ALGEBRAIC THEORY OF LINEAR MULTIVARIABLE CONTROL SYSTEMS

A THESIS
SUBMITTED TO THE DEPARTMENT OF ELECTRICAL AND
ELECTRONICS ENGINEERING
AND THE INSTITUTE OF ENGINEERING AND SCIENCE
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
Sevgi Babacan Çetin

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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Prof. Dr. A. Bülent Özgüler(Supervisor)

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ABSTRACT

ALGEBRAIC THEORY OF LINEAR MULTIVARIABLE CONTROL SYSTEMS

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M.S. in Electrical and Electronics Engineering

Supervisor: Prof. Dr. A. Bülent Özgüler

September 1998

The theory of linear multivariable systems stands out as the most developed and sophisticated among the topics of system theory. In the literature, many different solutions are presented to the linear multivariable control problems using three main approaches : geometric approach, fractional approach and polynomial model based approach. This thesis is a first draft for a textbook on linear multivariable control which contains a description of solutions to the most of the standard algebraic feedback control problems using simple linear algebra and a minimal amount of polynomial algebra. These problems are internal stabilization, disturbance decoupling by state feedback and measurement feedback, output stabilization, tracking with regulation in a scalar system, regulator problem with a single output channel and decentralized stabilization.

Keywords: Multivariable control, fractional approach, internal stabilization, disturbance decoupling, tracking and regulation, decentralized stabilization.

ÖZET

ÇOK DEĞİŞKENLİ DOĞRUSAL KONTROL SİSTEMLERİNİN CEBİRSEL TEORİSİ

Sevgi Babacan Çetin
Elektrik ve Elektronik Mühendisliği Bölümü Yüksek Lisans
Tez Yöneticisi: Prof. Dr. A. Bülent Özgüler
Eylül 1998

Çokdeğişkenli doğrusal sistemler, sistem teorisinin en karmaşık ve en fazla işlenmiş alanını oluşturmaktadır. Literatürde birçok çokdeğişkenli doğrusal sistem problemi şu üç yöntemden biri kullanılarak çözülmüştür : geometrik yaklaşım, kesir yaklaşımı ve polinom modellere dayalı yaklaşım. Çokdeğişkenli doğrusal kontrol üzerine bir ders kitabı taslağı olarak hazırlanan bu çalışma, birçok cebirsel geribeslemeli denetim probleminin çözümünü, basit doğrusal cebir yöntemleri ve minimum miktarda polinom cebiri kullanarak sunmaktadır. Ele alınan başlıca problemler şunlardır : içsel kararlılaştırma, durum ve ölçüm geribeslemesiyle bozanetkeni ortadan kaldırma, çıktı kararlılaştırılması, sayıl bir sistemde düzenleme ile izleme, tek çıktı kanallı düzenleme problemi ve dağıtılmış (özeksiz) kararlılaştırma.

Anahtar Kelimeler: Çokdeğişkenli kontrol, kesir yaklaşımı, içsel kararlılaştırma, bozanetkeni ortadan kaldırma, düzenleme ve izleme, dağıtılmış (özeksiz) kararlılaştırma.

ACKNOWLEDGMENTS

I would like to express my sincere gratitude to Prof. Dr. A. Bülent Özgüler for his supervision, guidance, suggestions, and especially encouragement throughout the development of this thesis.

I would like to thank Prof. Dr. Erol Sezer and Assoc. Prof. Dr. Ömer Morgül for reading and commenting on the thesis.

I express my special thanks to İsmail and my parents for their constant support, patience and sincere love.

Finally, many thanks to every close friend.

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Chapter 1

INTRODUCTION

The theory of linear multivariable systems stands out as the most developed and sophisticated among the topics of system theory. The structure of a multivariable dynamic system and the limitations it may impose on the success of a feedback control applied on this system in order to satisfy certain design specifications are well-investigated and well-understood for a large class of idealized control problems.

Yet, reference by control system designers to very basic and relevant results the theory has accumulated still remains surprisingly limited. Some quick explanations for this are that the real systems are too complex to yield to linear analysis, or that the design specifications are usually many more in number than any theory can anticipate. Such explanations can be discarded on the grounds that Newton's theory of motion is indispensible to designers of automobiles although an automobile is in fact far more complex than a point-mass. The reason for the limited attention the theory has received perhaps lies elsewhere.

If we consider one of the simplest feedback control problems of multivariable systems, say the disturbance decoupling problem, then we may better understand the source of difficulty. The problem at its most generality is the following: A system with two groups of outputs and two groups of inputs are given. One group of inputs, called the disturbance inputs, consists of variables with unwanted influence on one group of outputs, called the regulated outputs. The second groups of inputs and outputs consist of the inputs available for control, called the control inputs, and the outputs that can be measured, called the measured outputs. The problem is to determine a feedback controller which processes the measured outputs to produce values for the control inputs such that in the closed loop system the disturbance inputs have no influence on the regulated outputs. Since the introduction of a feedback loop into any system may cause instabilities of the signals around the loop, the satisfaction of stability of the feedback loop is another specification on the controller to be determined. The problem is a very basic one in many disciplines where a formal model is used for describing the system at hand. The disturbance inputs sum up the variables that are external to the model which are known to influence certain states or outputs. But the dynamics with which these variables are generated the model builder has little or no knowledge of.

The multivariable control systems literature contains many different solutions to this problem. The differences are in the language used in formulating the condition for solvability as well as in the technique of construction of a controller whenever one exists. One class of solutions uses the language of the so-called geometric approach. The condition for solvability is formulated in terms of (A, B) and (C, A) invariant subspaces which require an advanced knowledge of linear algebra. The construction of a controller in this approach is via the construction of a state-feedback matrix F and output injection matrix K which make certain subspaces $(A + BF)$-invariant and $(A + KC)$-invariant. The procedure is anything but straightforward. The second class of solutions uses the language of fractional approach or factorization approach. The condition for solvability, in its cleanest form, is formulated in terms of system matrices and as the existence of zero-cancellations among these system matrices. The construction of a solution requires an advanced knowledge of polynomial, or alternatively stable proper rational, matrix algebra. The procedure again is not at all simple. The link between the two approaches is not always obvious. It is quite usual that the specialists in one approach cannot follow the details of construction or even cannot fully grasp the limitations imposed by the solvability condition given by the other approach. In fact, a full exploration of the link between the two approaches is a speciality of another area of research known as the theory of polynomial models.

The picture drawn above may indeed look complicated to a researcher outside the area of control theory as well as to a designer of control systems. Although the rather heavy specialization in certain techniques is not a weakness of a theory, the fact that each particular approach demands a sophisticated mathematical background even at the descriptive level of stating the condition for solvability does shy away the potential appliers of the theory.

Having identified the source of difficulty as such, what needs to be done is clear. The results obtained by the theory must be presented in as simple a manner as possible, eliminating the need for a sophisticated mathematical background. This thesis attempts to present some of the known solutions to a number of standard problems of multivariable control systems with this objective in mind. The proofs given for Theorems (8.2.3) and (8.2.4) of the solution to disturbance decoupling problem without and with stability and the proof given for Theorem (9.1.1) of the output stabilization problem, a prototype problem of regulation, use simple linear algebra and a minimal amount of polynomial algebra. The link between the two solvability conditions, one coming from geometric approach and the other from fractional approach, are made explicit without recource to the theory of polynomial models. As they stand, the results and the construction of the controllers in these theorems should be easy to follow for anyone with a basic background in linear systems. This necessary background is given in Chapters 2-5 with as little technical demand from the reader as possible. Other standard problems of linear multivariable control are also presented. These include disturbance decoupling problem with measurement feedback as posed above, regulation and tracking problems, and decentralized stabilization problem. These incorporated, the thesis is a first draft for a textbook on linear multivariable control.

The thesis is organized as follows; Chapter 2 is devoted to some basic concepts of linear time invariant systems. Stability, reachability, equivalence of linear time invariant systems are presented. In Chapter 3, state feedback is introduced and the procedure for eigenvalue assignment is given. The concept of stabilizability also mentioned. Chapter 4 includes the concept of observability and synthesis of dynamic asymptotic observers and functional observers.

In Chapter 5, Kalman Canonical Decomposition theorem is given and the separation principle for feedback controllers is establihed. In Chapter 6, stable proper factorizations are examined. Chapter 7 contains the parametrization of all controllers in terms of a free parameter. In Chapter 8, the problem of cancelling the effect of disturbances using state feedback and output measurement feedback is examined. Two main approaches to this problem namely the geometric approach and the transfer matrix approach are reviewed and the solution techniques of these two approaches are illustrated. Output stabilization, tracking and regulation problem in the scalar system and regulator problem with a single output channel is considered in Chapter 9. In Chapter 10, we show how the decentralized stabilization problem can be transformed into a "make-coprime" problem. Finally concluding remarks are given.

Chapter 2

LINEAR TIME INVARIANT SYSTEMS: SOME BASIC CONCEPTS

In this chapter, we shall introduce the state variable description of linear time invariant systems. Then, the definitions of equilibrium state, asymptotic and exponential stability are given. Finally, the concept of reachability, equivalent dynamical representations and how we can separate the reachable part are presented.

A linear time invariant (LTI) system is defined by a pair of equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t),\ t \geq 0, \end{aligned} \tag{2.1}$$

where A, B, C, D are constant, real matrices of sizes $n \times n$, $n \times m$, $p \times n$, $p \times m$, respectively. For every $t \geq 0$, $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$, and $y(t) \in \mathbf{R}^p$. The components $u_i(t)$, $i = 1, ..., m$ of $u(t)$ are assumed to be piecewise continuous functions on the interval $[0, \infty)$. The vectors $x(t)$, $y(t)$, and $u(t)$ are called the state, output, and input of (2.1), respectively. Occasionally, the notation $\Sigma = (A, B, C, D)$ will be used to denote the LTI system (2.1) with all the associated restrictions.

2.1 State and Output Trajectories

Given an initial time $t_0 \geq 0$ and an initial state $x(t_0) =: x_0$, let $\varphi(t; t_0, x_0, u(.))$ be defined by

$$\varphi(t; t_0, x_0, u(.)) := e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau, \tag{2.2}$$

where

$$e^{At} = \sum_{i=0}^{\infty} \frac{(At)^i}{i!}$$

is the exponential matrix function of At. Clearly, $\varphi(t_0; t_0, x_0, u(.)) = x_0$. Moreover, using the differentiation and transition properties of the exponential matrix function, it is easy to verify that

$$\dot{\varphi}(t; t_0, x_0, u(.)) = A\varphi(t; t_0, x_0, u(.)) + Bu(t).$$

Thus, $\varphi(t; t_0, x_0, u(.))$ is a solution of (2.1) for the initial time $t_0 > 0$ and the initial state $x(t_0) = x_0$. Note that this solution is a continuous function of time t and, by the theory of ordinary differential equations, it is unique. The solution $\varphi(t; t_0, x_0, u(.))$ for $t \geq t_0$ is called the *state trajectory* of (2.1). The output expression for the initial time $t_0 \geq 0$ and the initial state $x(t_0) = x_0$ is obtained by substituting $x(t) = \varphi(t; t_0, x_0, u(.))$ into the output equation as

$$y(t) = Ce^{A(t-t_0)}x_0 + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du, \; t \geq t_0. \tag{2.3}$$

The more explicit notation $\eta(t, \varphi(t; t_o, x_o, u(.)), u(.))$ is used to denote the value $y(t)$ of the output at $t \geq t_0$, resulting from application of the input $u(.)$ in $[t_0, t]$ starting with the initial state $x_0 = x(t_0)$. The set of points $\eta(t, \varphi(t; t_o, x_o, u(.)), u(.))$, $t \geq t_0$ in $\mathbf{R}^p$ is called the *output trajectory* of the LTI system.

Alternatively using Laplace transform, we can also show that (2.2) is a solution of (2.1). Let $X(s)$, $U(s)$ be the Laplace transforms of $x(t)$ and $u(t)$ respectively. By setting $t_0 = 0$ and taking the Laplace transform of (2.1), we have

$$sX(s) - x(0) = AX(s) + BU(s),$$

so that,

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s).$$

Taking inverse Laplace transform of both sides of this equality,

$$\begin{aligned} x(t) &= L^{-1}\{(sI - A)^{-1}x_0\} + L^{-1}\{(sI - A)^{-1}BU(s)\}, \\ x(t) &= e^{At}x_0 + (e^{At}B * u(t)), \end{aligned}$$

where " $*$ " denotes time convolution. Hence,

$$\begin{aligned} x(t) &= e^{At}x_0 + \textstyle\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\ &= \varphi(t; 0, x_0, u(.)). \end{aligned}$$

This shows that $\varphi(t_0, x_0, u(.))$ is a solution of (2.1) for the initial time $t_0 = 0$ and the initial state x_0. The general solution for $t_0 > 0$ can be obtained using the following time-invariance property of any solution of (2.1)

$$\varphi(t; t_0, x_0, u(\tau - t_0)) = \varphi(t - t_0; 0, x_0, u(\tau)), \tag{2.4}$$

which is a consequence of the fact that both A and B in (2.1) are constant matrices.

Note that any solution (2.1) can be written as

$$\varphi(t; t_0, x_0, u(\tau)) = x_{zi}(t) + x_{zs}(t),$$

where

$$x_{zi}(t) := \varphi(t; t_0, x_0, 0) = e^{A(t-t_0)}x_0, \;\; x_{zs}(t) := \varphi(t; t_0, 0, u(\tau)) = \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau.$$

The term $x_{zi}(t)$ is called the *zero-input solution* and $x_{zs}(t)$ is called the *zero-state solution* of (2.1). More generally, the state trajectory has the following linearity property: For all $\alpha, \beta \in \mathbf{R}$, for all states $x, z \in \mathbf{R}^n$, and all inputs $u(.), v(.)$,

$$\varphi(t; t_0, \alpha x + \beta z, \alpha u(.) + \beta v(.)) = \alpha\varphi(t; t_0, x, u(.)) + \beta\varphi(t; t_0, z, v(.)). \quad (2.5)$$

2.2 Stability of LTI Systems

Consider the unforced system

$$\dot{x}(t) = Ax(t), \quad (2.6)$$

where $A \in \mathbf{R}^{n\times n}$ and $x(t) \in \mathbf{R}^n$ for each $t \geq 0$. This is a special case of (2.1) in which the input is set to zero so that

$$\varphi(t; t_0, x_0, 0) = e^{A(t-t_0)}x_0, \;\; t \geq t_0 \quad (2.7)$$

is the unique solution of (2.6) for the initial state $x(t_0) = x_0$.

The point $0 \in \mathbf{R}^n$ is an equilibrium point of (2.6) since any state trajectory starting at 0 at $t = t_0$ stays at 0 for all $t \geq t_0$.

Definition 2.2.1. *An equilibrium point* $\bar{x} \in \mathbf{R}^n$ *of (2.6) is called a* **stable equilibrium point** *if for all* $\epsilon > 0$, $t_0 \geq 0$, *there exists* δ *possibly depending on* ϵ *and* t_0 *such that for all* $x_0 \in \mathbf{R}^n$ *and all* $t \geq t_0$ *the implication*

$$\|x_0 - \bar{x}\| < \delta \Rightarrow \|\varphi(t; t_0, x_0, 0) - \bar{x}\| < \epsilon$$

holds.

In other words, $\bar{x}$ is a stable equlibrium point if small perturbations on the initial state $\bar{x}$ results in small perturbations on the trajectory. It is not difficult to show that the equilibrium point 0 of (2.6) is stable if and only if there exists $M > 0$ such that

$$\|\varphi(t; t_0, x_0, 0)\| \leq M\|x_0\|, \ \forall t \geq t_0. \tag{2.8}$$

Definition 2.2.2. *If the trajectory "approaches" the equilibrium as time progresses, then the equilibrium point is called asymptotically stable. More formally,* $\bar{x}$ *is an* **asymptotically stable equilibrium point** *if it is stable and for all* $t_0 \geq 0$, *there exists* δ *possibly depending on* t_0 *such that*

$$\|x_0 - \bar{x}\| < \delta \Rightarrow \lim_{t\to\infty} \|\varphi(t; t_0, x_0, 0) - \bar{x}\| = 0.$$

A third concept of stability relevant to (2.6) is *exponential stability.*

Definition 2.2.3. *The system (2.6) is called* **exponentially stable** *if for all* $t_0 \geq 0$ *and all* $x_0 \in \mathbf{R}^n$, *there exist* $M > 0$ *and* $\gamma > 0$ *such that*

$$\|\varphi(t; t_0, x_0, 0)\| \leq M\|x_0\|e^{-\gamma(t-t_0)} \ \forall t \geq t_0. \tag{2.9}$$

The constant γ *as above, if it exists, is called the decay rate.*

By the particular form of the solution (2.7) of (2.6), it turns out that asymptotic and exponential stability are equivalent requirements for (2.6).

Let $\mathbf{C}$ denote the set of complex numbers. By $\mathbf{C}_{-}$, $\mathbf{C}_{0}$, and $\mathbf{C}_{+}$, we denote the points in the open left half complex plane, imaginary axis, and the open right half complex plane, respectively. The points in the closed left and right half complex plane are denoted respectively by $\mathbf{C}_{0-}$ and $\mathbf{C}_{0+}$. Let $\mathbf{C}_{+e}$ denote $\mathbf{C}_{0+}$ together with the point at infinity.

Fact 2.2.1. *(i) The equilibrium point 0 of (2.6) is asymptotically stable, equivalently, the system (2.6) is exponentially stable, if and only if all eigenvalues of A have negative real parts, i.e., $\sigma(A) \subseteq \mathbf{C}_{-}$. (ii) The equilibrium point 0 of (2.6) is stable if and only if $\sigma(A) \subseteq \mathbf{C}_{0-}$ and an eigenvalue of A with zero real part has multiplicity at most one as a root of the minimal polynomial of A.*

Proof. Let $\lambda_1, ..., \lambda_r$ be the eigenvalues of A with multiplicities $m_1, ..., m_r$ as roots of the minimal polynomial of A. Then,

$$e^{At} = \sum_{i=1}^{r} \sum_{j=1}^{m_i} t^{j-1} e^{\lambda_i t} P_{ij}(A), \tag{2.10}$$

where $Pij(A)$ is the ij-th interpolating polynomial of A. The solution $\varphi(t; t_0, x_0, 0)$ is given by

$$e^{A(t-t_0)} x_0 = \sum_{i=1}^{r} \sum_{j=1}^{m_i} (t - t_0)^{j-1} e^{\lambda_i (t-t_0)} P_{ij}(A) x_0. \tag{2.11}$$

(i) If $Re(\lambda_i) < 0$ for all $i = 1, ..., r$, then by (2.11),

$$\begin{aligned}
\|e^{A(t-t_0} x_0\| &= \| \textstyle\sum_{i=1}^{r} \sum_{j=1}^{m_i} (t - t_0)^{j-1} e^{\lambda_i (t-t_0)} P_{ij}(A) x_0 \| \\
&\leq \textstyle\sum_{i=1}^{r} \sum_{j=1}^{m_i} (t - t_0)^{j-1} e^{\lambda_i (t-t_0)} \|P_{ij}(A) x_0\| \\
&\leq \textstyle\sum_{i=1}^{r} \sum_{j=1}^{m_i} (t - t_0)^{j-1} e^{\lambda_i (t-t_0)} \|P_{ij}(A)\| \, \|x_0\| \\
&\leq M e^{-\gamma (t-t_0)} \|x_0\|
\end{aligned}$$

for some sufficiently large $M > 0$ and $\gamma := \max_i\{-Re(\lambda_i)\}$. It follows that the equilibrium point 0 is asymptotically and the system is exponentially stable. Conversely, if some eigenvalue λ_i is such that $Re(\lambda_i) \geq 0$, then let x_0 in (2.11) be an eigenvector corresponding to λ_i so that (2.11) gives

$$\|e^{A(t-t_0)}x_0\| = \|e^{\lambda_i(t-t_0)}x_0\|.$$

Hence,

$$\lim_{t\to\infty} \|e^{A(t-t_0)}x_0\| \neq 0 \tag{2.12}$$

and one has neither asymptotic nor exponential stability.

(ii) If all eigenvalues have nonpositive real parts and those with zero real parts, say $\lambda_{i_1}, ..., \lambda_{i_k}$ are such that $m_{i_1} = ... = m_{i_k} = 1$, then in (2.11) the terms containing $e^{\lambda_{i_j}}$ $j = 1, ..., k$ have coefficients independent of $t - t_0$. It follows that for any $x_0 \in \mathbf{R}^n$

$$\|e^{A(t-t_0}x_0\| \leq M\|x_0\|$$

for some sufficiently large $M > 0$, i.e., (2.6) is stable. Conversely, if $m_{i_j} > 1$ for some j, then the term containing $e^{\lambda_{i_j}}$ in (2.11) has a nonconstant polynomial coefficient in $t - t_0$. Hence, also in this situation we get (2.12) and stability is not possible. $\square$

We will call the forced system (2.1) (asymptotically) stable if the equilibrium point 0 of the corresponding unforced system (2.6) is (asymptotically) stable.

Note by Fact(2.2.1) and its proof that *if the system (2.1) is asymptotically stable, then it is exponentially stable with decay rate* $\gamma = \max_i\{-Re(\lambda_i)\}$.

2.3 Reachability of LTI Systems

Given a LTI system (2.1) and two points in $\mathbf{R}^n$, when does there exist a suitable input such that the resulting state trajectory passes through the two given points? The concept of reachability, studied in this section, is essential for answering this question.

Definition 2.3.1. *A state $x \in \mathbf{R}^n$ is* **reachable at time t from x_0** *if there exist $t_0 \geq 0$ with $t \geq t_0$ and $u(\tau)$ with $t \geq \tau \geq t_0$ such that $\varphi(t; t_0, x_0, u(.)) = x$. A state $x_0 \in \mathbf{R}^n$ is* **controllable at time t_0 to x** *if there exist $t \geq t_0$ and $u(\tau)$ with $t \geq \tau \geq t_0$, such that $\varphi(t; t_0, x_0, u(.)) = x$.*

By time-invariance property (2.4) of (2.1), it is easy to see that x is reachable at time t from x_0 if and only if it is reachable at time $t - t_0$ from x_0. Similarly, x_0 is controllable at time t_0 to x if and only if it is controllable at time 0 to x. It follows that in studying the sets of reachable and controllable states of (2.1), there is no loss of generality in considering reachability and controllability at time 0. Moreover, by linearity property (2.5) of (2.1) and by invertibility of the exponential matrix function, x is reachable at time t from x_0 if and only if $x - e^{A(t-t_0)}x_0$ is reachable at time t from the state 0. Similarly, x_0 is controllable at time t_0 to x if and only if $x_0 - e^{-A(t-t_0)}x$ is controllable at time t_0 to state 0. It follows that, one can focus on reachable states from the origin and controllable states to the origin. Finally, note that any state is reachable from the zero state if and only if any state can be controlled to any other state as a consequence of linearity. By the invertibility of the exponential matrix function, it also follows that, any state can be controlled to the zero state if and only if any state can be reached from any other state.

These considerations allow us to concentrate on the set of reachable states from the zero state at time zero. i.e., the set

$$\mathcal{R}_0 := \{x \in \mathbf{R}^n : x \text{ is reachable from the zero state}\}.$$

This is a linear subspace of $\mathbf{R}^n$ since if $\varphi(t_1; 0, 0, u(.)) = x$ and $\varphi(t_2; 0, 0, v(.)) = z$, then by the linearity property (2.5), we have $\varphi(t; 0, 0, \alpha\bar{u}(.) + \beta\bar{v}(.)) = \alpha x + \beta z$, where $t := max\{t_1, t_2\}$, $\bar{u}(.)$ is equal to $u(.)$ in the interval $[0, t_1]$ and zero otherwise, and $\bar{v}(.)$ is equal to $v(.)$ in the interval $[0, t_2]$ and zero otherwise. We now give an explicit expression for the control input which drives the state trajectory to a given state starting at the origin. It will be seen that controls which achieve the task in an arbitrarily small time exist (provided there are no bounds on the control input). We first prove the following fact. Let us denote

$$< A \mid ImB >:= ImB + ImAB + ... + ImA^{n-1}B = Im[B\ AB.....\ A^{n-1}B].$$

Lemma 2.3.1. *Let*

$$W_t := \int_0^t e^{A\tau}BB'e^{A'\tau}d\tau,\ t > 0,$$

where A' denotes the transpose of A. Then,

$$ImW_t =< A \mid ImB >$$

for all positive t.

Proof. We show that $< A \mid ImB >^{\perp} = (ImW_t)^{\perp}$ where for a subspace $\mathcal{R} \subset \mathbf{R}^n$, $\mathcal{R}^{\perp}$ denotes the orthogonal complement of $\mathcal{R}$ in $\mathbf{R}^n$. First, suppose that $x \in < A \mid ImB >^{\perp}$, then

$$x'B = 0,\ \ x'AB = 0,\ \ \cdots\ x'A^{n-1}B = 0.$$

By Cayley Hamilton Theorem, $x'A^kB = 0$, $\forall k \geq 0$. Thus

$$x'e^{At}B = \sum_{k=0}^{\infty} \frac{x'A^k t^k B}{k!} = 0.$$

It follows that

$$x'W_t = \int_0^t x'e^{A\tau}BB'e^{A'\tau}d\tau = 0, \ \forall t > 0.$$

Hence $x \in (ImW_t)^{\perp}$. Now suppose that $x \in (ImW_t)^{\perp}$. Then $x'W_t x = 0$ so that

$$\int_0^t \| B'e^{A'\tau}x \|^2 \, d\tau = 0.$$

Therefore, $B'e^{A'\tau}x = 0$, $\forall \tau \in (0,t)$. Now repeated differentiation at $\tau = 0$ yields $B'(A')^k x = 0$ for $k = 0, 1, ..., n-1$ which implies $x \in < A \mid ImB >^{\perp}$. $\square$

Theorem 2.3.1. *The set of reachable states is given by*

$$\mathcal{R}_0 = < A \mid ImB > .$$

If $x \in < A|ImB >$, then there exists a z_x such that $x = W_t z_x$ and x is reachable from zero by the application of the input

$$u(\tau) := B'e^{A'(t-\tau)}z_x, \ \tau \in [0,t] \tag{2.13}$$

for any $t > 0$.

Proof. Let $x \in \mathcal{R}_0$ so that for some $t \geq 0$ and some $u(\tau)$, $\tau \in [0,t]$, we have $x = \varphi(t; 0, 0, u(\tau))$, i.e.,

$$x = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \int_0^t \sum_{k=0}^{\infty} \frac{A^k(t-\tau)^k}{k!}Bu(\tau)d\tau = \sum_{k=0}^{\infty} A^k B \int_0^t \frac{(t-\tau)^k}{k!}u(\tau)d\tau.$$

By Cayley-Hamilton theorem, $x \in ImB + ImAB + ... + ImA^{n-1}B$. It follows that $< A \mid ImB >$ contains $\mathcal{R}_0$. Conversely, suppose that $x \in ImB + ImAB + ... + ImA^{n-1}B$. Hence,

$$x = \int_0^t e^{A\tau}BB'e^{A'\tau}z_x d\tau = \int_0^t e^{A(t-\tau)}BB'e^{A'(t-\tau)}z_x d\tau = \varphi(t; 0, 0, B'e^{A'(t-\tau)}z_x).$$

Therefore, x is reachable from zero by the input $u(\tau) := B'e^{A'(t-\tau)}z_x$ for $\tau \in [0, t]$ for arbitrary $t > 0$. $\square$

We call the system (2.1) *completely reachable*, or simply *reachable* if $\mathcal{R}_0 = \mathbf{R}^n$, which is the case if and only if $< A \mid ImB >= \mathbf{R}^n$, by Theorem (2.3.1). If the system (2.1) is reachable, then any $x \in \mathbf{R}^n$ is reachable from zero by the input

$$u(\tau) := B'e^{A'(t-\tau)}W_t^{-1}x, \ \tau \in [0, t] \tag{2.14}$$

for any $t > 0$.

Note that smaller is the time during which a state is reached, larger is the magnitude of some entries in the control function due to the appearance of W_t^{-1} in (2.14). It might be wondered if there is some other bounded control transferring the zero state to a given nonzero state in arbitrarily small time. However, this is not in general possible. Using elementary methods of variational calculus, the control function (2.13) can be shown to be the minimizing function for the energy functional

$$\int_0^\infty u(t)'u(t)dt.$$

Thus, fast control requires large control energy and *vice versa*.

Alternatively, we can state the following rank condition for reachability.

Corollary 2.3.1. *The system (2.1) is (completely) reachable if and only if*

$$rank \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = n.$$

Since complete reachability is a property determined by the matrices A and B only, the phrase "(A, B) is reachable" is also used to refer to reachability of (2.1).

2.4 Transformation of Linear Systems

Many system properties remain unchanged under coordinate transformations in the states, inputs, and outputs.

Definition 2.4.1. *Two LTI systems* $\Sigma_1 := (A_1, B_1, C_1, D_1)$ *and* $\Sigma_2 := (A_2, B_2, C_2, D_2)$ *are called* **equivalent** *if there exist nonsingular matrices* $T \in \mathbf{R}^{n\times n}, R \in \mathbf{R}^{m\times m}, S \in \mathbf{R}^{p\times p}$ *such that*

$$A_2 = TA_1T^{-1}, B_2 = TB_1R^{-1}, C_2 = SC_1T^{-1}, D_2 = SD_1R^{-1}.$$

The systems Σ_1 and Σ_2 are thus equivalent if one can be obtained from the other by nonsingular coordinate transformations in the state space $\mathbf{R}^n$, the input space $\mathbf{R}^m$, and the output space $\mathbf{R}^p$. Interpreting the matrices as maps, we have the commutative diagram of Figure 2.1 for equivalence.

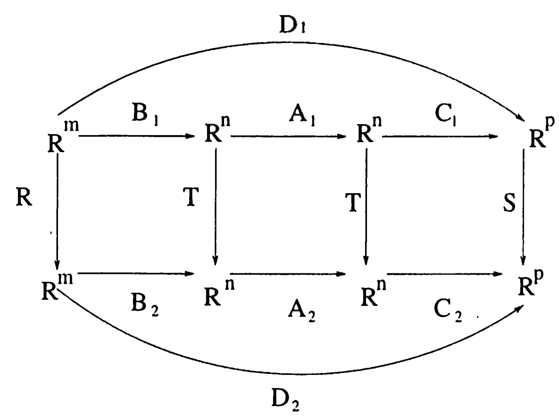

Figure 2.1: Equivalence of linear systems

Since A_1 and A_2 are related by a similarity transformation, the eigenvalues and their multiplicities remain unchanged under system equivalence. Reachability is also preserved under equivalence as expected. *Given the equivalent systems* Σ_1 *and* Σ_2, Σ_1 *is completely reachable if and only if* Σ_2 *is completely* *reachable.* To see this, note that

$$\left[\begin{array}{cccc} B_2 & A_2B_2 & \dots & A_2^{n-1}B_2 \end{array}\right] diag\left\{R, R, ..., R\right\} = T\left[\begin{array}{cccc} B_1 & A_1B_1 & \dots & A_1^{n-1}B_1 \end{array}\right].$$

Since R and T are nonsingular, the result follows by Corollary (2.3.1).

2.5 Separation of the Reachable Part

If a LTI system is not reachable, it is possible to identify a maximal "part" of the system which is reachable. The fact that $\mathcal{R}_0$ is an A-invariant subspace of $\mathbf{R}^n$ allows one to do this.

Definition 2.5.1. *Given any $A \in \mathbf{R}^{n\times n}$, a subspace $\mathcal{S} \subseteq \mathbf{R}^n$ is said to be* A-**invariant** *if $Ax \in \mathcal{S}$ for all $x \in \mathcal{S}$.*

Example 2.5.1. The subspaces $\{0\}$ and $\mathbf{R}^n$ are clearly A-invariant for any matrix A. If A has all its eigenvalues distinct and real, then the span of any collection of the corresponding eigenvectors is an A-invariant subspace of $\mathbf{R}^n$. Given $A \in \mathbf{R}^n$, there is a one-dimensional A-invariant subspace of $\mathbf{R}^n$ if and only if A has a real eigenvalue.

Fact 2.5.1. *The reachable subspace $\mathcal{R}_0$ is the smallest A-invariant subspace containing ImB of $\mathbf{R}^n$.*

Proof. Let $x \in \mathcal{R}_0$ so that $x = x_0 + x_1 + ... + x_{n-1}$, where $x_i \in A^iImB$ for $i = 0, 1, ..n-1$. Now $Ax_i \in (A^{i+1}ImB) \subseteq \mathcal{R}_0$ for $i = 0, 1, ..n-2$ and $Ax_{n-1} \in (A^nImB) \subseteq \mathcal{R}_0$, where the last inclusion is by Cayley-Hamilton theorem. Hence $\mathcal{R}_0$ is a A-invariant subspace containing ImB. Any A-invariant subspace containing ImB in $\mathbf{R}^n$ should contain A^iImB for $i = 0, 1, ..n-1$. Therefore, $\mathcal{R}_0$ is the smallest A-invariant subspace containing ImB. $\square$

Let the columns of a matrix R_0 be a basis for $\mathcal{R}_0$ and let a matrix R_1 be such that $T := [R_0\, R_1]$ is nonsingular. Since $\mathcal{R}_0$ is A-invariant containing ImB, there exist matrices A_1, A_2, A_3 and B_1 such that

$$A[R_0\ R_1] = [R_0\ R_1]\begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix},\ B = [R_0\, R_1]\begin{bmatrix} B_1 \\ 0 \end{bmatrix}.$$

We thus have

$$T^{-1}AT = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix},\ T^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}. \tag{2.15}$$

Note that

$$\begin{aligned} dim\mathbf{R}_0 &= rank\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = rank\left(T^{-1}\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}\right) \\ &= rank\begin{bmatrix} B_1 & A_1B_1 & \dots & A_1^{n-1}B_1 \\ 0 & 0 & & 0 \end{bmatrix} = rank\begin{bmatrix} B_1 & A_1B_1 & \dots & A_1^{n-1}B_1 \end{bmatrix}. \end{aligned}$$

By Cayley-Hamilton Theorem, this is equal to $rank\begin{bmatrix} B_1 & A_1B_1 & \dots & A_1^{n_1-1}B_1 \end{bmatrix}$, where $n_1 := sizeA_1$. We have thus shown: *If $dim\,\mathcal{R}_0 = n_1 < n$, then there exists a nonsingular matrix T such that (2.15) holds for some matrices A_i, $i = 1,2,3$ and B_1 such that $size\,A_1 = n_1$ and (A_1, B_1) is a reachable pair.* The particular form (2.15) attained by system equivalence is called the *reachable normal form.*

An alternative criterion for reachability is as follows.

Corollary 2.5.1. *The system (2.1) is (completely) reachable if and only if*

$$rank\begin{bmatrix} sI - A & B \end{bmatrix} = n,\ \forall s \in \mathbf{C}.$$

Proof. **[Only If]** Suppose $rank\begin{bmatrix} sI - A & B \end{bmatrix} \neq n$, for some $s \in \mathbf{C}$, then there exists a nonzero $q \in \mathbf{C}^n$ such that $q'(sI - A) = 0$, $q'B = 0$. This gives

$$q'\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = 0.$$

Since $q \neq 0$, there exists a nonzero vector in the left null space of the reachability matrix. By Corollary (2.3.1), (A, B) is not reachable.

[If] If (A, B) is not reachable, there exists a nonsingular $T \in \mathbf{R}^{n\times n}$ such that (2.15) holds. Now, for $s \in \sigma(A_3)$, we have $det(sI - A_3) = 0$ and hence

$$rank \begin{bmatrix} sI - A_1 & -A_2 & B_1 \\ 0 & sI - A_3 & 0 \end{bmatrix} < n \text{ then, } rank\, T^{-1} \begin{bmatrix} (sI - A)T & B \end{bmatrix} < n \quad (2.16)$$

which implies that $rank \begin{bmatrix} sI - A & B \end{bmatrix} < n$. □

We close this chapter by the following definition, the terminology being explained in Section (3.3).

Definition 2.5.2. *The pair (A, B) is called* **stabilizable** *if either (A, B) is reachable or $\sigma(A_3) \subset \mathbf{C}_-$, where A_3 is defined by the reachable normal form (2.15).*

Corollary 2.5.2. *(A, B) is stabilizable if and only if*

$$rank \begin{bmatrix} sI - A & B \end{bmatrix} = n, \ \forall s \in \mathbf{C}_{0+}. \quad (2.17)$$

Proof. **[If]** If (A, B) is not stabilizable, there exists $T \in \mathbf{R}^{n\times n}$ as in (2.15), where (A_1, B_1) reachable and $\sigma(A_3) \not\subset \mathbf{C}_-$. Hence there exists $s \in \mathbf{C}_{0+} \cap \sigma(A_3)$ such that $det(sI - A_3) = 0$ so that (2.16) holds and $rank \begin{bmatrix} sI - A & B \end{bmatrix} < n$ for this $s \in \mathbf{C}_{0+} \cap \sigma(A_3)$.

[Only If] If $rank \begin{bmatrix} sI - A & B \end{bmatrix} < n$ for some $s \in \mathbf{C}_{0+}$, then

$$rank \begin{bmatrix} sI - A_1 & -A_2 & B_1 \\ 0 & sI - A_3 & 0 \end{bmatrix} < n = rank\, T^{-1} \begin{bmatrix} (sI - A)T & B \end{bmatrix} < n.$$

Let $0 \neq q \in \mathbf{C}^n$ be such that $q' = \begin{bmatrix} q_1' & q_2' \end{bmatrix}$ and

$$\begin{bmatrix} q_1' & q_2' \end{bmatrix} \begin{bmatrix} sI - A_1 & -A_2 & B_1 \\ 0 & sI - A_3 & 0 \end{bmatrix} = 0. \tag{2.18}$$

By (2.18), $q_1'(sI - A_1) = 0$, $q_1' B_1 = 0$ so that $q_1 = 0$ by reachability of (A_1, B_1). Hence, again by (2.18), $q_2'(sI - A_3) = 0$ for $q_2 \neq 0$. Therefore, $s \in \sigma(A_3) \cap \mathbf{C}_{0+}$. □

2.6 Notes and References

The definition of state is due to Belman *et al.* [1], [2]. The theorem for the existence and uniqueness of the solution to (2.1) can be found in [3]. The invariance property (2.4) of (2.1) is explained in detail in Callier and Desoer [4]. The books [5], [6], [7] can be consulted for further background on stability of systems. The concept of reachability and controllability is introduced by Kalman [8] in the context of optimal control. The idea of the separation of the reachable part is due to Kalman [9], [10]. The criterian for reachability in Corollary (2.5.1) is known as Hautus Belevich Popov (HBP) test because original sources include [11], [12] and [13]. More fundamental facts of linear algebra used such as Cayley Hamilton theorem and their proofs can be found in [14].

Chapter 3

STATE FEEDBACK

A primary objective of control theory is the relocation of the system eigenvalues in order to achieve desired characteristics such as stability, satisfactory transient response. We assume in this chapter that all state variables are available for control purposes and show that if system is completely reachable, then any desired characteristic polynomial can be obtained by state feedback.

Consider the state equation in (2.1),

$$\dot{x}(t) = Ax(t) + Bu(t),\ t \geq 0. \tag{3.1}$$

If the input is a linear constant function of the states, then we can write

$$u(t) = Fx(t),\ F \in \mathbf{R}^{m\times n}. \tag{3.2}$$

The two equations (3.1) and (3.2) lead to a closed-loop unforced system

$$\dot{x}(t) = (A + BF)x(t) \tag{3.3}$$

driven only by the initial state $x_0 = x(0)$ as shown in Figure 3.1.

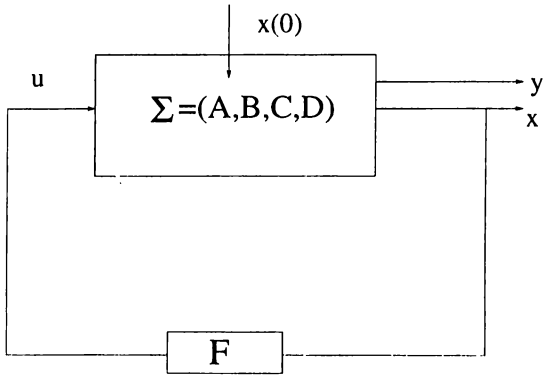

Figure 3.1: The system (2.1) with state feedback

3.1 Reachability and Feedback

In this section, it will be shown that exponential stability of the closed loop system with arbitrarily large decay rate can be achieved by state feedback if and only if (3.1) is reachable.

We first note the following properties of the induced matrix norm.

Fact 3.1.1. *For every* $\lambda \in \sigma(A)$, $\| A \| := sup_{x \neq 0} \frac{\| Ax \|}{\| x \|} \geq | \lambda |$.

Proof. Let λ be an eigenvalue of A and x_1 be a corresponding eigenvector. Then, $\| Ax_1 \| = | \lambda | \| x_1 \|$ so that $\frac{\| Ax_1 \|}{\| x_1 \|} = | \lambda |$. It follows that

$$sup_{x \neq 0} \frac{\| Ax \|}{\| x \|} \geq | \lambda | .$$

□

Fact 3.1.2. *For every* $\lambda \in \sigma(A)$,

$$\| e^{At} \| \geq | e^{\lambda t} | = e^{Re\{\lambda\} t}.$$

Proof. Let x_1 be a corresponding eigenvector of $\lambda \in \sigma(A)$. By the series expansion of the exponential matrix function

$$\begin{aligned} e^{At}x_1 &= x_1 + Ax_1t + \ldots.. + \frac{A^n x_1 t^n}{n!} + \ldots \\ &= x_1 + \lambda x_1 t + \ldots.. + \frac{\lambda^n x_1 t^n}{n!} + \ldots \end{aligned} \tag{3.4}$$

By (3.4), $e^{\lambda t}$ is an eigenvalue of e^{At}. The result follows using Fact (3.1.1). □

Theorem 3.1.1. *The pair (A, B) is reachable if and only if $\forall\gamma > 0$, $\exists F_\gamma \in \mathbf{R}^{m\times n}$ and $M_\gamma > 0$ such that*

$$\| e^{(A+BF_\gamma)t} \| \leq M_\gamma e^{-\gamma t}, \ \forall t \geq 0. \tag{3.5}$$

Proof. **[If]** Suppose (A, B) is not reachable. Let $T \in \mathbf{R}^{n\times n}$ be nonsingular putting (A, B) into reachable normal form as in (2.15). For any $F \in \mathbf{R}^{m\times n}$,

$$T^{-1}(A+BF)T = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \begin{bmatrix} F_1 & F_2 \end{bmatrix},$$

where $FT = \begin{bmatrix} F_1 & F_2 \end{bmatrix}$. Then

$$T^{-1}(A+BF)T = \begin{bmatrix} A_1 + B_1F_1 & A_2 + B_2F_2 \\ 0 & A_3 \end{bmatrix}.$$

Let

$$\hat{A} = \begin{bmatrix} A_1 + B_1F_1 & A_2 + B_2F_2 \\ 0 & A_3 \end{bmatrix}.$$

By Fact (3.1.2), $\| e^{(A+BF)t} \| \geq e^{Re\{\lambda_3\}t}$, where $\lambda_3 \in \sigma(A_3)$. Suppose now that $\forall\gamma > 0$, there exist F_γ and $M_\gamma > 0$ such that (3.5) holds. Choosing $\gamma > -Re\{\lambda_3\}$, we have

$$M_\gamma e^{-\gamma t} \geq \| e^{(A+BF_\gamma)t} \| \geq e^{Re\{\lambda_3\}t}.$$

It follows that

$$M_\gamma \geq e^{(Re\{\lambda_3\}+\gamma)t}. \tag{3.6}$$

Since $Re\{\lambda_3\} + \gamma > 0$, (3.6) fails for large t. Hence, if (A, B) is not reachable, then (3.5) is not satisfied for some $\gamma > 0$.

[Only if] Let

$$\bar{W}_\varepsilon := \int_0^\varepsilon e^{-A\tau} BB' e^{-A'\tau} d\tau = e^{-A\varepsilon} W_\varepsilon e^{-A'\varepsilon},$$

where W_ε is as defined in Lemma (2.3.1) and $F := -B'\bar{W}_\varepsilon^{-1}$. Since (A, B) is reachable, by Lemma (2.3.1), $\bar{W}_\varepsilon^{-1}$ exists. Consider the candidate Lyapunov function $V(x) := x'\bar{W}_\varepsilon x$ for $\dot{x} = (A + BF)'x$. We have

$$\begin{aligned} \dot{V}(x) &= (\tfrac{dV}{dx})^T \tfrac{dx}{dt} = 2x'\bar{W}_\varepsilon(A + BF)'x \\ &= 2\int_0^\varepsilon x'e^{-A\tau}BB'e^{-A'\tau}A'x d\tau + 2x'\bar{W}_\varepsilon F'B'x. \end{aligned}$$

Hence,

$$\dot{V}(x) = -\int_0^\varepsilon \frac{d}{d\tau} \| B'e^{-A'\tau}x \| \, d\tau - 2x'BB'x. \tag{3.7}$$

By (3.7), $\dot{V}(x) \leq 0$. Using Lyapunov Theorem, there exists $M > 0$ such that

$$\| e^{(A+BF)'t} \| = \| e^{(A+BF)t} \| \leq M, \; \forall t > 0.$$

Note that, using Corollary (2.3.1) it can easily be shown that, (A, B) is reachable if and only if $(A + \gamma I, B)$ is reachable. Hence, given any $\gamma > 0$, there exist F_γ, M_γ such that $\| e^{(A+\gamma I+BF_\gamma)t} \| \leq M_\gamma$. So, $\| e^{(A+BF_\gamma)t} \| \leq M_\gamma e^{-\gamma t}$, $\forall t > 0$. Then

$$F_\gamma = -B'W_{\varepsilon,\gamma}^{-1}, \tag{3.8}$$

where

$$W_{\varepsilon,\gamma}^{-1} := \int_0^\varepsilon e^{-(A+\gamma I)\tau} BB' e^{-(A+\gamma I)'\tau} d\tau.$$

The state feedback F_γ of (3.8) achieves the desired decay rate in the closed loop system (3.3). $\square$

The expression (3.8) for the state feedback shows that a small amplitude closed loop state and/or a large decay rate, can be achieved at the expense of allowing large magnitudes in the entries of the state feedback, i.e., at the expense of a "high-gain" state feedback.

3.2 Eigenvalue Assignment

If (A, B) is reachable, not only can one achieve an arbitrary decay rate for the closed loop system of Figure 3.1, but can also assign the spectrum of the closed loop system at any given n-points in the complex plane, the only restriction arising due to the fact that the state feedback F is a real matrix. In this section, we first show that, using state feedback, a reachable system can be made reachable from a single input, usually any of the m input components with nonzero effect on the state. The eigenvalue assignment result for single-input systems is then used to construct a state feedback achieving the desired spectrum for the original multi-input system.

Lemma 3.2.1. *Consider (3.1). If (A, B) is reachable, then for some $b = Bv \neq 0$, there exist $u_1, u_2, ...u_{n-1}$ such that the vectors*

$$\begin{aligned} x_1 &:= b = Bv \\ x_2 &:= Ax_1 + Bu_1 \\ \\ x_n &:= Ax_{n-1} + Bu_{n-1} \end{aligned} \tag{3.9}$$

are linearly independent.

Proof. The proof uses induction on n. For $n = 1$, $x_1 = b \neq 0$ and the statement is true. Suppose that $x_1, ...x_k$ are linearly independent. We show that there exists u_k such that $x_1, ...x_k, x_{k+1}$ are linearly independent for $k < n$. Suppose that such u_k does not exist. So $x_{k+1} = Ax_k + Bu_k$ is in $span\{x_1, ...x_k\}$. Let $L := span\{x_1, ...x_k\}$. Then $Ax_k + Bu_k \in L$, $\forall u_k$. By setting $u = 0$, $Ax_k \in L$. It follows that $Bu_k \in L$. Hence L is an A-invariant subspace containing ImB. Therefore $L \supseteq \mathbf{R}_o$ by Fact (2.5.1). This implies that $k = n$. □

Lemma 3.2.2. *If (A, B) is reachable and $b = Bv \neq 0$, then there exists F such that $(A + BF, b)$ is reachable.*

Proof. By Lemma (3.2.1), there exist $u_1, ..u_{n-1}$ such that $x_1, ..., x_n$ of (3.9) are linearly independent. Let F be chosen such that $Fx_k = u_k$ for $k = 1, ..n - 1$. Then

$$(A + BF)x_k = Ax_k + Bu_k = x_{k+1},\ k = 1, 2..n - 1.$$

So, $x_{k+1} = (A + BF)^k b$, $k = 1, 2..n - 1$. Therefore

$$rank \begin{bmatrix} b & (A + BF)b & \dots & (A + BF)^{n-1}b \end{bmatrix} = n.$$

Hence $(A + BF, b)$ is reachable. □

Theorem 3.2.1. *The following are equivalent.*

(i) *(A,B) is reachable.*

(ii) *For every set $\Gamma := \{\gamma_1, ...\gamma_n\} \subseteq \mathbf{C}$ which is symmetric with respect to the real axis, there exists F such that $\sigma(A + BF) = \Gamma$.*

Proof. **(ii) ⇒ (i)** Suppose that (A,B) is not reachable, and let $T \in \mathbf{R}^{n \times n}$ put (A, B) into reachable normal form (2.15). Then

$$\sigma(A + BF) = \sigma(T^{-1}(A + BF)T) = \sigma\left(\begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} + \begin{bmatrix} B_1F_1 & B_2F_2 \\ 0 & 0 \end{bmatrix}\right),$$

where $FT = \begin{bmatrix} F_1 & F_2 \end{bmatrix}$. It follows that

$$\begin{aligned}\sigma(A+BF) &= \sigma\left(\begin{bmatrix} A_1+B_1F_1 & A_2+B_2F_2 \\ 0 & A_3 \end{bmatrix}\right) \\ &= \sigma(A_1+B_1F_1)\cup\sigma(A_3). \end{aligned} \tag{3.10}$$

By (3.10), the eigenvalues of A_3 are in $\sigma(A+BF)$. So $\sigma(A+BF)$ can not be an arbitrary set.

(i) $\Rightarrow$ (ii) If (A,B) is reachable, then by Lemma (3.2.2), there exists F_1 such that $(A+BF_1, Bv)$ is reachable for some nonzero Bv. Let $\hat{A} := A+BF_1$ and $\hat{b} := Bv$. Note that there exists a transformation matrix T putting (A,b) into control canonical form, i.e., T is such that

$$T\hat{A}T^{-1} = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & 0 & \\ & & & & \\ & & & & \\ -\hat{a}_o & -\hat{a}_1 & & & -\hat{a}_{n-1} \end{bmatrix},\ T\hat{b} = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 1 \end{bmatrix}, \tag{3.11}$$

where $\hat{a}_i$'s are determined from

$$det(sI-\hat{A}) = s^n + \hat{a}_{n-1}s^{n-1} + ... + \hat{a}_1 s + \hat{a}_0.$$

Given any $\Gamma = \{\gamma_1, ..., \gamma_n\} \in \mathbf{C}$, let $(s-\gamma_1)(s-\gamma_2)...(s-\gamma_n) =: s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0$, $\hat{F}_2 := \begin{bmatrix} \hat{a}_0 - a_0 & . & . & . & \hat{a}_{n-1} - a_{n-1} \end{bmatrix}$, and $F_2 := \hat{F}_2 T$. If $F := F_1 + vF_2$, then the spectrum of the closed loop system is

$$\sigma(A+B(F_1+vF_2)) = \sigma(T(A+B(F_1+vF_2))T^{-1}) = \sigma(T\hat{A}T^{-1} + T\hat{b}\hat{F}_2).$$

By (3.11),

$$\sigma(A+BF) = \sigma\left(\begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & 0 & \\ & & & & \\ & & & & \\ -\hat{a}_o & -\hat{a}_1 & & & -\hat{a}_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \\ \\ 1 \end{bmatrix} \begin{bmatrix} \hat{a}_o - a_o & & & a_{n-1} - a_{n-1} \end{bmatrix}\right)$$

$$= \sigma\left(\begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & 0 & \\ & & & & \\ & & & & \\ -a_o & -a_1 & & & -a_{n-1} \end{bmatrix}\right) = \{\gamma_1, ..., \gamma_n\} = \Gamma. \tag{3.12}$$

□

The state feedback F constructed by the algebraic method in Theorem (3.2.1) achieves the decay rate $\max_i\{-Re\{\gamma_i\}\}$ for the closed loop system. Similar to the state feedback of Theorem (3.1.1), this feedback matrix also has entries of large magnitude if it achieves a large decay rate since $a_0 = \gamma_1 ... \gamma_n$ appears in its expression.

3.3 Stabilizability

If (A, B) is not reachable, then neither eigenvalue assignment nor exponential stability with arbitrary decay rate is possible in the closed loop system. We show in this section that it is still possible to achieve exponential stability with decay rate being determined by the eigenvalues of the "unreachable part" of (A, B) provided (A, B) is stabilizable.

Theorem 3.3.1. *There exists a state feedback $F \in \mathbf{R}^{m\times n}$ such that $\sigma(A + BF) \subseteq \mathbf{C}_-$ if and only if (A, B) is stabilizable.*

Proof. **[if]** Suppose that (A, B) is stabilizable. If (A, B) is reachable, the result follows by Theorem (3.2.1) on letting Γ be any symmetric subset of $\mathbf{C}_-$. If (A, B) is not reachable, then in the reachable normal form, A_3 has all its eigenvalues in $\mathbf{C}_-$. Note that for any F,

$$\begin{aligned} \sigma(A+BF) &= \sigma(T^{-1}(A+BF)T) = \sigma\left(\begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix}\begin{bmatrix} F_1 & 0 \end{bmatrix}\right) \\ &= \sigma\left(\begin{bmatrix} A_1 + B_1F_1 & A_2 + B_1F_2 \\ 0 & A_3 \end{bmatrix}\right) = \sigma(A_1 + B_1F_1) \cup \sigma(A_3), \end{aligned} \tag{3.13}$$

where $F := \begin{bmatrix} F_1 & F_2 \end{bmatrix} T^{-1}$. Since (A_1, B_1) is reachable, there exists F_1 such that $\sigma(A_1 + B_1F_1) \subseteq \mathbf{C}_-$. Hence, by (3.13), $\sigma(A + BF) \subseteq \mathbf{C}_-$.
[Only If] Suppose that for some F, one has $\sigma(A+BF) \subseteq \mathbf{C}_-$. Then by (3.13), $\sigma(A_3) \subseteq \mathbf{C}_-$. □

By Theorem (3.3.1), the eigenvalues of A_3 can not be shifted by any state feedback. For this reason, the matrix A_3 of the reachable normal form is sometimes referred to as the "unreachable part" of (A, B). Also note that all the eigenvalues of A_1 of the reachable normal form can be assigned arbitrarily.

3.4 Notes and References

For Lyapunov theorem and its proof, the book [6] can be referred to. As reported by Kailath [15], Bertram perhaps was the first one who realized in

1959 that if the system was controllable, the desired characteristic polynomial could be obtained by state variable feedback [15]. In 1962, Rosenbrock [16] discussed this problem but a complete statement and result was not given. Eigenvalue assignment problem and complete solution was first published by Rissanen [17], in a similar way, Popov [18] also obtained the same result for the multivariable systems. Lemma (3.2.1) and (3.2.2) in Section (3.2) are due to Heymann [19] and Wonham and Morse [20]. The control canonical form for a single input system is first published by Popov [21]. A more detailed discussion of stabilizability can be found in [22].

Chapter 4

OBSERVABILITY AND OBSERVERS

When the states are not available for measurements or when engagement of all the components of the states for feedback is not desirable, the simplest approach to achieving the control objectives would be to reconstruct the states from available mesurements, the outputs, and then apply the known state feedback techniques. The most direct means of reconstructing the states is to design an observer. Whether the states can be reconstructed at all is an issue that must first be studied. This gives rise to the concept of observability. In this chapter, we discuss the concept of observability and the design of dynamic and functional observers for a LTI system.

4.1 Observability

Consider the LTI system (2.1) with $\varphi(t; t_o, x_o, u(.))$ denoting the value at time $t > 0$ of the trajectory resulting by the initial state x_0 at t_0 and by the application of input $u(.)$ in $[t_0, t]$. Also let $\eta(t, \varphi(t; t_o, x_o, u(.)), u(.))$ be the value of the output at time $t > 0$ resulting from application of the input $u(.)$ in $[t_0, t]$ starting with the initial state x_0.

Definition 4.1.1. *A state $x_0 \in \mathbf{R}^n$ is said to be* **unobservable in** $[t_0, t_1]$ *if*

$$\eta(t, \varphi(t; t_o, x_o, 0), 0) = 0, \ \forall t \in [t_0, t_1], \ t_0 < t_1 \tag{4.1}$$

i.e., when no input is applied, the state x_0 gives rise to zero output at all times in $[t_0, t_1]$.

Note that if x_0 is unobservable in $[t_0, t_1]$, then its effect on the output is *indistinguishable* from that of the zero initial state $x(t_0) = 0$ since $\eta(t, \varphi(t; t_o, 0, 0), 0) = 0$. In what follows, we show that, as a consequence of time-invariance of (2.1), the interval of unobservability is immaterial.

Theorem 4.1.1. *The following are equivalent:*

(i) *A state $x_0 \in \mathbf{R}^n$ is unobservable in $[t_0, t_1]$ for some $t_0 < t_1$.*

(ii) $x_0 \in \bigcap_{i=0}^{n-1} Ker\, CA^i$.

(iii) *x_0 is unobservable in $[t, s]$ for any $t < s$.*

Proof. It is obvious that **(iii)** $\Rightarrow$ **(i)**. **(i)** $\Rightarrow$ **(ii)**: If x_0 is unobservable in $[t_0, t_1]$, then by (4.1) and (2.3),

$$\eta(t, \varphi(t; t_o, x_o, u(.)), 0) = Ce^{A(t-t_0)}x_0 = 0, \ \forall t \in [t_0, t_1].$$

The last equality evaluated at $t = t_0$ gives $Cx_0 = 0$. Taking the derivative of both sides of this equality successively and evaluating at $t = t_0$, we obtain

$$CA^i x_0 = 0, \; i = 0, 1, ..., n-1.$$

Hence, $x_0 \in \bigcap_{i=0}^{n-1} Ker\, CA^i$.

(ii) $\Rightarrow$ (iii) If (ii) holds, then by Cayley-Hamilton theorem, $CA^i x_0 = 0$ for all $i \geq 0$. It follows that for any $t < s$,

$$\eta(t, \varphi(\tau; s, x_o, u(.)), 0) = Ce^{A(\tau - s)} x_0 = 0, \; \forall \tau \in [t, s],$$

by the series expansion of the matrix exponential. Thus, x_0 is unobservable in $[t, s]$. □

Let us define the unobservable subspace of (2.1) by

$$\eta_0 := \{x_0 \in \mathbf{R}^n : \; x_0 \text{ is unobservable in } [0, t] \text{ for some } t > 0\}.$$

By Theorem (4.1.1),

$$\eta_0 = \bigcap_{i=0}^{n-1} Ker\, CA^i,$$

so that η_0 is a subspace of $\mathbf{R}^n$.

Fact 4.1.1. *The unobservable subspace η_0 is the largest A-invariant subspace contained in $Ker\, C$ of $\mathbf{R}^n$.*

Proof. By Cayley-Hamilton theorem, η_0 is A-invariant. Moreover, it is the largest A-invariant subspace contained in $Ker\, C$. To see this, note that any other such subspace η satisfies $\eta \subseteq Ker\, C$ and, by A-invariance, it also satisfies $A^i \eta \subseteq Ker\, C$ which implies $\eta \subseteq Ker\, CA^i$ for $i = 0, ..., n-1$. Hence, $\eta \subseteq \eta_0$. □

Suppose $\eta_0 \neq \{0\}$. Let the columns of a matrix N_0 be a basis for η_0 and let a matrix N_1 be such that $T := [N_0 \; N_1]$ is nonsingular. Since $span\, N_0$ is A-invariant in $Ker\, C$, for some matrices A_i, $i = 1,2,3$ and C_1, the following equalities hold:

$$AT = T\begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \; CT = \begin{bmatrix} 0 & C_1 \end{bmatrix}, \tag{4.2}$$

where $size\, A_3 = dim\, \eta_0$. Since $span\, N_0$ is the largest A-invariant subspace in $Ker\, C$, it also follows that

$$dim \bigcap_{i=0}^{n-1} Ker\, C_1 A_3^i = \bigcap_{i=0}^{n_3-1} Ker\, C_1 A_3^i = size\, A_3 = n - dim\, \eta_0.$$

Alternatively, note that

$$n - dim\, \eta_0 = rank\left(\begin{bmatrix} C \\ CA \\ \\ CA^{n-1} \end{bmatrix} T\right) = rank \begin{bmatrix} 0 & C_1 \\ 0 & C_1 A_3 \\ 0 & \vdots \\ 0 & C_1 A_3^{n-1} \end{bmatrix} = rank \begin{bmatrix} C_1 \\ C_1 A_3 \\ \vdots \\ C_1 A_3^{n-1} \end{bmatrix}.$$

By Cayley-Hamilton theorem,

$$rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = rank \begin{bmatrix} C_1 \\ C_1 A_3 \\ \\ C_1 A_3^{n_3-1} \end{bmatrix}$$

so that $dim \bigcap_{i=0}^{n_3-1} Ker\, C_1 A_3^i = size\, A_3 = n - dim\, \eta_0$.

Definition 4.1.2. *We call the system (2.1) or the pair (C, A)* **observable** *if $\eta_0 = \{0\}$. The system (2.1) or the pair (C, A) is called* **detectable** *if $\sigma(A_1) \subseteq$* $\mathbf{C}_-$, *where A_1 is as defined by (4.2).*

Note that $\eta_0 = \{0\}$ if and only if $rank \begin{bmatrix} C \\ CA \\ \\ CA^{n-1} \end{bmatrix} = n.$

We have shown above that if a system (2.1) is not observable, then it is equivalent to a system

$$(\begin{bmatrix} 0 & C_1 \end{bmatrix}, \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, T^{-1}B, D), \tag{4.3}$$

where (C_1, A_3) is observable and $size\, A_1 = dim\, \eta_0$. The sytem (4.3) is referred to as the *observable normal form.*

Corollary 4.1.1. *The pair (C, A) is observable if and only if*

$$rank \begin{bmatrix} sI - A \\ C \end{bmatrix} = n, \text{ for all } s \in \mathbf{C}.$$

Proof. **[Only If]** If $rank \begin{bmatrix} sI - A \\ C \end{bmatrix} < n$, for some $s \in \mathbf{C}$, then there exists a nonzero $q \in \mathbf{C}^n$ such that $q'(sI - A') = 0,\ q'C' = 0$. This gives

$$q' \begin{bmatrix} C' & A'C' & \dots & A'^{(n-1)}C' \end{bmatrix} = 0.$$

Since $q' \neq 0$, $rank \begin{bmatrix} C \\ CA \\ \\ CA^{n-1} \end{bmatrix} < n.$

[If] If (C, A) is not observable, there exists a nonsingular $T \in \mathbf{R}^{n \times n}$ such that

(4.2) holds where (C_1, A_3) is observable. For $s \in \sigma(A_1)$,

$$rank \begin{bmatrix} sI - A_1 & -A_2 \\ 0 & sI - A_3 \\ 0 & C_1 \end{bmatrix} < n, \text{ then } rank \begin{bmatrix} T^{-1}(sI - A) \\ C \end{bmatrix} T < n. \tag{4.4}$$

Hence $rank \begin{bmatrix} sI - A \\ C \end{bmatrix} < n.$ □

4.2 Dynamic Asymptotic Observers

Except in some trivial cases where the matrix C has full row rank, any reconstruction of the state $x(0)$ from the measurements $y(t)$, $t \in [0, t_1]$ in the LTI system (2.1) necessitates dynamic processing. An asymptotic observer is a LTI system, the output of which asymptotically tracks the states of (2.1). It is driven by the available system inputs and outputs. A block diagram of the asymptotic state reconstruction process considered in this section is given in Figure 4.1.

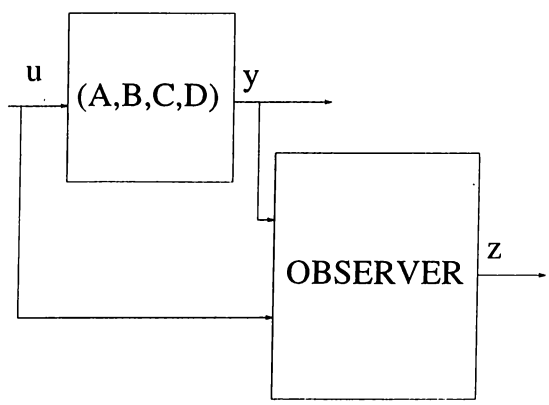

Figure 4.1: Open loop system state reconstruction

Consider a LTI system

$$\dot{z}(t) = Jz(t) + Ky(t) + Lu(t), \; t \geq 0, \tag{4.5}$$

where $J \in \mathbf{R}^{n_o \times n_o}$, $K \in \mathbf{R}^{n_o \times p}$, and $L \in \mathbf{R}^{n_o \times m}$. The system (4.5) is a candidate observer for (2.1) and the vector $z(t)$ is called the observer state. The system (4.5) is called a *full-state observer* if for all initial states $x_0, z_0 \in \mathbf{R}^n$, and for every input $u(t)$,

$$\lim_{t \to \infty} \| x(t) - z(t) \| = 0,$$

where $x(t)$ and $z(t)$ are the solutions of (2.1) and (4.5), respectively. Let us define the error vector by

$$e(t) = z(t) - x(t), \; t \geq 0.$$

The error obeys the equation

$$\begin{aligned} \dot{e}(t) &= Jz(t) + Ky(t) + Lu(t) - Ax(t) - Bu(t) \\ &= Jz(t) + KCx(t) + KDu(t) - Ax(t) - Bu(t) \\ &= (A - KC)e(t) + (J - A + KC)z(t) + (L - B + KD)u(t). \end{aligned}$$

Setting

$$J = A - KC, \tag{4.6}$$

$$L = B - KD, \tag{4.7}$$

the error equation simplifies to $\dot{e}(t) = Je(t)$. If $\sigma(J) \subseteq \mathbf{C}_-$, then $\lim_{t \to \infty} \| e(t) \| = 0$ for all $e_0 = z_0 - x_0$. Any observer satisfying the special choice (4.6) is called a **Luenberger observer.** It is clear that (4.5) is a Luenberger observer if and only if there exists $K \in \mathbf{R}^{n \times n}$ such that $\sigma(A - KC) \subseteq \mathbf{C}_-$. The order of the Luenberger observer is thus equal to the order of the system it observes. The decay rate of the Luenberger observer, when it exists, is defined as the decay rate of the (exponentially stable) error system $\dot{e}(t) = (A - KC)e(t)$. The crucial matrix K is called an *output injection* matrix for the system (2.1).

We now examine the conditions under which a Luenberger observer exists.

Theorem 4.2.1. *The following are equivalent:*

(i) (C, A) *is observable.*

(ii) *For every* $\gamma > 0$, *there exists a Luenberger observer for (2.1) achieving a decay rate* γ *for the error system.*

(iii) *For all symmetric family of* n *complex numbers* Λ, *there exists* K *such that* $\sigma(A - KC) = \Lambda$.

Proof. Note that (C, A) is observable if and only if (A', C') is reachable, as

$$rank \begin{bmatrix} C \\ CA \\ \\ CA^{n-1} \end{bmatrix} = rank \begin{bmatrix} C' & A'C' & & (A')^{n-1}C' \end{bmatrix}.$$

Hence, the result follows from Theorem (3.1.1) and Theorem (3.2.1) upon replacing A, B, and F by A', C', and $-K$, respectively. $\square$

Thus, observability of (2.1) is a necessary and sufficient condition for the existence of a Luenberger observer of arbitrarily large decay rate. Note that the same comment in Chapter 3 concerning large decay rate also applies here, i.e., a large decay rate in the observer is achieved by high gain output injection. If the decay rate is of no particular concern, then an observer exits under weaker condition.

Corollary 4.2.1. *The following are equivalent:*

(i) (C, A) *is detectable.*

(ii) *There exists a Luenberger observer for (2.1).*

(iii) *There exists* K *such that* $\sigma(A - KC) \subseteq \mathbf{C}_-$.

Proof. **(i)⇒ (ii):** (C, A) is detectable if and only if $\sigma(A_1) \in \mathbf{C}_-$, where A_1 is in (4.2). Note that

$$\sigma(J) = \sigma(A - KC) = \sigma\left(T^{-1}(A - KC)T \right) = \sigma\left(\begin{bmatrix} A_1 & A_2 - K_1C_1 \\ 0 & A_3 - K_2C_1 \end{bmatrix}\right), \tag{4.8}$$

where $K := \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$ and T, A_2, A_3, C_1 are as in (4.2). Since (C_1, A_3) is observable, there exists K_2 such that $\sigma(A_3 - K_2C_1) \subseteq \mathbf{C}_-$. Hence, $\sigma(J) \subseteq \mathbf{C}_-$ and $\lim_{t\to\infty} \| e(t) \| = 0$, for all initial states $e_0 = z_0 - x_0$. So, there exists a Luenberger observer.

(ii)⇒ (iii): If there exists a Luenberger observer, then the error system is asymptotically stable and

$$\sigma(J) = \sigma(A - KC) \subseteq \mathbf{C}_-$$

for some K.

(iii)⇒ (i): If there exists K such that $\sigma(A - KC) \subseteq \mathbf{C}_-$, then using (4.8) $\sigma(A_1) \subseteq \mathbf{C}_-$. This implies that (C, A) is detectable. □

4.3 Functional Observers

As an application of ideas used in obtaining a Luenberger observer, we now consider construction of functional observers which find applications in fault diagnosis. See e.g., [23].

If not the whole state but the reconstruction of some linear combinations of state components $x_i(t) \in \mathbf{R}^n$, $i = 1, ..., n$ is desired, then the order of observer can be less than n. A LTI system (4.5) is called a *functional observer* if for some $T \in \mathbf{R}^{n_o \times n}$,

$$\lim_{t \to \infty} \| z(t) - Tx(t) \| = 0, \ \forall x_0, z_0 \text{ and for all inputs } u(t),$$

where $z(t)$ and $x(t)$ are solutions to (4.5) and (2.1). The error vector in this case is defined as

$$e(t) := z(t) - Tx(t),$$

which obeys

$$\dot{e}(t) = Je(t) + (JT - TA + KC)x(t) + (L - TB + KD)u(t).$$

By analogy with the synthesis of the Luenberger observer, suppose it is possible to satisfy

$$L = TB - KD, \tag{4.9}$$

$$JT - TA = -KC, \tag{4.10}$$

$$\sigma(J) \subseteq \mathbf{C}_-. \tag{4.11}$$

Then, the error system $\dot{e}(t) = Je(t)$ would be asymptotically stable and the error would converge to zero exponentially fast for all initial states $e_0 = z_0 - Tx_0$ and for all input $u(t)$.

We examine the equations (4.9), (4.10), and (4.11). Clearly, if there exist matrices $T \in \mathbf{R}^{n_o \times n}$, $J \in \mathbf{R}^{n_o \times n_o}$, and $K \in \mathbf{R}^{n_o \times p}$ satisfying (4.10) and (4.11), then L can be defined by (4.9), and the functional observer synthesis is complete.

Fact 4.3.1. *Given any matrices A, B, and C of compatible sizes, the matrix equation $AX - XB = C$ has a unique solution X if and only if the eigenvalues of A and B are disjoint.*

Proof. See [24, pages 220–225]. $\square$

By Fact (4.3.1), for every K, there exists a unique nonzero $T \in \mathbf{R}^{l \times n}$ satisfying (4.10), provided J is chosen such that

$$\sigma(J) \cap \sigma(A) = \emptyset. \tag{4.12}$$

One way of synthesizing a functional observer is as follows:

(i) Choose any positive integer n_0 and a matrix $K \in \mathbf{R}^{n_o \times p}$.

(ii) Choose any $J \in \mathbf{R}^{n_o \times n_o}$ fulfilling (4.11) and (4.12).

(iii) Determine a $T \in \mathbf{R}^{n_o \times n}$ satisfying (4.10). Here, $T \neq 0$ provided $C \neq 0$.

Note that since $\sigma(A)$ is a finite number of points in the complex plane, fulfilling (4.11) and (4.12) simultaneously is easy. The resulting functional observer reconstructs at least one linear combination of the state components. The decay rate of the error system can be made as large as desired due to the freedom in assigning the spectrum of J in step (ii). The drawback of this procedure is that one does not have control on the number of linear combinations of state components that can be reconstructed. While one would like to choose a matrix T that has as many linearly independent rows as possible, this matrix is obtained through the solution of the matrix equation (4.10) *after* fixing K and J. Determining the exact relation between $rank\, T$ and matrices K and J seems to be a difficult problem. However note that, if (C, A) is observable, then the choice $T = I$ is possible while satisfying all three conditions (4.9), (4.10), and (4.11).

4.4 Notes and References

As reported by Kailath [15], definitions of observability and controllability and the duality between them were worked out by Kalman in 1959-1960 given in [8]. Observer theory dates from the paper [25] by Luenberger. Minimal order state observers for LTI systems is first introduced by Luenberger [25]. Murdoch [26] described a procedure to obtain a minimal order observer providing a specified linear functional of the state vector. Unknown input observers are discussed by Basile and Marro [27]. O'Reilly [28] provides further background on observers for linear systems.

Chapter 5

DYNAMIC STABILIZING CONTROLLER

The state feedback and the observer syntheses of Sections (3.2) and (4.2) can be combined to obtain a LTI system which stabilizes the original system (2.1). In this chapter, we first study the Kalman canonical decomposition which can be viewed as a more detailed normal form that can be attained by simple equivalence than the reachable and observable normal forms. We then give a procedure of synthesizing an observer plus state feedback scheme which leads to a dynamic controller that stabilizes the system (2.1).

5.1 Kalman Canonical Decomposition

Given (2.1), consider the reachable subspace $\mathcal{R}_0$ and the unobservable subspace η_0 which are both A-invariant subspaces of $X = \mathbf{R}^n$. Let

$$X_1 := \mathcal{R}_0 \cap \eta_0,$$

which is also an A-invariant subspace of X. Let X_2 be the complement of X_1 in $\mathcal{R}_0$, X_3 be the complement of X_1 in η_0, and X_4 be the complement of $X_1 \oplus X_2 \oplus X_3$ in $\mathbf{R}^n$. We thus have

$$X = X_1 \oplus X_2 \oplus X_3 \oplus X_4.$$

If T_i, $i = 1,2,3,4$ are basis matrices for X_i, $i = 1,2,3,4$, respectively, then $T := \begin{bmatrix} T_1 & T_2 & T_3 & T_4 \end{bmatrix}$ is a basis for $X = \mathbf{R}^n$. Note that

$$\begin{aligned}
A\begin{bmatrix} T_1 & T_2 & T_3 & T_4 \end{bmatrix} &= \begin{bmatrix} T_1 & T_2 & T_3 & T_4 \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}, \\
B &= \begin{bmatrix} T_1 & T_2 & T_3 & T_4 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}, \qquad (5.1) \\
C\begin{bmatrix} T_1 & T_2 & T_3 & T_4 \end{bmatrix} &= \begin{bmatrix} 0 & C_2 & 0 & C_4 \end{bmatrix},
\end{aligned}$$

where the zero entries in the new matrix representations of A, B, and C are obtained by A-invariance of X_1, $X_1 \oplus X_2$, $X_1 \oplus X_3$ and by the facts that $X_1 \oplus X_3 \subseteq KerC$, $ImB \subseteq X_1 \oplus X_2$.

The block diagram shown in Figure 5.1 is obtained from the equations in (5.1). As seen in this block diagram, the input affects only the subsystems $\Sigma_1 = (A_{11}, B_1, 0, D)$ and $\Sigma_2 = (A_{22}, B_2, C_2, D)$ and the output is affected only by the subsystems Σ_2 and $\Sigma_4 = (A_{44}, 0, C_4, D)$.

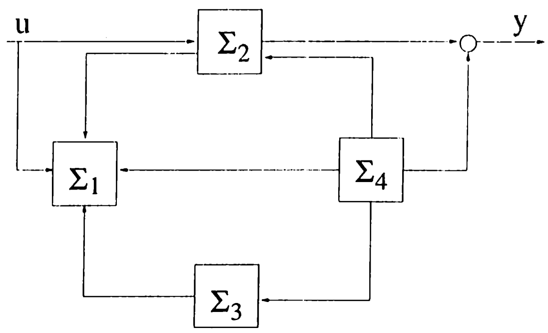

Figure 5.1: Kalman decomposition

Theorem 5.1.1. *Given the system (A, B, C, D) and the decomposition (5.1), the following hold:*

(i) $\left(\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \right)$ *is reachable.*

(ii) $\left(\begin{bmatrix} C_1 & C_2 \end{bmatrix}, \begin{bmatrix} A_{22} & A_{24} \\ 0 & A_{44} \end{bmatrix} \right)$ *is observable.*

(iii) (C_2, A_{22}) *is observable and* (A_{22}, B_2) *is reachable.*

(iv) $Z(s) = C(sI - A)^{-1}B + D = C_2(sI - A_{22})^{-1}B_2 + D.$

Proof. **(i):** Note that

$$\begin{aligned} rank \begin{bmatrix} B & AB & . & A^{n-1}B \end{bmatrix} &= rank \left(T^{-1} \begin{bmatrix} B & AB & . & A^{n-1}B \end{bmatrix} \right) \\ &= rank \begin{bmatrix} \hat{B} & \hat{A}\hat{B} & . \; . & \hat{A}^{n-1}\hat{B} \\ 0 & 0 & & 0 \end{bmatrix}, \end{aligned}$$

where $\hat{A} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ and $\hat{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$. Hence

$$\begin{aligned} rank\begin{bmatrix} B & AB & & A^{n-1}B \end{bmatrix} &= rank\begin{bmatrix} \hat{B} & \hat{A}\hat{B} & & \hat{A}^{n-1}\hat{B} \end{bmatrix} = dim\mathcal{R}_0 \\ &= n_1 + n_2, \end{aligned}$$

where $n_1 := size(A_{11})$ and $n_2 := size(A_{22})$. By Cayley-Hamilton theorem

$$rank\begin{bmatrix} \hat{B} & \hat{A}\hat{B} & & \hat{A}^{n-1}\hat{B} \end{bmatrix} = rank\begin{bmatrix} \hat{B} & \hat{A}\hat{B} & . & \hat{A}^{\hat{n}-1}\hat{B} \end{bmatrix}, \qquad (5.2)$$

where $\hat{n} = n_1 + n_2$. By Corollary (2.3.1), (5.2) implies that $(\hat{A}, \hat{B})$ is reachable.

(ii): Let $\bar{T} := \begin{bmatrix} T_1 & T_3 & T_2 & T_4 \end{bmatrix}$, which is obtained by a permutation of the columns of T. Then, $C\bar{T} = \begin{bmatrix} 0 & 0 & C_2 & C_4 \end{bmatrix}$ and

$$\bar{T}^{-1}A\bar{T} = \begin{bmatrix} A_{11} & A_{13} & A_{12} & A_{14} \\ 0 & A_{33} & 0 & A_{34} \\ 0 & 0 & A_{22} & A_{24} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}.$$

Setting $\tilde{A} = \begin{bmatrix} A_{22} & A_{24} \\ 0 & A_{44} \end{bmatrix}$, $\tilde{C} = \begin{bmatrix} C_2 & C_4 \end{bmatrix}$ we have

$$rank\begin{bmatrix} C \\ CA \\ \\ \\ CA^{n-1} \end{bmatrix} = rank\begin{bmatrix} 0 & \tilde{C} \\ 0 & \tilde{C}\tilde{A} \\ . & \\ . & . \\ 0 & \tilde{C}\tilde{A}^{n-1} \end{bmatrix} = n - dim\{\eta_0\} = n - (n_1 + n_3),$$

where $n_1 := size(A_{11})$ and $n_3 := size(A_{33})$. Thus

$$rank \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \\ \\ \tilde{C}\tilde{A}^{n-1} \end{bmatrix} = n - (n_1 + n_3) = n_2 + n_4,$$

where $n_4 := size(A_{44})$ and n_2 is as defined before. Using again Cayley-Hamilton theorem, $(\tilde{C}, \tilde{A})$ is observable.

(iii): By (5.2), the rows of $\begin{bmatrix} \hat{B} & \hat{A}\hat{B} & . & \hat{A}^{n-1}\hat{B} \end{bmatrix}$ are linearly independent. Note that

$$rank \begin{bmatrix} \hat{B} & \hat{A}\hat{B} & . & \hat{A}^{n-1}\hat{B} \end{bmatrix} = rank \begin{bmatrix} B_1 & * & & * \\ B_2 & A_{22}B_2 & . \; . & A_{22}^{\hat{n}-1}B_2 \end{bmatrix},$$

where "*" denotes entries whose values need not be written explicitly. By this equality, the rows of $\begin{bmatrix} B_2 & A_{22}B_2 & . & A_{22}^{\hat{n}-1}B_2 \end{bmatrix}$ are linearly independent. Once again by Cayley-Hamilton theorem,

$$rank \begin{bmatrix} B_2 & A_{22}B_2 & . \; . & A_{22}^{\hat{n}-1}B_2 \end{bmatrix} = rank \begin{bmatrix} B_2 & A_{22}B_2 & . & A_{22}^{n_2-1}B_2 \end{bmatrix},$$

so that (A_{22}, B_2) is reachable. Following a similar reasoning, it can be shown that (C_2, A_{22}) is observable.

(iv): Note that, the transfer matrix of the system (A, B, C, D) can be written as

$$Z(s) = C(sI - A)^{-1}B + D = CT(sI - T^{-1}AT)^{-1}T^{-1}B + D.$$

Using the equations in (5.1), we have

$$
\begin{aligned}
Z(s) &= \begin{bmatrix} 0 & C_2 & 0 & C_4 \end{bmatrix} \begin{bmatrix} sI - A_{11} & -A_{12} & -A_{13} & -A_{14} \\ 0 & sI - A_{22} & 0 & -A_{24} \\ 0 & 0 & sI - A_{33} & -A_{34} \\ 0 & 0 & 0 & sI - A_{44} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix} + D \\
&= \begin{bmatrix} 0 & C_2 \end{bmatrix} \begin{bmatrix} sI - A_{11} & -A_{12} \\ 0 & sI - A_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + D = C_2(sI - A_{22})^{-1} B_2 + D.
\end{aligned}
$$

□

By **(iii)** and **(iv)**, the transfer matrix $Z(s)$ depends only on the controllable and observable subsystem of (2.1). This explains why the input-output description is sometimes insufficient as a representation of the system since the uncontrollable and/or unobservable parts do not appear in the transfer matrix description.

5.2 Combined Observer and State Feedback Controllers

We can now consider how to combine an observer and a state feedback to construct a dynamic stabilizing controller for (2.1).

The observer described in Section (4.2) is used to reconstruct the states. If a constant feedback $u(t) = Fz(t)$ is utilized, with the estimated state z as its input instead of the state x, then Figure (5.2) is obtained.

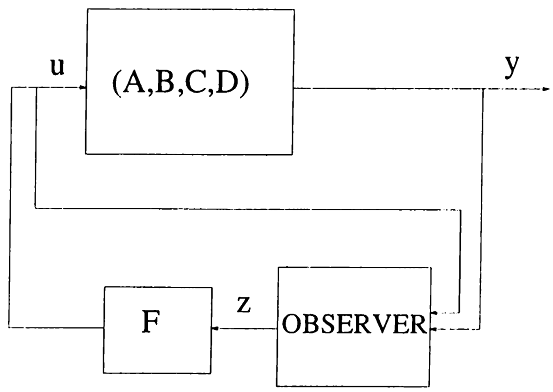

Figure 5.2: Closed loop observer plus state feedback configuration

Using (2.1), (4.5) and $u(t) = Fz(t)$, we have

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & BF \\ KC & J + (L + KD)F \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}.$$

If (4.6) and (4.7) are satisfied, the unforced closed loop system of Figure 5.2 is describe by the state equation

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & BF \\ KC & A - KC + BF \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}. \tag{5.3}$$

The eigenvalues of the combined system matrix in (5.3) are the roots of the determinant of

$$\begin{bmatrix} \lambda I - A & -BF \\ -KC & \lambda I - A + KC - BF \end{bmatrix}.$$

Note that

$$\begin{aligned} det \begin{bmatrix} \lambda I - A & -BF \\ -KC & \lambda I - A + KC - BF \end{bmatrix} &= det \begin{bmatrix} \lambda I - A & -BF \\ -(\lambda I - A + KC) & \lambda I - A + KC \end{bmatrix} \\ &= det \begin{bmatrix} \lambda I - A - BF & -BF \\ 0 & \lambda I - A + KC \end{bmatrix}. \end{aligned}$$

Thus, the spectrum of the combined system matrix in (5.3) coincides with

$$\sigma(A + BF) \cup \sigma(A - KC).$$

A consequence of this expression is that the state feedback and observer can be designed independently of each other. As far as the eigenvalues are concerned, it is immaterial to the state feedback whether the estimated state z or the actual state x are available. The eigenvalues of the entire system are the union of those of the closed loop system obtained by a state feedback and those of the error system obtained by an observer. This property is called the *separation principle* of the state feedback-observer design procedure.

Hence, we can state

Theorem 5.2.1. *The following hold:*

(i) *There exists an observer plus state feedback scheme which asymptotically stabilizes the closed loop system with arbitrary decay rate if and only if* (C, A) *is observable and* (A, B) *is reachable.*

(ii) *There exists an observer plus state feedback scheme which asymptotically stabilizes the plant if and only if* (C, A) *is detectable and* (A, B) *is stabilizable.*

Proof. **(i)** The spectrum of the closed loop system matrix coincides with $\sigma(A + BF) \cup \sigma(A - KC)$. Using Theorem (3.2.1) and (4.2.1), the closed loop system is stabilized with arbitrary decay rate if and only if (C, A) is observable and (A, B) is reachable.

(ii) Using Theorem (3.3.1) and Corollary (4.2.1), the plant is asymptotically stabilized by an observer plus state feedback scheme if and only if (C, A) is detectable and (A, B) is stabilizable. □

5.3 Notes and References

Kalman Canonical Decomposition originates from the papers [9] and [10]. The computation of the reachable/observable canonical form is presented by [29]. Dynamic stabilizing controller construction via observer plus state feedback scheme is first given by Brasch and Pearson [30]. The separation principle for feedback controllers is due to Luenberger [25], [31]. Our presentation of the basic properties of linear systems made use of Chapters 1-4 of [32].

Chapter 6

FRACTIONAL REPRESENTATIONS

Given a transfer function of a scalar LTI system, or more generally, any real rational function of s, it can be written as the ratio of a numerator and a denominator polynomial in s. Such a representation is a polynomial fractional representation or a polynomial factorization of the given rational function. Extensions of this idea in two different directions are possible. First, the numerator and denominator entries can be elements of any subring of the field of rational functions rather than the ring of polynomials. Second, similar fractional representations can be obtained for rational matrices of s and not only for scalar rational functions. In this chapter, we give a construction for fractional representations of a transfer matrix, where the numerator and denominator entries themselves are transfer matrices of stable LTI systems. Such fractional representations are known as stable proper fractional representations or as stable proper factorizations. The particular construction given in this chapter is based on a state space representation of the transfer matrix and on state feedback and output injection matrices.

6.1 Right and Left Coprime Fractional Representations

Let $\mathbf{S}$ denote the set of stable transfer functions, i.e.,

$$\mathbf{S} := \{\frac{p(s)}{q(s)} : \ p, q \text{ are polynomials such that } deg\ p \leq \ deg\ q \text{ and}$$
$$q(s) \ \text{ has all its roots in } \mathbf{C}_-\}.$$

A rational function $g(s) = \dfrac{p(s)}{q(s)}$ is said to be proper if $deg\ p \leq deg\ q$ and strictly proper if $deg\ p < deg\ q$. Consider a transfer matrix $G(s)$ of a LTI system with r inputs and p outputs.

Definition 6.1.1. (i) *An ordered pair* (N, M)*, where* $N \in \mathbf{S}^{p\times r}$, $M \in \mathbf{S}^{r\times r}$, M *is nonsingular, and* M^{-1} *is proper, is called a* **right coprime factorization** *over* $\mathbf{S}$ *of* G *if* $G = NM^{-1}$ *and there exist* $Q \in \mathbf{S}^{r\times p}$, $P \in \mathbf{S}^{r\times r}$ *such that*

$$QN + PM = I. \tag{6.1}$$

(ii) *Similarly, an ordered pair* $(\tilde{M}, \tilde{N})$*, where* $\tilde{N} \in \mathbf{S}^{p\times r}$, $\tilde{M} \in \mathbf{S}^{p\times p}$, $\tilde{M}$ *is nonsingular, and* $\tilde{M}^{-1}$ *is proper, is called a* **left coprime factorization** *over* $\mathbf{S}$ *of* G *if* $G = \tilde{M}^{-1}\tilde{N}$ *and there exist* $\tilde{Q} \in \mathbf{S}^{r\times p}$, $\tilde{P} \in \mathbf{S}^{p\times p}$ *such that*

$$\tilde{N}\tilde{Q} + \tilde{M}\tilde{P} = I. \tag{6.2}$$

(iii) $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ *with* N, M, $\tilde{M}$, $\tilde{N}$ *over* $\mathbf{S}$ *and where* M, $\tilde{M}$ *are nonsingular and have proper inverses, is a* **doubly coprime factorization** *over* $\mathbf{S}$ *if there exist* Q, P, $\check{Q}$, $\tilde{P}$ *over* $\mathbf{S}$ *such that*

$$\begin{bmatrix} P & Q \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & -\check{Q} \\ N & \check{P} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{6.3}$$

Note that if right and left coprime factorizations are given, then a doubly coprime factorization can also be obtained. To do this, let (N, M), $(\check{M}, \check{N})$ be any right coprime factorization (r.c.f) and left coprime factorization (l.c.f) of G. Then there exists Q, P, $\tilde{Q}$, $\tilde{P}$ over $\mathbf{S}$ such that (6.1) and (6.2) hold. Let $\Delta := Q\tilde{P} - P\tilde{Q}$, $\tilde{P}_n := \tilde{P} - N\Delta$, $\tilde{Q}_n := \check{Q} + M\Delta$. Thus (6.3) is satisfied with these new $\tilde{P}_n$ and $\tilde{Q}_n$.

Fact 6.1.1. (i)(A, B) *is stabilizable if and only if there exist stable rational matrices* $R_1(s)$, $R_2(s)$ *such that,*

$$(sI - A)R_1(s) + BR_2(s) = I, \tag{6.4}$$

(ii) (C, A) *is detectable if and only if there exist stable rational matrices* $R_1(s)$, $R_2(s)$ *such that,*

$$R_1(s)(sI - A) + R_2(s)C = I,$$

Proof. Only **(i)** is proved since (C, A) is detectable if and only if (A', C') is stabilizable. So, **(ii)** follows by **(i)**.

[If] Suppose (6.4) can be written but (A, B) is not stabilizable. Then, by Corollary (2.5.2), there exist $s \in \mathbf{C}_{0+}$ and a nonzero vector $q \in \mathbf{C}^n$ such that $q'(sI - A) = 0$, $q'B = 0$. For this s, $R_1(s)$ and $R_2(s)$ are both well-defined (i.e., not ∞) since all poles are in $\mathbf{C}_{-}$. Hence, (6.4) yields $q = 0$, which is a contradiction.

[Only If] If (A, B) is stabilizable, there exists a stable feedback F such that

$\sigma(A+BF) \subseteq \mathbf{C}_-$, or equivalently, $(sI-A-BF)^{-1}$ is a stable rational matrix. But then,

$$(sI-A)(sI-A-BF)^{-1} - BF(sI-A-BF)^{-1} = I$$

and (6.4) is satisfied with $R_1 := (sI-A-BF)^{-1}$, $R_2 := -F(sI-A-BF)^{-1}$. □

6.2 Coprime Factorization From State-Space Description

Consider the system (2.1). The transfer function of this system is

$$Z(s) = C(sI-A)^{-1}B + D.$$

Suppose (A,B) is stabilizable and (C,A) is observable. The objective is to obtain a doubly coprime factorization of $Z(s)$. In Theorem (6.2.1), one such factorization is given. To prove this theorem Lemma (6.2.1) and Lemma (6.2.2) are used.

Lemma 6.2.1. *Let $G(s) = C(sI-A)^{-1}B$. A doubly coprime factorization of G, the matrix equality (6.3) in which M, $\tilde{M}$ have proper inverses , is obtained by the matrices*

$$\begin{aligned} N &= C(sI-A-BF)^{-1}B \\ M &= I + F(sI-A-BF)^{-1}B \\ Q &= -F(sI-A+KC)^{-1}K \\ P &= I - F(sI-A+KC)^{-1}B \end{aligned}$$

$$
\begin{aligned}
\tilde{N} &= C(sI - A + KC)^{-1}B \\
\tilde{M} &= I - C(sI - A + KC)^{-1}K \\
\tilde{P} &= I + C(sI - A - BF)^{-1}K \\
\tilde{Q} &= -F(sI - A - BF)^{-1}K,
\end{aligned}
$$

where F *and* K *are any matrices such that* $\sigma(A+BF) \subseteq \mathbf{C}_-$ *and* $\sigma(A+KC) \subseteq \mathbf{C}_-$.

Proof. We first verify that $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ using the matrix identity $Y(I+XY)^{-1} = (I+YX)^{-1}Y$:

$$
\begin{aligned}
NM^{-1} &= C(sI - A - BF)^{-1}B\left[I + F(sI - A - BF)^{-1}B\right]^{-1} \\
&= C(sI - A)^{-1}(I - BF(sI - A)^{-1})^{-1}B\left[I + F(sI - A)^{-1}(I - BF(sI - A)^{-1})^{-1}B\right]^{-1} \\
&= C(sI - A)^{-1}B(I - F(sI - A)^{-1}B)^{-1}\left[I + F(sI - A)^{-1}B(I - F(sI - A)^{-1}B)^{-1}\right]^{-1} \\
&= C(sI - A)^{-1}B(I - F(sI - A)^{-1}B)^{-1}(I - F(sI - A)^{-1}B) \\
&= C(sI - A)^{-1}B \\
&= G.
\end{aligned}
$$

The verification that $G = \tilde{M}^{-1}\tilde{N}$ is entirely similar. $\tilde{N}M + \tilde{M}N = 0$ since $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$.

The next part of the proof shows that (6.1) and (6.2) hold. We have

$$
\begin{aligned}
QN + PM &= -F(sI - A + KC)^{-1}KC(sI - A - BF)^{-1}B \\
&\quad + \left[I - F(sI - A + KC)^{-1}B\right]\left[I + F(sI - A - BF)^{-1}B\right] \\
&= -F(sI - A + KC)^{-1}KC(sI - A - BF)^{-1}B + I + F(sI - A - BF)^{-1}B \\
&\quad - F(sI - A + KC)^{-1}B - F(sI - A + KC)^{-1}BF(sI - A - BF)^{-1}B.
\end{aligned}
$$

Note that the following term on the right hand side of the above equality

$$
F(sI - A + KC)^{-1}[-KC + (sI - A + KC) - (sI - A - BF) - BF](sI - A - BF)^{-1}B \tag{6.5}
$$

is zero since the term inside the brackets is zero. Thus (6.1) is satisfied. (6.2) follows in the same manner. Finally,

$$\begin{aligned}
-P\tilde{Q}+Q\tilde{P} &= [I-F(sI-A+KC)^{-1}B]F(sI-A-BF)^{-1}K \\
&\quad -F(sI-A+KC)^{-1}K[I+C(sI-A-BF)^{-1}K] \\
&= F(sI-A-BF)^{-1}K-F(sI-A+KC)^{-1}BF(sI-A-BF)^{-1}K \\
&\quad -F(sI-A+KC)^{-1}K-F(sI-A+KC)^{-1}KC(sI-A-BF)^{-1}K.
\end{aligned}$$

The right hand side of the last equality is equal to (6.5). Hence $-P\check{Q}+Q\tilde{P}=$ 0 □

Lemma 6.2.2. *Let N, M, $\tilde{M}$, $\tilde{N}$, Q, P, $\tilde{Q}$, $\check{P}$ be a doubly coprime factorization of $G(s)$ and satisfy (6.3). Then a doubly coprime factorization of $G(s)+D=Z(s)$ can be obtained by the following replacements:*

$$\begin{aligned}
N &\leftarrow N+DM \\
P &\leftarrow P-QD \\
\tilde{N} &\leftarrow \check{N}+\tilde{M}D \\
\tilde{P} &\leftarrow \tilde{P}-D\tilde{Q}.
\end{aligned}$$

Proof. Note that $(N+DM,M)$ is the right coprime factorization for $G(s)+D$ and $(N+DM,M)$ satisfy (6.1) with $(Q,P-QD)$. Similarly $(\tilde{M},\check{N}+\tilde{M}D)$ is the left coprime factorization for $G(s)+D$ and with $(\tilde{P}-D\tilde{Q},\tilde{Q})$, (6.2) is satisfied. In addition,

$$\begin{aligned}
-(P-QD)\tilde{Q}+Q(P-D\tilde{Q}) &= -P\tilde{Q}+QD\check{Q}+QP-QD\check{Q} \\
&= -P\check{Q}+QP \\
&= 0.
\end{aligned}$$

□

The following result is an immediate consequence of Lemma (6.2.1) and Lemma (6.2.2).

Theorem 6.2.1. *Given the system (2.1) with (A, B) stabilizable and (C, D) detectable, a doubly coprime factorization of $Z(s)$ is obtained by the matrices*

$$\begin{aligned}
N &= (C+DF)(sI-A-BF)^{-1}B+D \\
M &= I+F(sI-A-BF)^{-1}B \\
Q &= -F(sI-A+KC)^{-1}K \\
P &= I-F(sI-A+KC)^{-1}(B-KD) \\
\tilde{N} &= C(sI-A+KC)^{-1}(B-KD)+D \\
\tilde{M} &= I-C(sI-A+KC)^{-1}K \\
\tilde{P} &= I+(C+DF)(sI-A-BF)^{-1}K \\
\tilde{Q} &= -F(sI-A-BF)^{-1}K,
\end{aligned}$$

where F and K are any matrices satisfying $\sigma(A+BF) \subseteq \mathbf{C}_-$ and $\sigma(A+KC) \subseteq \mathbf{C}_-$.

6.3 Common Factors and Unimodular Matrices

The stable proper fractional representations constructed in the previous section are such that (6.1) and (6.2) hold. These linear equations over $\mathbf{S}$ ensure that the fractions are coprime. It is our purpose in this section to clarify the relation between coprimeness and equations (6.1) and (6.2).

Definition 6.3.1. *Let* $N \in \mathbf{S}^{p\times r}$ *and* $M \in \mathbf{S}^{r\times r}$. *A square matrix* $E \in \mathbf{S}^{r\times r}$ *is called a* **common right factor** *of the pair* (N, M) *if there exist* $\hat{N} \in \mathbf{S}^{p\times r}$, $\hat{M} \in \mathbf{S}^{r\times r}$ *such that*

$$N = \hat{N}E, \ M = \hat{M}E.$$

A square matrix $D \in \mathbf{S}^{r\times r}$ *is a* **greatest common right factor** *of the pair* (N, M) *if*

(i) *D is a common right factor of (N, M) and*

(ii) *any other common right factor of E is a right factor of D, i.e., there exists* $\hat{D} \in \mathbf{S}^{r\times r}$ *such that*

$$D = \hat{D}E.$$

The pair (N, M) is called **right coprime** *over* $\mathbf{S}$ *if there exist matrices Q, P over* $\mathbf{S}$ *such that*

$$QN + PM = I.$$

Definition 6.3.2. *Let* $\tilde{N} \in \mathbf{S}^{p\times r}$ *and* $\tilde{M} \in \mathbf{S}^{p\times p}$. *A square matrix* $E \in \mathbf{S}^{p\times p}$ *is called a* **common left factor** *of the pair* $(\tilde{M}, \tilde{N})$ *if there exist* $\hat{N} \in \mathbf{S}^{p\times r}$, $\hat{M} \in \mathbf{S}^{p\times p}$ *such that*

$$\tilde{N} = E\hat{N}, \ \tilde{M} = E\hat{M}.$$

A square matrix $D \in \mathbf{S}^{p\times p}$ *is a* **greatest common left factor** *of the pair* $(\tilde{M}, \tilde{N})$ *if*

(i) *D is a common left factor of $(\tilde{M}, \tilde{N})$ and*

(ii) *any other common left factor of E is a left factor of D, i.e., there exists* $\hat{D} \in \mathbf{S}^{p\times p}$ *such that*

$$D = E\hat{D}.$$

The pair $(\tilde{M}, \tilde{N})$ is called **left coprime** *over* $\mathbf{S}$ *if there exist matrices $\tilde{Q}$, $\tilde{P}$ over* $\mathbf{S}$ *such that*

$$\tilde{M}\tilde{P} + \tilde{N}\tilde{Q} = I.$$

Fact 6.3.1. *Suppose that $N \in \mathbf{S}^{p\times r}$, $M \in \mathbf{S}^{r\times r}$ where M is nonsingular and M^{-1} is proper. Then, (N, M) is right coprime over $\mathbf{S}$ if and only if a greatest common right factor $D \in \mathbf{S}^{r\times r}$ of (N, M) is such that $D^{-1} \in \mathbf{S}^{r\times r}$.*

Proof. **[Only if]** if (N, M) is right coprime then there exists $Q \in \mathbf{S}^{r\times p}$ and $P \in \mathbf{S}^{r\times r}$ such that $QN + PM = I$. Let D be a greatest common right factor of (N, M), then

$$N = \hat{N}D, \ M = \hat{M}D,$$

where $\hat{N} \in \mathbf{S}^{p\times r}$, $\hat{M} \in \mathbf{S}^{r\times r}$. So $Q\hat{N}D + P\hat{M}D = I$. Multiplying D^{-1} from the right, we obtain $Q\hat{N} + P\hat{M} = D^{-1}$. This implies that $D^{-1} \in \mathbf{S}$.
[If] Let $G = NM^{-1}$. Construct a right coprime representation as $G = \tilde{N}\tilde{M}^{-1}$. So there exist P, $Q \in \mathbf{S}$ such that $P\tilde{M} + Q\tilde{N} = I$. Then

$$PM + QN = \tilde{M}^{-1}M.$$

Let $D := \tilde{M}^{-1}M$. Note that D is a greatest common right factor of (N, M). If the inverse of all greatest common right factors must be over $\mathbf{S}$ since they can only differ by a left factor whose inverse is also over $\mathbf{S}$. Thus, by hypothesis, $D^{-1} \in \mathbf{S}$. Then,

$$D^{-1}PM + D^{-1}QN = I,$$

which implies that (N, M) is right coprime. □

Fact 6.3.2. *Suppose that $\tilde{N} \in \mathbf{S}^{p\times r}$, $\tilde{M} \in \mathbf{S}^{p\times p}$ where $\tilde{M}$ is nonsingular and $\tilde{M}^{-1}$ is proper. Then, $(\tilde{M}, \tilde{N})$ is left coprime if and only if a greatest common left factor $D \in \mathbf{S}^{p\times p}$ of $(\tilde{M}, \tilde{N})$ is such that $D^{-1} \in \mathbf{S}^{p\times p}$.*

Definition 6.3.3. *Any matrix $D \in \mathbf{S}^{r\times r}$ which is nonsingular is called* **unimodular** *over $\mathbf{S}$ if $D^{-1} \in \mathbf{S}^{r\times r}$.*

By Fact (6.3.1) and Fact (6.3.2), (N, M) is right coprime if and only if a greatest common right factor of (N, M) is unimodular and $(\tilde{M}, \tilde{N})$ is left coprime over $\mathbf{S}$ if and only if a greatest common left factor of $(\tilde{M}, \tilde{N})$ is unimodular.

6.4 Some Properties of Polynomial Matrices

In this section, we present some basic facts about polynomial matrices and give some results on coprimeness of polynomial matrices.

Definition 6.4.1. **(i)** *Two polynomial matrices $M(s) \in \mathbf{R}[s]^{r\times r}$ and $N(s) \in \mathbf{R}[s]^{r\times p}$ are* **right coprime** *over $\mathbf{R}[s]$ if every common polynomial right factor of M and N is a unimodular polynomial matrix.*
(ii) *Two polynomial matrices $M(s) \in \mathbf{R}[s]^{p\times p}$ and $N(s) \in \mathbf{R}[s]^{p\times r}$ are* **left coprime** *over $\mathbf{R}[s]$ if every common polynomial left factor of M and N is a unimodular polynomial matrix.*

Fact 6.4.1. **(i)** *Given two polynomial matrices $M(s) \in \mathbf{R}[s]^{r\times r}$ and $N(s) \in \mathbf{R}[s]^{r\times p}$ with $M(s)$ nonsingular, $(N(s), M(s))$ is right coprime over $\mathbf{R}[s]$ if and only if there exist polynomial matrices $P(s)$, $Q(s)$ satisfying*

$$P(s)M(s) + Q(s)N(s) = I_r.$$

(ii) *Given two polynomial matrices $M(s) \in \mathbf{R}[s]^{p\times p}$ and $N(s) \in \mathbf{R}[s]^{p\times r}$ with $M(s)$ nonsingular, $(M(s), N(s))$ is left coprime over $\mathbf{R}[s]$ if and only if there exist polynomial matrices $P(s)$, $Q(s)$ satisfying*

$$M(s)P(s) + N(s)Q(s) = I_p.$$

Proof. Kailath [15]. □

Fact 6.4.2. *Let K, L, M be polynomial matrices with M nonsingular. If $LM^{-1}K$ is polynomial (respectively, stable rational)*
(i) *(L, M) right coprime over $\mathbf{R}[s]$ implies that $K = MX$ for some polynomial (respectively, stable rational) matrix X.*
(ii) *(M, K) left coprime over $\mathbf{R}[s]$ implies that $L = XM$ for some polynomial (respectively, stable rational) matrix X.*

Proof. Let $Y = LM^{-1}K$. We only prove **(i)** as **(ii)** follows by duality (by transposition of matrices). If (L, M) is right coprime over $\mathbf{R}[s]$, then by Fact (6.4.1), there exists polynomial matrices $P(s)$, $Q(s)$ such that $PL + QM = I$. Hence, $PLM^{-1}K + QK = M^{-1}K$ or $PY + QK = M^{-1}K := X$. If Y is polynomial (respectively, stable rational), X is also polynomial (respectively, stable rational). □

Fact 6.4.3. *Let $\mathcal{V}$ be a subspace of $\mathbf{R}^n$ and let V be a basis matrix for $\mathcal{V}$. Then, $\mathcal{V} \subseteq \mathcal{R}_0$ if and only if there exist polynomial matrices $P(s)$, $Q(s)$ satisfying*

$$V = (sI - A)P(s) + BQ(s). \tag{6.6}$$

Proof. **[If]** Suppose (6.6) holds so that

$$(sI - A)^{-1}V = P(s) + (sI - A)^{-1}BQ(s). \tag{6.7}$$

Let us write $Q(s) = Q_0 + Q_1 s + ... + Q_k s^k$ for constant matrices Q_i, $i = 1, ..., k$. Equating the coefficients of s^{-1} on both sides of the equality (6.7), we obtain

$$V = BQ_0 + ABQ_1 + ...A^{k-1}BQ_{k-1}.$$

Hence, $span\{V\} = \mathcal{V} \subseteq \mathcal{R}_0$.

[Only If] Suppose $\mathcal{V} \subseteq \mathcal{R}_0$ so that $spanV \subseteq < A|ImB >$. There thus exist constant matrices Q_0, $Q_1, ..., Q_{n-1}$ such that

$$V = BQ_0 + ABQ_1 + ...A^{n-1}BQ_{n-1}. \tag{6.8}$$

Multiplying each term in (6.8) on the left by A^i, we have

$$A^iV = A^iBQ_0 + A^{i+1}BQ_1 + ... + A^{i+n-1}BQ_{n-1}$$

for $i = 0, 1, 2....$ Thus,

$$\begin{aligned}
\sum_{i=1}^{\infty} A^{i-1}Vs^{-i} &= \sum_{i=1}^{\infty} A^{i-1}BQ_0s^{-i} + \sum_{i=1}^{\infty} A^iBQ_1s^{-i} + ... + \sum_{i=1}^{\infty} A^{i+n-2}BQ_{n-1}s^{-i} \\
&= (sI - A)^{-1}BQ_0 + [s(sI - A)^{-1}BQ_1 - BQ_1] + ... \\
&+ [s^{n-1}(sI - A)^{-1}BQ_{n-1} - \sum_{i=2-n}^{\infty} A^{i+n-2}BQ_{n-1}s^i] \\
&= [(sI - A)^{-1}BQ_s]_-,
\end{aligned}$$

where $Q(s) := Q_0 + Q_1s + ... + Q_{n-1}s^{n-1}$ and where $[Y(s)]_-$ denotes the strictly proper part in the Laurent series expansion of $Y(s)$. Therefore,

$$(sI - A)^{-1}V = (sI - A)^{-1}BQ(s) - [(sI - A)^{-1}BQ(s)]_+$$

or equivalently

$$V = BQ(s) + (sI - A)P(s),$$

with $P(s) = -[(sI - A)^{-1}BQ(s)]_+$, where $[Y(s)]_+$ denotes the polynomial part in the Laurent series expansion of $Y(s)$. □

Fact 6.4.4. *The following hold:*

(i) (A, B) *is reachable if and only if* $((sI - A), B)$ *is left coprime over* $\mathbf{R}[s]$.

(ii) (C, A) *is observable if and only if* $(C, (sI - A))$ *is right coprime over* $\mathbf{R}[s]$.

Proof. **(i): [If]** If $((sI - A), B)$ is left coprime, then there exist polynomial matrices $X_1(s), X_2(s)$ such that

$$(sI - A)X_1 + BX_2(s) = I_n. \tag{6.9}$$

Then

$$\begin{bmatrix} sI - A & B \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = I_n.$$

So, $rank \begin{bmatrix} sI - A & B \end{bmatrix} = n, \forall s \in \mathbf{C}$. By Corollary (2.5.1), (A, B) is reachable.
[Only if] If (A, B) is reachable, then by Corollary (2.5.1),

$$rank \begin{bmatrix} sI - A & B \end{bmatrix} = n, \text{ for all } s \in \mathbf{C}. \tag{6.10}$$

Then every nonsingular left polynomial factor of $(sI - A)$ and B must be a unimodular polynomial matrix. To see this, suppose there exists $D(s)$, a nonsingular polynomial matrix with $detD(s)$ nonconstant, such that

$$sI - A = D(s)M(s), \; B = D(s)N(s)$$

for some polynomial matrices $M(s)$ and $N(s)$. Then, evaluating at a zero s_0 of $detD(s)$, we see that (6.10) fails for $s = s_0$. Hence, (6.10) implies that $D(s)$ has constant determinat, and every common left factor of $(sI - A)$ and B is a unimodular polynomial matrix. This implies that $((sI - A), B)$ is left coprime.
(ii): Note that (C, A) is observable if and only if (A', C') is reachable. Then, the proof follows easily by **(i)**. □

6.5 Notes and References

The idea of "factoring" the transfer matrix of a system as the "ratio" of two stable rational matrices was first used in [33] by Vidyasagar, but the analysis of the stability of a given plant was considered rather than the synthesis of control systems. Definitions of coprimeness, common factors and unimodular matrices over any principe ideal domain, in particular, over $\mathbf{S}$ and over $\mathbf{R}[s]$, can be found in [34]. Coprime factorization from state space description in Section (6.2) is presented by Khargonekar and Sontag [35] and by Nett, Jacobson and Balas [36]. Polynomial factorizations and the polynomial system matrix are introduced by Rosenbrock [37]. Wolovich [38] and Kučera [39] contributes to this approach. Rosenbrock [40] also presented design methods based on polynomial factorizations.

Chapter 7

ALL INTERNALLY STABILIZING CONTROLLERS

The main advantage of working with stable proper fractional representations consists of yielding a simple parametrization of all controllers that stabilize a given plant. The parametrization is obtained in terms of a free matrix parameter and it is especially suitable for considering additional design specifications to closed loop stability. In Chapter 5, we studied the construction of a particular internally stabilizing controller for (2.1). In this chapter we use the fractional representation developed in Chapter 6 to describe the set of all possible internally stabilizing controllers. For this purpose in Section (7.1), we first clarify the link between input-output stability of a feedback interconnection and its internal stability.

7.1 Closed Loop Stability

Consider the feedback system shown in Figure 7.1. In this figure, $P(s)$ is the transfer matrix of a given plant, $K(s)$ is the transfer matrix of a dynamic feedback controller applied to the plant, u_1 and u_2 are externally applied inputs, e_1 and e_2 are the inputs to the plant and controller respectively, and y_1 and y_2 are the outputs of the plant and controller respectively.

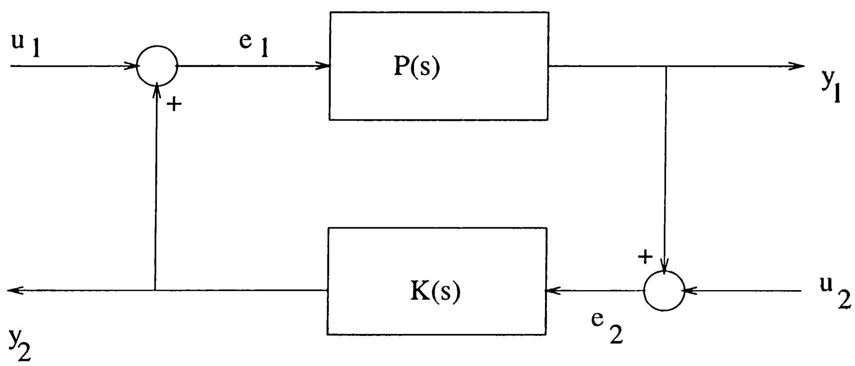

Figure 7.1: Feedback loop for internal stability

The system of Figure 7.1 is then described by

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} I & -K \\ -P & I \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}; \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \tag{7.1}$$

It should be guaranteed that there is a unique solution for the inputs to the plant and controller in terms of the external inputs and that the transfer matrix from (u_1, u_2) to (e_1, e_2) is proper rational. If these two conditions are satisfied, the system is called *well-posed.*

If the interconnection is well-posed, then

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} I & -K \\ -P & I \end{bmatrix}^{-1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Let $L := \begin{bmatrix} I & -K \\ -P & I \end{bmatrix}$. It is possible to obtain several equivalent expressions for L^{-1}. One of them is

$$L^{-1} = \begin{bmatrix} I + K(I-PK)^{-1}P & K(I-PK)^{-1} \\ (I-PK)^{-1}P & (I-PK)^{-1} \end{bmatrix}. \tag{7.2}$$

This is obtained by writing L as a multiplication of two triangular matrices as $L^{-1} = \left(\begin{bmatrix} I & 0 \\ -P & I \end{bmatrix} \begin{bmatrix} I & -K \\ 0 & I-PK \end{bmatrix} \right)^{-1}$. Using the following matrix identities

$$(I-PK)^{-1} = I + P(I-KP)^{-1}K, \quad K(I-PK)^{-1} = (I-KP)^{-1}K, \tag{7.3}$$

another expression for L^{-1} is obtained as

$$L^{-1} = \begin{bmatrix} (I-KP)^{-1} & (I-KP)^{-1}K \\ P(I-KP)^{-1} & I + P(I-KP)^{-1}K \end{bmatrix}. \tag{7.4}$$

Fact 7.1.1. *If $P(s)$ and $K(s)$ are proper, then the interconnection is well-posed if and only if*

$$det(I - P_{(\infty)}K_{(\infty)}) \neq 0,$$

where $P_{(\infty)} := \lim_{s\to\infty} P(s)$, $K_{(\infty)} := \lim_{s\to\infty} K(s)$.

Proof. If $P(s)$ and $K(s)$ are proper, then $(I-PK)$ is also proper and

$$I - PK = A_0 + A_1 s^{-1} + A_2 s^{-2} + \ldots$$

for constant matrices A_k, $k = 0, 1, 2\ldots$

[Only if] If the interconnection is well-posed, then by (7.2), $(I-PK)^{-1}$ exists and is proper. Let $(I-PK)^{-1} = B_0 + B_1 s^{-1} + B_2 s^{-2} + \ldots$ for constant matrices B_k, $k = 0, 1, 2\ldots$ Then

$$(A_0 + A_1 s^{-1} + A_2 s^{-2} + \ldots)(B_0 + B_1 s^{-1} + B_2 s^{-2} + \ldots) = I$$

so that

$$\begin{aligned} A_0B_0 &= I \\ A_0B_k + A_1B_{k-1} + ...A_kB_0 &= 0, \quad k = 1, 2... \end{aligned} \tag{7.5}$$

Note that $A_0 = I - P_{(\infty)}K_{(\infty)}$. Thus $det(A_0) = det(I - P_{(\infty)}K_{(\infty)}) \neq 0$ by (7.5). **[If]** If $det(I - P_{(\infty)}K_{(\infty)}) \neq 0$, then A_0 is nonsingular. Hence B_0 exists. Then by (7.5), B_k for $k = 1, 2, 3..$ can be determined uniquely. Hence $(I - PK)^{-1}$ exists and is proper. Therefore the interconnection (7.1) is well-posed. □

The interconnection is called *stable* if it is well-posed and the transfer matrix from (u_1, u_2) to (e_1, e_2) is stable rational. Note that stability is defined in terms of the transfer matrix from u to e not u to y. The reason is that both notions of stability are equivalent.

Fact 7.1.2. *Suppose the system (7.1) is well-posed. The interconnection is* **stable** *if and only if the transfer matrix from* (u_1, u_2) *to* (y_1, y_2) *is stable.*

Proof. Observe that

$$\begin{aligned} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} P & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \\ &= \begin{bmatrix} P & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} I + K(I - PK)^{-1}P & K(I - PK)^{-1} \\ (I - PK)^{-1}P & (I - PK)^{-1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= \begin{bmatrix} P(I - KP)^{-1} & PK(I - PK)^{-1} \\ K(I - PK)^{-1}P & K(I - PK)^{-1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \end{aligned} \tag{7.6}$$

Using the matrix identities in (7.3),

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} (I - PK)^{-1}P & -I + (I - PK)^{-1} \\ -I + (I - KP)^{-1} & K(I - PK)^{-1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{7.7}$$

It is immediate by a comparison of (7.4) and (7.7) that the feedback interconnection is stable if and only if the transfer matrix from (u_1, u_2) to (y_1, y_2) is stable. □

Suppose now that P and K are described by the state space equations of the form

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_p e_1, \;\; y_1 = C_p x_p + D_p e_1, & (7.8) \\ \dot{x}_c &= A_c x_c + B_c e_2, \;\; y_2 = C_c x_c. & (7.9) \end{aligned}$$

For simplicity, we assume that K is strictly proper. Then, the overall system is described as

$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A_p & B_p C_c \\ B_c C_p & A_c + B_c D_p C_c \end{bmatrix} \begin{bmatrix} x_p \\ x_c \end{bmatrix} + \begin{bmatrix} B_p & 0 \\ B_c D_p & B_c \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 & C_c \\ C_p & D_p C_c \end{bmatrix} \begin{bmatrix} x_p \\ x_c \end{bmatrix} + \begin{bmatrix} I & 0 \\ D_p & I \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Let

$$\hat{A} := \begin{bmatrix} A_p & B_p C_c \\ B_c C_p & A_c + B_c D_p C_c \end{bmatrix}. \tag{7.10}$$

The feedback system (7.1) is *internally stable* if the overall state equation is asymptotically stable, i.e., $\sigma(\hat{A}) \subseteq \mathbf{C}_-$.

Theorem 7.1.1. *Suppose that the systems (7.8) and (7.9) are stabilizable, detectable, and the system (7.1) is well-posed. Under these conditions, the system (7.1) is internally stable if and only if it is stable.*

Proof. **[Only if]** If the interconnection is internally stable, then $\sigma(\hat{A}) \subseteq \mathbf{C}_-$. Hence the overall system is asymptotically stable and in particular the transfer function from u to e is stable rational.

[If] By Kalman Canonical Decomposition, if a system (A, B, C, D) is stabilizable and detectable, then stability of the transfer function $G(s) = C(sI - A)^{-1}B + D$ implies that $\sigma(A) \subseteq \mathbf{C}_-$. Note that the overall state representation is stabilizable if the plant and controller representations are stabilizable since with

$$\hat{B} := \begin{bmatrix} B_p & 0 \\ B_c D_p & B_c \end{bmatrix}$$

and $\hat{A}$ in (7.10),

$$\begin{aligned} rank \begin{bmatrix} sI - \hat{A} & \hat{B} \end{bmatrix} &= rank \begin{bmatrix} sI - A_p & -B_p C_c & B_p & 0 \\ -B_c C_p & sI - A_c - B_c D_p C_c & B_c D_p & B_c \end{bmatrix} \\ &= rank \begin{bmatrix} sI - A_p & B_p & 0 & 0 \\ 0 & 0 & sI - A_c & B_c \end{bmatrix}. \end{aligned}$$

Similarly if the plant and controller representations are detectable, overall state representation is also detectable. Thus the system (7.1) is internally stable. □

After defining the internal stability of the closed loop system, we characterize the stability of (7.1) in terms of the coprime factorizations over $\mathbf{S}$ of the plant and controller. This is the principal result of this section.

Lemma 7.1.1. *Let (N, M) be any right coprime factorization of P over $\mathbf{S}$, and let (Y, X) be any left coprime factorization of K over $\mathbf{S}$. Then the following are equivalent:*

(i) *The feedback interconnection (7.1) is (internally) stable.*

(ii) $\begin{bmatrix} YM & -X \\ -N & I \end{bmatrix}^{-1}$ *is a stable rational matrix.*

(iii) $(YM - XN)^{-1}$ *is a stable rational matrix.*

Proof. **(ii)$\Rightarrow$ (i)** By the definition of L, we have

$$L^{-1} = \begin{bmatrix} I & -Y^{-1}X \\ -NM^{-1} & I \end{bmatrix}^{-1} = \left(\begin{bmatrix} Y^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} YM & -X \\ -N & I \end{bmatrix} \begin{bmatrix} M^{-1} & 0 \\ 0 & I \end{bmatrix} \right)^{-1}$$
$$= \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} YM & -X \\ -N & I \end{bmatrix}^{-1} \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix}, \quad (7.11)$$

which shows that L^{-1} is stable since $\begin{bmatrix} YM & -X \\ -N & I \end{bmatrix}^{-1}$ is stable. This implies that the feedback interconnection is stable.

(i)$\Rightarrow$ (iii) From (7.4),

$$L^{-1} = \begin{bmatrix} M(YM - XN)^{-1}Y & M(YM - XN)^{-1}X \\ N(YM - XN)^{-1}Y & I + N(YM - XN)^{-1}X \end{bmatrix}.$$

If the feedback interconnection is stable, then each entry is stable. Since (N, M) is right coprime, $(YM - XN)^{-1}Y$ and $(YM - XN)^{-1}X$ are also stable. By left coprimeness of (Y, X), it follows that $(YM - XN)^{-1}$ is stable.

(iii)$\Rightarrow$ (ii) Observe that

$$\begin{bmatrix} YM & -X \\ -N & I \end{bmatrix}^{-1} = \begin{bmatrix} (YM - XN)^{-1} & (YM - XN)^{-1}X \\ N(YM - XN)^{-1} & I + N(YM - XN)^{-1}X \end{bmatrix}.$$

Thus, if $(YM - XN)^{-1}$ is stable, then left hand side is also stable. $\square$

Lemma 7.1.2. *Let $(\tilde{M}, \tilde{N})$ be any left coprime factorization of P over* **S**, *and let $(\tilde{X}, \tilde{Y})$ be any right coprime factorization of K over* **S**. *Then the following are equivalent:*

(i) *The feedback interconnection (7.1) is (internally) stable.*

(ii) $\begin{bmatrix} I & -\bar{X} \\ -\tilde{N} & \tilde{M}\bar{Y} \end{bmatrix}^{-1}$ *is a stable rational matrix.*

(iii) $(\tilde{M}\bar{Y} - \tilde{N}\bar{X})^{-1}$ *is a stable rational matrix.*

Proof. The proof follows along similar lines of Lemma (7.1.1). $\square$

7.2 Parametrization Of All Stabilizing Controllers

The main objective of this section is to parametrize all controllers that stabilize P in terms of a free parameter which ranges over stable rational matrices.

Theorem 7.2.1. *Let (N, M), $(\tilde{M}, \tilde{N})$ be any right and left coprime factorizations of P, respectively. Select matrices $\bar{P}$, $\bar{Q}$, $\hat{P}$, $\hat{Q}$ $\in$ $\mathbf{S}$ with compatible dimensions such that*

$$\begin{bmatrix} \bar{P} & \bar{Q} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & -\hat{Q} \\ N & \hat{P} \end{bmatrix} = I. \tag{7.12}$$

Then any controller K which stabilizes P is of the form

$$K = (\bar{P} + W\tilde{N})^{-1}(W\tilde{M} - \bar{Q}) \tag{7.13}$$

for some stable W such that $(\bar{P} + W\tilde{N})^{-1}$ exists and is proper. If P is strictly proper, any controller given in (7.13) stabilizes the plant for some stable W. Conversely, let W be any stable matrix of appropriate dimensions such that $(\bar{P} + W\tilde{N})^{-1}$ exists and is proper, then the corresponding controller $K = (\bar{P} + W\tilde{N})^{-1}(W\tilde{M} - \bar{Q})$ stabilizes P.

Proof. By Lemma (7.1.1), any stabilizing controller $K(s) = Y^{-1}X$ should be such that $(YM - XN)^{-1}$ exists and is stable. With $U := (YM - XN)$, $\tilde{Y} = U^{-1}Y$, $\tilde{X} = U^{-1}X$, any stabilizing controller $K(s) = \tilde{Y}^{-1}\tilde{X}$ should satisfy

$$\tilde{Y}M - \tilde{X}N = I. \tag{7.14}$$

Then

$$(\tilde{Y} - \bar{P})M - (\tilde{X} + \bar{Q})N = 0.$$

Thus $\tilde{Y} - \bar{P} = W\tilde{N}$, $\tilde{X} + \bar{Q} = W\tilde{M}$ for some stable W given by $W = \tilde{Y}\check{Q} + \tilde{X}\check{P}$. This follows since by (7.12) and (7.14), $W\tilde{N} = \tilde{Y} - \bar{P}$, $W\tilde{M} = \tilde{X} + \bar{Q}$. Therefore any stabilizing controller can be expressed by (7.13) for some stable W such that $(\bar{P} + W\tilde{N})^{-1}$ exists and is proper.

If $P(s)$ is strictly proper, then N and $\tilde{N}$ are also strictly proper. From $\bar{P}M + \bar{Q}N = I$, we have $\bar{P}_{(\infty)}M_{(\infty)} = I$ so that $\bar{P}_{(\infty)}$ is nonsingular. Note that $(\bar{P} + W\tilde{N})_{(\infty)} = \bar{P}_{(\infty)}$ and thus $(\bar{P} + W\tilde{N})^{-1}$ exists and is proper.

Conversely by Lemma (7.1.1), any controller stabilizes the system if and only if $(YM - XN)^{-1}$ exists and is stable, where $K(s) = Y^{-1}X$. If $K(s) = (\bar{P} + W\tilde{N})^{-1}(W\tilde{M} - \bar{Q})$, then

$$\begin{aligned}(YM - XN)^{-1} &= \left[\ (\bar{P} + W\tilde{N})^{-1}M - (W\tilde{M} - \bar{Q})M\ \right]^{-1} = \left[\ I + W(\tilde{N}M - \tilde{M}N)\ \right]^{-1} \\ &= I.\end{aligned}$$

Thus any K given by (7.13) stabilizes P. $\square$

In Theorem (7.2.1), there is a one-to-one correspondence between the parameter and the controller in the following sense: Suppose equation (7.12) holds. Then corresponding to each controller K, there is a unique W such that $K(s) = (\bar{P} + W\tilde{N})^{-1}(W\tilde{M} - \bar{Q})$. In fact, if $K = (\bar{P} + W_1\tilde{N})^{-1}(W_1\tilde{M} - \bar{Q}) =$

$(\bar{P} + W_2\tilde{N})^{-1}(W_2\tilde{M} - \bar{Q})$, then by the fact that both factorizations are left coprime, we have

$$\bar{P} + W_1\tilde{N} = U(\bar{P} + W_2\tilde{N}),\ W_1\tilde{M} - \bar{Q} = U(W_2\tilde{M} - \bar{Q})$$

for some unimodular U. These give $W_1 = UW_2$, $\bar{P} = U\bar{P}$, $\bar{Q} = U\bar{Q}$. By the last two equalities $U = I$ and $W_1 = W_2$.

7.3 Strong Stabilization

As a first application of the parametrization in Theorem (7.2.1), we consider the following problem called **strong stabilization problem**: *Given a plant P, when does there exist a stable controller K such that the feedback interconnection of Figure 7.1 is (internally) stable?* If such a stable controller exists, then P is called **strongly stabilizable**. One motivation for the strong stabilization problem is that if P is strongly stabilizable, then the resulting transfer matrix has the same $\mathbf{C}_{+e}$-zeros as P and no others. Refer to (7.6) and letting $YM - XN = I$,

$$\begin{aligned} P(I - KP)^{-1} &= NM^{-1}(I - Y^{-1}XNM^{-1})^{-1} \\ &= N(YM - XN)^{-1}Y = NY \end{aligned}$$

Since K is a stable controller, then Y is unimodular. As a result N and NY have the same $\mathbf{C}_{+e}$-zeros. On the other hand, stabilization by an unstable controller always introduce additional $\mathbf{C}_{+e}$-zeros in the closed loop transfer matrix beyond those of P. Since the ability of a plant to track reference signal and reject disturbances is affected by the $\mathbf{C}_{+e}$-zeros, it is desirable to use a stable controller in such situations.

By Theorem (7.2.1), any internally stabilizing controller is given by (7.13) for some stable matrix W. Such a controller is stable if and only if its denominator matrix $\bar{P} + W\tilde{N}$ has a stable inverse, or equivalently, if and only if $\bar{P} + W\tilde{N}$ is unimodular. The strong stabilization problem hence has a solution if and only if there exists a stable matrix W such that

$$U := \bar{P} + W\tilde{N}$$

is unimodular, where the stable rational matrices $\bar{P}$, $\tilde{N}$ satisfy (7.13). Note that $\tilde{N}$ is the numerator matrix of a left coprime factorization of the plant transfer matrix and $\bar{P}$ is a stable matrix that figures in the equation $\bar{P}M + \bar{Q}N = I$ for a right coprime factorization (N, M) of the plant transfer matrix. Both matrices $\tilde{N}$ and $\bar{P}$ are hence determined once the plant transfer matrix is given. The mathematical problem of choosing W such that $\bar{P} + W\tilde{N}$ is unimodular is solved in Youla, Bongiorno, Lu [41] and Vidyasagar [42]. Here, we give their result without proof.

Given a transfer matrix $P(s)$, a complex number s_0 is a **blocking zero** of P if $P(s_0) = 0$.

The following theorem states necessary and sufficient conditions for strong stabilizability based on the locations of real unstable poles and blocking zeros of the plant.

Theorem 7.3.1. *Given $P(s)$, there exists a stable $K(s)$ which stabilizes $P(s)$ if and only if the real unstable poles of $P(s)$ and the real unstable blocking zeros of $P(s)$ have a* **parity interlacing property**, *i.e., between every two real unstable blocking zeros of $P(s)$ there exists even number of poles of $P(s)$.*

Proof. [41]. □

7.4 Notes and References

Desoer and Chan [43] presented closed loop stability and its relation to state space internal stability. The definition and discussion of well-posedness of a composite system can be found in [44] and [45]. The idea which is the characterization of all controllers that stabilizes a given plant is first presented by Youla, Jabr and Bongiorno [46]. This characterization is different from the one in Section (7.2) since it contains both stable rational functions and polynomials. The characterization in Theorem (7.2.1) of all internally stabilizing controllers is due to Desoer, Liu, Murray and Saeks [47]. The book by Kučera [39] contains a characterization of all proper controllers for a particular Hurwitz set. In [41], strong stabilizability is defined and the necessary and sufficient conditions are given. Anderson [48] also presented a set of conditions for strong stabilizability based on Cauchy indeces. Simultaneous stabilization that is designing a controller which stabilizes each of a given family of plants involves the strong stabilization problem and is developed by Vidyasagar and Viswanadham [42].

Chapter 8

DISTURBANCE DECOUPLING

The unknown inputs to a system are generally termed as disturbance inputs. Disturbances are physical inputs to the system such as wind gusts influencing an aircraft or fluctuations of the feed stream in a distillation column. In control system design one of the additional objectives to stabilization or pole assignment is decoupling the effect of disturbances acting on the system from certain system outputs. Disturbance decoupling becomes especially significant when the designer has no knowlegde of the dynamics of these undesirable inputs. We study the problem of cancelling the effect of disturbances using state feedback and output measurement feedback. Two main approaches to this problem namely the geometric approach and the transfer matrix approach are reviewed and the solution techniques of these two approaches are illustrated.

8.1 A Disturbance Decoupled System

Consider a system with control input u, disturbance input w and output y

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew, \\ y &= Cx, \ \ t \geq 0, \end{aligned} \tag{8.1}$$

where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $E \in \mathbf{R}^{n\times q}$, $C \in \mathbf{R}^{p\times n}$. In this section, we investigate under what conditions on the matrices A, B, E, C the output $y(t)$ is independent of $w(t)$. The output of (8.1) can be written as

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + \int_0^t Ce^{A(t-\sigma)}Ew(\sigma)d\sigma.$$

It is independent of $w(t)$ if and only if

$$C\int_0^t e^{A(t-\sigma)}Ew(\sigma)d\sigma = 0, \ \forall t \geq 0, \tag{8.2}$$

or equivalently, if and only if

$$C(sI - A)^{-1}E = 0, \ \forall s \in \mathbf{C},$$

i.e., the transfer matrix from w to y is identically zero.

The following lemma gives a set of necessary and sufficient conditions to have a disturbance decoupled system.

Lemma 8.1.1. *The system (8.1) is disturbance decoupled if and only if one of the following equivalent conditions hold:*

(i) $Ce^{At}E = 0, \ \forall t \geq 0.$

(ii) $C(sI - A)^{-1}E = 0, \ \ \forall s \in \mathbf{C}.$

(iii) $CA^kE = 0, \ \forall k = 0, 1, ...$

(iv) $< A|ImE > \subseteq KerC.$

(v) $ImE \subseteq \bigcap_{i=1}^{n} Ker(CA^{i-1}).$

Proof. The equivalence of the conditions **(i)**-**(ii)** are easy to see. We show the equivalence of **(iii)**-**(v)**.

(iii) $\Leftrightarrow$ **(iv)** : By Cayley-Hamilton theorem, **(iii)** is equivalent to $CA^kE = 0$ for $k = 1, 2, ...n-1$, where $n = size(A)$. Hence,

$$< A|ImE >= Im[E\ AE\ ...\ A^{n-1}E] \subseteq KerC \Leftrightarrow CA^kE = 0,\ k = 0, 1, ...$$

(iv) $\Leftrightarrow$ **(v)** : $< A|ImE >= Im[E\ AE\ ...\ A^{n-1}E] \subseteq KerC \Leftrightarrow CA^kE = 0$, for $k = 0, 1, ... \quad \Leftrightarrow ImE \subseteq \bigcap_{i=1}^{n} Ker(CA^{i-1})$. □

By **(iv)** in Lemma (8.1.1), disturbance decoupling problem is solvable if and only if the largest A-invariant subspace containing ImE is contained in $KerC$. This turns out to be a useful condition in the solution of disturbance decoupling problem via state feedback.

8.2 Disturbance Decoupling By State Feedback

Suppose that the system (8.1) is not disturbance decoupled. The objective of this section is to find necessary and sufficient conditions for disturbance decoupling using state feedback.

Suppose that state feedback has been incorporated into (8.1) such that $u(t) = Fx(t)$, where $F \in \mathbf{R}^{m \times n}$. The closed loop system under state feedback is

$$\begin{aligned} \dot{x} &= (A + BF)x + Ew, \qquad (8.3) \\ y &= Cx,\ t \geq 0. \end{aligned}$$

Thus, disturbance decoupling by state feedback is possible if and only if one of the equivalent conditions of Lemma (8.1.1) with A replaced by $A + BF$ holds for some F. We now investigate when such an F exists by two different tecniques.

8.2.1 Geometric Approach

This approach is based on the use of certain spaces that are (A, B)-invariant which will be defined later. The following lemma states the necessary and sufficient condition to have disturbance decoupling system in a different manner.

Lemma 8.2.1. *The system (8.1) can be disturbance decoupled by state feedback if and only if there exists $F \in \mathbf{R}^{m\times n}$ and an $(A + BF)$-invariant subspace $\mathcal{V} \subseteq \mathbf{R}^n$ such that*

$$ImE \subseteq \mathcal{V} \subseteq KerC. \tag{8.4}$$

Proof. If the system is disturbance decoupled, then by Lemma (8.1.1) $< A + BF|ImE >\subseteq KerC$. Let $\mathcal{V} :=< A + BF|ImE >$. This subspace is $(A+BF)$-invariant and satisfies (8.4). Conversely, if (8.4) is satisfied by an $(A+BF)$-invariant subspace $\mathcal{V} \subseteq \mathbf{R}^n$ and $F \in \mathbf{R}^{m\times n}$, then since, by Fact (2.5.1), $< A + BF|ImE >$ is the smallest $(A + BF)$-invariant subspace containing Im E, we have $< A + BF|ImE >\subseteq \mathcal{V}$. By (8.4), $< A + BF|ImE >\subseteq KerC$, so the system (8.3) is disturbance decoupled by Lemma (8.1.1). □

In order to obtain a condition purely in terms of the problem data A, B, E, C, we examine $(A + BF)$-invariance more closely.

Lemma 8.2.2. *The following three statements on a subspace $\mathcal{V} \subseteq \mathbf{R}^n$ are equivalent:*

(i) *$\mathcal{V}$ is $(A+BF)$-invariant for some $F \in \mathbf{R}^{m\times n}$.*

(ii) *$A\mathcal{V} \subseteq \mathcal{V} + ImB$.*

(iii) *For every $v_0 \in \mathcal{V}$, there exists an input $u(t)$, $t \geq 0$, such that the solution of the system*

$$\dot{x} = Ax + Bu, \; t \geq 0, \tag{8.5}$$

with the initial condition $x(0) = v_0$ satisfies $x(t) \in \mathcal{V}, \forall t \geq 0$.

Proof. **(ii)** $\Rightarrow$ **(i)** : Let V be a basis matrix for the subspace $\mathcal{V}$. Then

$$AV = VA_0 + BB_0$$

for some matrices $A_0 \in \mathbf{R}^{dim\mathcal{V}\times dim\mathcal{V}}$, $B_0 \in \mathbf{R}^{m\times dim\mathcal{V}}$. Since V has full column rank it has a left inverse $V^{\#}$ satisfying $V^{\#}V = I$. Let $F = -B_0V^{\#}$. Then,

$$(A+BF)V = (A - BB_0V^{\#})V = AV - BB_0 = VA_0.$$

and hence $\mathcal{V}$ is $(A+BF)$-invariant.

(i) $\Rightarrow$ **(iii)** : If $\mathcal{V}$ is $(A+BF)$-invariant, then for $v_0 \in \mathcal{V}$,

$$e^{(A+BF)t}v_0 \in \mathcal{V}.$$

Letting $u = Fx$, the solution of the system (8.5) with initial condition v_0 remains in $\mathcal{V}$ by the application of this input.

(iii) $\Rightarrow$ **(ii)** : Since both v_0 and $x(t)$ $\forall t \geq 0$ are in $\mathcal{V}$, it follows that $\dot{x}(0) = \lim_{t\to 0} \frac{1}{t}[x(t) - v_0]$ is also in $\mathcal{V}$. But then, by (8.5) and by **(iii)**,

$$\dot{x}(0) = Av_0 + Bu_0 \in \mathcal{V}, \; \forall v_0 \in \mathcal{V}.$$

Hence, $A\mathcal{V} \in \mathcal{V} + ImB$. □

Definition 8.2.1. *Let $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$. We say that a subspace $\mathcal{V} \in \mathbf{R}^{n\times m}$ is (A,B)-**invariant** if one of the conditions in Lemma (8.2.2) holds.*

Note that any A-invariant subspace is automatically (A,B)-invariant by putting $F = 0$ in condition **(i)** of Lemma (8.2.2). Let us denote the class of (A,B)-invariant subspaces of $\mathbf{R}^n$ contained in a subspace S by $\mathcal{V}\{A,B;S\}$. The following property about (A,B)-invariant subspaces in S is important for our problem. Let $\mathcal{V}^*(A,B;S)$ be the largest or supremal element of $\mathcal{V}\{A,B;S\}$.

Lemma 8.2.3. *For every subspace $S \subseteq \mathbf{R}^n$, $\mathcal{V}\{A,B;S\}$ contains a supremal element $\mathcal{V}^*(A,B;S)$.*

Proof. From Lemma (8.2.2), if $\mathcal{V}_1, \mathcal{V}_2 \in \mathcal{V}\{A,B;S\}$, then

$$\begin{aligned} A(\mathcal{V}_1+\mathcal{V}_2) &= A\mathcal{V}_1 + A\mathcal{V}_2 \\ &\subseteq \mathcal{V}_1+\mathcal{V}_2+ImB, \end{aligned}$$

hence, $\mathcal{V}_1 + \mathcal{V}_2 \in \mathcal{V}(A,B;S)$. Since S is finite dimensional, there exists a supremal element of $\mathcal{V}\{A,B;S\}$ which contains every other element in $\mathcal{V}\{A,B;S\}$. □

The following theorem which gives a condition for the solvability of our problem in terms of problem data follows immediately from Lemma (8.2.2) and the definition of $\mathcal{V}^*(A,B;S)$.

Theorem 8.2.1. *Disturbance decoupling problem by state feedback (DDPSF) is solvable if and only if*

$$ImE \subseteq \mathcal{V}^*(A,B;KerC).$$

An algorithm for the computation of $\mathcal{V}^*(A, B; KerC)$ is given next.

Theorem 8.2.2. *Let $A \in \mathbf{R}^{n\times n}, B \in \mathbf{R}^{n\times m}$ and let S be a subspace of $\mathbf{R}^n$. Define a sequence of subspaces $\mathcal{V}^i$ by*

$$\begin{aligned}\mathcal{V}^0 &:= S,\\ \mathcal{V}^i &:= S \cap A^{-1}(\mathcal{V}^{i-1} + ImB);\ i = 1, 2, \ldots.\end{aligned}$$

where for a subspace τ, $A^{-1}(\tau) := \{x \in \mathbf{R}^n : Ax \in \tau\}$. There exists $k \leq n$ such that

$$\mathcal{V}^i = \mathcal{V}^*(A, B; S)$$

for all $i \geq k$.

Proof. Observe that the sequence $\mathcal{V}^i$ is nonincreasing. The proof is by induction. Clearly $\mathcal{V}^1 \subseteq \mathcal{V}^0$. If $\mathcal{V}^i \subseteq \mathcal{V}^{i-1}$, then

$$\begin{aligned}\mathcal{V}^{i+1} &= S \cap A^{-1}(\mathcal{V}^i + ImB)\\ &\subseteq S \cap A^{-1}(\mathcal{V}^{i-1} + ImB)\\ &= \mathcal{V}^i.\end{aligned}$$

Thus, for some k, $\mathcal{V}^i = \mathcal{V}^k$ $(i \geq k)$. Moreover, $\mathcal{V}^k$ is (A, B)-invariant and contained in S. In fact, since $\mathcal{V}^k \subseteq S$, we have

$$\begin{aligned}A\mathcal{V}^k = A\mathcal{V}^{k+1} &\subseteq A(S \cap A^{-1}(\mathcal{V}^k + ImB))\\ &\subseteq \mathcal{V}^k + ImB.\end{aligned}$$

Now, $\mathcal{V} \in \mathcal{V}\{A, B; S\}$ if and only if

$$\mathcal{V} \subseteq S,\ \mathcal{V} \subseteq A^{-1}(\mathcal{V} + ImB). \tag{8.6}$$

From (8.6), $\mathcal{V} \subseteq \mathcal{V}^0$, and if $\mathcal{V} \subseteq \mathcal{V}^{i-1}$,

$$\begin{aligned}
\mathcal{V} &\subseteq S \cap A^{-1}(\mathcal{V} + ImB) \\
&\subseteq S \cap A^{-1}(\mathcal{V}^{i-1} + ImB) \\
&= \mathcal{V}^i.
\end{aligned}$$

Hence, $\mathcal{V} \subseteq \mathcal{V}^k$. Since $\mathcal{V}$ is arbitrary, $\mathcal{V}^k = \mathcal{V}^*(A, B; S)$. □

Theorems (8.2.1) and (8.2.2) give a constructive solution to the disturbance decoupling problem by state feedback based on the geometric approach.

8.2.2 Transfer Matrix Approach

In this section, we examine the same problem by using a different technique. We show that the problem can be transformed to existence of a solution to a matrix equation of the form $G_2(s) + G_1(s)Y(s) = 0$. We also incorporate stability requirement and investigate the construction of a solution.

Theorem 8.2.3. *Consider the system (8.3) and suppose that (C, A) is observable. Let $G_1(s) := C(sI - A)^{-1}B$, $G_2(s) := C(sI - A)^{-1}E$. DDPSF has a solution if and only if the equation*

$$G_1(s)Y(s) + G_2(s) = 0 \tag{8.7}$$

has a strictly proper rational matrix solution $Y(s)$.

Proof. **[Only if]** If DDPSF has a solution, then by Lemma (8.1.1), there exists a state feedback $F \in \mathbf{R}^{m \times n}$ such that

$$C(sI - A - BF)^{-1}E = 0.$$

Then

$$C(sI - A)^{-1}\left[I - BF(sI - A)^{-1} \right]^{-1} E = 0.$$

Using the matrix identities in (7.3),

$$\begin{aligned} C(sI - A)^{-1}\left\{ I + BF(sI - A)^{-1}\left[I - BF(sI - A)^{-1} \right]^{-1} \right\} E &= 0 \\ C(sI - A)^{-1}E + C(sI - A)^{-1}BF(sI - A - BF)^{-1}E &= 0. \end{aligned}$$

With $G_1(s)$, $G_2(s)$ defined above and with

$$Y(s) := F(sI - A - BF)^{-1}E, \tag{8.8}$$

(8.7) is satisfied.

[if] Let $Y(s)$ be a strictly proper solution of (8.7) and let F_0, K_0, L_0 be such that

$$Y(s) = F_0(sI - K_0)^{-1}L_0,$$

where (F_0, K_0) is observable and (K_0, L_0) is reachable. In what follows we construct the required F out of F_0. Let $N(s)$, $M(s)$ be right coprime polynomial matrices such that

$$(sI - K_0)^{-1}L_0 = N(s)M(s)^{-1}. \tag{8.9}$$

By (8.7) and (8.9),

$$C(sI - A)^{-1}[BF_0N + EM] = 0.$$

By observability of (C, A), and by Fact (6.4.4), we have

$$\begin{aligned} EM + BF_0N &= (sI - A)T, \\ CT &= 0 \end{aligned} \tag{8.10}$$

for some polynomial matrix $T(s)$. As (N, M) is right coprime, there exist polynomial matrices P, Q such that

$$PM + QN = I$$

and

$$TPM + TQN = T. \tag{8.11}$$

Dividing TQ on the right by $(sI - K_0)$, we have

$$TQ = \tilde{T}(sI - K_0) + V, \tag{8.12}$$

where V is constant and $\tilde{T}$ is a polynomial matrix. By (8.11) and (8.12),

$$(TP + \tilde{T}L_0) + VNM^{-1} = TM^{-1}.$$

Since TM^{-1}, VNM^{-1} are strictly proper and $(TP + \tilde{T}L_0)$ is a polynomial matrix,

$$\begin{aligned} TP + \tilde{T}L_0 &= 0 \\ T &= VN. \end{aligned} \tag{8.13}$$

By (8.10) and (8.13),

$$E = VL_0. \tag{8.14}$$

We now show that N has **R**-linearly independent rows.

If $gN(s) = 0$ for some constant g, then $gNM^{-1} = g(sI - K_0)^{-1}L_0 = 0$. By reachability of (K_0, L_0) and by Fact (6.4.4), we have

$$(sI - K_0)\tilde{P} + L_0\tilde{Q} = I$$

for some polynomial matrices $\tilde{P}$ and $\tilde{Q}$. Hence,

$$g\tilde{P}(s) = g(sI - K_0)^{-1}$$

so that $g = 0$. This proves that N has **R**-linearly independent rows. Thus by (8.10) and (8.13),

$$CV = 0$$

and by (8.9), (8.10), (8.13)and (8.14)

$$AV = VK_0 + BF_0.$$

It follows that $span\{V\}$ is an (A, B)-invariant subspace containing ImE, and by (8.14), contained in $KerC$. If V has linearly independent columns, the required state feedback is constructed as

$$F := -F_0 V^{\#},$$

where $V^{\#}V = I$. If V does not have full column rank, then let U be a nonsingular constant matrix such that $VU = \begin{bmatrix} V_1 & 0 \end{bmatrix}$ with V_1 having full column rank. It is easy to see that there exist constant matrices L_1, K_{11}, F_1 satisfying

$$\begin{aligned} E &= V_1 L_1 \\ CV_1 &= 0 \\ AV_1 &= V_1 K_{11} + BF_1. \end{aligned}$$

Hence, $F := -F_1 V_1^{\#}$, where $V_1^{\#} V_1 = I$ is a solution. □

We now examine the problem of DDPSF with stability. The following theorem gives the necessary and sufficient condition.

Theorem 8.2.4. *Consider the system (8.3). Suppose (C, A) is observable. Let $G_1(s) = C(sI - A)^{-1}B$, $G_2(s) = C(sI - A)^{-1}E$. DDPSF with stability has a solution if and only if (A, B) is stabilizable and the equation*

$$G_1(s)Y(s) + G_2(s) = 0 \tag{8.15}$$

has a strictly proper and stable rational matrix solution $Y(s)$.

Proof. **[Only if]** If DDPSF with stability has a solution, then there exists a state feedback $F \in \mathbf{R}^{m \times n}$ such that

$$C(sI - A - BF)^{-1}E = 0$$

and

$$\sigma(A + BF) \subseteq \mathbf{C}_-.$$

Hence (A, B) is stabilizable, $Y(s)$ in (8.8) is a strictly proper and stable solution of (8.15).

[if] Let $Y(s) = F_0(sI - K_0)^{-1}L_0$ be as in the proof of Theorem (8.2.3) with (F_0, K_0) observable and (K_0, L_0) reachable. In addition, since $Y(s)$ is a stable rational matrix, $\sigma(K_0) \subseteq \mathbf{C}_-$. We obtain a full column rank constant matrix V such that

$$\begin{aligned} AV &= VK_0 + BF_0 \\ CV &= 0 \\ E &= VL_0 \end{aligned} \tag{8.16}$$

as in the proof of Theorem (8.2.3). Since V has full column rank, there exist matrices W, $\tilde{W}$, $\tilde{V}$ of compatible sizes such that

$$\begin{bmatrix} W \\ \tilde{W} \end{bmatrix} \begin{bmatrix} V & \tilde{V} \end{bmatrix} = I.$$

Let $\tilde{A} := A - BF_0W$, then for some constant matrices K_1, K_2, B_1 and B_2, we have

$$\begin{aligned} \tilde{A} \begin{bmatrix} V & \tilde{V} \end{bmatrix} &= \begin{bmatrix} V & \tilde{V} \end{bmatrix} \begin{bmatrix} K_0 & K_1 \\ 0 & K_2 \end{bmatrix}, \\ B &= \begin{bmatrix} V & \tilde{V} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \end{aligned}$$

By stabilizability of (A, B), the pair $(A - BF_0W, B)$ is also stabilizable. Thus, by Corollary (2.5.2),

$$rank \begin{bmatrix} sI - K_0 & -K_1 & B_1 \\ 0 & sI - K_2 & B_2 \end{bmatrix} = n, \ \forall s \in \mathbf{C}_{0+},$$

where $n = size(\check{A})$. This implies that (K_2, B_2) is also stabilizable. So there exists F_2 with compatible size such that

$$\sigma(K_2 + B_2F_2) \subseteq \mathbf{C}_-.$$

Now,

$$\begin{aligned}(\check{A} + BF_2\tilde{W})\begin{bmatrix} V & \tilde{V} \end{bmatrix} &= \begin{bmatrix} V & \tilde{V} \end{bmatrix}\begin{bmatrix} K_0 & K_1 \\ 0 & K_2 \end{bmatrix} + \begin{bmatrix} V & \tilde{V} \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\begin{bmatrix} 0 & F_2 \end{bmatrix} \\ &= \begin{bmatrix} V & \tilde{V} \end{bmatrix}\begin{bmatrix} K_0 & K_1 + B_1F_2 \\ 0 & K_2 + B_2F_2 \end{bmatrix}.\end{aligned}$$

Hence $\sigma(\check{A} + BF_2\tilde{W}) \subseteq \mathbf{C}_-$. Therefore, $F := F_2\tilde{W} - F_0W$ is a stabilizing feedback achieving decoupling for the system (8.3). □

8.3 Disturbance Decoupling By Measurement Feedback

If the whole state is not available for feedback but some prescribed output is, then disturbance decoupling can be achieved under more restrictive conditions. In this section we formulate and solve a general disturbance decoupling problem by measurement feedback (DDPMF) without and with stability.

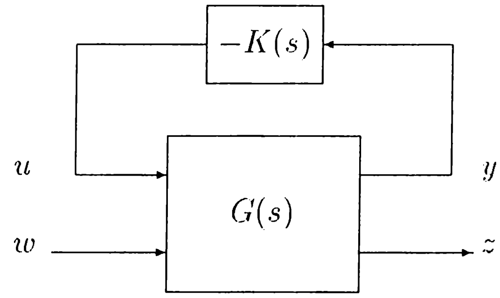

Figure 8.1: Two channel system with measurement feedback

Consider a plant $G(s)$ having two vector inputs and two vector outputs. Control input, disturbance input, measured output, and controlled output are represented by u, w, y and z respectively. A dynamic feedback controller is applied at the input u and output y so that the resulting closed loop system has input w and output z. The first problem considered is to determine such a controller $K(s)$ such that the controlled output z is independent of the disturbance input w. The closed loop system is illustrated in Figure 8.1. We have

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix}, \quad u = -K(s)y, \tag{8.17}$$

where we assume that $G_{11}(s)$ is strictly proper for well-definedness of the feedback loop. For decoupling, the disturbance input to control output transfer matrix

$$T_{zw} = G_{22} - G_{21}K(I + G_{11}K)^{-1}G_{12} \tag{8.18}$$

should be identically zero.

Lemma 8.3.1. *Consider the system (8.17). Assume that G_{11} is strictly proper. DDPMF is solvable if and only if there exists a proper rational matrix*

$Y(s)$ *satisfying*

$$G_{22} = G_{21} Y G_{12}. \tag{8.19}$$

Proof. **[Only if]** If DDPMF is solvable, then there exists a controller K such that (8.18) is satisfied. Let $Y(s) := K(I + G_{11}K)^{-1}$. Then $Y(s)$ is proper and satisfies (8.19).

[If] If there exists a proper $Y(s)$ such that (8.19) is satisfied, then let $K := (I - YG_{11})^{-1}Y$. Thus (8.18) is satisfied. Note that since G_{11} is strictly proper and $Y(s)$ is proper, K is also proper. □

We now impose the additional requirement that the closed loop is internally stable. Thus, DDPMF with stability requires the existence of a controller $K(s)$ which solves internal stability problem for the plant $G_{11}(s)$ and, T_{zw} in (8.18) is identically zero. Let

$$G_{11} = NM^{-1} = \tilde{M}^{-1}\tilde{N}$$

be a doubly coprime stable factorization of G_{11}. Thus, there exist stable matrices P, Q, $\tilde{P}$, $\tilde{Q}$ such that

$$\begin{bmatrix} P & Q \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & -\tilde{Q} \\ N & \tilde{P} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

By Theorem (7.2.1), any K which internally stabilizes G_{11} is necessarily of the form

$$K(s) = (P - \Theta\tilde{N})^{-1}(Q + \Theta\tilde{M}) \tag{8.20}$$

for some stable proper rational matrix Θ.

Lemma 8.3.2. *Consider the system (8.17). DDPMF with stability is solvable if and only if the following equation*

$$G_{22} - G_{21}MQG_{12} = G_{21}M\Theta\tilde{M}G_{12} \tag{8.21}$$

has a stable proper rational matrix solution Θ.

Proof. **[Only if]** If DDPMF with stability is solvable, then there exists a controller $K(s)$ of the form (8.20) which satisfies (8.18). Let $\hat{P} := P - \Theta\tilde{N}$ and $\hat{Q} := Q + \Theta\tilde{M}$. Then

$$\begin{aligned} G_{22} &= G_{21}K(I + G_{11}K)^{-1}G_{12} \\ &= G_{21}\hat{P}^{-1}\hat{Q}(I + NM^{-1}\hat{P}^{-1}\hat{Q})^{-1}G_{12} \\ &= G_{21}(I + \hat{P}^{-1}\hat{Q}NM^{-1})^{-1}\hat{P}^{-1}\hat{Q}G_{12} \\ &= G_{21}M(\hat{P}M + \hat{Q}N)^{-1}\hat{Q}G_{12} \\ &= G_{21}M\hat{Q}G_{12} \\ &= G_{21}M(Q + \Theta\tilde{M})G_{12}. \end{aligned}$$

Thus (8.21) is satisfied.

[If] if (8.21) is satisfied for some stable proper Θ, then with this Θ, $K(s)$ in (8.20) is a stabilizing controller satisfying (8.18). $\square$

We now state a more compact solvability condition to disturbance decoupling problem by measurement feedback without and with stability in terms of polynomial system matrices.

Let

$$\dot{x} = Ax + \begin{bmatrix} B & E \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix}, \quad \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} C \\ D \end{bmatrix} x + \begin{bmatrix} 0 & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix}$$

be a canonical realization of the system (8.17).

Theorem 8.3.1. *Suppose that (A, B) is stabilizable and (C, A) is detectable. Consider the equation*

$$\begin{bmatrix} sI - A & E \\ -D & W_{22} \end{bmatrix} = \begin{bmatrix} sI - A & B \\ -D & W_{21} \end{bmatrix} X(s) \begin{bmatrix} sI - A & E \\ -C & W_{12} \end{bmatrix}. \tag{8.22}$$

(a) Disturbance decoupling problem by measurement feedback without stability is solvable if and only if the equation (8.22) has a proper rational matrix solution $X(s)$.

(b) Disturbance decoupling problem by measurement feedback with stability is solvable if and only if the equation (8.22) has a stable, proper rational matrix solution $X(s)$.

Proof. See [49] □

The matrices that figure in (8.19) are over the principle domain of proper rational matrices. The equation (8.21) can on the other hand be transformed to an equation over the principle domain of stable proper rational matrices. The solvability conditions to matrix equations in terms of system zeros can be found in Özgüler [50]. The solvability of (8.22) on the other hand in terms of system zeros is discussed in [49].

8.4 Notes and References

The idea of (A, B)-invariant subspaces and the results given in Theorem (8.2.1) and Theorem (8.2.2) were presented independently by Basile and Marro [51], [52] and by Wonham and Morse [20]. Numerical aspects of the computation of supremal element $\mathcal{V}^*(A, B; KerC)$ are discussed by Moore and Laub [53] and by Linnemann [54]. The transfer matrix approach to DDPSP has been given by [55] in a model matching context. The solution of DDP by measurement feedback via transfer matrix approach is obtained by Ohm, Howze and Bhattacharyya [56]. In Özgüler and Eldem [49], a polynomial fractional approach has been used to obtain a solution to DDP with measurement feedback without or with stability in terms of matrix equations involving polynomial system matrices.

Chapter 9

TRACKING AND REGULATION

A general control problem requires the design of a controller such that the closed loop system is internally stable,

(i) The output of the plant tracks a desired reference signal, and

(ii) The output of the plant rejects the effect of a disturbance signal.

We shall assume that the reference and disturbance variables satisfy known, time invariant, linear differential equations so that the combined system can be obtained. This combined system need not be controllable and observable or even stabilizable and detectable.

In this chapter, we first consider a prototype regulation problem known as output stabilization problem. We then consider a scalar system and obtain a necessary and sufficient condition for the solvability of a tracking with regulation problem for this scalar system. We then discuss regulator problem with internal stability (RPIS) for a multivariable system with a single output channel.

9.1 Output Stabilization Problem

We consider the system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ z &= Dx, \quad t \geq 0, \end{aligned} \tag{9.1}$$

where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, and $D \in \mathbf{R}^{p\times n}$. The problem is to stabilize the output $z(t)$ using state feedback. Hence, the aim is to determine a state feedback F, such that $\forall x_0$,

$$\|z(t)\| = \|D\ e^{(A+BF)t}\ x_0\| \to 0\ as\ t \to \infty. \tag{9.2}$$

Note that (9.2) is equivalent to the requirement that $D(sI - A - BF)^{-1}$ is a stable transfer matrix.

The conditions for the existence of state feedback F such that (9.2) is satisfied are stated in this section. Now we have

Theorem 9.1.1. *Given the system (9.1). Suppose* (D, A) *is observable and* $\sigma(A) \subseteq \mathbf{C}_{0+}$. *The following are equivalent statements:*

(i) *Output stabilization problem (OSP) is solvable.*

(ii) *There exists a stable transfer matrix X(s) and a strictly proper transfer matrix* $Y(s)$ *such that*

$$X(s) = D(sI - A)^{-1} + D(sI - A)^{-1}BY(s). \tag{9.3}$$

(iii) $< A|ImB > +\mathcal{V}^*(A, B; KerD) = \mathbf{R}^n$.

Proof. **(i) $\Rightarrow$ (ii):** If OSP is solvable, then there exists F such that $D(sI - A - BF)^{-1}$ is a stable rational matrix. Let $X(s) = D(sI - A - BF)^{-1}$ so that

$$\begin{aligned} X(s) &= D(sI-A)^{-1}[I - BF(sI-A)^{-1}]^{-1} \\ &= D(sI-A)^{-1}\left\{ I + BF(sI-A)^{-1}[I - BF(sI-A)^{-1}]^{-1} \right\} \\ &= D(sI-A)^{-1} + D(sI-A)^{-1}BF(sI-A-BF)^{-1}. \end{aligned}$$

Thus, with $Y(s) = F(sI - A - BF)^{-1}$, (ii) holds.

(ii) $\Rightarrow$ (iii): Let $Y(s) = F_0(sI - K_0)^{-1}L_0$ be a canonical realization of $Y(s)$ and let

$$(sI - K_0)^{-1}L_0 = N(s)M(s)^{-1} \tag{9.4}$$

for right coprime polynomial matrices N, M. Then by (9.3) and (9.4),

$$XM = D(sI - A)^{-1}[M + BF_0N].$$

By the assumption of observability of (D, A) and by Fact (6.4.4), D and $(sI-A)$ are right coprime. By Fact (6.4.2), it follows that

$$XM = D\bar{X} \tag{9.5}$$

for a stable rational matrix

$$\bar{X} = (sI - A)^{-1}[M + BF_0N]. \tag{9.6}$$

Since $\sigma(A) \subseteq \mathbf{C}_{0+}$, $\bar{X}$ is also an unstable rational matrix. It follows that $\bar{X}$ should be a polynomial matrix. As (N, M) is right coprime, there exist polynomial matrices P, Q such that

$$PM + QN = \bar{X}.$$

Dividing Q by $(sI - K_0)$ on the right, we have

$$Q = \bar{Q}(sI - K_0) + V$$

for a polynomial matrix $\bar{Q}$ and a constant matrix V. Hence

$$PM + \bar{Q}(sI - K_0)N + VN = \bar{X}.$$

Since $\bar{X}M^{-1}$ is strictly proper by (9.6), from this equality we obtain

$$\bar{X} = VN. \tag{9.7}$$

Substituting (9.7) in (9.6) and (9.5),

$$\begin{aligned} X &= DV(sI - K_0)^{-1}L_0 \\ I &= [(sI - A)V - BF_0](sI - K_0)^{-1}L_0. \end{aligned}$$

By reachability of (K_0, L_0), $(sI - K_0)$ and L_0 are left coprime. Fact (6.4.2) gives

$$\begin{aligned} (sI - A)V - BF_0 &= V(sI - K_0) && (9.8) \\ I &= VL_O && (9.9) \\ \hat{X} &:= DV(sI - K_0)^{-1} \text{ is stable rational.} && (9.10) \end{aligned}$$

By (9.8), $span\{V\}$ is (A, B)-invariant and by (9.9), V is of full row rank.

We now identify the part of $span\{V\}$ which is in $KerD$. Let T be a nonsingular matrix such that $(DVT, T^{-1}K_0T)$ is in observable canonical form, i.e.

$$DVT = \begin{bmatrix} D_1 & 0 \end{bmatrix}, T^{-1}K_0T = \begin{bmatrix} K_1 & 0 \\ K_3 & K_2 \end{bmatrix}, \tag{9.11}$$

where (D_1, K_1) is observable. Let

$$VT = \begin{bmatrix} V_1 & V_2 \end{bmatrix}, \; T = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \tag{9.12}$$

with T_1, V_1 having the same number of columns as D_1. By (9.8), (9.11) and (9.12), we have

$$
\begin{aligned}
(sI - A)V_1 - BF_0T_1 &= V_1(sI - K_1) - V_2K_3 &\quad (9.13)\\
(sI - A)V_2 - BF_0T_2 &= V_2(sI - K_2) &\quad (9.14)\\
DV_2 &= 0 &\quad (9.15)
\end{aligned}
$$

and by (9.10) and (9.11),

$$\hat{X}T = D_1(sI - K_1)^{-1}.$$

By observability of (D_1, K_1) and by stability of $\hat{X}T$,

$$\sigma(K_1) \subseteq \mathbf{C}_-.$$

Since V is of full row rank,

$$\mathbf{R}^n = span\{V_1\} + span\{V_2\},$$

where by (9.14) and (9.15)

$$span\{V_2\} \subseteq \mathcal{V}^*(A, B; KerD). \qquad (9.16)$$

We now show that $span\{V_1\} \subseteq< A|ImB > +span\{V_2\}$ which will complete the proof that $\mathbf{R}^n =< A|ImB > +\mathcal{V}^*(A, B; KerD)$. Let $\check{V}_2$ be a basis matrix for $span\{V_2\}$ and let $\check{V}_1$ be such that $[\check{V}_1\ \check{V}_2]$ is nonsingular. Hence there exist $\hat{V}_1, \hat{V}_2$ such that

$$\begin{bmatrix} \check{V}_1\check{V}_2 \end{bmatrix} \begin{bmatrix} \hat{V}_1 \\ \hat{V}_2 \end{bmatrix} = I.$$

Note that $V_1 = \check{V}_1W_1 + \check{V}_2W_2$ for some W_2 and full row rank W_1. By (9.14) and (9.15),

$$(sI - A)\check{V}_2 - BF_0\check{T}_2 = \check{V}_2(sI - \check{K}_2),\ D\check{V}_2 = 0$$

for some suitable $\tilde{T}_2, \tilde{V}_2$. Thus,

$$(sI - A - BF_0\tilde{T}_2\hat{V}_2)\tilde{V}_2 = \tilde{V}_2(sI - \tilde{K}_2) \quad (9.17)$$

$$D\tilde{V}_2 = 0. \quad (9.18)$$

Substituting $V_1 = \tilde{V}_1 W_1 + \tilde{V}_2 W_2$ in (9.13) and adding $BF_0\tilde{T}_2\hat{V}_2\tilde{V}_1 = 0$,

$$(sI - A - BF_0\tilde{T}_2\hat{V}_2)(\tilde{V}_1 W_1 + \tilde{V}_2 W_2) - BF_0\tilde{T}_1 = (\tilde{V}_1 W_1 + \tilde{V}_2 W_2)(sI - K_1) - \tilde{V}_2\tilde{K}_3$$

for some matrices $\tilde{T}_1$ and $\tilde{K}_3$. Using (9.17),

$$(sI - A - BF_0\tilde{T}_2\hat{V}_2)\tilde{V}_1 W_1 = \tilde{V}_2(\tilde{K}_2 W_2 - W_2 K_1 - \tilde{K}_3) + \tilde{V}_1 W_1(sI - K_1) + BF_0\tilde{T}_1. \quad (9.19)$$

Multiplying every term on the right by a right inverse $\hat{W}_1$ for W_1, we have

$$(sI - A - BF_0\tilde{T}_2\hat{V}_2)\tilde{V}_1 = \tilde{V}_2\theta + \tilde{V}_1(sI - W_1 K_1 \hat{W}_1) + BF_0\tilde{T}_1\hat{W}_1 \quad (9.20)$$

for some constant θ. The next step is to show that $(W_1 K_1 \hat{W}_1, \hat{V}_1 B)$ is reachable. Suppose not, so that for some eigenvalue λ of $(W_1 K_1 \hat{W}_1)$ and for some nonzero vector y, we have

$$y^T(\lambda I - W_1 K_1 \hat{W}_1) = 0, \; y^T\hat{V}_1 B = 0.$$

Then by (9.20) and (9.17),

$$y^T\hat{V}_1(\lambda I - A - BF_0\tilde{T}_2\hat{V}_2)\tilde{V}_1 = y^T\hat{V}_1(\lambda I - A)\tilde{V}_1 = 0$$

$$y^T\hat{V}_1(\lambda I - A)\tilde{V}_2 = 0.$$

Hence λ is an eigenvalue of A, since $y^T\hat{V}_1 \neq 0$. By (9.19),

$$y^T W_1(\lambda I - K_1) = 0.$$

Therefore λ is an eigenvalue of K_1. However $\sigma(A) \cap \sigma(K_1) = \emptyset$ which yields a contradiction. Hence $(W_1 K_1 \hat{W}_1, \hat{V}_1 B)$ is reachable and by Fact (6.4.3), there exist polynomial matrices Q_2, P_2 such that

$$(sI - W_1 K_1 \hat{W}_1) P_2(s) + \hat{V}_1 B Q_2(s) = I.$$

Using $\tilde{V}_1 \hat{V} = I - \tilde{V}_2 \hat{V}_2$ and multiplying on the left by $\tilde{V}_1$,

$$\tilde{V}_1 (sI - W_1 K_1 \hat{W}_1) P_2(s) + B Q_2(s) - \tilde{V}_2 \hat{V}_2 B Q_2(s) = \tilde{V}_1.$$

By (9.20) and (9.17),

$$\begin{aligned} \tilde{V}_1 &= (sI - A)\tilde{V}_1 P_2(s) + BQ_3(s) + \tilde{V}_2 Q_4(s) \\ &= (sI - A - BF_0 \tilde{T}_2 \hat{V}_2)\tilde{V}_1 P_2(s) + BQ_3(s) + \tilde{V}_2 Q_4(s) \\ &= (sI - A - BF_0 \tilde{T}_2 \hat{V}_2) P_3(s) + BQ_3(s) + \tilde{V}_2 L_1 \end{aligned}$$

for some polynomial matrices $Q_4(s)$, $Q_3(s)$, $P_3(s)$, and a constant matrix L_1. By Fact (6.4.3), it follows that

$$span\{\tilde{V}_1\} \subseteq < A|ImB > + span\{\tilde{V}_2\}$$

and

$$span\{V_1\} \subseteq < A|ImB > + span\{V_2\}.$$

Therefore, by (9.16),

$$\mathbf{R}^n = < A|ImB > + \mathcal{V}^*(A, B; KerD).$$

(iii)$\Rightarrow$ (i): Let V_2 be a basis matrix for $\mathcal{V}^*(A, B; KerD)$ and V_1 be such that $\begin{bmatrix} V_1 & V_2 \end{bmatrix}$ is nonsingular. Then, by **(iii)**,

$$span\{V_1\} \subseteq < A|ImB > + \mathcal{V}^*(A, B; KerD)$$

i.e, $span\{V_1\}$ is either reachable or contained in $span\{V_2\}$. Therefore by Fact (6.4.3), there exists a constant matrix K_4 and polynomial matrices $P_2(s)$, $Q_2(s)$ such that

$$V_1 = (sI - A)P_2(s) + BQ_2(s) + V_2K_4. \tag{9.21}$$

Since $\begin{bmatrix} V_1 & V_2 \end{bmatrix}$ is nonsingular, we have

$$AV_1 = V_1K_1 - V_2K_3 \tag{9.22}$$

for some constant matrices K_3, K_1. By the fact that V_2 is a basis for $\mathcal{V}^*(A, B; KerD)$,

$$AV_2 = V_2K_2 - BF_0, \ DV_2 = 0 \tag{9.23}$$

for some constant matrices K_2 and F_0. Let $\begin{bmatrix} \hat{V}_1 \\ \hat{V}_2 \end{bmatrix}$ be the inverse of $\begin{bmatrix} V_1 & V_2 \end{bmatrix}$. Then, by (9.23), (9.22) and (9.21),

$$(sI - A - BF_0\hat{V}_2)V_2 = V_2(sI - K_2) \tag{9.24}$$

$$DV_2 = 0 \tag{9.25}$$

$$(sI - A - BF_0\hat{V}_2)V_1 = V_1(sI - K_1) + V_2K_3 \tag{9.26}$$

$$(sI - A - BF_0\hat{V}_2)P_2(s) + BQ_3(s) + V_2K_4 = V_1 \tag{9.27}$$

for some polynomial matrix $Q_3(s)$.

We now show that $(K_1, \hat{V}_1B)$ is reachable. By multiplying (9.27) on the left by $\hat{V}_1$, and by (9.24) and (9.26), we obtain

$$(sI - K_1)\hat{V}_1P_2(s) + \hat{V}_1BQ_3(s) = I$$

so that by Fact (6.4.3), $(K_1, \hat{V}_1B)$ is indeed reachable. Hence, there exists F_1 such that $\sigma(K_1 + \hat{V}_1BF_1) \subseteq \mathbf{C}_-$. Let $\hat{K}_1 := K_1 + \hat{V}_1BF_1$. By (9.26),

$$(sI - A - BF_0\hat{V}_2)V_1 = V_1(sI - \hat{K}_1) + B\theta_1 + V_2K_3$$

for some constant matrix θ_1. Or,

$$(sI - A - BF_0\hat{V}_2 - B\theta_1\hat{V}_1)V_1 = V_1(sI - \hat{K}_1) + V_2K_3. \tag{9.28}$$

By (9.24),

$$(sI - A - BF_0\hat{V}_2 - B\theta_1\hat{V}_1)V_2 = V_2(sI - K_2). \tag{9.29}$$

With $\psi := sI - A - BF_0\hat{V}_2 - B\theta_1\hat{V}_1$ and by (9.28), (9.29) and (9.25),

$$\begin{aligned} D\psi^{-1}\begin{bmatrix} V_1 & V_2 \end{bmatrix} &= \begin{bmatrix} DV_1(sI - \hat{K}_1)^{-1} - D\psi^{-1}V_2K_4(sI - K_1)^{-1} & D\psi^{-1}V_2 \end{bmatrix} \\ &= \begin{bmatrix} DV_1(sI - \hat{K}_1)^{-1} - DV_2(sI - K_2)^{-1}K_3(sI - \hat{K}_1)^{-1} & DV_2(sI - K_2)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} DV_1(sI - \hat{K}_1)^{-1} & 0 \end{bmatrix} \end{aligned}$$

so that $D\psi^{-1}$ is stable rational. □

Note that the proof **(ii)** $\Rightarrow$ **(iii)** can be extended to construct a feedback F such that $D(sI - A - BF)^{-1}$ is stable rational. Since $(W_1K_1\hat{W}_1, \hat{V}_1B)$ is reachable, there exists F_1 such that

$$\sigma(W_1K_1\hat{W}_1 + \hat{V}_1BF_1) \subseteq \mathbf{C}_-.$$

Let $\tilde{K}_1 := W_1K_1\hat{W}_1 + \hat{V}_1BF_1$, by (9.17), (9.18)and (9.20),

$$\begin{aligned} (sI - A - BF_0\tilde{T}_2\hat{V}_2 - B\theta_1\hat{V}_1)\hat{V}_2 &= \tilde{V}_2(sI - \tilde{K}_2) \\ (sI - A - BF_0\tilde{T}_2\hat{V}_2 - B\theta_1\hat{V}_1)\hat{V}_1 &= \tilde{V}_1(sI - \tilde{K}_1) + \tilde{V}_2\theta_2 \\ D\hat{V}_2 &= 0 \end{aligned}$$

for some suitable θ_1 and θ_2. These three equalities yield that $D(sI-A-BF)^{-1}$ is stable rational with $F := F_0\tilde{T}_2\hat{V}_2 + \theta_1\hat{V}_1$

9.2 Tracking with Regulation: Scalar Case

Consider the system in Figure 9.1. The problem is to determine a proper controller such that the tracking error (the difference between the reference input and the corresponding output) converges to zero from arbitrary initial values and the feedback system consisting of the plant and controller is internally stable.

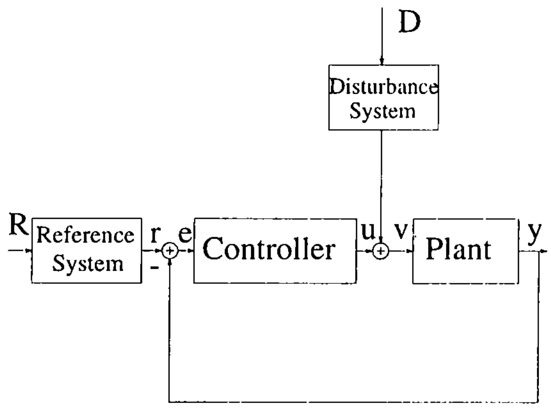

Figure 9.1: Tracking with regulation

Let the plant, controller, reference system and disturbance system transfer functions be written in coprime stable proper fractional representations as p/q, p_c/q_c, p_r/q_r and p_d/q_d respectively. The disturbance and reference systems are supposed to be driven by D and R which are arbitrary unknown constants.

Tracking error in Figure 9.1 is

$$e = \frac{qq_c}{qq_c + pp_c} \begin{bmatrix} p_r/q_r & -pp_d/qq_d \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}. \tag{9.30}$$

By Lemma (7.1.1), the pair $(p/q, p_c/q_c)$ is internally stable if and only if

$$\phi := qq_c + pp_c \tag{9.31}$$

has a stable inverse. Note that if

$$\frac{qq_cp_r}{\phi q_r} \text{ and } \frac{pq_cp_d}{\phi q_d} \text{ are stable rational,} \tag{9.32}$$

then the tracking error in (9.30) converges to zero for all constants R, D. Since the pairs $(q_r, \phi^{-1}p_r)$ and $(q_d, \phi^{-1}p_d)$ are coprime, (9.32) holds if and only if

$$q_r \text{ divides } qq_c \text{ and } q_d \text{ divides } pq_c. \tag{9.33}$$

Let the greatest common divisors of (q, q_r) and (q_d, p) be Q_r and Q_d, respectively. Hence, there exist stable proper rational functions $\bar{q}_r$, $\bar{q}$, $\bar{q}_d$ and $\bar{p}$ such that

$$\begin{aligned} q_r &= Q_r\bar{q}_r,\ q = Q_r\bar{q}, \\ q_d &= Q_d\bar{q}_d,\ p = Q_d\bar{p}, \end{aligned} \tag{9.34}$$

where the pairs $(\bar{q}_r, \bar{q})$ and $(\bar{q}_d, \bar{p})$ are coprime. Let the least common multiple (lcm) of $\bar{q}_r$ and $\bar{q}_d$ be q_{dr}. Thus, for stable transfer functions $\tilde{q}_r$ and $\tilde{q}_d$,

$$q_{dr} = \bar{q}_r\tilde{q}_r,\ q_{dr} = \bar{q}_d\tilde{q}_d,$$

where the pair $(\tilde{q}_r, \tilde{q}_d)$ is coprime.

We can now state

Theorem 9.2.1. *Tracking with regulation in the scalar system is possible if and only if (q_{dr}, p) is coprime.*

Proof. **[Only if]** If the tracking with regulation problem in the scalar system is solvable then (9.33) holds. Hence by (9.33) and (9.34), $\bar{q}_r$ divides q_c and $\bar{q}_d$ divides q_c. Therefore q_{dr} divides q_c and there exists a stable proper rational function $\bar{q}_c$ such that

$$q_c = q_{dr}\bar{q}_c. \tag{9.35}$$

By (9.31) and (9.35), we have

$$q\bar{q}_c q_{dr} + pp_c = \phi$$

and

$$q\phi^{-1}\bar{q}_c q_{dr} + p_c\phi^{-1}p = 1.$$

Hence the pair (q_{dr}, p) is coprime.

[If] By the hypothesis, there exist stable proper rational functions x, y such that

$$q_{dr}x + py = 1. \tag{9.36}$$

Since (p, q) is coprime

$$qa + pb = 1 \tag{9.37}$$

for some stable proper rational functions a and b. Multiplying (9.36) by q a, we have

$$qq_{dr}xa + pqya = qa. \tag{9.38}$$

By (9.37) and (9.38),

$$qq_{dr}xa + p(b + qya) = 1.$$

Let

$$q_c := q_{dr}xa \text{ and } p_c := b + qya. \tag{9.39}$$

Note that q_r divides $(q_{dr}Q_r)$ and q_d divides $(q_{dr}Q_d)$. By (9.39), q_{dr} divides q_c so that q_r divides $q_c q$ and q_d divides pq_c. Therefore by the choice of dynamic controller in (9.39), the tracking error converges to zero for arbitrary R, D while assuring the internal stability. $\square$

9.3 Regulator Problem with a Single Output Channel

In the configuration of Figure 9.2, the plant has two input channels, a control input u, a disturbance input w, and only one output channel y.

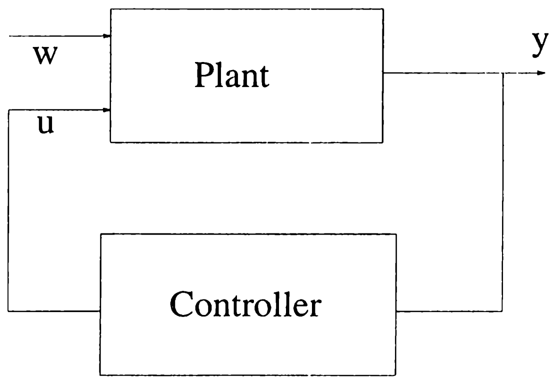

Figure 9.2: RPIS with a single output channel

The problem is to design a dynamic controller, $K(s)$ such that the disturbance to output transfer matrix T_{wy} is stable and the feedback loop is internally stable. The output channel processed by the controller and the one to be regulated hence coincide. We now obtain a solution to this problem using the technique of stable proper factorization.

Let

$$y = M^{-1} \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix},$$

where M, N_1, N_2 are stable proper rational matrices and (M, N_2) is left coprime. The disturbance to output transfer matrix is

$$T_{wy} = (M - N_1 K)^{-1} N_2. \tag{9.40}$$

Let

$$M^{-1}N_1 = \bar{M}^{-1}\bar{N}$$

for left coprime $(\bar{M}, \bar{N})$. Then

$$M = D\bar{M}, \ N_1 = D\bar{N},$$

for some stable proper rational matrix D. Note that, $T_{wy} = (\bar{M} - \bar{N}_1 K)^{-1} D^{-1} N_2$ and the zeros of the greatest common left factor D appears among the poles of T_{wy} unless they are cancelled by the controller $K(s)$.

Theorem 9.3.1. *The RPIS with a single output channel is solvable if and only if there exist stable proper rational matrices X and Y satisfying*

$$XD + \bar{N}Y = I. \tag{9.41}$$

Proof. **[Only if]** If RPIS is solvable then the feedback loop is internally stable. So by Lemma (7.1.2), there exists right coprime fractional representation for the controller $K(s) = QP^{-1}$, where Q and P are stable proper rational matrices such that

$$\bar{M}P - \bar{N}Q = I. \tag{9.42}$$

The disturbance to output transfer matrix T_{wy} in (9.40) becomes

$$\begin{aligned} T_{wy} &= (M - N_1 QP^{-1})^{-1} N_2 \\ &= (D\bar{M} - D\bar{N}QP^{-1})^{-1} N_2 \\ &= P(\bar{M}P - \bar{N}Q)^{-1} D^{-1} N_2. \end{aligned}$$

Using (9.42), we have $T_{wy} = PD^{-1}N_2$. Since T_{wy} is a stable transfer matrix, there exists stable proper rational matrix $\tilde{P}$ such that

$$P = \tilde{P}D.$$

Hence (9.42) gives

$$\bar{M}\tilde{P}D - \bar{N}Q = I.$$

With $X = \bar{M}P$ and $Y = -Q$, (9.41) is satisfied.

[If] Since $(\bar{M}, \bar{N})$ is left coprime, there exist stable proper rational matrices $\bar{Q}$, $\bar{P}$ such that

$$\bar{M}\bar{P} - \bar{N}\bar{Q} = I. \tag{9.43}$$

Using (9.43) and (9.41),

$$\begin{aligned}
(\bar{M}\bar{P} - \bar{N}\bar{Q})XD + \bar{N}Y &= I \\
\bar{M}\bar{P}XD + \bar{N}(Y - \bar{Q}XD) &= I.
\end{aligned}$$

Let

$$\hat{P} := \bar{P}XD, \quad \hat{Q} := -Y + \bar{Q}XD \quad \text{and} \quad K(s) = \hat{Q}\hat{P}^{-1}. \tag{9.44}$$

By this choice of dynamic compensator $K(s)$,

$$\bar{M}\hat{P} - \bar{N}\hat{Q} = I.$$

So, internal stability of $(M^{-1}N_1, K(s))$ is satisfied. Moreover, the disturbance to output transfer matrix T_{wy} can be written as

$$\begin{aligned}
T_{wy} &= \hat{P}D^{-1}N_2 \\
&= \bar{P}XN_2,
\end{aligned}$$

which is a stable proper rational matrix. □

The solvability of the matrix equation (9.41) requires that the matrices D and $\bar{N}$ are "coprime" in some sense. To make this precise, we examine this equation more closely.

Fact 9.3.1. *There exist stable proper rational matrices X, Y such that (9.41) holds if and only if*

$$D\bar{N} = N\bar{D} \tag{9.45}$$

for some stable proper rational matrices N, $\bar{D}$ with the left coprime pair (D, N) and right coprime pair $(\bar{N}, \bar{D})$.

Proof. **[Only if]** If (9.41) is satisfied then (Y, D) is right coprime. Let

$$YD^{-1} = \bar{D}^{-1}\bar{Y} \tag{9.46}$$

for left coprime $(\bar{D}, \bar{Y})$. By (9.41),

$$DX + D\bar{N}\bar{D}^{-1}\bar{Y} = I.$$

Since $(\bar{D}, \bar{Y})$ is left coprime, $\bar{D}$ is a right factor of $D\bar{N}$, i.e.,

$$D\bar{N} = N\bar{D}$$

for some stable proper rational matrix N. Note that (D, N) is left coprime. In (9.46), since both fractions are coprime, $detD = u\ det\bar{D}$, for some unit u, i.e., u and u^{-1} are both stable proper functions. This implies that in (9.45), $(\bar{N}, \bar{D})$ is right coprime.
[If] If (9.45) holds, then $D^{-1}N = \bar{N}\bar{D}^{-1}$ with left coprime (D, N) and right coprime $(\bar{N}, \bar{D})$. Hence there exist stable proper matrices $\tilde{X}$, $\tilde{Y}$ such that

$$D\tilde{X} + N\hat{Y} = I.$$

So by (9.45)

$$\tilde{X}D + \bar{N}\bar{D}^{-1}\tilde{Y}D = I.$$

Since the pair $(\bar{N}, \bar{D})$ is right coprime, by Fact (6.4.2),

$$\tilde{Y}D = \bar{D}Y$$

for some stable proper rational matrix Y. Let $X := \tilde{X}$, then (9.41) is satisfied.

□

In Theorem (9.3.1), it is shown that by choosing dynamic compensator $K(s) = (-Y + \bar{Q}XD)(\bar{P}XD)^{-1}$ in (9.44), tracking error converges to zero and the system is internally stable. Note that this controller has the matrix D as a factor of its denominator matrix. It follows that any controller which solves RPIS should contain the matrix D which is a greatest common left factor of M and N_1, in its denominator. This fact is known as the *internal model principle* for RPIS considered here.

9.4 Notes and References

The solution to output stabilization problem in Section (9.1) has been obtained in a geometric framework by Bhattacharrya, Pearson and Wonham [57]. Transfer matrix condition given to this problem in Theorem (9.1.1) is new. The solutions to some special regulation and tracking problems were first given by Bengtsson [58] and Francis [59]. The simple problem of tracking with regulation in the scalar system in Section (9.2) is also new. The concept of skew-primeness was first used by Wolovich and Ferreira [60]. Pernebo [61], Cheng and Pearson [62] presented the solution of the regulation problem stated in Wonham [63] through a stable rational (not necessarily proper) factorization approach. Solvability conditions to this problem through polynomial system matrix approach were obtained by Khargonekar and Özgüler [64]. The regulator problem with a single output channel is a special case of the general regulator problem as posed by Wonham [63].

Chapter 10

DECENTRALIZED STABILIZATION

In large scale or geographically distributed systems, it is more a rule than exception that the control configuration has structural limitations. One of the most common structural limitations is that certain inputs can only be controlled by certain specified outputs. The control configuration that results by such a restriction is called a decentralized control. Most commonly, a decentralized control configuration is one in which the controller is restricted to be block diagonal with fixed sized blocks.

In this chapter, we illustrate how the fractional representation technique of Chapter 7 can be used to tackle the decentralized stabilization problem where the controller is constrained to be a 2×2 block diagonal transfer matrix. Rather than presenting a complete solution, we show how the problem can be transformed into a "make-coprime" problem, a solution to which is more transparent than a solution to the original problem.

10.1 Decentralized Stabilization Problem

Consider the plant having the following state space description

$$\begin{aligned} \dot{x} &= Ax + \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x, \end{aligned} \tag{10.1}$$

where $A \in \mathbf{R}^{n \times n}$, $B_1 \in \mathbf{R}^{n \times r_1}$, $B_2 \in \mathbf{R}^{n \times r_2}$, $C_1 \in \mathbf{R}^{p_1 \times n}$ and $C_2 \in \mathbf{R}^{p_2 \times n}$. Let $B := \begin{bmatrix} B_1 & B_2 \end{bmatrix}$, $C := \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$. We would like to determine a controller of the form

$$K(s) = \begin{bmatrix} K_{11} & 0 \\ 0 & K_{22} \end{bmatrix} \tag{10.2}$$

such that $K(s)$ internally stabilizes the system (10.1). Suppose that the plant is stabilizable and detectable. Note that this is a necessary condition for the existence of even a centralized controller. In order to have a stable rational matrix description of the plant, let

$$Q(s) := \frac{sI - A}{s+1}, \quad R := B, \quad P(s) := \frac{C}{s+1}. \tag{10.3}$$

Note that by Fact (6.1.1), if (A, B) is stabilizable, then (Q, R) is left coprime over $\mathbf{S}$ and if (C, A) is detectable, then (P, Q) is right coprime over $\mathbf{S}$. The relation from u to y is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} Q^{-1} \begin{bmatrix} R_1 & R_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

where R and P are partitioned according to the partition of B and C. The closed loop system is shown in Figure 10.1.

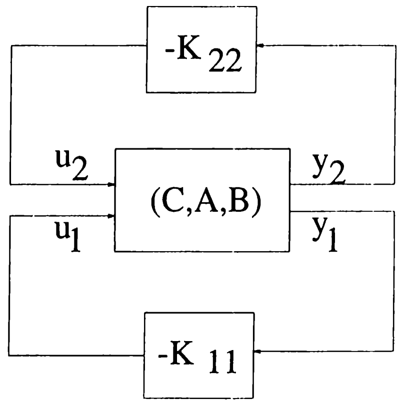

Figure 10.1: Closed loop system for DSP

The following result concerns internal stabilization from two-sided fractional representations.

Lemma 10.1.1. *Let a plant transfer matrix $\hat{G}$ be given in a fractional representation over* **S** *by*

$$\hat{G} := \hat{N}\hat{M}^{-1}\hat{L},$$

where $(\hat{N}, \hat{M})$ is right coprime and $(\hat{M}, \hat{L})$ is left coprime. Then a controller $K = \hat{P}\hat{Q}^{-1}$ internally stabilizes $\hat{G}$ if and only if

$$\hat{\Phi} := \begin{bmatrix} \hat{M} & \hat{L}\hat{P} \\ -\hat{N} & \hat{Q} \end{bmatrix} \tag{10.4}$$

is unimodular.

Proof. Let $\hat{N}\hat{M}^{-1} = \bar{M}^{-1}\bar{N}$, where $(\bar{M}, \bar{N})$ is left coprime. Then, $\hat{G} = \bar{M}^{-1}\bar{N}\hat{L}$ for a left coprime pair $(\bar{M}, \bar{N}\hat{L})$. Note that the controller $K = \hat{P}\hat{Q}^{-1}$ internally stabilizes $\hat{G}$ if and only if

$$\bar{M}\hat{Q} + \bar{N}\hat{L}\hat{P} \text{ is unimodular.}$$

Let the doubly-coprime representation of $\hat{Z} := \hat{N}\hat{M}^{-1} = \bar{M}^{-1}\bar{N}$ be

$$\begin{bmatrix} X_1 & Y_1 \\ \bar{N} & \bar{M} \end{bmatrix} \begin{bmatrix} \hat{M} & X_2 \\ -\hat{N} & Y_2 \end{bmatrix} = I,$$

where X_1, Y_1, X_2, Y_2 are over $\mathbf{S}$. Thus,

$$\begin{bmatrix} X_1 & Y_1 \\ \bar{N} & \bar{M} \end{bmatrix} \begin{bmatrix} \hat{M} & \hat{L}\hat{P} \\ -\hat{N} & \hat{Q} \end{bmatrix} = \begin{bmatrix} I & X_1\hat{L}\hat{P} + Y_1\hat{Q} \\ 0 & \bar{N}\hat{L}\hat{P} + \bar{M}\hat{Q} \end{bmatrix}.$$

It follows that $K = \hat{P}\hat{Q}^{-1}$ internally stabilizes $\hat{G}$ if and only if $\hat{\Phi}$ in (10.4) is unimodular. □

The following result gives a first necessary and sufficient condition to have a solution to decentralized stabilization problem(DSP).

Lemma 10.1.2. *DSP is solvable if and only if there exist coprime factorizations*

$$K_1 = P_{c_1}Q_{c_1}^{-1},\ K_2 = P_{c_2}Q_{c_2}^{-1}$$

such that the matrix Φ *below is unimodular*

$$\Phi := \begin{bmatrix} Q & R_1 P_{c_1} & R_2 P_{c_2} \\ -P_1 & Q_{c_1} & 0 \\ -P_2 & 0 & Q_{c_2} \end{bmatrix}. \qquad (10.5)$$

Proof. Let $G := PQ^{-1}R$ and $M^{-1}N = PQ^{-1}$ with left coprime pair (M, N). Then $G = M^{-1}NR$, where (M, NR) is left coprime. Let $P_c := \begin{bmatrix} P_{c_1} & 0 \\ 0 & P_{c_2} \end{bmatrix}$, $Q_c := \begin{bmatrix} Q_{c_1} & 0 \\ 0 & Q_{c_2} \end{bmatrix}$, so $K(s) = P_c Q_c^{-1}$ and (P_c, Q_c) is a right coprime factorization of K. We now apply Lemma (10.1.1) to abtain the result. $\square$

Theorem 10.1.1. *DSP has a solution if and only if the following rank conditions are satisfied for all* $s \in \mathbf{C}_{0+}$:

$$(i)\ rank \begin{bmatrix} sI - A & B_1 & B_2 \end{bmatrix} = n,$$

$$(ii)\ rank \begin{bmatrix} sI - A' & C_1' & C_2' \end{bmatrix} = n,$$

$$(iii)\ rank \begin{bmatrix} sI - A & B_2 \\ -C_1 & 0 \end{bmatrix} \geq n, \tag{10.6}$$

$$(iv)\ rank \begin{bmatrix} sI - A & B_1 \\ -C_2 & 0 \end{bmatrix} \geq n.$$

Proof. **[Only if]** If DSP is solvable, then by Lemma (10.1.2), there exist $K_i(s) = P_{c_i} Q_{c_i}^{-1}$, $i = 1, 2$ such that Φ in (10.5) is unimodular. Thus,

$$rank\Phi(s) = n + p_1 + p_2,\ \forall s \in \mathbf{C}_{0+}. \tag{10.7}$$

This implies that

$$rank \begin{bmatrix} Q & R_1 P_{c_1} & R_2 P_{c_2} \\ -P_1 & Q_{c_1} & 0 \end{bmatrix} = n + p_1,\ \forall s \in \mathbf{C}_{0+}.$$

Hence

$$rank \begin{bmatrix} Q & R_2 P_{c_2} \\ -P_1 & 0 \end{bmatrix} \geq n, \ \forall s \in \mathbf{C}_{0+}. \tag{10.8}$$

Similarly by (10.7),

$$rank \begin{bmatrix} Q & R_1 P_{c_1} \\ -P_1 & Q_{c_1} \\ -P_2 & 0 \end{bmatrix} = n + p_1, \ \forall s \in \mathbf{C}_{0+}.$$

So

$$rank \begin{bmatrix} Q & R_1 P_{c_1} \\ -P_2 & 0 \end{bmatrix} \geq n, \forall s \in \mathbf{C}_{0+}. \tag{10.9}$$

By (10.8) and (10.9),

$$rank \begin{bmatrix} Q & R_2 \\ -P_1 & 0 \end{bmatrix} \geq n, \ rank \begin{bmatrix} Q & R_1 \\ -P_2 & 0 \end{bmatrix} \geq n, \ \forall s \in \mathbf{C}_{0+}. \tag{10.10}$$

Using (10.3) and (10.10), last two conditions in (10.6) is satisfied. Since we assume that the plant is stabilizable and detectable, first two conditions also hold.

[If] By Lemma (10.1.3) stated below there exists $K_1(s) = P_{c_1} Q_{c_1}^{-1}$ such that the closed loop system Z_f in Figure 10.2, with input u_2 and output y_2 is stabilizable and detectable.

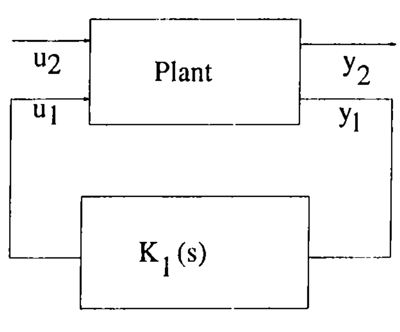

Figure 10.2: The closed loop system Z_f

The relation between u_2 and y_2 is

$$y_2 = (Z_{22} - Z_{21}K_1(I - Z_{11}K_1)^{-1}Z_{12})u_2.$$

This can be written as

$$y_2 = \begin{bmatrix} P_2 & 0 \end{bmatrix} \begin{bmatrix} Q & R_1 P_{c_1} \\ -P_1 & Q_{c_1} \end{bmatrix}^{-1} \begin{bmatrix} R_2 \\ 0 \end{bmatrix} u_2$$

with left coprime pair $\left(\begin{bmatrix} Q & R_1 P_{c_1} \\ -P_1 & Q_{c_1} \end{bmatrix}, \begin{bmatrix} R_2 \\ 0 \end{bmatrix} \right)$ over $\mathbf{S}$ and right coprime pair $\left(\begin{bmatrix} P_2 & 0 \end{bmatrix}, \begin{bmatrix} Q & R_1 P_{c_1} \\ -P_1 & Q_{c_1} \end{bmatrix} \right)$ over $\mathbf{S}$, by the fact that Z_f is stabilizable from u_2 and detectable at y_2. By Lemma (10.1.1) there exist a controller $K_2(s) = P_{c_2}Q_{c_2}^{-1}$ satisfying $u_2 = K_2(s)y_2$ such that the new closed loop system is internally stable, i.e., Φ in (10.5) is unimodular. It follows that $K(s)$ defined by $K(s) := \begin{bmatrix} P_{c_1} & 0 \\ 0 & P_{c_2} \end{bmatrix} \begin{bmatrix} Q_{c_1} & 0 \\ 0 & Q_{c_2} \end{bmatrix}^{-1}$ internally stabilizes the plant. □

The crucial result used in the contruction of a stabilizing decentralized controller of Theorem (10.1.1) is the following solution to a "make-coprime" problem.

Lemma 10.1.3. *Suppose the plant in Figure 10.2 is stabilizable and detectable. The closed loop system of Figure 10.2 can be made stable from u_2 and detectable at y_2 by a choice of a controller $K_1(s)$ if and only if the condition* **(iii)** *and* **(iv)** *of Theorem (10.1.1) hold for all $s \in \mathbf{C}_{0+}$.*

Proof. Özgüler [50]. □

By Theorem (10.1.1) to solve the DSP, rank conditions in (10.6) should be satisfied. Note that these rank conditions are related to the state space description of the plant not the controller. If DSP is solvable, the points s, which fail to satisfy the requirement (10.6) should be in the left half plane.

10.2 Decentralized Fixed Modes

Consider the system (10.1) and the set of controllers $\mathcal{K}$ which contains compensators of the form $K(s) = \begin{bmatrix} K_{11} & 0 \\ 0 & K_{22} \end{bmatrix}$. Then decentralized fixed modes of (10.1) with respect to $\mathcal{K}$

$$\Lambda(C, A, B, \mathcal{K}) := \bigcap_{K \in \mathcal{K}} \sigma(A + BKC).$$

Hence, decentralized fixed modes are equal to the eigenvalues of closed loop system which are common with the eigenvalues of A and independent of the particular controller used.

These modes may be thought of as a generalization of uncontrollable modes and unobservable modes that occur in the centralized control but generally include other modes of the system also. Let

$$\mathcal{Z} := \{s \in \mathbf{C} : rank \begin{bmatrix} sI - A & B_1 & B_2 \end{bmatrix} < n \text{ or } rank \begin{bmatrix} sI - A & B_1 \\ -C_2 & 0 \end{bmatrix} < n \text{ or}$$

$$rank \begin{bmatrix} sI - A & B_2 \\ -C_1 & 0 \end{bmatrix} < n \text{ or } rank \begin{bmatrix} sI - A \\ C_1 \\ C_2 \end{bmatrix} < n.\}$$

Theorem 10.2.1. *Consider the system (10.1). Then a necessary and sufficient condition for* $\lambda \in \sigma(A)$ *to be a decentralized fixed mode of (10.1) is that* $\lambda \in \mathcal{Z}$.

Proof. [65]. □

This result gives an algebraic characterization of decentralized fixed modes. By Theorem (10.1.1) and (10.2.1), there exists a decentralized compensator that internally stabilizes the plant (10.1) if and only if the system (10.1) has no unstable decentralized fixed modes.

10.3 Notes and References

The solvability conditions to DSP given in Theorem (10.1.1) is due to Corfmat and Morse [66], [67]. The make-coprime problem stated in Lemma (10.1.3) is considered in a slightly different manner by [66]. Wang and Davison [68] introduced the fundamental notion of decentralized fixed modes and presented a first solution to DSP which was extended by [69]. Anderson and Clements [65] presented an algebraic characterization of decentralized fixed modes via polynomial matrix fraction representations. The identification of decentralized fixed modes from the plant transfer matrix is given in [70]. Vidyasagar and Viswanadham [71] solved the synthesis problem for DSP via fractional factorizations over a principle ideal domain. A characterization of decentralized fixed modes for interconnected systems was studied by [72].

Chapter 11

CONCLUSIONS

We have presented a first draft of a book on linear multivariable control which contains a description of solutions to most of the standard algebraic feedback control problems. These problems include internal stabilization, disturbance decoupling by state feedback and measurement feedback, output stabilization, tracking with regulation in a scalar system, regulator problem with a single output channel and decentralized stabilization.

In order to complete the project started in this thesis, examples on applications need to be added, the proof of Theorem (9.1.1) should be given when the assumptions fail, regulator problem with more than one output channel need to be studied and the dependencies among various sections need to be more carefully organized.

The proofs given for Theorem (8.2.3), (8.2.4) and (9.1.1) implicitly employ ideas from the theory of polynomial models of Fuhrmann and makes the relation between the geometric and fractional concepts explicit. Although elegance of solutions provided by a pure geometric or pure fractional approach is no longer there, the presentation here demands a moderate amount of knowledge of linear algebra from the reader.

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