias [21] provides approximate solutions for the periodic review case where the lead time is random and partial backordering is possible.

Archibald [1] proposes a method which minimizes the average stationary cost for continuous review inventory systems under discrete compound Poisson demand and one order outstanding assumption. He defines a cycle as the time between the arrivals of successive supplier shipments. He first calculates the expected cost and the length of a cycle for a given starting inventory. Then, by using the fact that discrete compound Poisson demand is memoryless and there is no outstanding order at the start of a cycle, transition from a starting inventory to the next is defined to be Markovian. The expected cycle cost is expressed as the weighted sum of cycle costs corresponding to every possible starting inventory level.

Ravichandran [28] studies the stochastic process induced by a continuous review (s, S) inventory model with Poisson demands and a random lead time with phase type distribution. The stationary distribution of the stock level is obtained as a closed form expression for the unit demands case.

Hill [14] considers the (Q, r) policy for continuous review inventory systems where demands follow a Poisson process and at most two orders may be outstanding. He describes a numerical procedure for computing steady state values of two key measures of system performance, namely the percentage of satisfied demand and the average stock level. Buchanan and Love [5] also consider the (Q, r) inventory model with lost sales but they assume that the lead time has an Erlang distribution.

The literature that we reviewed so far assume that the items in the inventories have infinite lifetimes. The conventional ordering policies that are discussed in the previous paragraphs may not be appropriate when applied to perishable inventory systems. Therefore, distinct models are developed for these kind of systems. The literature on ordering policies for perishable inventories can be classified into two categories. The first category considers items with continuous decay (e.g. radioactive materials, photographic films). The second category includes the cases where the lifetime of products is a known constant independent of all other parameters of the system (e.g. blood inventories, foodstuffs).

Raafat [27] presents an exhaustive review on continuously deteriorating inventory models. The first study in this area which considers random demands belong to Shah and Jaiswal [37]. In their paper, they develop an order-level inventory model by assuming instantenaous delivery and constant rate of deterioration.

Nahmias and Wang [24] derive a heuristic lot size reorder policy for an exponential decay problem and discuss some of the difficulties that arise due to the presence of positive lead time.

Liu [18] studies the (s, S) model with random lifetimes and discusses the difference between the proportional inventory decay and the finite lifetime of a product. He also assumes that the lead time is zero.

Kalpakam and Sapna [15] analyze the (s, S) model for inventory systems with Poisson demands, exponentially distributed lead times and items with exponential lifetimes. The steady state operating characteristics are obtained explicitly and analytical properties of the long run expected cost rate is discussed. Later, Kalpakam and Sapna [16] consider a one-to-one ordering, perishable inventory model with renewal demands and exponential lifetimes. In the paper, the problem of minimizing the long-run expected cost rate is discussed and a non-exhaustive numerical study is provided.

A recent study by Liu and Cheung [19] investigates base-stock policies with unit demands, exponentially distributed lifetimes and a positive lead time. They provide the expression of the operating characteristics for complete backorders, complete lost sales and partial backorders. They optimize the system parameters subject to fill rate and waiting time constraints.

Nahmias [22] provides a comprehensive survey and reviews the relevant literature on the problem of determining suitable ordering policies for fixed life perishable products. He also considers a limited number of models where the products are subject to continuous exponential decay.

The first analysis for fixed life perishability belongs to Van Zyl [40]. He considers a periodic review inventory problem and computes the optimal ordering policies assuming that the lifetime of items is exactly two periods. Nahmias [23] extends this study for items that may have lifetimes of more than two periods.

Weiss [42] considers a continuous review inventory system where the products have fixed lifetimes and there is an instantenaous delivery of orders. He presents that an optimal policy for the lost sales case is of the type "never order" or the type "order up to S at the instant that the inventory level reaches zero". He also proves that for the full backlogging case, there exists an optimal policy that is of the type "order up to S as soon as the marginal shortage cost of not ordering is greater than the optimal expected average cost". The major assumption in this study is that the penalty cost incurred, even if the length of time the system is short is zero. In the paper, the cost expression for the lost sales model is developed as a function of S. Some computational results are also provided.

Schmidt and Nahmias [39] study $(S-1, S)$ policies with positive lead time for a single item whose lifetime is fixed. This study provides the first analysis for perishables with a positive lead time. They assume that the inventory is monitored continuously, demands follow a Poisson process and unmet demands are lost. They comment that the form and structure of an optimal policy for a continuous review perishable inventory system with positive lead time appears to be extremely complex and it is unlikely that anyone would be able to find or to use an optimal policy.

Chiu [7] proposes an approximate continuous review perishable inventory model which operates under the (Q, r) policy. He assumes a positive order lead time and a fixed shelf life for products. The paper provides an approximate solution by assuming that no undershoot occurs at the reorder point r and by using only the total beginning stock instead of the state vector that denotes the remaining lifetime of items. The approximation is verified by a comparison with the Weiss [42] model. The computational results reveal that the mean absolute deviation is 0.58%. The paper also compares the approximate model to the conventional (Q, r) model with no perishability. A simulation model of the real system is also developed to validate the approximate results.

Ravichandran [29] studies a continuous review perishable inventory system of (S, s) type. He assumes that the demands are governed by a Poisson process and there is a positive lead time with an arbitrary distribution. He presents an expression for the stationary distribution of the inventory level process under a specified aging phenomena. The specific aging phenomena assumes that the aging of a fresh batch does not begin until all units of the previous batch are exhausted either by demand or decay. He derives the cost rate function by making use of the stationary distribution of the inventory level process.

Chapter 3

THE TIME-BASED INVENTORY CONTROL POLICY

3.1 Description Of The Model

In this study, we consider a single-item, single-location continuous review inventory system where the products have fixed shelf lifes and unmet demands are lost.

In the inventory theory literature, the systems with fixed product lifetimes are considered to be difficult to analyze when replenishment lead time is positive. The possible difficulties which arise in analyzing such systems are discussed in Chapter 1. The first study which considers a positive lead time for perishable products belong to Schmidt and Nahmias (1985) and during the last thirteen years, there have been very few reported research in this area. Moreover, the proposed models do not address the issues regarding the optimal ordering policy with reasonably general assumptions. The existing studies consider that the inventory system under consideration operates under an (s, S) type policy and develop either approximate models (e.g. Ravichandran [29], Chiu [7] or models for a prespecified class of (s, S) policies such as $(S-1, S)$ policy (Schmidt and

Nahmias [39]).

The major reason for using (s, S) type policies for perishable inventories is that many practical replenishment problems that assume infinite lifetime for the products satisfy the mathematical conditions under which (s, S) policies are optimal. However, this is not necessarily true when perishability is introduced to the problem. Let us consider an inventory level process where products are demanded in discrete units. According to the (s, S) policy, an order is placed when the inventory position hits s units. If the items in the inventory are subject to decay after a constant time, (s, S) policy is not optimal because the nature of the policy necessitates to wait until the inventory position becomes s units even though it may be more beneficial to order between demand arrivals. Hence, it is reasonable to think that the optimal inventory control policy for perishables should incorporate the information of the remaining lifetimes of items.

With this motivation, we propose a new time-based policy for controlling perishable inventory systems. Our model provides a starting point for the analysis of perishables with a different approach other than the conventional policies. As will be discussed in Chapter 4, the time-based policy is more robust against the perishability of goods for some cases and it performs better than the conventional policies.

The time-based policy is applicable to the inventory systems where all transactions are monitored continuously and inventory ordering decisions are made as soon as a transaction occurs. The products in the inventory have a constant lifetime and our model assumes that the aging of a fresh batch does not begin until all units of the previous batch are exhausted either by demand or decay. This specified aging phenomena was first introduced by Ravichandran [29]. The main motivation for the specified aging of a batch is from production or inventory environments in which goods are protected enough not to decay until they are unpacked. A new batch is unpacked when the goods from the previous batch are either used up by demand or decay. Some composite raw materials which are preserved in the refrigerators until they go through the manufacturing process are examples for inventories with this specified aging pattern. Besides the applicability of the specified aging phenomena to various inventory systems, it can also be a good approximation for the systems in which the products begin aging as soon as they arrive to the system. The performance of this approximation will be discussed later in Chapter 4.

Direct costs associated to the system are the linear holding cost, linear perishing cost and the fixed ordering cost. Inventory investment is based on a service level criterion rather than on the classical cost minimization approach. In other words, no explicit value is assigned to the lost sales cost. It is often difficult for management to accurately estimate lost sales costs since it is generally not a direct out of pocket cost but a cost of loosing goodwill of a customer. The consequences of loss of customer goodwill are hard to evaluate and hence, the cost minimization approach may not be feasible. On the contrary, using a service level criterion generates useful managerial insights. Therefore, we optimize the relevant costs subject to the constraint that the average fraction of lost sales is not greater than a fixed value.

Having stated the main motivation and basic characteristics of our model, we next list the assumptions our model. Some of these assumptions are mentioned in the previous paragraphs but we express them below in order to be more explicit.

Assumptions

1. Demands arrive to the inventory system one at a time.
2. Demands are governed by a Poisson process.
3. Demands that cannot be met are lost.
4. The inventory levels are monitored continuously.
5. There is a positive lead time.
6. There exists at most one order outstanding at any time.
7. Direct costs associated to the system are the linear holding cost, linear perishing cost and the fixed ordering cost.

8. The products in the inventory have a constant lifetime and aging of a fresh batch does not begin until all units of the previous batch are exhausted either by demand or through decay.

Under these assumptions, we propose the following time-based policy.

The Control Policy A replenishment order of Q units is placed either when the inventory drops to r or after T units of time have elapsed since the last instance at which the inventory level hit Q, whichever occurs first.

The above policy will be referred to as (Q, r, T) policy. The decision variables for this policy are the order quantity (Q), the reorder point (r) and the time spent in the system since the last instance at which the inventory position is Q units (T). The ordering decision is based on the relationship between the variables r and T. If we denote the time at which an order is given by $O(t)$, r can be considered as the inventory threshold for reorder, i.e. inventory position at $O(t) \geq r$. T indicates the upper bound for $O(t)$, i.e. $O(t) \leq T$. Thus, we call T as the time threshold for reorder.

Our aim is to derive explicit expressions of the key operating characteristics of the model and determine the optimal values of the decision variables Q, r and T for given cost parameters and the service level constraint. The operating characteristics for the described system can be listed as the expected on hand inventory per unit time, expected number of lost sales per unit time and expected number of units that perish per unit time. We also derive the long run average cost rate function by making use of these quantities and the renewal reward theorem.

3.2 Notation and Preliminaries

In this section, we present the necessary notation and the preliminary analysis of the model under consideration. In particular, typical behaviour of the inventory process is displayed in detail which will form the basis of the cost expressions that will be derived in Section 2.3.

Notation

Q	=	Order quantity.
r	=	Inventory threshold for reorder.
T	=	Time threshold for reorder.
λ	=	Demand arrival rate.
L	=	Lead time.
τ	=	Constant lifetime for a batch of Q units. $\tau > T$
h	=	Holding cost per unit per unit time.
π	=	Lost sales cost per unit.
p	=	Perishing cost per unit.
K	=	Fixed ordering cost.
α	=	Prespecified value for the average fraction of lost sales.
X_n	=	Random variable representing the arrival time of n^{th} consecutive demand.
$f(t)$	=	Pdf of the time interval between successive demands.
$N(t)$	=	Counting process associated with demand process in $(0, t)$.
$F_n(t)$	=	$\Pr(N(t) \leq n)$
$\overline{F}_n(t)$	=	$1 - F_n(t)$
$\Delta(n)$	=	$F_n(T + L) - F_n(T)$
$J(a, b)$	=	$\int_a^b t f_Q(t) dt$
$\overline{J}(a)$	=	$\int_a^\infty t f_Q(t) dt$
$G(i, k)$	=	$\int_0^{\tau-L} F_{r+i}(\tau - t) f_{Q-r+k}(t) dt$
$H(i, k)$	=	$\int_T^{T+L} F_{r+i}(T + L - t) f_{Q-r+k}(t) dt$
$E(CL)$	=	Expected cycle length.
$E(OH)$	=	Expected on hand inventory per cycle.
$E(LS)$	=	Expected number of lost sales per cycle.
$E(P)$	=	Expected number of units that perish in a cycle.

Under the specific aging pattern, the instances at which the inventory level hits Q units are the regenerative epochs. As the system regenerates itself on these epochs, we can derive the operating characteristics by employing the renewal reward theorem [30]. For this purpose, we define a regenerative cycle as follows.

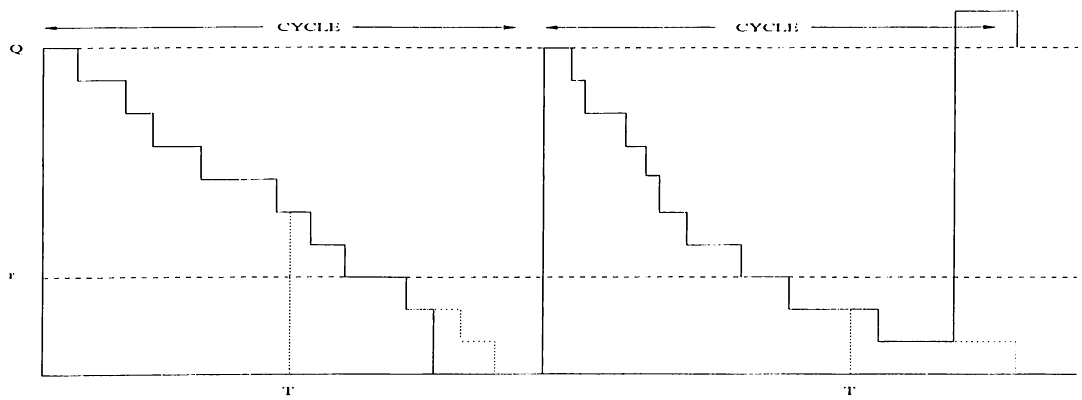

Figure 3.1: A Typical Realization of the Model

Cycle definition A cycle is the time between two consecutive instances at which inventory level hits Q.

Figure 3.1 presents a typical realization of the inventory level process. Each cycle begins with a fresh batch of Q units. Units are withdrawn from stock according to Poisson arrivals and one at a time. A replenishment order is given either at time T or when the inventory level drops to r. A regenerative cycle may end in two ways. The inventory level may drop to zero either by demand arrivals or by decay of units as illustrated in the first cycle of Figure 3.1. In this case, the cycle ends when the inventory position is increased to Q units by the arrival of a fresh batch. The next cycle begins with this batch of Q units which has a useful lifetime of τ. If the outstanding order arrives when there are still some items in the inventory, the inventory position increases above Q units. At this instance, inventory on hand is composed of a number of items from the previous batch which are subject to decay and an unpacked batch of Q units. The cycle ends either by demand arrivals or decay of the items from the previous batch.

Based on the relations among T, X_{Q-r}, X_Q and τ, there exist eight possible realizations for a cycle. Note here that X_{Q-r} and X_Q are random variables where T and τ are nonnegative constants. The possible realizations are illustrated in Figure 3.2 and Figure 3.3. In order to avoid repetition, we will only explain four

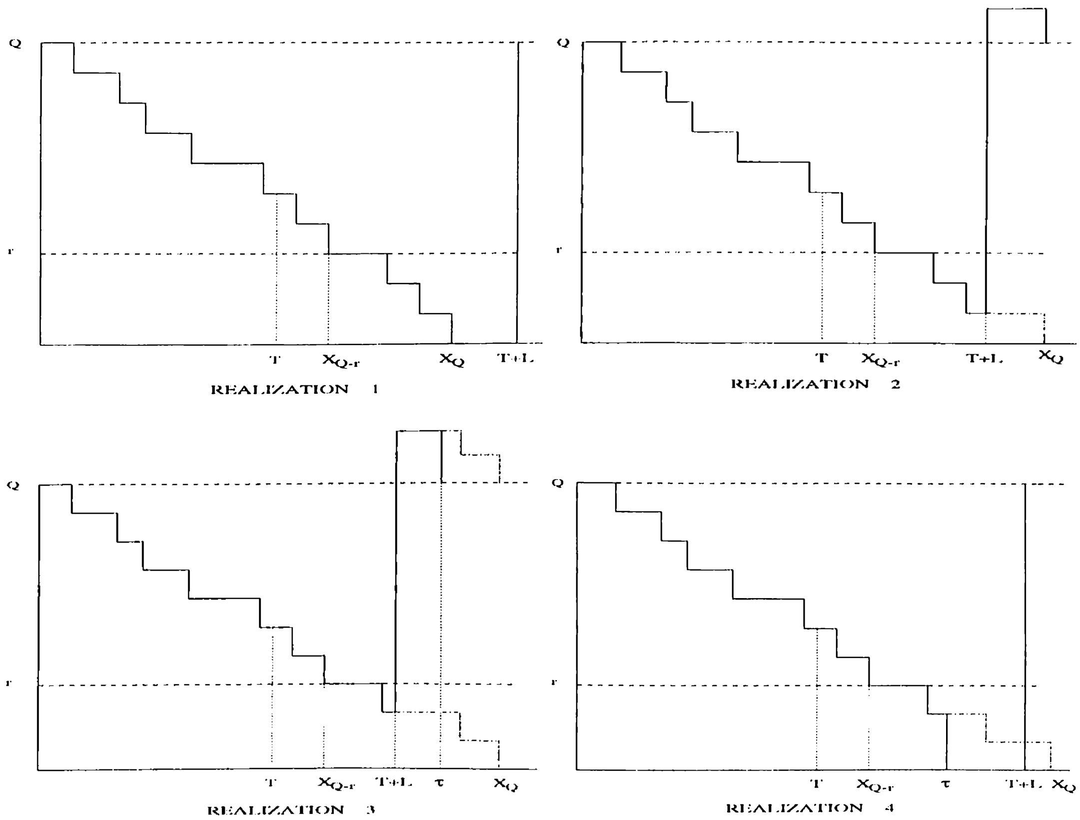

Figure 3.2: Possible Realizations when $T < X_{Q-r}$

of these realizations where the order is given at the time threshold for reorder T ($T < X_{Q-r}$). The other cycle realizations follow the same pattern except that for these cases, the order is placed when the inventory position decreases to r ($X_{Q-r} < T$).

Realization 1 The inventory position drops to zero during the lead time. Lost sales are incurred until a batch of Q units arrives. Note that no items perish in this case. The life time of a batch is greater than the time of last demand arrival.

Realization 2 A batch of Q units arrives when there are still some items in the inventory. Therefore, the inventory position increases above Q units after the lead time. The inventory level decreases to Q units by demand arrivals and the cycle ends. Again, the life time of a batch is greater than the time of last

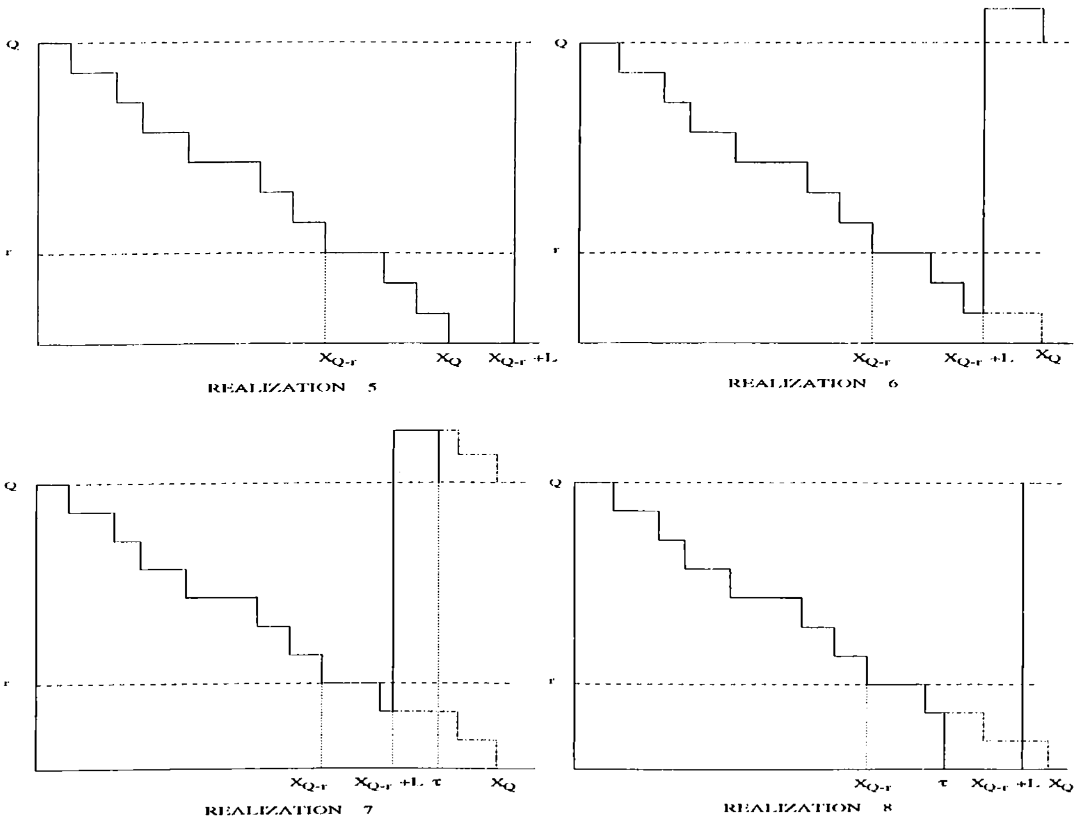

Figure 3.3: Possible Realizations when $X_{Q-r} < T$

demand arrival.

Realization 3 Similar to Realization 2, the arrival of an order increases the inventory position above Q units. The cycle ends by perishing of the items from the previous batch. At this point, the new batch is unpacked and a new cycle begins with the fresh batch.

Realization 4 Some of the items from a batch of Q units perish before a new order arrives decreasing the inventory level to zero. The cycle ends by the arrival of a new batch.

We develop the expressions for the operating characteristics with respect to the stochastic processes associated with each of these possible realizations. The following section presents the expressions for the expected cycle length and the operating characteristics of the time-based inventory control policy.

3.3 Derivation of The Operating Characteristics

In this section, we derive the expressions for the expected values of the cycle length, on hand inventory, number of lost sales and the number of items that perish in a cycle as a function of the decision variables Q, r and T. These expressions are then used to construct the average cost function which is explicitly discussed below.

Let us denote expected cycle length, expected on hand inventory, expected number of lost sales and expected number of units that perish per cycle by $E(CL)$, $E(OH)$, $E(LS)$ and $E(P)$, respectively. We consider the optimization of the following problem.

$$\min C(Q,r,T) = \frac{K + hE(OH) + pE(P)}{E(CL)} \tag{3.1}$$

subject to

$$\frac{E(LS)}{\lambda E(CL)} \leq \alpha \tag{3.2}$$

where α is the maximum allowed value for the average fraction of time the system is out of stock.

We know from the theory of Langrange multipliers that we can form the function

$$\psi(Q,r,T,\beta) = C(Q,r,T) + \beta(\frac{E(LS)}{E(CL)} - \alpha\lambda) \tag{3.3}$$

where β is the Langrange multiplier to minimize Equation 3.1 subject to the constraint 3.2 and minimizing $\psi(Q,r,T,\beta)$ for a given β will yield the same $Q^*(\beta)$, $r^*(\beta)$, $T^*(\beta)$ as minimizing

$$AC(Q,r,T) = \frac{K + hE(OH) + pE(P) + \beta E(LS)}{E(CL)} \tag{3.4}$$

Hence, an interesting observation is that in order to determine Q^*, r^*, T^*, we can first determine $Q^*(\beta)$, $r^*(\beta)$, $T^*(\beta)$ by minimizing Equation 3.4 and then selecting the β^* for which $\frac{E(LS)}{E(CL)} = \lambda\alpha$. The values of $Q^*(\beta)$, $r^*(\beta)$, $T^*(\beta)$ evaluated at β^* are Q^*, r^*, T^*, respectively. $AC(Q, r, T)$ is simply the long run average cost rate function and β^* corresponds to the lost sales cost per unit. Thus, minimizing Equation 3.1 subject to the Equation 3.2 is equivalent to minimizing the long run average cost given by Equation 3.4. But, in the former case, we do not need to assign an explicit value for the lost sales cost.

After setting our problem, now we need to derive the expressions for $E(CL)$, $E(OH)$, $E(LS)$ and $E(P)$. In the analysis, we shall not let T be greater than τ, since postponing an ordering decision until the batch has completely decayed makes no sense. The expressions of the operating characteristics differ for the cases when $T < \tau < T + L$ and $\tau > T + L$. Because, some of the realizations which are observed in one case cannot be observed in the other. For instance, if we consider the case $\tau > T + L$, for the cases where an ordering decision is made at time T, no items perish before the order arrival. However, this particular realization is observed when $T < \tau < T + L$. Analysis of both cases is necessary for completeness of the model as we cannot guarantee that the optimal T always satisfies one condition but not the other for a given parameter set.

The following two theorems provide the expressions for the operating characteristics of the system for both cases where $T < \tau < T + L$ and $\tau > T + L$. First, let us define the following quantities.

$$\begin{aligned}
C_1(Q, r) &= [\tau - L\overline{F}_r(L) - \tfrac{r}{\lambda}F_{r+1}(L)]F_{Q-r}(\tau - L) \\
C_2(Q, r, T) &= [LF_r(L) - \tfrac{r}{\lambda}F_{r+1}(L)]F_{Q-r}(T) \\
\eta(Q, r, T) &= -\tau G(0,0) + \tfrac{Q-r}{\lambda}G(0,1) + \tfrac{r}{\lambda}G(1,0) \\
\zeta(Q, r, T) &= (T + L)H(0,0) - \tfrac{Q-r}{\lambda}H(0,1) - \tfrac{r}{\lambda}H(1,0)
\end{aligned}$$

Theorem 1 If $T < \tau < T + L$, the $E(CL)$, $E(OH)$, $E(LS)$ and $E(P)$ are given by the following equations, respectively.

$$\begin{aligned}
E(CL) = \; & L + \eta(Q, r, T) + T\overline{F}_{Q-r}(T) + \tfrac{Q-r}{\lambda}[F_{Q-r+1}(T) - F_{Q-r+1}(\tau - L)] \\
& + C_1(Q, r)
\end{aligned} \tag{3.5}$$

$$E(OH) = Q[\eta(Q,r,T) + \tfrac{Q+1}{2\lambda}F_{Q+1}(\tau) + \tau\overline{F}_Q(\tau) + C_1(Q,r) - \tfrac{Q-r}{\lambda}F_{Q-r+1}(\tau - L)] - \tfrac{\lambda\tau^2}{2}\overline{F}_{Q-1}(\tau) \quad (3.6)$$

$$E(LS) = \lambda[\eta(Q,r,T) + C_1(Q,r) - \tau\overline{F}_Q(\tau) - T\overline{F}_{Q-r}(T) + L] + (Q-r)[F_{Q-r+1}(T) - F_{Q-r+1}(\tau - L)] - QF_{Q+1}(\tau) \quad (3.7)$$

$$E(P) = Q\overline{F}_Q(\tau) - \lambda\tau\overline{F}_{Q-1}(\tau) \quad (3.8)$$

Proof: The proof for Theorem 1 is given in Appendix A.

Theorem 2 If $\tau > T + L$, the $E(CL)$, $E(OH)$, $E(LS)$ and $E(P)$ are given by the following equations, respectively.

$$E(CL) = \zeta(Q,r,T) + C_2(Q,r,T) + \tau\overline{F}_Q(\tau) + \frac{Q}{\lambda}F_{Q+1}(\tau) \quad (3.9)$$

$$E(OH) = Q[\zeta(Q,r,T) + C_2(Q,r,T) + TF_{Q-r}(T) - \tfrac{Q-r}{\lambda}F_{Q-r+1}(T) + \tfrac{3Q+1}{2\lambda}F_{Q+1}(\tau) + 2\tau\overline{F}_Q(\tau) - T - L] - \tfrac{\lambda\tau^2}{2}\overline{F}_{Q-1}(\tau) \quad (3.10)$$

$$E(LS) = \lambda[\zeta(Q,r,T) + C_2(Q,r,T)] \quad (3.11)$$

$$E(P) = Q\overline{F}_Q(\tau) - \lambda\tau\overline{F}_{Q-1}(\tau) \quad (3.12)$$

Proof: The proof for Theorem 2 is given in Appendix A.

Theorem 1 and Theorem 2 above are used to construct the objective function given in 3.1. Given the involved form of the expressions, it seems almost impossible to find explicit expressions for optimal values of Q, r and T. Furthermore, we have the constraint 3.2 in our optimization problem which makes the analysis even more difficult. Our observations from the computational study indicate that the average cost function given by Equation 3.1 is unimodal with respect to Q, r and T. When we fix r and investigate the average cost function by varying Q and T, we observe that there exist more than one value of T^* which

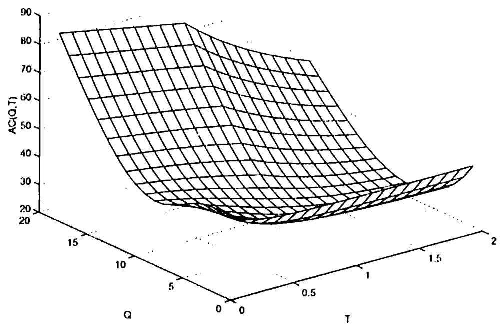

Figure 3.4: Shape of the Average Cost Function w.r.t. Q and T

results in the same optimal cost value (alternate optima). Figure 3.4 illustrates such behaviour of T when λ=5, p=10, K=50, L=1, h=1, α=0.01, τ=2 and r^*=0. Thus, we compute the optimal average cost and the corresponding values for Q, r and T by means of an exhaustive search. In order to make the search region smaller and hence speed up the procedure, we first solve the optimization problem for the (Q, r) model. As we search in a two dimensional space and both Q and r are discrete variables, exhaustive search results in a shorter time in this case. Then, we investigate the optimal values for Q, r and T in the vicinity of the (Q, r) pair obtained from the previous search. Note that T is always bounded by τ so, the search space for T changes according to the specified value for τ. For computing optimal (Q, r, T) values and the corresponding expected costs , we have developed a computer program in FORTRAN language. In order to facilitate the computations for the convolutions of $F(t)$ and $f(t)$, the necessary subroutines from the IMSL MATH/LIBRARY are linked to the program. Our numerical results are discussed in detail in Chapter 4.

3.4 Special Cases of the Model

In this section, we present the special cases of the (Q, r, T) model. Our model provides a rich and a flexible control policy which also induces insightful special cases. These cases are discussed below.

<u>**Case 1:**</u> (Q,r) **Model,** $T=\tau$

The perishable (Q,r) model for the inventory system under consideration places an order when the inventory position is exactly r units or when a decay of products take place at time τ. In our model, if we let $T=\tau$, we satisfy the ordering policy described above and obtain the expressions of the (Q,r) model for products which decay according to the specified aging phenomena.

Ravichandran [29] studies a similar model but instead of a constant lead time he assumes that the lead time is random with an arbitrary distribution. He derives the operating characteristics by defining the stationary distribution of the inventory level process. We consider a different approach by defining regenerative cycles and using the renewal reward theorem. The following cost expressions are obtained for the (Q,r) model which are also of interest since such explicit expressions for constant lead time are not provided in Ravichandran.

$$\begin{aligned} E(CL) &= L+\eta(Q,r,T)+\tau\overline{F}_{Q-r}(\tau)+\tfrac{Q-r}{\lambda}[F_{Q-r+1}(\tau)-F_{Q-r+1}(\tau-L)] \\ &\quad +C_1(Q,r) \end{aligned} \tag{3.13}$$

$$\begin{aligned} E(OH) &= Q[\eta(Q,r,T)+\tau\overline{F}_Q(\tau)-\tfrac{Q-r}{\lambda}F_{Q-r+1}(\tau-L)+\tfrac{(Q+1)}{2\lambda}F_{Q+1}(\tau) \\ &\quad +C_1(Q,r)]-\tfrac{\lambda\tau^2}{2}\overline{F}_{Q-1}(\tau) \end{aligned} \tag{3.14}$$

$$\begin{aligned} E(LS) &= \lambda[L+\eta(Q,r,T)-\tau(F_{Q-r}(\tau)-F_Q(\tau))-F_{Q-r}(\tau-L)] \\ &\quad +(Q-r)[F_{Q-r+1}(\tau)-F_{Q-r+1}(\tau-L)]-QF_{Q+1}(\tau) \end{aligned} \tag{3.15}$$

$$E(P) = Q\overline{F}_Q(\tau)-\lambda\tau\overline{F}_{Q-1}(\tau) \tag{3.16}$$

In the computational study that we present in the next chapter we make use of this model and compare the performances of the time-based policy and the (Q,r) policy.

<u>**Case 2:**</u> (Q,r,T) **Model,** $\tau\longrightarrow\infty$

As mentioned before, the literature on nonperishable inventories is mostly based on the (s,S) (or (Q,r)) type policies. Although the optimal policy for the lost sales case is not available yet, it is well known that (s, S) policies perform very well for a wide range of parameter settings. However, investigating different policies can still be a fruitful research area. In our model, when we let $\tau \to \infty$, the time-based model applies to the continuous review inventory system under consideration where products are assumed to have infinite life times and the operating characteristics for this case have the following expressions.

$$E(CL) = \frac{Q}{\lambda} + \zeta(Q,r,T) + C_2(Q,r,T) \tag{3.17}$$

$$\begin{aligned} E(OH) = {} & Q[\frac{(3Q+1)}{2\lambda} - (T+L) + \zeta(Q,r,T) - \frac{Q-r}{\lambda}F_{Q-r+1}(T) \\ & +(C_2(Q,r,T) + TF_{Q-r}(T))] \end{aligned} \tag{3.18}$$

$$E(LS) = \lambda[\zeta(Q,r,T) + C_2(Q,r,T)] \tag{3.19}$$

Case 3: (Q,T) **Model,** r **=0,** $\tau \longrightarrow \infty$

Another special case for infinite lifetime products is the (Q,T) policy which we obtain by setting the reorder point to zero. If we consider that the policies for inventory systems with nonperishable products are generally used for inventory systems with a large number of items such as department stores and mega markets, the computational savings become as important as the accuracy obtained. The expressions for the (Q,T) policy turn out to be straightforward and easily computable. Moreover, the analysis of this model results in interesting findings. Thus, we examine the (Q,T) policy in detail in Chapter 5.

We should note as a last remark that in our perishable time-based model if we take the limits $\tau \to \infty$, $T \to \infty$, respectively, the conventional (Q,r) model reported in Hadley and Whitin [12] is easily obtained.

Chapter 4

NUMERICAL ANALYSIS

In this chapter we present the results of an extensive search for the performance of the (Q, r, T) policy developed in the previous chapter.

Unfortunately, the related literature lacks a detailed computational analysis of the existing methods which would serve as a standard choice of system parameters. Therefore, we tried to select an informative subset of the parameter space which would reflect the performance of our model. The existing studies mainly consider the analytical aspects of the (s, S) or (Q, r) type policies (i.e. derivation of operating characteristics and the long run average cost function). The reasons for not having much numerical work may be that the existing models for perishables do not represent a system with considerably general assumptions but are based on some restrictive conditions and that they are complex in nature. We are aware of only two studies that present some computational results for the models with positive lead times. Schmidt and Nahmias [39] conduct a sensitivity analysis for their (s, S) model by computing the optimal S and the corresponding cost value on 166 different settings. Chiu [7] compares his approximate (Q, r) model with the Weiss [42] model which assumes zero lead time. He also validates his results by using simulation of the positive lead time case.

This computational study attempts to highlight the basic features of the (Q, r, T) policy. Based on a wide range of parameter settings, we first analyze the sensitivity of the model to various parameters of the inventory system under consideration. Second, we compare the performance of the (Q, r, T) policy with that of the (Q, r) policy. We observe that the time-based policy outperforms the classical (Q, r) by achieving a maximum improvement of 41.39% in the average cost value. Further details of such comparisons are discussed in the following section.

4.1 Sensitivity Analysis

We use two different experimental setups to analyze the sensitivity of the (Q, r, T) model to various parameters of the system under consideration. Fixed parameters are $\lambda = 5$ and $h = 1$ (The analysis is done for different λ values and λ=5 case is selected for illustrative purposes). In the first stage of the analysis, we also fix L=1 and use the following ranges for the rest of the parameters.

Parameter	Symbol	Values Tested
Frac. of Lost Sales	α	0.005,0.01,0.02,0.05,0.1
Ordering Cost	K	50,100
Perishing Cost	p	1,10,50
Shelf Life	τ	2,4,6

Table 4.1: Test Parameters

The focus of interest in this experimental search is to determine how optimal values for the decision variables and the average costs change with respect to ordering cost, perishing cost, average fraction of lost sales and shelf life. The experimental points selected above represent a broad range of cases which includes for instance, the case where there is almost no lost sales, high setup costs and a short shelf life as well as the case with lost sales of %10, low setup cost and a long shelf life.

For the inventory system under consideration, inventory level is depleted by the decay of the products in the inventory as well as by demand arrivals. Hence, the ordering decision depends not only on the demand rate but also on the lifetime of items. In order to avoid lost sales during the lead time, the time when the products perish is also as important as the number of demands during the lead

λ=5			p=1				p=10				p=50			
		α	Q_1^*	r_1^*	T^*	C_1	Q_1^*	r_1^*	T^*	C_1	Q_1^*	r_1^*	T^*	C_1
		0.005	13	9	1.00	37.24	10	9	0.23	45.01	10	9	0.23	73.46
		0.01	12	8	0.94	36.22	9	8	0.11	43.28	9	8	0.11	62.42
	τ=2	0.02	12	7	1.00	35.22	9	7	0.30	42.09	8	7	0.05	54.60
		0.05	11	5	0.98	33.45	9	6	0.84	39.53	7	6	0.10	49.18
		0.1	11	2	1.05	31.48	9	5	1.16	36.90	7	5	0.50	45.79
		0.005	21	8	2.95	27.56	16	9	4.00	29.63	13	9	4.00	31.78
		0.01	20	7	2.81	26.70	16	8	4.00	28.54	13	8	4.00	30.69
K=50	τ=4	0.02	20	6	3.01	25.61	16	7	4.00	27.41	13	7	4.00	29.52
		0.05	19	4	2.99	23.96	16	5	3.02	25.34	13	5	1.79	27.81
		0.1	18	2	3.02	22.32	16	3	2.89	23.58	13	4	3.19	25.48
		0.005	24	8	4.79	26.04	22	8	3.65	26.48	20	8	2.88	27.16
		0.01	24	7	4.57	25.17	22	7	3.70	25.61	20	7	2.99	26.29
	τ=6	0.02	23	6	4.61	24.08	22	6	4.10	24.47	20	6	3.30	25.15
		0.05	24	4	4.76	22.57	22	4	4.05	22.97	19	5	6.00	23.32
		0.1	20	3	6.00	20.92	20	3	6.00	21.02	19	3	4.61	21.37
		0.005	13	9	1.00	62.96	10	9	0.23	73.44	10	9	0.23	101.90
		0.01	14	7	1.00	61.92	10	8	0.38	72.29	9	8	0.11	92.58
	τ=2	0.02	13	6	1.00	60.67	10	7	0.57	70.72	8	7	0.05	87.11
		0.05	12	2	0.99	58.59	10	6	1.03	67.47	8	6	0.44	83.37
		0.1	12	4	1.15	55.49	10	3	0.97	63.35	8	5	1.03	78.13
		0.005	21	8	2.95	40.89	18	8	2.14	44.92	15	9	4.00	49.92
		0.01	21	7	3.00	40.01	18	8	3.20	43.87	15	8	4.00	48.74
K=100	τ=4	0.02	20	6	3.01	39.06	18	6	2.54	42.57	15	7	4.00	47.41
		0.05	20	4	3.05	37.36	17	5	3.11	40.20	15	5	2.57	44.82
		0.1	19	2	3.13	35.26	17	3	3.10	37.63	15	3	2.55	42.04
		0.005	27	8	5.03	36.09	25	8	5.02	37.19	22	8	3.65	39.13
		0.01	27	7	5.05	35.14	25	7	5.01	36.18	22	7	3.70	38.19
	τ=6	0.02	26	6	5.11	34.15	24	6	5.03	35.06	22	6	4.10	36.92
		0.05	26	4	5.13	32.29	25	4	5.04	33.19	22	4	4.05	35.05
		0.1	26	1	5.04	30.33	24	2	5.05	31.06	21	3	6.00	32.64

Table 4.2: Sensitivity Results w.r.t. K, τ, p, α

time. Thus, in the second step of the analysis, we are particularly interested in the interaction effects of the lead time and the product lifetime. For this purpose, we studied the case with K =50, τ=2 with different choices for L which take the values 0.25τ, 0.5τ and 0.75τ.

Table 4.2 presents our results for the first experimental setup. We note that most of the results agree with what one would intuitively expect. The optimal value for Q increases as K increases. However, this increase is not as significant as it is for nonperishable inventories. For instance, when τ =2, there is almost no change in the optimal value of Q as K increases. We also observe that for the cases where the optimal value for Q increases with the increase in K, the optimal values of r decreases and T increases. This means that for a fixed shelf life and an average fraction of lost sales increase in the order quantity allows the ordering decision to be made at a later time.

When we increase p, we observe that Q decreases in order to decrease the number of units that perish. One important observation is that as p increases, the policy sets the reorder decision either by decreasing T or increasing r. When products are subject to decay in a short time, for small values of α (when the average fraction of lost sales is forced to be less than %2) r does not change with the increase in p, but T decreases considerably. Average costs are more sensitive to T when a low fraction of lost sales is desired and the life time of the products is short. However, for higher values of α, no specific pattern is observed for T as p increases. Generally, the optimal value of r increases for these cases.

The change in the value of shelf life has a noticable effect on the optimal value of Q. As shelf life increases, Q increases considerably. At the same time, r tends to decrease but the change in r is almost within 1 unit. T generally increases with the increase in shelf life. It seems that the impact of T decreases with increasing shelf life. However, for the cases where perishing cost is high (p=10,50) and the average fraction of lost sales is low, we observe that the optimal value of T first increases and then decreases with increasing shelf life. As expected, the average costs increase with p and K. As the lifetime of the products increase, the costs decrease.

Table 4.3 displays the results when we change L. For fixed α, as lead time

λ=5		p=1				p=10				p=50			
τ=2	α	Q_1^*	r_1^*	T^*	C_1	Q_1^*	r_1^*	T^*	C_1	Q_1^*	r_1^*	T^*	C_1
L=0.5	0.005	12	5	1.50	36.12	9	5	0.82	43.09	7	5	0.28	52.93
	0.01	12	4	1.50	35.39	9	5	1.59	42.18	7	5	2.00	51.84
	0.02	12	3	1.50	34.77	9	4	1.52	40.93	7	4	0.77	50.76
	0.05	11	2	1.52	33.40	9	3	1.58	39.10	7	3	1.11	48.36
	0.1	11	0	1.58	31.49	9	1	1.45	36.64	7	2	1.34	45.30
L=1	0.005	13	9	1.00	37.24	10	9	0.23	45.01	10	9	0.23	73.46
	0.01	12	8	0.94	36.22	9	8	0.11	43.28	9	8	0.11	62.42
	0.02	12	7	1.00	35.22	9	7	0.30	42.09	8	7	0.05	54.60
	0.05	11	5	0.98	33.45	9	6	0.84	39.53	7	6	0.10	49.18
	0.1	11	2	1.05	31.48	9	5	1.16	36.90	7	5	0.50	45.79
L=1.5	0.005	13	12	0.04	38.41	13	12	0.04	53.79	13	12	0.04	122.12
	0.01	14	11	0.46	37.06	12	3	0.03	49.17	12	3	0.03	102.09
	0.02	13	10	0.49	35.56	11	10	0.04	45.05	11	10	0.04	84.27
	0.05	12	7	0.50	33.50	10	9	0.25	40.68	10	9	0.25	67.86
	0.1	11	5	0.54	31.50	9	8	1.13	37.13	9	8	1.13	54.53

Table 4.3: Sensitivity Results w.r.t. L, p, α

increases, Q and r increases and T decreases. Because, the increase in the lead time increases the risk of having lost sales during the lead time. The average costs also increase with L. When lead time is large with respect to τ, we observe that the optimal r does not increase with p but T decreases. The policy parameter T makes the (Q, r, T) policy more proactive against the risk of losing sales during longer lead times.

4.2 Comparison with the (Q, r) Model

The existing literature on perishable inventories is mostly devoted to the investigations of several forms of (s, S) or (Q, r) models. As we propose a new policy for controlling perishable inventories, it is of interest to compare the performance of the (Q, r, T) policy with the conventional (Q, r) policy. For this purpose, we tested both policies in a wide range of parameter settings.

We perform our analysis for different demand rates such as λ=0,25,0.5,5,10. With $\lambda < 1$, we consider the inventory systems with slow moving products. λ=5,10 corresponds to the case where the products in the inventory system are subject to relatively high demand rates. We vary the shelf life of items (τ) as follows.

λ	τ
0.25,0.5	12,15,20
5,10	2,4,6

The parameter values for the shelf life are selected in a way that we are able to observe the effects of perishability for each demand rate we consider. The fixed parameters are K =50, L=1 and h=1. We vary perishing cost and average fraction of lost sales as presented in Table 4.1.

Without loss of generality, we will base our discussions on the cases where λ=0.25 and 5. We present the results for these parameters in Table 4.4. The results for λ=0.5 and 10 are provided in Appendix B.

λ=5		p=1								p=10								p=50							
	α	Q_1^*	r_1^*	T^*	C_1	Q_2^*	r_2^*	C_2	$\Delta\%$	Q_1^*	r_1^*	T^*	C_1	Q_2^*	r_2^*	C_2	$\Delta\%$	Q_1^*	r_1^*	T^*	C_1	Q_2^*	r_2^*	C_2	$\Delta\%$
τ=12	0.005	4	1	9.84	8.19	5	4	11.11	26.28	4	1	9.84	9.29	5	4	12.78	27.31	2	1	1.12	11.84	5	4	20.2	41.39
	0.01	4	1	11.09	7.93	3	2	9.01	11.99	3	1	11.02	8.68	3	2	9.66	10.14	2	1	3.47	11.42	3	2	12.53	8.86
	0.02	4	1	11.33	7.85	3	2	9.01	12.87	3	1	11.43	8.60	3	2	9.66	10.97	2	1	11.58	10.91	3	2	12.53	12.93
	0.05	4	0	11.34	7.28	4	1	7.64	4.71	3	0	10.03	7.97	3	1	8.49	6.12	2	0	4.65	10.57	2	1	10.88	2.85
	0.1	3	0	12.00	6.87	3	0	6.87	0.00	3	0	12.00	7.45	3	0	7.45	0.00	2	0	10.92	9.42	3	0	10.06	6.36
τ=15	0.005	5	1	14.02	7.53	3	2	8.53	11.72	4	1	11.04	8.29	3	2	8.88	6.64	3	1	5.77	10.22	3	2	10.47	2.39
	0.01	4	1	14.14	7.24	3	2	8.53	15.12	3	1	14.02	7.86	3	2	8.88	11.49	3	1	14.02	9.44	3	2	10.47	9.84
	0.02	4	1	14.56	7.17	4	2	8.08	11.26	3	1	14.88	7.78	4	2	8.72	10.78	3	1	14.88	9.34	2	1	9.68	3.51
	0.05	4	0	14.30	6.51	4	1	7.10	8.31	4	0	14.30	7.13	4	1	7.73	7.76	3	0	10.94	8.77	3	1	9.33	6.00
	0.1	4	0	15.00	6.34	4	0	6.34	0.00	3	0	15.00	6.68	3	0	6.68	0.00	3	0	15	8.13	3	0	8.13	0.00
τ=20	0.005	5	1	19.00	6.80	4	2	7.67	11.34	5	1	19.00	7.28	4	2	7.95	8.43	3	1	6.29	8.59	3	2	8.89	3.37
	0.01	4	1	19.31	6.70	5	2	7.59	11.73	4	1	19.31	6.98	4	2	7.95	12.20	3	1	19.08	7.85	3	2	8.89	11.70
	0.02	4	1	20.00	6.66	4	1	6.66	0.00	4	1	20.00	6.93	4	1	6.93	0.00	3	1	20	7.82	3	1	7.82	0.00
	0.05	4	0	19.11	5.85	4	1	6.66	12.16	4	0	19.11	6.11	4	1	6.93	11.83	4	0	19.11	7.28	3	1	7.82	6.91
	0.1	4	0	20.00	5.76	4	0	5.76	0.00	4	0	20.00	6.01	4	0	6.01	0.00	3	0	20	6.63	3	0	6.63	0.00

λ=5		p=1								p=10								p=50							
	α	Q_1^*	r_1^*	T^*	C_1	Q_2^*	r_2^*	C_2	$\Delta\%$	Q_1^*	r_1^*	T^*	C_1	Q_2^*	r_2^*	C_2	$\Delta\%$	Q_1^*	r_1^*	T^*	C_1	Q_2^*	r_2^*	C_2	$\Delta\%$
τ=2	0.005	13	9	1.00	37.24	11	10	38.67	3.70	10	9	0.23	45.01	11	10	47.65	5.54	10	9	0.23	73.46	11	10	87.54	16.08
	0.01	12	8	0.94	36.22	12	10	38.15	5.06	9	8	0.11	43.28	10	9	44.58	2.92	9	8	0.11	62.42	10	9	72.99	14.48
	0.02	12	7	1.00	35.22	10	8	36.74	4.14	9	7	0.30	42.09	9	8	42.47	0.89	8	7	0.05	54.60	9	8	61.50	11.22
	0.05	11	5	0.98	33.45	10	7	34.95	4.29	9	6	0.84	39.53	9	7	40.75	2.99	7	6	0.10	49.18	8	7	53.15	7.47
	0.1	11	2	1.05	31.48	12	7	32.41	2.87	9	5	1.16	36.90	9	6	38.55	4.28	7	5	0.50	45.79	7	6	47.73	4.06
τ=4	0.005	21	8	2.95	27.56	17	9	28.16	2.13	16	9	4.00	29.63	16	9	29.63	0.00	13	9	4.00	31.78	13	9	31.78	0.00
	0.01	20	7	2.81	26.70	16	8	27.38	2.48	16	8	4.00	28.54	16	8	28.54	0.00	13	8	4.00	30.69	13	8	30.69	0.00
	0.02	20	6	3.01	25.61	16	7	26.26	2.48	16	7	4.00	27.41	16	7	27.41	0.00	13	7	4.00	29.52	13	7	29.52	0.00
	0.05	19	4	2.99	23.96	18	6	24.58	2.52	16	5	3.02	25.34	16	6	26.22	3.36	13	5	1.79	27.81	13	6	28.25	1.56
	0.1	18	2	3.02	22.32	17	4	22.5	0.80	16	3	2.89	23.58	15	4	23.72	0.59	13	4	3.19	25.48	14	4	25.80	1.24
τ=6	0.005	24	8	4.79	26.04	22	9	26.9	3.20	22	8	3.65	26.48	21	9	27.11	2.32	20	8	2.88	27.16	19	9	27.51	1.27
	0.01	24	7	4.57	25.17	22	8	25.88	2.74	22	7	3.70	25.61	21	8	26.09	1.84	20	7	2.99	26.29	19	8	26.48	0.72
	0.02	23	6	4.61	24.08	22	7	24.86	3.14	22	6	4.10	24.47	21	7	25.06	2.35	20	6	3.30	25.15	19	7	25.44	1.14
	0.05	24	4	4.76	22.57	21	5	22.85	1.23	22	4	4.05	22.97	20	5	22.99	0.09	19	5	6.00	23.32	19	5	23.32	0.00
	0.1	20	3	6.00	20.92	20	3	20.92	0.00	20	3	6.00	21.02	20	3	21.02	0.00	19	3	4.61	21.37	20	3	21.46	0.42

Table 4.4: (Q,r,T) vs. (Q,r) $(\lambda=0.25,5)$

The results in Table 4.4 should be considered as separate analyses as we consider different values for the shelf life of the products. Our aim is to provide a general idea on the parameter range where the (Q,r,T) policy outperforms the (Q,r) policy. The percentage improvement in the average costs obtained by using the (Q,r,T) policy is computed as follows.

$$\Delta\% = \frac{C_2(Q_2^*,r_2^*) - C_1(Q_1^*,r_1^*,T^*)}{C_2(Q_2^*,r_2^*)} \times 100$$

where C_1 corresponds the average cost function of (Q,r,T) model and C_2 corresponds to that of (Q,r) model.

The set for λ=0.25 is the one from which we observe the most significant improvement of the (Q,r,T) policy. The main observations from this set of results are the following.

1. The improvement obtained by the (Q,r,T) policy is higher when α is low (i.e. α=0.005,0.01,0.02) which corresponds to high service levels. When the constraint on average fraction of lost sales is tight, (Q,r) policy increases r in order not to take the risk of losing sales during the lead time. However, (Q,r,T) policy operates with smaller r and prefers to order at time T which results in lower average costs.

2. As α increases, the difference between the two policies does not follow a linear pattern. Generally, when we let the average fraction of lost sales to be higher, the difference between the two policies becomes smaller.

3. When α is low, the parameters of (Q,r) policy are not sensitive to changes in p. On the other hand, those of the (Q,r,T) policy depend highly on the value of p. Thus, (Q,r,T) policy always results in an improvement for these cases which means that it is more robust when there is a trade off between the tight constraint on the average number of lost sales and the risk of incurring high perishing costs.

4. When the lifetime of items is short ($\tau = 12$) and the average fraction of lost sales cannot be more than 0.5%, we observe the highest percentage

difference in the average costs of two policies. For these cases, the parameters of the (Q, r) policy does not change as p increases. The reason for this phenomenon is the one order outstanding assumption. (Q, r) policy keeps r as high as possible in order to satisfy the constraint on the average number of lost sales and hence operates with large Q. But, the (Q, r, T) policy can easily relax this assumption by keeping r low and placing orders at time T. Thus, it can decrease the value of Q in order to avoid the risk of perishing especially when p is large. For large p, an improvement of 41.39% is observed.

5. We cannot observe a monotonic behaviour in the percentage differences between the two policies with respect to τ, p and α. The reason for this is that the (Q, r) policy selects the optimal values of Q and r from a discrete space. For instance, when τ =20 and α=0.02, the average costs for the two policies are the same. When we increase α to 5%, the percentage deviation between the two becomes 12.16%. Here, we observe that the (Q, r) policy is the same for both α=2% and 5% which means that it cannot find any better values for Q and r. However, as T is a continuous variable, (Q, r, T) policy attains a lower average cost by adjusting T.

The observations for λ=0.25 case are also valid for λ=5. However, in the latter case the difference between the two policies is not as significant as it is for λ=0.25. A reasonable explanation for this may be the following. Suppose the optimal reorder point for the (Q, r) policy is selected to be r^*. Thus, the policy waits until the inventory level depletes from r^*+1 to r^* even though it may be more beneficial to order during the inter-demand time. When the mean of inter-demand times are large, it is more probable that such an event occurs. (Q, r, T) policy can handle this weakness of the (Q, r) policy by placing orders at the time threshold for inventory. Therefore, it performs much better than the (Q, r) policy when demands are low. However, the percentage deviation between the average costs of the two policies decreases with increasing demand rate. Our computational results also confirm this explanation. Figure 4.1 and Figure 4.2

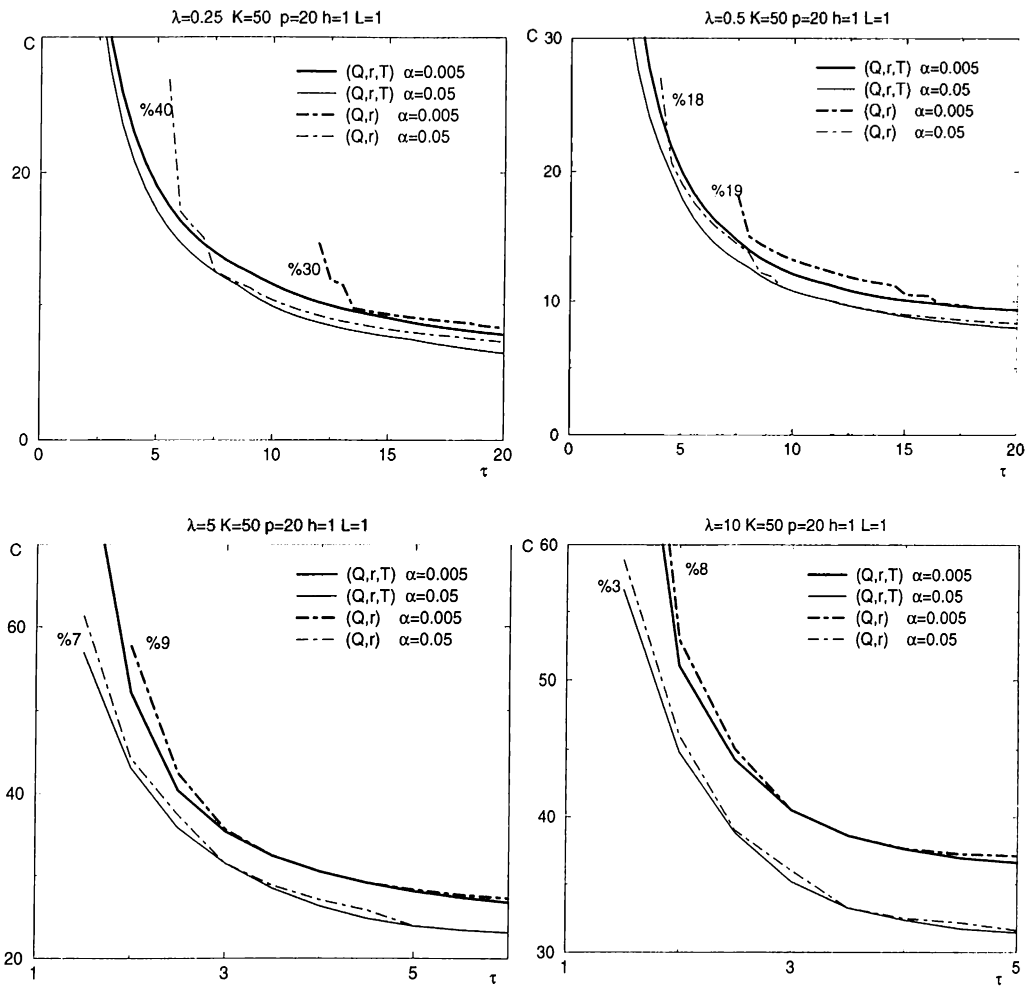

Figure 4.1: Average Cost vs. Shelf Life

present how the average costs of the two polices change for different demand rates with respect to shelf life and perishing cost, respectively.

As seen from Figure 4.1, the difference between the two policies decreases with increasing shelf life. This means that for the systems in which the products have infinite lifetimes, ordering at T does not result in considerable savings. However, when the products have shorter perishing times, the results show that using (Q, r, T) policy becomes more advantageous. When the demand rate for items is low, savings up to 41% is achieved. As demand rate increases, it is observed that the percentage savings decrease. In general, for a given demand rate, the difference between the average costs also decrease when we increase α.

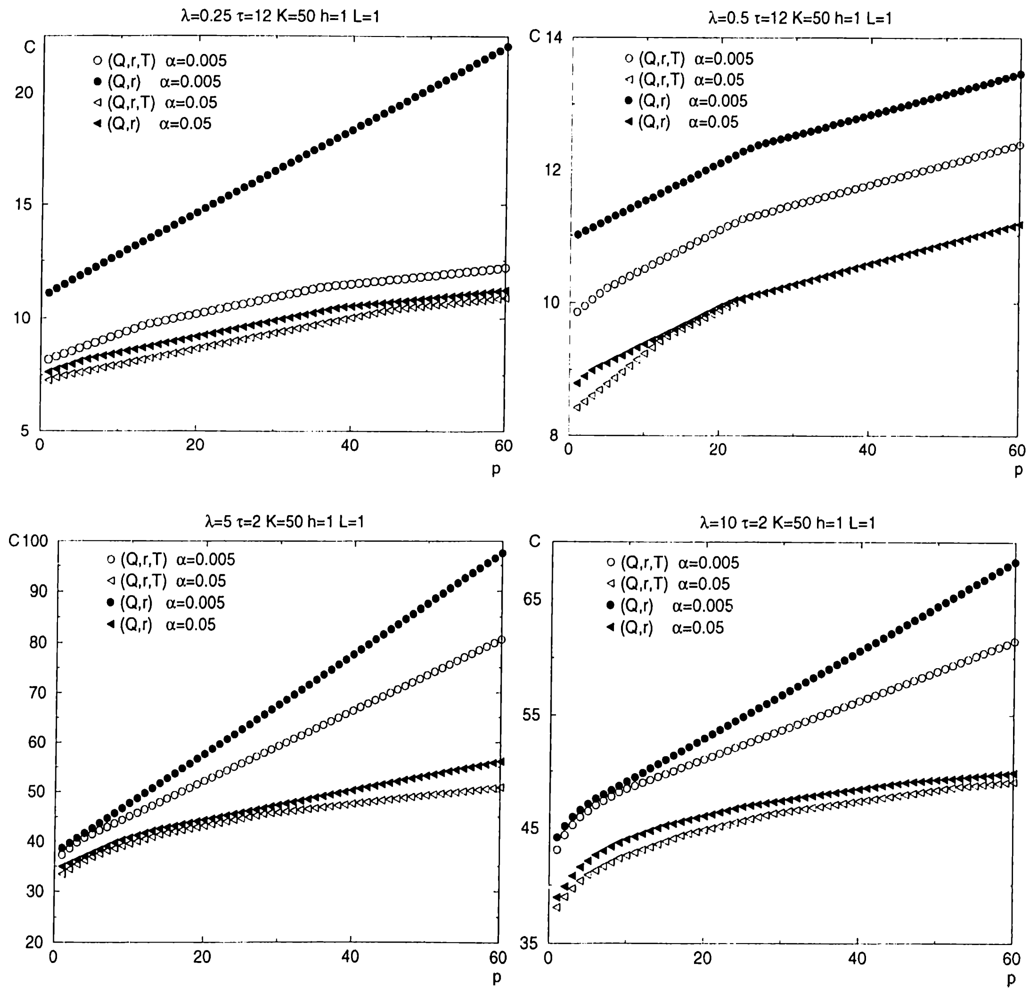

Figure 4.2: Average Cost vs. Perishing Cost

Figure 4.2 shows that for small α, as the perishing cost increases the (Q, r, T) policy performs much better than the (Q, r) policy. When the constraint on the average fraction of lost sales is relaxed, the (Q, r) policy performs as good as the (Q, r, T) policy.

The main conclusions of our experimental study can be summarized as follows. The (Q, r, T) policy outperforms the classical (Q, r) policy in 140 experimental points out of 180. The maximum savings are observed for the cases where the demand for items is low and the average fraction of lost sales is less than 2%. The mean improvement for these cases is approximately 12%. It is observed that when shelf life for products is long (the cases for which the infinite lifetime assumption

						(Q,r,T)				Simulation				
λ	τ	p	Q_1^*	r_1^*	T^*	C_1	HC	PC	OC	C_2	HC	PC	OC	Δ %
0.25	12	1	4	1	11.09	7.93	3.19	0.12	4.62	7.89	3.14	0.14	4.61	0.51
		5	3	1	11.02	8.33	2.65	0.36	5.32	8.44	2.58	0.48	5.38	-1.30
		20	3	1	11.02	9.40	2.65	1.43	5.32	9.90	2.58	1.93	5.39	-5.05
	15	1	4	1	14.14	7.24	3.18	0.07	3.99	7.22	3.13	0.09	4.00	0.28
		5	4	1	14.14	7.53	3.18	0.36	3.99	7.57	3.13	0.45	4.00	-0.53
		20	3	1	14.02	8.26	2.68	0.79	4.78	8.70	2.62	1.21	4.87	-5.06
0.5	12	1	7	1	11.00	9.43	4.73	0.14	4.56	9.42	4.70	0.16	4.57	0.11
		5	7	1	11.00	10.01	4.73	0.72	4.56	10.05	4.70	0.78	4.57	-0.40
		20	5	2	11.78	11.02	4.36	1.14	5.52	12.03	4.14	2.17	5.72	-8.40
	15	1	8	1	14.02	9.04	5.21	0.10	3.73	9.03	5.19	0.11	3.72	0.11
		5	7	1	12.03	9.36	5.00	0.34	4.02	9.39	4.81	0.44	4.14	-0.32
		20	5	2	14.99	10.08	4.44	0.45	5.20	10.73	4.28	1.11	5.34	-6.06
2	4	1	9	4	3.02	20.67	6.62	0.46	13.59	20.88	6.26	0.61	14.01	-1.01
		5	8	4	3.01	22.19	6.20	1.61	14.38	23.40	5.81	2.51	15.08	-5.17
		20	7	4	2.18	25.78	6.01	4.15	15.62	30.76	5.57	8.30	16.89	-16.19
	6	1	11	4	5.14	17.77	7.78	0.18	9.81	17.91	7.48	0.30	10.14	-0.78
		5	10	4	5.20	18.43	7.34	0.59	10.49	19.11	7.05	1.18	10.88	-3.56
		20	9	4	5.29	19.78	6.91	1.47	11.41	22.09	6.65	3.59	11.85	-10.46
5	2	1	12	8	0.94	36.22	8.76	1.32	26.14	37.06	8.29	1.70	27.08	-2.27
		5	10	8	0.38	40.46	8.63	3.54	28.29	46.36	7.98	6.88	31.50	-12.73
		20	9	8	0.11	48.07	8.34	9.57	30.16	66.07	7.74	24.23	34.10	-27.24
	4	1	20	7	2.81	26.70	12.64	0.48	13.58	26.79	12.27	0.66	13.86	-0.34
		5	17	8	3.40	27.85	11.79	0.95	15.12	29.22	11.38	2.19	15.64	-4.69
		20	14	8	3.99	29.40	10.47	1.04	17.88	33.08	10.23	4.45	18.40	-11.12

Table 4.5: (Q, r, T) as an Approximation to a Generalized Case

may hold), the policy parameter T is not effective in determining an optimal policy. However, when we consider a relatively short shelf life for products, the (Q, r, T) policy has an outstanding performance because the policy parameter T allows the model to incorporate the remaining shelf life of products at an arbitrary instance of the inventory level process. For instance, the maximum improvement of 41.39% is achieved when λ=0.25, α=0.005 and τ=12. In general, the difference between the two policies decreases with the increase in demand rate and shelf life. Therefore, the mean percentage difference between the two policies is 5% when all experiments are considered.

Consequently, we would like to note some remarks about the specified aging assumption. Although there are perishable inventories which satisfy this assumption, in many other perishable inventories products begin aging as soon as they arrive to the system. In order to investigate under which conditions the specified aging phenomena can be a good approximation to these cases, we have developed a simulation program in FORTRAN programming language. In this program, we consider the (Q, r, T) policy for products that begin aging as soon as they arrive. The regenerative approach is used in simulating the system. The regeneration epochs are defined as the instances at which a fresh batch of Q units arrives when the inventory level is zero. For a given parameter set, a single simulation run of 50000 regenerative cycles are made. In order to compare the (Q, r, T) policy that operates under the specified aging assumption with the explained case, we simulated the system at the optimal values of the policy parameters for different demand rates and cost parameters. Fixed parameters are L=1 and α=0.01. Table 4.5 presents our results.

In Table 4.5, HC, PC, and OC represent the average holding cost, average perishing cost and average ordering cost per unit time, respectively. The results indicate that when the perishing cost is low, the (Q, r, T) policy under the specified aging assumption can be a good approximation for the case where products begin aging as soon as they arrive to the system. In the cases where demand rate is low (i.e. λ=0.25 and 0.5), the percentage difference between the two average costs is within 1%. The percentage deviation increases with the increase in demand rate. As perishing cost increases, the performance of the approximation decreases. Because, when we assume that products follow a specified aging pattern we underestimate the perishing costs and the magnitude of this underestimation increases as perishing cost increases.

Chapter 5

The (Q, T) MODEL

In this chapter, we will consider a time-based inventory control policy for a single-item, single location continuous review inventory problem.

5.1 Description of the (Q, T) Policy

A special case of the (Q, r, T) policy is the (Q, T) policy which is obtained in the limiting case when $\tau \to \infty$ and r=0. The (Q, T) model is applicable to the inventory systems where all transactions are monitored continuously and the products in the inventory are assumed to have infinite lifetimes. Also, in many continuous review systems, demand process can be characterized by single units separated by random intervals. Household appliances, prescription glasses and many consumer durables in a department store with an optical scanner can be considered as examples for such kind of inventories.

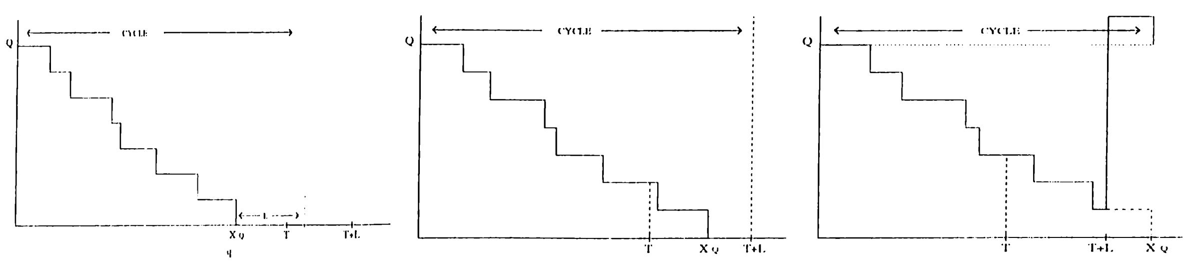

Figure 5.1: Three Possible Realizations of the Model

The (Q,T) policy places an order at the time threshold for inventory instead of the reorder point. According to the policy, a replenishment order of Q is placed either when inventory level drops to zero or after T units of time have elapsed since the last instance at which the inventory level hit Q, whichever occurs first. The possible realizations of the model is given in Figure 5.1.

In this chapter, our aim is to investigate how the lost sales problem operates under the (Q,T) policy. We first present the operating characteristics of the (Q,T) model. Second, we propose an iterative procedure to determine the optimal order quantity (Q) and the time threshold for inventory (T). Lastly, we discuss our computational results. When we compare the performance of the (Q,T) policy with that of the classical (Q,r) policy for Poisson demands, we observe that the time-based policy finds only near optimal solutions to the problem. On the other hand, the analytic expressions for the (Q,r) policy where demands follow an arbitrary distribution is not available in the literature. The (Q,T) model is interesting in the sense that by a simple heuristic we are able to compute the order quantity (Q) and the reorder point (r) for the cases where demands are governed by an arbitrary distribution. Of the 174 experimental points tested, the results reveal that the average cost of ordering policies determined by the heuristic has a mean deviation of only %0.92 from the optimal values that are provided by the (Q,r) model.

5.2 Operating Characteristics

The key operating characteristics of the system are the expected cycle length, expected on hand inventory and expected number of lost sales. The expressions derived for the (Q,r,T) policy in Chapter 3 reduce to the following quantities in the limiting case when $\tau \to \infty$ and r=0.

Result 1

$$E(CL) = J(0,T) + \overline{J}(T+L) + T\Delta(Q) + LF_Q(T+L) \tag{5.1}$$

Result 2

$$E(OH) = \frac{Q(Q+1)}{2\lambda} - Q(T+L)\overline{F}_Q(T+L) + Q\overline{J}(T+L) \tag{5.2}$$

Result 3

$$E(LS) = \lambda[T\Delta(Q) - J(T, T+L) + LF_Q(T+L)] \tag{5.3}$$

By using the expressions for the operating characteristics, we obtain the long run average cost rate function $AC(Q,T)$ by using renewal reward theorem. $AC(Q,T)$ is given by Equation 5.4.

$$AC(Q,T) = \frac{E(CC(Q,T)}{E(CL(Q,T))} = \frac{K + hE(OH) + \pi E(LS)}{E(CL)} \tag{5.4}$$

Unlike the previous case, for the analysis of the (Q,T) model we assign an explicit value to the lost sales cost which is denoted by π.

The above expressions assume that demand process is governed by Poisson arrivals. When demand process has an arbitrary demand distribution, we cannot use our previous definition for a cycle. Non-Poisson demands introduce a memory to the process. Therefore, the epochs at which the inventory position increases or decreases to Q units are not regeneration points any longer.

The exact analysis for this case can still be carried out by making use of the renewal reward theorem but it requires a different cycle definition. There may be more than one alternative for the cycle definition in non-Poisson demand case. One definition is that a regenerative cycle starts whenever the inventory level decreases to $Q-1$ units. We should note that when the inventory level increases to Q units by an order arrival, the random time until the next demand arrival is the forward recurrence time. As the policy starts to keep track of the time spent in the system whenever the inventory level decreases or increases to Q units, the limiting distribution of this residual lifetimes should be determined to find the exact solution to the system. Another cycle definition may be to start a cycle whenever inventory level drops to zero and there is no outstanding order. In this case, a cycle is composed of several subcycles which makes the analysis rather difficult.

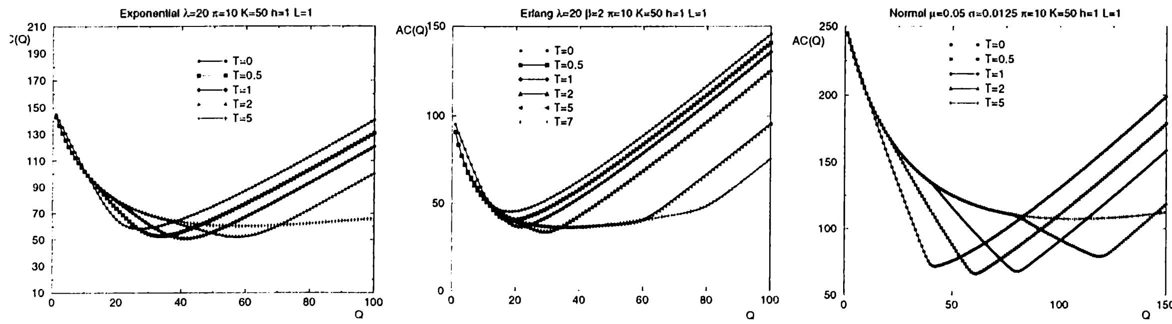

Figure 5.2: Behaviour of the Cost Rate Function w.r.t. Q

Although it can be manageable to derive the exact expression for the long run average cost rate function for non-Poisson demands, the resulting formulation will not be so simple because of the facts discussed in the previous paragraphs. Therefore, it is worthwile to use our model as an approximation to non-Poisson demands. With this approximation, we make the assumption that the epochs at which inventory level increases to Q units are regeneration points although the inventory level process does not regenerate itself at these epochs.

5.3 Optimization Algorithm

After stating the objective function of our model, we now propose an iterative procedure to determine the optimal order quantity (Q) and reorder time (T). In order to develop such an algorithm, we should first investigate the behaviour of the long run average cost rate function with respect to the decision variables Q and T. The analytical properties of the cost function $AC(Q,T)$ are difficult to discuss. Therefore, we illustrated the function for a wide range of parameter settings (48 experimental points for each demand distribution that we consider) and observed its behaviour.

Figure 5.2 shows how the cost rate function changes with respect to Q for fixed values of T. In these cases, demands follow Poisson, Erlang and Normal demands, respectively. As seen from the figures, the cost rate function appears convex with respect to Q. As T increases, the cost rate function becomes extremely flat. In

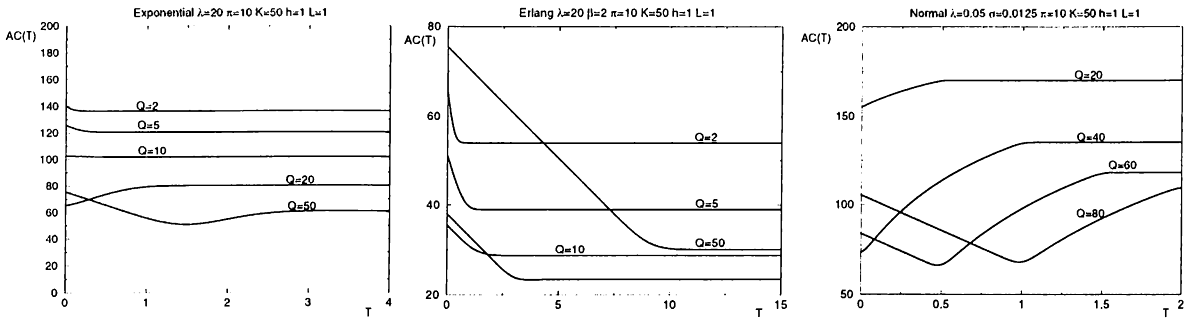

Figure 5.3: Behaviour of the Cost Rate Function w.r.t. T

the next stage of the analysis, we fix Q and observe the behaviour of the cost rate function with respect to T. As seen from Figure 5.3, the cost function can be an increasing, decreasing and a convex function of T. In the cases where it is convex, we observe a single minimum with respect to T. But when it is decreasing or increasing with respect to T, we observe alternate optima. However, the first order condition of the cost rate function $AC(Q,T)$ with respect to T satisfies the following equality at T^*.

$$AC'(T) = \frac{\frac{\partial E(CC(Q,T^*))}{\partial T}}{\frac{\partial E(CL(Q,T^*))}{\partial T}} = \frac{E(CC(Q,T^*))}{E(CL(Q,T^*))} \tag{5.5}$$

An observation from our test experiments is that for Q^*, this equality has only one solution which means that there is a single value T^* which optimizes the cost rate function. Therefore, we can consider that $AC(Q,T)$ is a unimodal function. For illustrative purposes,we present Figure 5.4 for which the parameters are $\lambda = 10$, $\pi = 5$, $K = 100$ and $Q^* = 45$.

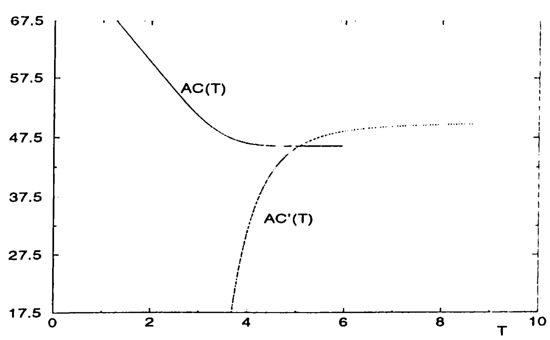

Figure 5.4: An Illustrative Example For Unimodality

Parameter	Symbol	Values Tested
Holding Cost	h	1
Lost sales Cost	π	2,5,10
Ordering Cost	K	10,20,50,100
Arrival Rate	λ	5,10,20,40
Lead Time	L	1

Table 5.1: Parameters Tested for (Q,T) Model

An iterative linear search procedure is used to optimize our objective function (Equation 5.4). The initial approximation to Q_0 is obtained by computing the nearest integer to conventional EOQ which is equal to $\sqrt{2\lambda K/h}$. By using Q_0, we determine the corresponding optimal T_0 by a linear search. Then, we continue iteratively until no improvement in the objective function is achieved. The solution algorithm has been programmed in the FORTRAN language. The FORTRAN subroutines CDFGAM and CDFNOR are linked to the program to facilitate the computations of the numerical analysis. The algorithm converges to the optimal value in seconds on a Pentium 200 MMX with 64 MBytes of Random Access Memory under operating system LINUX.

After computing (Q_T^*, T^*) values by the above algorithm, we apply a simple heuristic to convert these values to $(\hat{Q}^*, \hat{r}^*)$. As T^* is the time spent in the system until an order is given, on the average λT^* demands occur in this interval. Therefore, we can approximate the reorder point suggested by the time-based policy as the nearest integer to $Q^* - \lambda T^*$. Then, the heuristic order quantity and reorder point are expressed as follows.

$$\hat{r}^* = \max(\lfloor Q - \lambda T \rfloor, 0) \quad \text{and} \quad \hat{Q}^* = Q_T^* \tag{5.6}$$

In the next section, we compare performance of this heuristic $(\hat{Q}^*, \hat{r}^*)$ pair with the optimal (Q,r) pair.

5.4 Computational Results

In this section, we test the performance of the heuristic for 48 different experimental points for each demand distribution which are exponential, Erlang and Normal. 16 experiments for the exponential distribution are not considered in the analysis because average expected costs come out to be greater than $\lambda\pi$ which means that losing all demands result in less costs. Therefore, our experiment set consists of 174 points. For the exponential case, the optimal average costs for the (Q,r) model is computed by using the expressions developed by Hadley and Whitin [12]. For Erlang and Normal demand distributions, a simulation model is developed.

No.	λ	π	K	$\Delta_1\%$	$\Delta_2\%$	$\Delta_3\%$	$\Delta_4\%$
1	5	5	10	5.92	7.84	6.79	5.64
2	5	5	20	5.06	3.41	3.98	4.44
3	5	5	50	0.39	1.15	1.80	2.54
4	5	10	10	4.81	10.77	15.16	7.68
5	5	10	20	8.24	9.55	9.05	6.15
6	5	10	50	6.88	4.57	3.29	3.90
7	10	5	10	2.35	6.85	9.19	4.38
8	10	5	20	4.64	5.86	4.64	3.55
9	10	5	50	3.70	1.63	1.64	2.55
10	10	10	10	1.02	7.05	9.73	3.91
11	10	10	20	4.90	7.83	9.70	4.40
12	10	10	50	6.73	6.88	5.55	3.59
13	20	2	10	0.44	2.95	2.42	1.98
14	20	2	20	1.48	0.93	1.58	1.96
15	20	5	10	0.22	3.42	6.81	1.97
16	20	5	20	2.09	4.75	5.59	2.48
17	20	5	50	3.68	3.62	2.32	2.01
18	20	10	20	1.28	5.17	7.68	2.47
19	20	10	50	4.71	5.84	5.98	2.51
20	20	10	100	5.15	4.96	4.02	2.27
21	40	2	20	0.22	1.79	1.32	1.05
22	40	2	50	0.93	0.38	0.67	0.94
23	40	5	50	2.19	3.29	3.33	1.44
24	40	5	100	2.72	2.51	1.45	1.27
25	40	10	50	2.33	4.2	4.93	1.69
26	40	10	100	3.66	4.29	4.07	1.63

Table 5.2: (Q,T) vs. (Q,r)

In our experimental study, we take the holding cost (h) and the lead time (L) as unity. The ranges for the demand arrival rate, lost sales cost and ordering cost are presented in Table 5.1. We consider two different values of shape parameters for the Erlang distribution which are β=2,4. The standard deviation for the

No.	$\Delta_1\%$	$\Delta_2\%$	$\Delta_3\%$	$\Delta_4\%$	No.	$\Delta_1\%$	$\Delta_2\%$	$\Delta_3\%$	$\Delta_4\%$
1	1.44	0	0	0	14	0.09	0	0	0.01
2	0.97	1.07	0	0.9	15	0.38	1.79	0	0
3	0.39	0	0	0.39	16	0.74	0.79	0.69	1.15
4	3.80	0	0	6.91	17	0.26	0.20	0.68	0
5	2.02	0.86	3.04	5.53	18	0.91	2.35	1.78	0
6	0.67	2.82	0	0	19	1.04	1.08	0.12	0.36
7	1.48	0.09	0	4.01	20	0.80	0.51	0	0.63
8	0.36	0.1	1.48	0	21	0.11	0.05	0.08	0.02
9	0.50	0.66	0.01	0	22	0.06	0	0	0.01
10	0.60	3.57	0.24	1.4	23	0.34	0.35	0	0.27
11	2.06	3.48	2.51	2.04	24	0.07	0	0.37	0
12	1.41	0.15	1.14	2	25	0.69	1.38	0.41	0.53
13	0.27	0.07	0	0.06	26	0.08	0.89	0.04	0.59

Table 5.3: Performance of the Heuristic

Normal distribution is taken as $\frac{1}{4\lambda}$.

Before discussing the results of the heuristic, we will present the percentage deviation of the optimal costs determined by the (Q,T) model from that of (Q,r) model. As mentioned earlier, the (Q,T) policy does not perform better than the (Q,r) policy. Table 5.2 presents 26 selected experimental points out of 48. The whole data set is provided in Appendix C. The percentage error of the suboptimal (Q,T) policy is computed by

$$\Delta\% = \frac{AC(Q_T^*,T^*) - TC(Q^*,r^*)}{TC(Q^*,r^*)} \times 100$$

where $TC(Q,r)$ corresponds to the average cost function of the (Q,r) policy.

We observe from Table 5.2 † that the time-based model performs better for high demands. In general, the percentage deviation is higher in the cases where the lost sales cost is high (i.e. π=10). When all experiments are considered, the mean percentage deviation is 3.24%.

When we compute the average cost of the (Q,r) policy by using the optimal $(\hat{Q}^*,\hat{r}^*)$ pairs, we observe that the suggested heuristic has an outstanding performance. In Table 5.3 we present the percentage errors of the heuristic for the parameter set stated in Table 5.2. Of the 174 experimental points tested, the mean percentage error of the heuristic is only 0.92% and the heuristic finds the exact ordering policy for 63 cases. Especially, for the cases where demand rate is

†1,2,3,4 refers to the exponential, Erlang(λ,2), Erlang(λ,4) and Normal demand distributions, respectively.

high (i.e. λ=40), the percentage error is less than %1 for almost all distributions. The maximum deviation of 6.91% is observed for the case where the demand distribution is Normal(0.2,0.05), π=10 and K=10.

Chapter 6

CONCLUSION

In this thesis, we propose a time-based control policy, the (Q, r, T) policy, for continuous review inventory systems where the products have constant shelf lifes and unmet demands are lost. The model admits a positive lead time. The aging of products in the inventory follows a specified aging pattern such that the aging of a fresh batch does not begin until all the products from the previous batch are exhausted either by demand or decay. The expressions for the key operating characteristics are derived for Poisson demands case by using the renewal reward theorem. Based on these performance measures, the problem is analyzed under a service level criterion, namely the average fraction of lost sales.

In the inventory theory literature, the systems with constant product lifetimes are considered to be difficult to analyze especially when the lead time is positive. Therefore, there is not much done in this area. The existing studies are based on the classical (s, S) or (Q, r) type policies. Even for these policies a model that satisfies considerably general assumptions have not been developed yet.

The (Q, r, T) model provides a starting point for the analysis of perishable inventories with a different approach other than the conventional policies. The model introduces an additional decision variable to the (Q, r) model which is the time threshold for inventory. This time-based decision variable makes the model more proactive against the perishability of products. The results of a performance comparison with the (Q,r) model indicate that the proposed time-based approach outperforms the (Q,r) model in the 140 experimental points out of 180. Of the 180 experimental points, a mean improvement of 5% is obtained in average costs. The maximum savings (i.e. 41.39%) are observed for the cases where demand rate is low (i.e. λ=0.25,0.5) and the average fraction of lost sales is less than 2%. This computational study is the first in the sense that it involves comparison of two different policies for controlling perishable inventories.

The (Q,r,T) model also provides a rich and flexible control policy which induces insightful special cases such as the perishable (Q,r) model (Ravichandran [29]) and the conventional (Q,r) model (Hadley and Whitin [12]). An interesting special case is the (Q,T) model which is obtained in the limiting case where $\tau \rightarrow \infty$ and r=0. The (Q,T) policy applies for the inventory systems with nonperishable products. Although it provides suboptimal results when compared to the (Q,r) policy, it facilitates a quick and efficient heuristic solution procedure to the problem where demands follow an arbitrary distribution. Since the general solution does not exist for the (Q,r) model, the heuristic that makes use of the (Q,T) model can be one of the methods to be applied to nonperishable inventories. Of the 174 experimental points tested for exponential, Erlang and Normal demand distributions, the mean deviation of the heuristic order quantity and reorder point determined by the (Q,T) policy deviates from the optimal (Q,r) pair by only 0.92% and, in 63 of the test experiments it finds the optimal solution.

Most of the existing research in inventory theory literature is based on the assumption that the products in the inventories have infinite lifetimes. Especially, for the full backlogging case the exact analysis of the (s,S) policy is available and hence there is a vast literature on the algorithms and approximations for computing optimal or near optimal solution of (s,S) policies. On the other hand, for the cases where unsatisfied demands are lost, the optimal ordering policy has not been computed by analytical means. Moreover, when perishability is included in the analysis the problem becomes even more complex. The literature in this area is restricted to Poisson demands, one order outstanding assumption and some other simplifying assumptions regarding the aging of products in the inventory. A model with considerably general assumptions has not been developed yet.

The (Q, r, T) model that we propose in this thesis also lacks this generality. Therefore considerable improvements in the model is possible. One extension may be the incorporation of random perishing times, which is not too hard to do by using our current setup of the model. However, other extensions such as relaxing the assumption on the specified aging, arbitrary demand arrivals, more than one order outstanding at a time necessitate a distinct and complex analysis of the model which can be a fruitful research area for future studies.

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Appendix

A.1 APPENDIX A

A.1.1 Proof of Theorem 1

Derivation of the Expected Cycle Length

$$CL = \begin{cases} X_{Q-r} + L & X_{Q-r} < T, X_{Q-r} < X_Q < \min(T, X_{Q-r} + L) \\ X_{Q-r} + L & \tau - L < X_{Q-r} < T, X_Q > \tau \\ X_Q & X_{Q-r} < T, X_{Q-r} + L < X_Q < \tau \\ \tau & X_{Q-r} < \tau - L, X_Q > \tau \\ T + L & X_{Q-r} > T, X_Q > \tau \\ T + L & T < X_{Q-r} < X_Q < \tau \end{cases} \tag{A.1.1}$$

$$\begin{aligned} E[CL] = \; & E[(X_{Q-r} + L)I(X_{Q-r} < T, X_{Q-r} < X_Q < \min(X_{Q-r} + L, \tau))] \\ & + E[(X_{Q-r} + L)I(\tau - L < X_{Q-r} < T, X_Q > \tau)] \\ & + E[X_Q I(X_{Q-r} < T, X_{Q-r} + L < X_Q < \tau)] \\ & + E[\tau I(X_{Q-r} < \tau - L, X_Q > \tau)] \\ & + E[(T + L)I(X_{Q-r} > T, X_Q > \tau)] \\ & + E[(T + L)I(T < X_{Q-r} < X_Q < \tau)] \end{aligned} \tag{A.1.2}$$

Then

$$
\begin{aligned}
E[CL] = \; & \int_{x=0}^{\tau-L}\int_{y=0}^{L}(x+L)f_{Q-r}(x)f_r(y)\;dy\;dx \\
& +\int_{x=\tau-L}^{T}\int_{y=0}^{\tau-x} f_{Q-r}(x)f_r(y)\;dy\;dx \\
& +\int_{x=\tau-L}^{T}\int_{y=\tau-x}^{\infty} f_{Q-r}(x)f_r(y)\;dy\;dx \\
& +\int_{x=0}^{\tau-L}\int_{y=L}^{\tau-x} f_{Q-r}(x)f_r(y)\;dy\;dx \\
& +\tau\int_{x=0}^{\tau-L}\int_{y=\tau-x}^{\infty} f_{Q-r}(x)f_r(y)\;dy\;dx \\
& +(T+L)\int_{x=T}^{\tau}\int_{y=\tau-x}^{\infty} f_{Q-r}(x)f_r(y)\;dy\;dx \\
& +(T+L)\int_{x=T}^{\tau}\int_{y=0}^{\tau-x} f_{Q-r}(x)f_r(y)\;dy\;dx \\
= \; & \tfrac{Q-r}{\lambda}F_r(L)F_{Q-r+1}(\tau-L)+LF_r(L)F_{Q-r}(\tau-L) \\
& +\tfrac{Q-r}{\lambda}\int_{\tau-L}^{T} F_r(\tau-x)f_{Q-r+1}(x)\;dx \\
& +L\int_{\tau-L}^{T} F_r(\tau-x)f_{Q-r}(x)\;dx \\
& +\tfrac{Q-r}{\lambda}[F_{Q-r+1}(T)-F_{Q-r+1}(\tau-L)] \\
& -\tfrac{Q-r}{\lambda}\int_{\tau-L}^{T} F_r(\tau-x)f_{Q-r+1}(x)\;dx+L[F_{Q-r}(T)F_{Q-r}(\tau-L)] \\
& -L\int_{\tau-L}^{T} F_r(\tau-x)f_{Q-r}(x)\;dx \\
& -\tfrac{Q-r}{\lambda}\int_{0}^{\tau-L} F_r(\tau-x)f_{Q-r+1}(x)\;dx \\
& -\tfrac{Q-r}{\lambda}F_r(L)F_{Q-r+1}(\tau-L) \\
& +\tfrac{r}{\lambda}\int_{0}^{\tau-L} F_{r+1}(\tau-x)f_{Q-r}(x)\;dx \\
& -\tfrac{r}{\lambda}F_{r+1}(L)F_{Q-r}(\tau-L)+\tau F_{Q-r}(\tau-L) \\
& -\tau\int_{0}^{\tau-L} F_r(\tau-x)f_{Q-r}(x)\;dx \\
& -(T+L)-(T+L)F_{Q-r}(T) \\
& (T+L)\int_{T}^{\tau} F_r(\tau-x)f_{Q-r}(x)\;dx \\
& (T+L)\int_{T}^{\tau} F_r(\tau-x)f_{Q-r}(x)\;dx \\
= \; & (T+L)+\tfrac{Q-r}{\lambda}\int_{0}^{\tau-L} F_r(\tau-x)f_{Q-r+1}(x)\;dx \\
& +\tfrac{r}{\lambda}\int_{0}^{\tau-L} F_{r+1}(\tau-x)f_{Q-r}(x)\;dx \\
& -\tau\int_{0}^{\tau-L} F_r(\tau-x)f_{Q-r}(x)\;dx \\
& +L[F_r(L)-\tfrac{r}{\lambda}F_{r+1}(L)+\tau-L]F_{Q-r}(\tau-L) \\
& +\tfrac{Q-r}{\lambda}F_{Q-r+1}(T)-\tfrac{Q-r}{\lambda}F_{Q-r+1}(\tau-L)-TF_{Q-r}(T)
\end{aligned}
\tag{A.1.3}
$$

Derivation of Expected On Hand Inventory

$$OH = \begin{cases} \sum_{i=1}^{Q} X_i & X_{Q-r} < T, X_{Q-r} < X_Q < X_{Q-r} + L, X_Q < \tau \\ \sum_{i=1}^{N(\tau)} X_i + \tau[Q - N(\tau)] & \tau - L < X_{Q-r} < T, X_Q > \tau \\ \sum_{i=1}^{Q} X_i + Q[X_Q - X_{Q-r} - L] & X_{Q-r} < T, X_{Q-r} + L < X_Q < \tau \\ \sum_{i=1}^{N(\tau)} X_i + \tau[Q - N(\tau & X_{Q-r} < \tau - L, X_Q > \tau \\ \sum_{i=1}^{N(\tau)} X_i + \tau[Q - N(\tau)] & X_{Q-r} > T, X_Q > \tau \\ \sum_{i=1}^{Q} X_i & T < X_{Q-r} < X_Q < \tau \end{cases} \tag{A.1.4}$$

$$\begin{aligned} E[OH] = \quad & E[\textstyle\sum_{i=1}^{N(\tau)} X_i I(X_Q > \tau)] \\ & + E[\textstyle\sum_{i=1}^{Q} X_i I(X_Q < \tau)] \\ & + E[\tau(Q - N(\tau))I(X_Q > \tau)] \\ & + E[Q(X_Q - X_{Q-r} - L)I(X_{Q-r} < T, X_{Q-r} + L < X_Q < \tau)] \\ & + E[Q(\tau - X_{Q-r} - L)I(X_{Q-r} < \tau - L, X_Q > \tau)] \end{aligned} \tag{A.1.5}$$

$$\begin{aligned} E[OH]_1 &= E[\textstyle\sum_{i=1}^{N(\tau)} X_i I(X_Q > \tau)] \\ &= E[\textstyle\sum_{i=1}^{N(\tau)} X_i I(N(\tau) < Q)] \\ &= E[E(\textstyle\sum_{i=1}^{N(\tau)} X_i I(N(\tau) < Q)|N(\tau) = n] \\ &= \textstyle\sum_{i=1}^{Q-1} n\frac{\tau}{2} e^{-\lambda\tau} \frac{(\lambda\tau)^n}{n!} \\ &= \frac{\lambda\tau^2}{2}[1 - F_{Q-1}(\tau)] \end{aligned} \tag{A.1.6}$$

In order to find $E[h\sum_{i=1}^{Q} X_i I(X_Q < \tau)]$, denote the joint density of $(X_1, X_2, ..., X_Q)$ by $f(x_1, x_2, .., x_Q)$. $(X_1, X_2, ..., X_Q) = (x_1, x_2, ..., x_Q)$ is equivalent to $(T_1, T_2, ..., T_Q) = (x_1, x_2 - x_1, x_3 - x_2, .., x_Q - x_{Q-1})$ where T_i's are inter-demand times.

$$f(x_1, x_2, .., x_Q) = \lambda e^{-\lambda x_1} \lambda e^{-\lambda(x_2 - x_1)} ... \lambda e^{-\lambda(x_Q - x_{Q-1})} = \lambda^Q e^{-\lambda x_Q}$$

$$
\begin{aligned}
E[OH]_2 = \;& E[\textstyle\sum_{i=1}^{Q} X_i I(X_Q < \tau)] \\
= \;& \textstyle\int_{x_Q=0}^{\tau} \int_{x_{Q-1}=0}^{x_Q} .. \int_{x_1=0}^{x_2} (x_1 + x_2 + .. + x_Q)\lambda^Q e^{-\lambda x_Q} dx_1 dx_2 .. dx_Q \\
= \;& \textstyle\sum_{i=1}^{Q} \int_{x_Q=0}^{\tau} \int_{x_{Q-1}=0}^{x_Q} .. \int_{x_i=0}^{x_{i+1}} .. \int_{x_1=0}^{x_2} x_i \lambda^Q e^{-\lambda x_Q} dx_1 .. dx_i .. dx_Q \\
= \;& \textstyle\sum_{i=1}^{Q} \frac{i}{\lambda} F_{Q+1}(\tau) \\
= \;& \frac{Q(Q+1)}{2\lambda} F_{Q+1}(\tau)
\end{aligned} \tag{A.1.7}
$$

$$
\begin{aligned}
E[OH]_3 = \;& E[\tau(Q - N(\tau))I(X_Q > \tau)] \\
= \;& Q\tau[1 - F_Q(\tau)] - \tau E[N(\tau)I(N(\tau) < Q)] \\
= \;& Q\tau[1 - F_Q(\tau)] - \lambda\tau^2[1 - F_{Q-1}(\tau)]
\end{aligned} \tag{A.1.8}
$$

$$
\begin{aligned}
E[OH]_4 = \;& E[Q(X_Q - X_{Q-r} - L)I(X_{Q-r} < T, X_{Q-r} + L < X_Q < \tau)] \\
= \;& Q \textstyle\int_{x=0}^{\tau-L} \int_{y=L}^{\tau-x} (y - L) f_{Q-r}(x) f_r(y)\ dy\ dx \\
= \;& Q \frac{r}{\lambda} \textstyle\int_{x=0}^{\tau-L} F_{r+1}(\tau - x) f_{Q-r}(x)\ dx \\
= \;& Q \frac{r}{\lambda} F_{r+1}(L) F_{Q-r}(\tau - L) + QLF_r(L)F_{Q-r}(\tau - L) \\
& \quad -QL \textstyle\int_{x=0}^{\tau-L} F_r(\tau - x) f_{Q-r}(x)\ dx
\end{aligned} \tag{A.1.9}
$$

$$
\begin{aligned}
E[OH]_5 = \;& E[Q(\tau - X_{Q-r} - L)I(X_{Q-r} < \tau - L, X_Q > \tau)] \\
= \;& Q(\tau - L) \textstyle\int_{x=0}^{\tau-L} \int_{y=\tau-x}^{\infty} f_{Q-r}(x) f_r(y)\ dy\ dx \\
& \quad -Q \textstyle\int_{x=0}^{\tau-L} \int_{y=\tau-x}^{\infty} x f_{Q-r}(x) f_r(y)\ dy\ dx \\
= \;& Q(\tau - L)F_{Q-r}(\tau - L) - Q(\tau - L) \textstyle\int_{x=0}^{\tau-L} f_{Q-r}(x) F_r(\tau - x)\ dx \\
& \quad -Q \frac{Q-r}{\lambda} F_{Q-r+1}(\tau - L) + Q \frac{Q-r}{\lambda} \textstyle\int_{x=0}^{\tau-L} f_{Q-r+1}(x) F_r(\tau - x)\ dx
\end{aligned} \tag{A.1.10}
$$

$$
\begin{aligned}
E[OH] = \; & E[OH]_1 + E[OH]_2 + E[OH]_3 + E[OH]_4 + E[OH]_5 \\
= \; & -\tfrac{h\lambda\tau^2}{2}(1 - F_{Q-1}(\tau)) + \tfrac{Q(Q+1)}{2\lambda} F_{Q+1}(\tau)] + Q\tau[1 - F_Q(\tau)] \\
& +Q\tfrac{r}{\lambda}\textstyle\int_{x=0}^{\tau-L} F_{r+1}(\tau - x) f_{Q-r}(x)\; dx \\
& -Q\tau \textstyle\int_{x=0}^{\tau-L} F_r(\tau - x) f_{Q-r}(x)\; dx \\
& +Q\tfrac{Q-r}{\lambda}\textstyle\int_{x=0}^{\tau-L} f_{Q-r+1}(x) F_r(tau - x)\; dx \\
& +F_{Q-r}(\tau - L)(Q(\tau - L) + QLF_r(L) - Q\tfrac{Q-r}{\lambda}F_{r+1}(L)) \\
& -Q\tfrac{Q-r}{\lambda}F_{Q-r+1}(\tau - L)
\end{aligned}
\tag{A.1.11}
$$

Derivation of Expected Number of Lost Sales

$$
LS = \begin{cases}
N(X_{Q-r} + L - X_Q) & X_{Q-r} < T, X_{Q-r} < X_Q < X_{Q-r} + L, X_Q < \tau \\
N(X_{Q-r} + L - \tau) & \tau - L < X_{Q-r} < T, X_Q > \tau \\
N(T + L - \tau) & X_{Q-r} > T, X_Q > \tau \\
N(T + L - X_Q) & T < X_{Q-r} < X_Q < \tau
\end{cases}
\tag{A.1.12}
$$

$$
\begin{aligned}
E[LS] = \; & E[N(X_{Q-r} + L - X_Q) I(X_{Q-r} < T, X_{Q-r} < X_Q < X_{Q-r} + L, X_Q < \tau)] \\
& +E[N(X_{Q-r} + L - \tau) I(\tau - L < X_{Q-r} < T, X_Q > \tau)] \\
& +E[N(T + L - \tau) I(X_{Q-r} > T, X_Q > \tau)] \\
& +E[N(T + L - X_Q) I(T < X_{Q-r} < X_Q < \tau)]
\end{aligned}
\tag{A.1.13}
$$

$$
\begin{aligned}
E[LS]_1 = \; & E[N(X_{Q-r} + L - X_Q) I(X_{Q-r} < T, X_{Q-r} < X_Q < X_{Q-r} + L, X_Q < \tau)] \\
= \; & \lambda \textstyle\int_{x=0}^{\tau-L} \int_{y=0}^{L} (L - y) f_{Q-r}(x) f_r(y)\; dy\; dx \\
& \quad +\lambda \textstyle\int_{x=\tau-L}^{T} \int_{y=0}^{\tau-x} (L - y) f_{Q-r}(x) f_r(y)\; dy\; dx \\
= \; & (\lambda L F_r(L) - r) F_{Q-r}(\tau - L) \\
& \quad +\lambda L \textstyle\int_{x=\tau-L}^{T} F_r(\tau - x) f_{Q-r}(x)\; dx \\
& \quad -r \textstyle\int_{x=\tau-L}^{T} F_{r+1}(\tau - x) f_{Q-r}(x)\; dx
\end{aligned}
\tag{A.1.14}
$$

$$\begin{aligned}
E[LS]_2 = & \; E[N(X_{Q-r} + L - \tau) I(\tau - L < X_{Q-r} < T, X_Q > \tau)] \\
= & \; \lambda \int_{x=\tau-L}^{T} \int_{y=\tau-x}^{\infty} (x + L - \tau) f_{Q-r}(x) f_r(y) \, dy \, dx \\
= & \; (Q-r)[F_{Q-r+1}(T) - F_{Q-r+1}(\tau - L)] \\
& \quad -(Q-r) \int_{x=\tau-L}^{T} F_r(\tau - x) f_{Q-r+1}(x) \, dx \\
& \quad -\lambda(\tau - L)[F_{Q-r}(T) - F_{Q-r}(\tau - L)] \\
& \quad +\lambda(\tau - L) \int_{x=\tau-L}^{T} F_r(\tau - x) f_{Q-r}(x) \, dx
\end{aligned} \tag{A.1.15}$$

$$\begin{aligned}
E[LS]_3 = & \; E[N(T + L - \tau) I(X_{Q-r} > T, X_Q > \tau)] \\
= & \; \lambda(T + L - \tau) \int_{x=T}^{\tau} \int_{y=\tau-x}^{\infty} f_{Q-r}(x) f_r(y) \, dy \, dx \\
& \quad +\lambda(T + L - \tau) \int_{x=\tau}^{\infty} \int_{y=0}^{\infty} f_{Q-r}(x) f_r(y) \, dy \, dx \\
= & \; -\lambda(T - \tau + L)[1 - F_{Q-r}(T) - \int_{x=T}^{\tau} F_r(\tau - x) f_{Q-r}(x) \, dx]
\end{aligned} \tag{A.1.16}$$

$$\begin{aligned}
E[LS]_4 = & \; E[N(T + L - X_Q) I(T < X_{Q-r} < X_Q < \tau)] \\
= & \; \lambda(T + L) \int_{x=T}^{\tau} \int_{y=0}^{\tau-x} f_{Q-r}(x) f_r(y) \, dy \, dx \\
& \quad -\lambda \int_{x=T}^{\tau} \int_{y=0}^{\tau-x} x f_{Q-r}(x) f_r(y) \, dy \, dx \\
& \quad -\lambda \int_{x=T}^{\tau} \int_{y=0}^{\tau-x} y f_{Q-r}(x) f_r(y) \, dy \, dx \\
= & \; \lambda(T + L) \int_{x=T}^{\tau} F_r(\tau - x) f_{Q-r}(x) \, dx \\
& \quad -(Q-r) \int_{x=T}^{\tau} F_r(\tau - x) f_{Q-r+1}(x) \, dx \\
& \quad -r \int_{x=T}^{\tau} F_{r+1}(\tau - x) f_{Q-r}(x) \, dx
\end{aligned} \tag{A.1.17}$$

$$\begin{aligned}
E[LS] = & \; E[LS]_1 + E[LS]_2 + E[LS]_3 + E[LS]_4 \\
= & \; -Q F_{Q+1}(\tau) + \lambda \tau F_Q(\tau) \\
& \quad +r \int_{x=0}^{\tau-L} F_{r+1}(\tau - x) f_{Q-r}(x) \, dx \\
& \quad -\lambda \tau \int_{x=0}^{\tau-L} F_r(\tau - x) f_{Q-r}(x) \, dx \\
& \quad +(Q-r) \int_{x=0}^{\tau-L} F_r(\tau - x) f_{Q-r+1}(x) \, dx \\
& \quad +[\lambda L F_r(L) - r F_{r+1}(L) + \lambda(\tau - L)] F_{Q-r}(\tau - L) \\
& \quad +(Q-r) F_{Q-r+1}(T) - (Q-r) F_{Q-r+1}(\tau - L) \\
& \quad -\lambda T F_{Q-r}(T) + \lambda(T + L - \tau)
\end{aligned} \tag{A.1.18}$$

Derivation of Expected Number of Items That Perish

$$\begin{aligned} E[P] =& \ E[(Q - N(\tau))I(X_Q > \tau)] \\ =& \ Q[1 - F_Q(\tau)] - \sum_{n=0}^{Q-1} nP(N(\tau) = n) \\ =& \ Q[1 - F_Q(\tau)] - [\lambda\tau - \lambda\tau \sum_{n=Q-1}^{\infty} nP(N(\tau) = n) \\ =& \ Q[1 - F_Q(\tau)] - \lambda\tau[1 - F_{Q-1}(\tau)] \end{aligned} \tag{A.1.19}$$

A.1.2 Proof of Theorem 2

Derivation of Expected Cycle Length

$$CL = \begin{cases} X_Q & X_{Q-r} < T, X_{Q-r} + L < X_Q < \tau \\ X_{Q-r} + L & X_{Q-r} < T, X_{Q-r} < X_Q > X_{Q-r} + L \\ \tau & X_{Q-r} < T, X_Q > \tau \\ \tau & X_{Q-r} > T, X_Q > \tau \\ X_Q & X_{Q-r} > T, T + L < X_Q < \tau \\ T + L & T < X_{Q-r} < X_Q < T + L \end{cases} \tag{A.1.20}$$

$$\begin{aligned} E[CL] =& \ E[X_Q I(X_{Q-r} < T, X_{Q-r} + L < X_Q < \tau)] \\ & + E[(X_{Q-r} + L)I(X_{Q-r} < T, X_{Q-r} < X_Q > X_{Q-r} + L)] \\ & + E[\tau \ X_Q > \tau] + E[X_Q I(X_{Q-r} > T, T + L < X_Q < \tau)] \\ & + E[(T + L)I(T < X_{Q-r} < X_Q < T + L)] \\ =& \ \int_{x=0}^{T} \int_{y=L}^{\tau-x} (x + y) f_{Q-r}(x) f_r(y) \ dy \ dx \\ & + \int_{x=0}^{T} \int_{y=0}^{L} (x + L) f_{Q-r}(x) f_r(y) \ dy \ dx \\ & + \tau E[I(X_Q > \tau)] \\ & + \int_{x=T}^{T+L} \int_{y=T+L-x}^{\tau-x} (x + y) f_{Q-r}(x) f_r(y) \ dy \ dx \\ & + \int_{x=T+L}^{\tau} \int_{y=0}^{\tau-x} (x + y) f_{Q-r}(x) f_r(y) \ dy \ dx \\ & + (T + L) \int_{x=T}^{T+L} \int_{y=0}^{T+L-x} f_{Q-r}(x) f_r(y) \ dy \ dx \end{aligned} \tag{A.1.21}$$

$$\begin{aligned} =& \ \frac{Q}{\lambda} F_{Q+1}(\tau) + \tau[1 - F_Q(\tau) \\ & - \frac{Q-r}{\lambda} \int_{T}^{T+L} F_r(T + L - x) f_{Q-r+1}(x) \ dx \\ & + (T + L) \int_{T}^{T+L} F_r(T + L - x) f_{Q-r}(x) \ dx \\ & + [LF_r(L) - \frac{r}{\lambda} F_{r+1}(L)] F_{Q-r}(T) \\ & - \frac{r}{\lambda} \int_{T}^{T+L} F_{r+1}(T + L - x) f_{Q-r}(x) \ dx \end{aligned} \tag{A.1.22}$$

Derivation of Expected On Hand Inventory

$$OH = \begin{cases} \sum_{i=1}^{Q} X_i + Q[X_Q - X_{Q-r} - L] & X_{Q-r} < T, X_{Q-r} + L < X_Q < \tau \\ \sum_{i=1}^{Q} X_i & X_{Q-r} < T, X_{Q-r} < X_Q > X_{Q-r} + L \\ \sum_{i=1}^{N(\tau)} X_i + \tau[Q - N(\tau)] + Q[\tau - X_{Q-r} - L] & X_{Q-r} < T, X_Q > \tau \\ \sum_{i=1}^{N(\tau)} X_i + \tau[Q - N(\tau)] + Q[\tau - T - L] & X_{Q-r} > T, X_Q > \tau \\ \sum_{i=1}^{Q} X_i + Q[X_Q - T - L] & X_{Q-r} > T, T + L < X_Q < \tau \\ \sum_{i=1}^{Q} X_i & T < X_{Q-r} < X_Q < T + L \end{cases} \tag{A.1.23}$$

$$\begin{aligned} E[OH] = \; & E[\textstyle\sum_{i=1}^{N(\tau)} X_i I(X_Q > \tau)] \\ & + E[\textstyle\sum_{i=1}^{Q} X_i I(X_Q < \tau)] \\ & + E[Q(X_Q - X_{Q-r} - L) I(X_{Q-r} < T, X_{Q-r} + L < X_Q < \tau)] \\ & + E[\tau(Q - N(\tau)) I(X_Q > \tau)] \\ & + E[Q(\tau - X_{Q-r} - L) I(X_{Q-r} < T, X_Q > \tau)] \\ & + E[Q(\tau - T - L) I(X_{Q-r} > T, X_Q > \tau)] \\ & + E[Q(X_Q - T - L) I(X_{Q-r} > T, T + L < X_Q < \tau)] \end{aligned} \tag{A.1.24}$$

$$\begin{aligned} E[OH]_1 = \; & E[\textstyle\sum_{i=1}^{N(\tau} X_i I(X_Q > \tau)] \\ = \; & E[E[\textstyle\sum_{i=1}^{N(\tau)} X_i I(N(\tau) < Q | N(\tau) = n)]] \\ = \; & \textstyle\sum_{n=0}^{Q-1} n \frac{\tau}{2} P(N(\tau) = n) \\ = \; & \frac{\tau}{2} \textstyle\sum_{n=0}^{Q-1} n P(N(\tau) = n) \\ = \; & \frac{\tau}{2}[\lambda\tau - \textstyle\sum_{n=Q}^{\infty} n P(N(\tau) = n) \\ = \; & \frac{\tau}{2}[\lambda\tau - \lambda\tau \textstyle\sum_{n=Q-1}^{\infty} P(N(\tau) = n) \\ = \; & \frac{\lambda\tau^2}{2}[1 - F_{Q-1}(\tau)] \end{aligned} \tag{A.1.25}$$

$$\begin{aligned} E[OH]_2 = \; & E[\textstyle\sum_{i=1}^{Q} X_i I(X_Q < \tau)] \\ = \; & \frac{Q(Q+1)}{2\lambda} F_{Q+1}(\tau) \end{aligned} \tag{A.1.26}$$

$$\begin{aligned}
E[OH]_3 = &\ E[Q(X_Q - X_{Q-r} - L)I(X_{Q-r} < T, X_{Q-r} + L < X_Q < \tau)] \\
= &\ Q \int_{x=0}^{T} \int_{y=L}^{\tau-x} (y - L) f_{Q-r}(x) f_r(y)\ dy\ dx \\
= &\ Q \int_{x=0}^{T} \int_{y=L}^{\tau-x} y f_{Q-r}(x) f_r(y)\ dy\ dx - QL \int_{x=0}^{T} \int_{y=L}^{\tau-x} f_{Q-r}(x) f_r(y)\ dy\ dx
\end{aligned} \tag{A.1.27}$$

$$\begin{aligned}
E[OH]_4 = &\ E[\tau(Q - N(\tau))I(X_Q > \tau)] \\
= &\ E[\tau Q I(X_Q > \tau)] - E[\tau N(\tau) I(X_Q > \tau)] \\
= &\ Q\tau[1 - F_Q(\tau)] - \tau E[N(\tau)I(N(\tau) < Q)] \\
= &\ Q\tau[1 - F_Q(\tau)] - \tau \textstyle\sum_{n=0}^{Q-1} nP(N(\tau) = n) \\
= &\ Q\tau[1 - F_Q(\tau)] - \tau[\lambda\tau - \textstyle\sum_{n=Q}^{\infty} nP(N(\tau) = n)] \\
= &\ Q\tau[1 - F_Q(\tau)] - \tau[\lambda\tau - \lambda\tau \textstyle\sum_{n=Q-1}^{\infty} P(N(\tau) = n)] \\
= &\ Q\tau[1 - F_Q(\tau)] - \lambda\tau^2[1 - F_{Q-1}(\tau)]
\end{aligned} \tag{A.1.28}$$

$$\begin{aligned}
E[OH]_5 = &\ E[Q(\tau - X_{Q-r} - L)I(X_{Q-r} < T, X_Q > \tau)] \\
= &\ Q(\tau - L) \int_{x=0}^{T} \int_{y=\tau-x}^{\infty} f_{Q-r}(x) f_r(y)\ dy\ dx \\
&\quad - Q \int_{x=0}^{T} \int_{y=\tau-x}^{\infty} x f_{Q-r}(x) f_r(y)\ dy\ dx \\
= &\ Q(\tau - L) \int_{x=0}^{T} f_{Q-r}(x)[1 - F_r(\tau - x)]\ dx \\
&\quad - Q \int_{x=0}^{T} x f_{Q-r}(x)[1 - F_r(\tau - x)]\ dx \\
= &\ Q(\tau - L)F_{Q-r}(T) - Q(\tau - L) \int_{x=0}^{T} f_{Q-r}(x) F_r(\tau - x)\ dx \\
&\quad - Q\tfrac{Q-r}{\lambda} F_{Q-r+1}(T) + Q\tfrac{Q-r}{\lambda} \int_{x=0}^{T} f_{Q-r+1}(x) F_r(\tau - x)\ dx
\end{aligned} \tag{A.1.29}$$

$$\begin{aligned}
E[OH]_6 = &\ E[Q(\tau - T - L)I(X_{Q-r} > T, X_Q > \tau)] \\
= &\ Q(\tau - T - L) \int_{x=T}^{\tau} \int_{y=\tau-x}^{\infty} f_{Q-r}(x) f_r(y)\ dy\ dx \\
&\quad + Q(\tau - T - L) \int_{x=\tau}^{\infty} \int_{y=0}^{\infty} f_{Q-r}(x) f_r(y)\ dy\ dx \\
= &\ Q(\tau - T - L) \int_{x=T}^{\tau} f_{Q-r}(x)[1 - F_r(\tau - x)]\ dx \\
&\quad + Q(\tau - T - L) \int_{x=\tau}^{\infty} f_{Q-r}(x)\ dx \\
= &\ Q(\tau - T - L) - Q(\tau - T - L)F_{Q-r}(T) \\
&\quad - Q(\tau - T - L) \int_{x=T}^{\tau} f_{Q-r}(x)[1 - F_r(\tau - x)]\ dx
\end{aligned} \tag{A.1.30}$$

Derivation of Number of Lost Sales

$$LS = \begin{cases} N(X_{Q-r} + L - X_Q) & X_{Q-r} < T, X_{Q-r} < X_Q < X_{Q-r} + L \\ N(T + L - X_Q) & T < X_{Q-r} < X_Q < T + L \end{cases} \tag{A.1.31}$$

$$\begin{aligned} E[LS] = & \; E[N(X_{Q-r} + L - X_Q)I(X_{Q-r} < T, X_{Q-r} < X_Q < X_{Q-r} + L] \\ & + E[N(T + L - X_Q)I(T < X_{Q-r} < X_Q < T + L)] \end{aligned} \tag{A.1.32}$$

$$\begin{aligned} E[LS]_1 = & \; E[N(X_{Q-r} + L - X_Q)I(X_{Q-r} < T, X_{Q-r} < X_Q < X_{Q-r} + L] \\ = & \; \lambda \int_{x=0}^{T} \int_{y=0}^{L} (L - y) f_{Q-r}(x) f_r(y) \; dy \; dx \\ = & \; \lambda L F_r(L) F_{Q-r}(T) - r F_{r+1}(L) F_{Q-r}(T) \end{aligned} \tag{A.1.33}$$

$$\begin{aligned} E[LS]_2 = & \; E[N(T + L - X_Q)I(T < X_{Q-r} < X_Q < T + L)] \\ = & \; \lambda(T + L) \int_{x=T}^{T+L} \int_{y=0}^{T+L-x} f_{Q-r}(x) f_r(y) \; dy \; dx \\ & - \lambda \int_{x=T}^{T+L} \int_{y=0}^{T+L-x} x f_{Q-r}(x) f_r(y) \; dy \; dx \\ & - \lambda \int_{x=T}^{T+L} \int_{y=0}^{T+L-x} y f_{Q-r}(x) f_r(y) \; dy \; dx \\ = & \; \lambda(T + L) \int_{x=T}^{T+L} F_r(T + L - x) f_{Q-r}(x) \; dx \\ & - (Q - r) \int_{x=T}^{T+L} F_r(T + L - x) f_{Q-r+1}(x) \; dx \\ & - r \int_{x=T}^{T+L} F_{r+1}(T + L - x) f_{Q-r}(x) \; dx \end{aligned} \tag{A.1.34}$$

$$\begin{aligned} E[LS] = & \; E[LS]_1 + E[LS]_2 \\ = & \; F_{Q-r}(T)[\lambda L F_r(L) - r F_{r+1}(L) \\ & + \lambda(T + L) \int_{x=T}^{T+L} F_r(T + L - x) f_{Q-r}(x) \; dx \\ & - r \int_{x=T}^{T+L} F_{r+1}(T + L - x) f_{Q-r}(x) \; dx \\ & - (Q - r) \int_{x=T}^{T+L} F_r(T + L - x) f_{Q-r+1}(x) \; dx \end{aligned} \tag{A.1.35}$$

Derivation of Expected Number of Items That Perish

$$\begin{aligned} E[P] = & \; E[(Q - N(\tau))I(X_Q > \tau)] \\ = & \; Q[1 - F_Q(\tau)] - \textstyle\sum_{n=0}^{Q-1} n P(N(\tau) = n) \\ = & \; Q[1 - F_Q(\tau)] - [\lambda\tau - \lambda\tau \textstyle\sum_{n=Q-1}^{\infty} n P(N(\tau) = n) \\ = & \; Q[1 - F_Q(\tau)] - \lambda\tau[1 - F_{Q-1}(\tau)] \end{aligned} \tag{A.1.36}$$

A.2 APPENDIX B

λ=0.5										
		α	Q_1^*	r_1^*	T^*	C_1	Q_2^*	r_2^*	C_2	Δ %
		0.005	6	2	11.12	9.86	5	3	11.01	10.45
		0.01	7	1	11	9.43	4	2	10.55	10.62
	τ=12	0.02	6	1	11.18	9.02	6	2	9.71	7.11
		0.05	6	0	11.03	8.43	6	1	8.8	4.20
		0.1	5	0	11.98	7.91	6	0	8.07	1.98
		0.005	6	2	14.32	9.43	4	2	10.34	8.80
		0.01	8	1	14.02	9.04	6	2	9.39	3.73
p=1	τ=15	0.02	6	1	14.28	8.49	7	2	9.37	9.39
		0.05	7	0	14.13	7.99	6	1	8.41	4.99
		0.1	6	0	15	7.52	6	0	7.52	0.00
		0.005	7	2	20	9.14	7	2	9.14	0.00
		0.01	9	1	19.02	8.69	7	2	9.14	4.92
	τ=20	0.02	7	1	20	8.16	7	1	8.16	0.00
		0.05	8	0	19.14	7.61	7	1	8.16	6.74
		0.1	6	0	20	7.18	6	0	7.18	0.00
		0.005	5	2	11.17	10.5	5	3	11.53	8.93
		0.01	5	2	11.78	10.45	4	2	10.83	3.51
	τ=12	0.02	5	1	11.04	9.55	5	2	10.44	8.52
		0.05	6	0	11.03	9.24	5	1	9.38	1.49
		0.1	5	0	11.98	8.38	6	0	8.85	5.31
		0.005	6	2	14.32	9.8	4	2	10.44	6.13
		0.01	7	1	12.03	9.69	6	2	9.76	0.72
p=10	τ=15	0.02	6	1	14.28	8.85	6	2	9.76	9.32
		0.05	6	0	11.87	8.48	6	1	8.77	3.31
		0.1	5	0	15	7.71	5	0	7.71	0.00
		0.005	6	2	20	9.29	6	2	9.29	0.00
		0.01	8	1	16.13	9	6	2	9.29	3.12
	τ=20	0.02	6	1	20	8.27	6	1	8.27	0.00
		0.05	7	0	15.92	7.85	6	1	8.27	5.08
		0.1	6	0	20	7.26	6	0	7.26	0.00
		0.005	4	2	11.23	12.08	4	3	13.15	8.14
		0.01	4	2	12	12.06	4	2	12.06	0.00
	τ=12	0.02	4	1	7.14	11.37	4	2	12.06	5.72
		0.05	4	1	12	10.87	4	1	10.87	0.00
		0.1	4	0	11.06	9.67	4	1	10.87	11.04
		0.005	5	2	14.61	10.77	4	2	10.87	0.92
		0.01	5	2	15	10.76	5	2	10.76	0.00
p=50	τ=15	0.02	5	1	13.58	9.77	5	2	10.76	9.20
		0.05	5	0	8.88	9.64	5	1	9.68	0.41
		0.1	5	0	15	8.53	5	0	8.53	0.00
		0.005	6	2	20	9.66	6	2	9.66	0.00
		0.01	6	1	9.64	9.59	6	2	9.66	0.72
	τ=20	0.02	6	1	20	8.64	6	1	8.64	0.00
		0.05	6	0	12.35	8.4	6	1	8.64	2.78
		0.1	5	0	20	7.52	5	0	7.52	0.00

Table A.1: (Q, r, T) vs. (Q, r) (λ=0.5)

λ=10										
		α	Q_1^*	r_1^*	T^*	C_1	Q_2^*	r_2^*	C_2	Δ %
		0.005	22	15	1	43.09	20	16	44.17	2.45
		0.01	21	14	1.01	41.91	20	15	42.98	2.49
	τ=2	0.02	21	12	0.92	40.46	20	14	41.72	3.02
		0.05	20	10	1.01	38.08	20	12	38.92	2.16
		0.1	19	7	1.02	35.63	19	10	36.36	2.01
		0.005	33	14	2.31	36.74	31	15	37.19	1.21
		0.01	32	13	3	35.21	31	14	36.17	2.65
p=1	τ=4	0.02	31	12	4	34.13	31	12	34.13	0.00
		0.05	32	9	2.88	31.54	30	10	32.11	1.78
		0.1	30	7	3.82	29.28	31	7	29.35	0.24
		0.005	36	14	3.04	36.51	38	14	36.66	0.41
		0.01	32	13	6	35.09	32	13	35.09	0.00
	τ=6	0.02	36	11	2.96	33.99	31	12	34.08	0.26
		0.05	34	9	4.02	31.35	35	9	31.47	0.38
		0.1	34	6	3.34	29.23	30	7	29.25	0.07
		0.005	16	15	0.06	48.45	17	16	49.09	1.30
		0.01	16	14	0.24	47.12	16	15	47.84	1.51
	τ=2	0.02	16	13	0.71	45.43	16	14	46.65	2.62
		0.05	16	11	0.97	42.6	16	12	44	3.18
		0.1	16	9	1.04	39.48	16	10	40.82	3.28
		0.005	30	14	1.82	37.34	29	15	37.5	0.43
		0.01	30	13	2.25	35.77	29	14	36.48	1.95
p=10	τ=4	0.02	28	12	4	34.42	28	12	34.42	0.00
		0.05	30	9	2.47	32.08	28	10	32.34	0.80
		0.1	29	7	3.14	29.53	31	7	29.83	1.01
		0.005	36	14	3.04	36.51	38	14	36.67	0.44
		0.01	32	13	6	35.09	32	13	35.09	0.00
	τ=6	0.02	36	11	2.96	33.99	31	12	34.08	0.26
		0.05	34	9	4.02	31.35	35	9	31.47	0.38
		0.1	34	6	3.34	29.24	30	7	29.25	0.03
		0.005	16	15	0.06	58.84	17	16	64.39	8.62
		0.01	15	14	0.07	54.42	16	15	58.21	6.51
	τ=2	0.02	14	13	0.12	51.33	15	14	53.81	4.61
		0.05	14	11	0.38	48.36	13	12	49.25	1.81
		0.1	14	9	0.58	44.92	13	10	45.56	1.40
		0.005	26	15	4	38.02	26	15	38.02	0.00
		0.01	27	13	1.66	36.58	26	14	36.99	1.11
p=50	τ=4	0.02	26	12	2.69	34.89	27	12	34.92	0.09
		0.05	26	10	4	32.74	26	10	32.74	0.00
		0.1	27	7	2.47	30.14	26	8	30.63	1.60
		0.005	35	14	2.75	36.51	38	14	36.69	0.49
		0.01	32	13	6	35.09	32	13	35.09	0.00
	τ=6	0.02	35	11	2.78	34	31	12	34.08	0.23
		0.05	34	9	4.02	31.36	35	9	31.47	0.35
		0.1	34	6	3.34	29.24	30	7	29.25	0.03

Table A.2: (Q, r, T) vs. (Q, r) (λ=10)

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No.	λ	π	K	Q_T^*	T^*	AC	Q^*	r^*	TC	%Δ
1	5	5	10	10	0.88	13.16	11	5	12.43	5.92
2	5	5	20	15	2.43	16.78	15	4	15.97	5.06
3	5	5	50	23	8.51	23.25	23	2	23.16	0.39
4	5	10	10	9	0.29	14.30	11	7	13.65	4.81
5	5	10	20	13	1.18	18.66	15	6	17.24	8.24
6	5	10	50	23	3.76	26.41	23	5	24.71	6.88
7	5	10	100	33	7.1	34.39	32	4	33.31	3.23
8	10	2	10	14	0.68	16.13	15	7	15.81	2.00
9	10	5	10	13	0.08	18.67	16	11	18.24	2.35
10	10	5	20	20	0.9	24.44	22	10	23.36	4.64
11	10	5	50	32	2.49	35.07	33	8	33.82	3.70
12	10	5	100	45	5.06	46.07	45	6	45.72	0.77
13	10	10	10	14	0.02	19.89	16	13	19.69	1.02
14	10	10	20	19	0.53	26.15	22	12	24.93	4.90
15	10	10	50	31	1.84	38.01	33	11	35.61	6.73
16	10	10	100	45	3.48	50.67	46	10	47.91	5.77
17	20	2	10	20	0.01	23.72	22	19	23.62	0.44
18	20	2	20	29	0.63	31.05	29	17	30.60	1.48
19	20	5	10	23	0	26.63	24	23	26.57	0.22
20	20	5	20	27	0.2	34.68	30	22	33.97	2.09
21	20	5	50	44	1.16	50.78	47	20	48.98	3.68
22	20	5	100	64	2.36	68.16	65	18	66.21	2.93
23	20	10	20	26	0.03	36.43	31	24	35.97	1.28
24	20	10	50	43	0.9	53.61	47	23	51.19	4.71
25	20	10	100	62	1.91	72.27	65	22	68.73	5.15
26	40	2	20	40	0.01	45.21	43	39	45.11	0.22
27	40	2	50	64	0.78	66.35	65	34	65.74	0.93
28	40	5	50	62	0.46	72.27	67	42	70.72	2.19
29	40	5	100	89	1.2	97.96	92	40	95.36	2.72
30	40	10	50	61	0.33	75.18	66	46	73.47	2.33
31	40	10	100	87	1	102.04	93	44	98.43	3.66

Table A.3: Comparison of Cost Values for Poisson Demands

No.	$\hat{Q}^*$	$\hat{r}^*$	Q^*	r^*	$\%\Delta$	No.	$\hat{Q}^*$	$\hat{r}^*$	Q^*	r^*	$\%\Delta$
1	10	6	11	5	1.44	16	45	10	46	10	0.03
2	15	3	15	4	0.97	17	21	20	22	19	0.27
3	23	0	23	2	0.39	18	29	16	29	17	0.09
4	9	8	11	7	3.80	19	23	22	24	23	0.38
5	13	7	15	6	2.02	20	27	23	30	22	0.74
6	23	4	23	5	0.67	21	44	21	47	20	0.26
7	33	0	32	4	3.57	22	64	17	65	18	0.11
8	14	7	15	7	0.21	23	26	25	31	24	0.91
9	13	12	16	11	1.48	24	43	25	47	23	1.04
10	20	11	22	10	0.36	25	62	24	65	22	0.80
11	32	7	33	8	0.50	26	41	40	43	39	0.11
12	45	0	45	6	0.80	27	64	33	65	34	0.06
13	14	13	16	13	0.60	28	62	44	67	42	0.34
14	19	14	22	12	2.06	29	89	41	92	40	0.07
15	31	13	33	11	1.41	30	61	48	66	46	0.69

Table A.4: Comparison of Ordering Policies for Poisson Demands

			$\alpha = 2$							$\alpha = 4$						
λ	π	K	Q_T^*	T^*	AC	Q^*	r^*	TC	$\%\Delta$	Q_T^*	T^*	AC	Q^*	r^*	TC	$\%\Delta$
5	2	10	6	24.62	6.88	7	0	6.75	1.97	4	15.88	4.88	5	0	4.73	3.15
5	2	20	9	15.35	9.35	9	0	9.25	1.03	6	17.88	6.78	7	0	6.67	1.55
5	2	50	14	19.29	14.7	14	0	14.64	0.42	10	22.01	10.72	11	0	10.66	0.61
5	2	100	20	22.01	21	20	0	20.94	0.27	15	25	15.27	15	0	15.21	0.38
5	5	10	7	2.18	8.61	7	2	7.99	7.84	5	5.98	5.65	5	0	5.29	6.79
5	5	20	10	5.36	10.9	10	1	10.54	3.41	7	17.89	7.37	7	0	7.09	3.98
5	5	50	15	18.46	15.79	15	0	15.61	1.15	11	23.99	11.13	11	0	10.93	1.8
5	5	100	21	23.33	21.8	21	0	21.67	0.6	15	25	15.56	15	0	15.42	0.9
5	10	10	7	1.41	9.57	7	3	8.64	10.77	5	2.97	6.42	5	1	5.57	15.16
5	10	20	10	2.94	12.37	11	2	11.29	9.55	7	5.48	8.24	7	1	7.55	9.05
5	10	50	17	7.46	17.42	16	2	16.66	4.57	11	18.44	11.77	11	0	11.39	3.29
5	10	100	23	17.59	23.08	23	1	22.77	1.33	16	25	16.04	15	0	15.76	1.74
10	2	10	10	24.62	10.33	10	0	10.16	1.71	6	24.62	6.88	7	0	6.69	2.95
10	2	20	13	24.62	13.39	13	0	13.24	1.09	9	10.55	9.35	9	0	9.19	1.67
10	2	50	20	15.1	20.4	20	0	20.3	0.48	14	15.88	14.7	14	0	14.59	0.72
10	2	100	28	13.6	28.97	28	0	28.88	0.3	20	17.81	21	20	0	20.91	0.41
10	5	10	10	0.91	12.46	11	5	11.66	6.85	7	1.97	8.37	7	2	7.67	9.19
10	5	20	14	1.95	16.29	15	4	15.38	5.86	10	3.97	10.87	10	1	10.39	4.64
10	5	50	23	5.31	23.24	23	3	22.87	1.63	15	14.47	15.79	16	0	15.53	1.64
10	5	100	31	14.56	31.14	30	0	30.95	0.62	21	18.28	21.8	21	0	21.61	0.89
10	10	10	9	0.43	13.32	11	6	12.45	7.05	7	1.51	9.03	8	2	8.23	9.73
10	10	20	14	1.5	17.58	15	5	16.31	7.83	10	2.89	11.9	10	2	10.85	9.7
10	10	50	22	3.37	25.58	23	5	23.93	6.88	16	6	17.27	16	2	16.36	5.55
10	10	100	32	5.97	34.08	32	4	32.74	4.08	23	11.39	23.08	22	1	22.62	2.01
20	2	10	14	0.58	15.74	15	8	15.29	2.95	10	24.62	10.33	10	0	10.09	2.42
20	2	20	20	24.62	20.33	20	0	20.15	0.93	13	24.62	13.39	13	0	13.18	1.58
20	2	50	29	24.9	29.1	28	0	28.97	0.45	20	8.9	20.4	20	0	20.25	0.75
20	2	100	40	8.9	40.4	40	0	40.28	0.29	28	15.1	28.97	29	0	28.86	0.38
20	5	10	14	0.25	17.59	16	11	17	3.42	10	0.93	11.92	10	5	11.16	6.81
20	5	20	20	0.93	23.36	21	10	22.3	4.75	14	1.88	15.84	15	4	15.01	5.59
20	5	50	32	2.37	34.31	32	9	33.11	3.62	23	4.45	23.21	22	3	22.68	2.32
20	5	100	45	4.28	45.97	46	4	45.54	0.94	31	11.63	31.14	30	0	30.88	0.85
20	10	10	13	0.03	18.46	15	12	17.96	2.81	10	0.74	12.57	10	6	11.8	6.55
20	10	20	19	0.64	24.62	21	11	23.41	5.17	14	1.59	16.77	14	5	15.57	7.68
20	10	50	31	1.91	36.38	33	11	34.37	5.84	22	3.35	24.85	23	5	23.44	5.98
20	10	100	45	3.45	49.21	46	10	46.88	4.96	32	5.67	33.65	32	4	32.35	4.02
40	2	10	20	0.01	22.84	21	19	22.57	1.2	14	0.52	15.45	9	14	14.96	3.25
40	2	20	28	0.52	30.44	29	18	29.9	1.79	20	24.62	20.33	2	20	20.07	1.32
40	2	50	43	24.9	43.59	43	0	43.42	0.38	29	24.9	29.1	0	29	28.91	0.67
40	2	100	57	24.62	57.83	56	0	57.69	0.25	40	15.1	40.4	0	39	40.25	0.37
40	5	10	22	0	24.89	23	22	24.61	1.15	14	0.29	16.77	10	15	16.14	3.88
40	5	20	28	0.29	33.05	30	21	32.23	2.55	20	0.95	22.56	10	21	21.59	4.5
40	5	50	45	1.22	49.22	46	20	47.66	3.29	32	2.31	33.7	9	32	32.62	3.33
40	5	100	64	2.31	66.93	64	19	65.29	2.51	45	3.98	45.81	7	46	45.15	1.45
40	10	10	24	0	26.25	24	23	26	0.98	13	0.1	17.43	11	16	16.8	3.76
40	10	20	27	0.15	34.36	30	23	33.56	2.37	19	0.71	23.48	11	21	22.31	5.23
40	10	50	44	1.02	51.27	46	22	49.2	4.2	31	1.97	35.17	10	33	33.52	4.93
40	10	100	63	2.02	69.86	66	21	66.99	4.29	45	3.46	48.08	10	46	46.2	4.07

Table A.5: Comparison of Cost Values for Erlang Demands

	$\alpha = 2$					$\alpha = 4$				
No.	$\hat{Q}^*$	$\hat{r}^*$	Q^*	r^*	$\%\Delta$	$\hat{Q}^*$	$\hat{r}^*$	Q^*	r^*	$\%\Delta$
1	6	0	7	0	0.04	4	0	5	0	0.52
2	9	0	9	0	0	6	0	7	0	0.13
3	14	0	14	0	0	10	0	11	0	0.06
4	20	0	20	0	0	15	0	15	0	0
5	7	2	7	2	0	5	0	5	0	0
6	10	0	10	1	1.07	7	0	7	0	0
7	15	0	15	0	0	11	0	11	0	0
8	21	0	21	0	0	15	0	15	0	0
9	7	3	7	3	0	5	1	5	1	0
10	10	3	11	2	0.86	7	0	7	1	3.04
11	17	0	16	2	2.82	11	0	11	0	0
12	23	0	23	1	0.21	16	0	15	0	0.01
13	10	0	10	0	0	6	0	7	0	0.12
14	13	0	13	0	0	9	0	9	0	0
15	20	0	20	0	0	14	0	14	0	0
16	28	0	28	0	0	20	0	20	0	0
17	10	5	11	5	0.09	7	2	7	2	0
18	14	4	15	4	0.1	10	0	10	1	1.48
19	23	0	23	3	0.66	15	0	16	0	0.01
20	31	0	30	0	0.05	21	0	21	0	0
21	9	7	11	6	3.57	7	3	8	2	0.24
22	14	7	15	5	3.48	10	3	10	2	2.51
23	22	5	23	5	0.15	16	1	16	2	1.14
24	32	2	32	4	1.83	23	0	22	1	0.38
25	14	8	15	8	0.07	10	0	10	0	0
26	20	0	20	0	0	13	0	13	0	0
27	29	0	28	0	0.05	20	0	20	0	0
28	40	0	40	0	0	28	0	29	0	0.04
29	14	12	16	11	1.79	10	5	10	5	0
30	20	11	21	10	0.79	14	5	15	4	0.69
31	32	8	32	9	0.2	23	1	22	3	0.68
32	45	2	46	4	0.38	31	0	30	0	0.02
33	13	12	15	12	1.08	10	6	10	6	0
34	19	13	21	11	2.35	14	6	14	5	1.78
35	31	12	33	11	1.08	22	5	23	5	0.12
36	45	11	46	10	0.51	32	4	32	4	0
37	20	19	21	19	0.15	14	9	9	14	0
38	28	18	29	18	0.05	20	0	2	20	0.08
39	43	0	43	0	0	29	0	0	29	0
40	57	0	56	0	0.06	40	0	0	39	0.01
41	23	22	23	22	0	14	11	10	15	0.42
42	28	22	30	21	0.38	20	11	10	21	1.5
43	45	21	46	20	0.35	32	9	9	32	0
44	64	18	64	19	0	45	5	7	46	0.37
45	24	23	24	23	0	13	12	11	16	1.6
46	27	24	30	23	0.79	19	12	11	21	2.09
47	44	24	46	22	1.38	31	11	10	33	0.41
48	63	23	66	21	0.89	45	10	10	46	0.04

Table A.6: Comparison of Ordering Policies for Erlang Demands

No.	Q_T^*	T^*	AC	Q^*	r^*	TC	%Δ
1	10	24.62	10.33	10	0	10.02	3.07
2	13	24.62	13.38	13	0	13.13	1.89
3	20	15.08	20.4	20	0	20.22	0.9
4	28	15.94	28.96	29	0	28.83	0.44
5	10	0.97	11.23	10	5	10.63	5.64
6	14	1.84	15.25	15	4	14.6	4.44
7	22	3.73	23.05	22	4	22.48	2.54
8	31	20.44	31.13	31	0	30.81	1.04
9	10	0.87	11.59	10	5	10.76	7.68
10	14	1.7	15.74	14	5	14.83	6.15
11	22	3.37	23.88	22	5	22.98	3.9
12	32	5.51	32.92	32	4	31.99	2.92
13	14	0.46	15.05	14	9	14.51	3.7
14	20	25.59	20.33	20	1	20.02	1.54
15	29	24.62	29.1	29	0	28.86	0.83
16	40	15.08	40.4	40	0	40.22	0.46
17	14	0.35	15.74	14	10	15.08	4.38
18	20	0.98	21.55	20	10	20.81	3.55
19	32	2.26	32.92	32	9	32.1	2.55
20	45	3.72	45.52	45	9	44.92	1.34
21	14	0.3	16.11	15	10	15.5	3.91
22	20	0.91	22.05	21	10	21.12	4.4
23	31	2.03	33.68	32	10	32.51	3.59
24	45	3.48	46.67	45	10	45.5	2.57
25	20	0	21.36	21	19	20.95	1.98
26	28	0.44	29.39	29	19	28.83	1.96
27	43	25.59	43.58	43	0	43.29	0.67
28	57	25.59	57.83	58	0	57.59	0.42
29	21	0	22.09	21	20	21.66	1.97
30	28	0.36	30.35	29	20	29.61	2.48
31	45	1.24	46.67	45	20	45.75	2.01
32	63	2.19	64.9	64	19	63.91	1.55
33	22	0.03	22.59	22	21	22.04	2.5
34	28	0.33	30.87	28	21	30.13	2.47
35	44	1.14	47.44	45	20	46.28	2.51
36	63	2.1	65.96	64	20	64.5	2.27
37	40	0	31.88	40	39	31.53	1.11
38	40	0	41.73	41	39	41.3	1.05
39	63	0.62	64.43	64	38	63.83	0.94
40	86	25.59	86.83	87	0	86.53	0.34
41	41	0	33.07	42	41	32.76	0.95
42	41	0	42.76	42	41	42.25	1.2
43	63	0.55	65.96	64	40	65.03	1.44
44	89	1.22	92.01	90	40	90.86	1.27
45	42	0	33.84	42	41	33.46	1.13
46	42	0	43.35	42	41	42.95	0.93
47	62	0.5	66.77	65	41	65.66	1.69
48	89	1.18	93.08	90	41	91.59	1.63

Table A.7: Comparison of Cost Values for Normal Demands

No.	$\hat{Q}^*$	$\hat{r}^*$	Q^*	r^*	$\%\Delta$	No.	$\hat{Q}^*$	$\hat{r}^*$	Q^*	r^*	$\%\Delta$
1	10	0	10	0	0.00	25	20	19	21	19	0.06
2	13	0	13	0	0.00	26	28	19	29	19	0.01
3	20	0	20	0	0.00	27	43	0	43	0	0.00
4	28	0	29	0	0.03	28	57	0	58	0	0.00
5	10	5	10	5	0.00	29	21	20	21	20	0.00
6	14	5	15	4	0.90	30	28	21	29	20	1.15
7	22	3	22	4	0.39	31	45	20	45	20	0.00
8	31	0	31	0	0.00	32	63	19	64	19	0.00
9	10	6	10	5	6.91	33	22	21	22	21	0.00
10	14	6	14	5	5.53	34	28	21	28	21	0.00
11	22	5	22	5	0.00	35	44	21	45	20	0.36
12	32	4	32	4	0.00	36	63	21	64	20	0.63
13	14	9	14	9	0.00	37	40	39	40	39	0.00
14	20	0	20	1	0.00	38	40	39	41	39	0.02
15	29	0	29	0	0.00	39	63	38	64	38	0.01
16	40	0	40	0	0.00	40	86	0	87	0	0.01
17	14	11	14	10	4.00	41	42	41	42	41	0.00
18	20	10	20	10	0.00	42	42	41	42	41	0.00
19	32	9	32	9	0.00	43	63	41	64	40	0.27
20	45	8	45	9	0.02	44	89	40	90	40	0.00
21	14	11	15	10	1.40	45	42	41	42	41	0.00
22	20	11	21	10	2.04	46	42	41	42	41	0.00
23	31	11	32	10	2.00	47	62	42	65	41	0.53
24	45	10	45	10	0.00	48	89	42	90	41	0.59

Table A.8: Comparison of Ordering Policies for Normal Demands