ON REAL ENRIQUES SURFACES

A THESIS
SUBMITTED TO THE DEPARTMENT OF MATHEMATICS
AND THE INSTITUTE OF ENGINEERING AND SCIENCES
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
Özgül Küçük
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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Asst. Prof. Dr. Alexander Degtyarev(Principal Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

ON REAL ENRIQUES SURFACES

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M.S. in Mathematics
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July, 1997

In this work we showed that the Pontrjagin-Viro form of a real Enriques surface satisfies the congruence relation stated as Proposition 3.5 and besides any quadratic form $P : H_*((E_{\mathbb{R}}^{(1)}) \oplus H_*(E_{\mathbb{R}}^{(2)})) \to \mathbb{Z}/4$ of a triad $(E_{\mathbb{R}}, E_{\mathbb{R}}^{(1)}, E_{\mathbb{R}}^{(2)})$ satisfying Proposition 3.5 can be realized as the Pontrjagin-Viro form of a real Enriques surface.

Keywords : Real algebraic surface, Real Enriques surface, Brown invariant, Spectral sequence, Rohklin-Guillou-Marin form, Pontrjagin-Viro form

ÖZET

GERÇEL ENRIQUES YÜZEYLER ÜZERİNE

Özgül Küçük
Matematik Bölümü Yüksek Lisans
Danışman: Assist. Prof. Alexander Degtyarev
Temmuz, 1997

Bu çalışmada herhangi bir gerçel Enriques yüzeyinin Pontrjagin-Viro formunun önerme 3.5 de ifade edilen uyumluluk bağıntısını sağladığını ve diğer yandan da herhangi bir $(E_{\mathbb{R}}, E_{\mathbb{R}}^{(1)}, E_{\mathbb{R}}^{(2)})$ üçül grubunun önerme 3.5'i saglayan her ikincil dereceli $P : H_*((E_{\mathbb{R}}^{(1)}) \oplus H_*(E_{\mathbb{R}}^{(2)})) \rightarrow \mathbb{Z}/4$ formunun bir gercel Enriques yüzeyinin Pontrjagin-Viro formu olarak ifade edilebilecegini gösterdik.

Anahtar Kelimeler : Gerçel cebirsel yüzey, Gerçel Enriques yüzeyi, Brown degişmezi, Spectral dizi, Rohklin-Guillou-Marin formu, Pontrjagin-Viro formu

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Chapter 1

Introduction

The theory of algebraic surfaces differs from the theory of Riemann surfaces and algebraic curves in many aspects. It is much more difficult and lacks cohesion. While curves have a natural continuous invariant their periods realized geometrically by the Jacobian, no fully satisfactory continuous invariant has been found for surfaces. As a result the theory of algebraic surfaces does not possess the natural cohesiveness of the theory of curves: it tends to concentrate mainly on the study of special classes of surfaces. Algebraic surfaces possess a variety of numerical invariants and are not so readily classified.

The classification of algebraic surfaces made by Enriques is extended by Kodaira to non-algebraic ones: surfaces are divided into ten classes, i.e., every surface has a minimal model in exactly one of classes 1) to 10) of table 2.1 (see Theorem 2.2).

As it is seen, birational classification of surfaces amounts to the biregular classification of minimal surfaces; however, since we deal with real algebraic surfaces we need to note that minimality over $\mathbb{R}$ may not always imply minimality over $\mathbb{C}$. More precisely, for surfaces of Kodaira dimension ≥ 0 there is always a unique way to blow the surface down to a minimal model and minimality over $\mathbb{R}$ implies minimality over $\mathbb{C}$, but if Kodaira dimension is < 0 then there are surfaces minimal over $\mathbb{R}$ but not minimal over $\mathbb{C}$, e.g., cubic surfaces with disconnected real part.

According to the Enriques-Kodaira classification of complex algebraic surfaces, there are five special classes of surfaces; abelian surfaces, surfaces with a pencil of rational curves, hyperelliptic surfaces, surfaces with a pencil of elliptic curves of Kodaira dimension 1, and Enriques surfaces. Abelian surfaces were classified by Comessatti (see [14]). Some results on the topology of hyperelliptic surfaces and real surfaces with a real pencil of rational curves and the classification of singular fibres of real pencils of eliptic curves were obtained by Silhol(see [2]). Hence it is quite natural to study the real Enriques surfaces, as their classification was left undone untill now .

The tools used in the classification of real Enriques surfaces: A real Enriques surface E is a complex Enriques surface E equipped with an anti-holomorphic involution $conj : E \to E$, called complex conjugation. The fixed point set $E_{\mathbb{R}} = Fix\ conj$ is called the real part of the surface.

Universal covering of an Enriques surface is a K3 surface, thus, the study of a real Enriques surface can be reduced to the study of a real K3 surface supplied with a holomorphic fixed-point free involution.

The real structure on E lifts to the covering K3-surface X , together with the deck translation involution this gives rise to a $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ -action on X. Hence, there is a natural decomposition of the real part into two disjoint halves $E_{\mathbb{R}}^{(1)}$, $E_{\mathbb{R}}^{(2)}$ which is called the sign decomposition, and which is a deformation invariant. (Recall that two real Enriques surfaces have the same deformation type if they can be included into a continous one-parameter family of real Enriques surfaces).

Complex Enriques surfaces are all diffeomorphic and their moduli space is irreducible. The moduli space of real Enriques surfaces is not connected. That is why they are more interesting to study. The real parts of real Enriques surfaces have several different types. The classification of the topological types of the real parts of real Enriques surfaces is given by Theorem 3.3. This classification, due to A. Degtyarev and V.Kharlamov (see [4]) not only completes Nikulin's classification (see [3]) but also gives all existing topological types.

Pontrjagin-Viro form P is a new invariant of a real algebraic surface introduced first in [6] and studied in details in [9] . This invariant is only well-defined in certain special cases. Let $E_{\mathbb{R}}$ be a real Enriques surface; then there is a necessary condition ($\chi(E_{\mathbb{R}}) = 0 \bmod 8$) and some sufficient conditions (Lemma 3.1) for P to be well-defined; when defined P satisfies Proposition 3.5.

In this thesis we proved the following theorem:

Theorem : Given a decomposition $E_{\mathbb{R}} = E_{\mathbb{R}}^{(1)} \sqcup E_{\mathbb{R}}^{(2)}$ from tables 1 and 2, the Pontrjagin-Viro form can take any value satisfying Proposition 3.5. Furthermore in all cases listed in the tables the Pontrjagin-Viro form is uniquely recovered (up to autohomemorphism of $E_{\mathbb{R}}$preserving the complex separation) from the complex seperation and $P(w_1)$ via Proposition 3.5.

In fact the Pontrajagin-Viro form determines a real Enriques M-surface up to deformation. Any quadratic form $P : H_*((E_{\mathbb{R}}^{(1)}) \oplus H_*(E_{\mathbb{R}}^{(2)})) \to \mathbb{Z}/4$ of a triad $(E_{\mathbb{R}}, E_{\mathbb{R}}^{(1)}, E_{\mathbb{R}}^{(2)})$ satisfying Proposition 3.5 can be realized as the Pontrjagin-Viro form of a real Enriques surface. But note that if $(E_{\mathbb{R}}, E_{\mathbb{R}}^{(1)}, E_{\mathbb{R}}^{(2)})$ does not satisfy the sufficient conditions of Lemma 3.1, it can also be realized by a real Enriques surface not admitting Pontrjagin-Viro form.

Some generalities: Each particular class of surfaces gives experimental material that helps discovering new general results. Possible applications of Theorem 3.6 are considered in chapter 3.

Chapter 2

Classification Of Algebraic Surfaces

In this chapter we will give a brief overview of the classification of algebraic surfaces.

Our main subject is real surfaces. However, we treat them from the complex point of view, i.e.,we will consider a real algebraic variety as a (complex) algebraic variety with an anti-holomorphic involution.

2.1 Real and Complex Surfaces

Definition 2.1 *An affine homogeneous algebraic variety is a subset of* $\mathbb{C}^n$ *which can be realized as the common zero locus of a collection of homogeneous polynomials in* $\mathbb{C}[x_1,x_2,\ldots,x_n]$ *; the polynomial ring over* $\mathbb{C}$ *with* n *variables.*

Throughout the text let $\mathbb{C}P^n$ denote the complex projective space of dimension n, the space of complex lines in $\mathbb{C}^{n+1}$.

Definition 2.2 *A projective algebraic variety is a subset of* $\mathbb{C}P^n$ *given by a homogeneous variety in* $\mathbb{C}^{n+1}$.

Since we deal with good objects, for the rest of the text we will not make any distinction between smooth complex analytic varieties and complex manifolds.

In this thesis we consider real Enriques surfaces, but as we have indicated at the beginning of the chapter, we will treat them from the complex point of view. Thus it is helpful to explain the relation between real and complex surfaces:

Nonsingular real algebraic surfaces are surfaces given in a real projective space by a nonsingular system of homogeneous polynomial equations with real coefficients. If we consider the complexification of these polynomial equations, the resulting complex surface, given by the same equations in the corresponding complex projective space is invariant under the complex conjugation involution, and the original real surface is its fixed point set.

We can take the surface as an abstract analytic manifold and thus arrive to the notion of complex analytic manifold equipped with a real structure. The latter is just an antiholomorphic involution on the manifold.

The methods used in the classification are topological; that is why we deal with analytic manifolds.

The following statements are found in [1] .

Lemma 2.1 *Every (smooth) compact abstract algebraic surface is projective.*

Lemma 2.2 *Let X be a compact surface and Y obtained from X by blowing up a point. Then X is projective iff Y is projective.*

Due to these results we can call a smooth projective algebraic surface simply an algebraic surface, and since Enriques surfaces are algebraic [11] , we will not distinguish them as projective.

2.2 Classification of Complex Algebraic Surfaces

At the beginning of this century Castelnuovo, Enriques and many others had succeeded in creating an impressive essentially geometric theory of birational classification for smooth algebraic surfaces. Kodaira extended the classical results on algebraic surfaces in an essential way and also treated non-algebraic surfaces. For these surfaces the plurigenera and Kodaira dimension can be defined in the same way as for algebraic surfaces, and thus the Enriques classification is extended to the Enriques-Kodaira classification of all compact complex surfaces.

Given n, the n-dimensional compact, connected complex manifolds X can be classified according to their Kodaira dimension Kod(X), which can assume the values -∞,0,1,...,n. In the case $n = 2$ the surfaces in the classes Kod(X) = $-\infty$ or Kod(X) = 0, and to a less extent those with Kod(X) = 1, can be classified in more details.

Theorem 2.1 *see [1] Every compact connected surface X has a minimal model.*

Starting from the rough classification by Kodaira dimension, surfaces are divided into ten classes. This classification is called the Enriques-Kodaira classification and is embodied by the following central result.

Theorem 2.2 *see [1] Every surface has a minimal model in exactly one of the classes* 1) *to* 10) *of Table* 2.1. *This model is unique (up to isomorphism) except for the surfaces with minimal models in the classes* 1) , 2) *and* 3).

The basic idea of the classification is as follows : first a classification according to Kodaria dimension and then a finer classification, biregular classification of minimal smooth algebraic surfaces is done. Since every algebraic surface is birationally equivalent to a smooth one we will consider only smooth surfaces.

Class of X	Kod X
1) minimal rational surfaces **2)** minimal surfaces of class VII **3)** ruled surfaces of genus $g \geq 0$	$-\infty$
4) Enriques surfaces **5)** hyperelliptic surfaces **6)** Kodaira surfaces **a)** primary **b)** secondary **7)** K3 surfaces **8)** tori	0
9) minimal properly elliptic surfaces	1
10) minimal surfaces of general type	2

Table 2.1

Thus, even from the biregular point of view it is sufficient to classify minimal surfaces, at least in the case of Kodaira dimension ≥ 0. If $\mathrm{Kod}(X) = -\infty$ then different Y's can give the same X.

Example: $\mathbb{C}P^2$ with two points blown-up is isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$ with one point blown-up. Both $\mathbb{C}P^2$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$ are minimal, but after some blow-ups they give isomorphic surfaces.

A birational transformation between two minimal surfaces of Kodaira dimension ≥ 0 is always an isomorphism, in other words for Kodaira dimension ≥ 0 birational classification of all surfaces amounts to biregular classification of minimal surfaces.

2.3 Minimality over $\mathbb{R}$ and over $\mathbb{C}$

Definition 2.3 *A smooth surface X is called minimal if any degree 1, regular map $X \to X'$ is an isomorphism.*

The following theorem is due to Castelnuovo:

Theorem 2.3 *A smooth surface is minimal over $\mathbb{C}$ if it does not contain any (-1) curve, or equivalently if it cannot be obtained from another smooth algebraic surface by blowing up a point.*

Definition 2.4 *A nonsingular surface X_{min} is called a minimal model of a nonsingular surface X, if X_{min} is minimal itself, and if there is a blow-down map from X onto X_{min}, i.e., if X is obtained from X_{min} by a sequence of blow ups.*

Every smooth surface X can be obtained from a minimal surface Y by blowing up. At first sight it might seem that classifying only minimal surfaces is not very satisfactory, because one and the same surface X might be obtained by blowing up different minimal surfaces Y. However, if Kod$(X) \geq 0$ then Y is determined by X up to isomorphism, as indicated in the following theorem, see [1]:

Theorem 2.4 *If X is a compact surface with Kod $(X) \geq 0$ then all minimal models of X are isomorphic.*

A minimal surface over $\mathbb{R}$ may not be minimal over $\mathbb{C}$, i.e, complexification of the surface may not always correspond to a minimal complex surface. The reason is that a blow-down of the complexification may not be defined over $\mathbb{R}$, (if the (-1)-curve blown down is not real). The following theorem due to Manin gives the criteria of minimality over $\mathbb{R}$.

Theorem 2.5 *A surface is minimal over $\mathbb{R}$ if it does not contain real (-1) curves or pairs of disjoint conjugate (-1) curves.*

However, if Kod$(X) \geq 0$, there always is a unique way to blow the surface down to a minimal model. As a consequence, both the minimal model X_{min} andthe blow-down map $X \to X_{min}$ are defined over $\mathbb{R}$, i.e., X_{min} has a real structure and the blow-down map is equivariant. We can reformulate this as the following corollary:

Corollary 2.1 *For every surface X with $Kod(X) \geq 0$ the minimal model of X over $\mathbb{R}$ is minimal over $\mathbb{C}$.*

If $\mathrm{Kod}(X) = -\infty$ then there are surfaces minimal over $\mathbb{R}$ but not minimal over $\mathbb{C}$. The simplest example is a cubic surface with disconnected real part $\mathbb{R}P^2 \sqcup S^2$). Cubic surfaces are known to be rational. On the other hand, all real rational surfaces minimal over $\mathbb{C}$ have connected real parts,and birational maps do not change the number of components.

2.4 Classification of Real Surfaces

The current state of the classification of real algebraic surfaces is as follows:

Kod $X = -\infty$: rational, ruled surfaces : done,

Kod $X = 0$: tori, K3 surfaces : done, Enriques surfaces : to be done.

Chapter 3

Real Enriques Surfaces

In this chapter we will investigate real Enriques surfaces and Pontrjagin-Viro form. We will state our main result and consider some possible applications.

Some preliminary definitions: The *geometric genus*, denoted by p_g, of a surface X is the dimension of the space of global holomorphic 2-forms on X.

Irregularity q is defined as the dimension of the space of global holomorphic 1-forms, $q = \dim H^0(\Omega)$.

If D is a divisor on X, then we can introduce the sheaves $\Omega^p(D)$ of meromorphic p-forms with all terms having poles bounded by D; it is more common to write $O(D)$ for $\Omega^0(D)$.

If $K = \mathrm{div}(w)$ is a canonical divisor, we may define the *plurigenera* of X to be the dimensions of the spaces $H^0(O[nK])$ as n varies; more precisely,

$P_n = \dim H^0(O[nK])$, for $n \geq 0$.

Throughout the text, unless stated otherwise, all cohomology and homology groups are with coefficients in $\mathbb{Z}/2$.

3.1 Real Enriques Surfaces

At the end of the last century it had been conjectured that a surface with $q = p_g = 0$ must be rational; no counter examples were known. Castelnuovo's rationality criterion (that $q = P_2 = 0$) is stronger, since $P_2 = 0$ implies $P_1 = p_g = 0$. It was Enriques who finally settled the question and constructed non-rational surfaces with $q = p_g = 0$, which are named after him. It turned out in the works of Enriques and Castelnuovo that Enriques surfaces play a special role in the classification of algebraic surfaces.

Definition 3.1 *An Enriques surface is a complex analytic surface E with $\pi_1(E) = \mathbb{Z}/2$ and $2c_1(E) = 0$.*

Definition 3.2 *A real Enriques surface is a complex Enriques surface E equipped with an anti-holomorphic involution $conj : E \to E$, called complex conjugation; the fixed point set $E_{\mathbb{R}} = Fix\ conj$ is called the* real part *of the surface or its set of* real points.

Theorem 3.1 *A complex analytic surface E with $\pi_1(E) = \mathbb{Z}/2$ is Enriques if and only if its universal covering is a K3 surface.*

From now on let E be real Enriques surface and X it's universal covering, which is a K3-surface. We denote by $\Gamma : X \to X$ the deck translation involution.

Theorem 3.2 *(see [6]): There are two and only two liftings $t^{(1)}, t^{(2)} : X \to X$ of $conj$ to X. Both the liftings are involutions. They are anti-holomorphic, commute with each other, and their composition is Γ. Both the real parts $X_{\mathbb{R}}^{(i)} = Fix t^{(i)}$, $i = 1,2$ and their images $E_{\mathbb{R}}^{(1)}$, $E_{\mathbb{R}}^{(2)}$ in E are disjoint, and $E_{\mathbb{R}}^{(1)} \cup E_{\mathbb{R}}^{(2)} = E_{\mathbb{R}}$.*

Thus, E is a real Enriques surface if and only if it is isomorphic to a quotient of a real K3 surface by a fixed point free holomorphic involution Γ commuting with the real structure. This reduces the theory of real Enriques surfaces to the study of certain group actions on K3 surfaces. Furthermore, the set of components of the real part of a real Enriques surface decomposes into two halves, denoted by $E_{\mathbb{R}}^{(1)}$, $E_{\mathbb{R}}^{(2)}$. The study of this decomposition was started by V.Nikulin [3] as part of his attempt to classify real Enriques surfaces and recently completed in [4], [5].

3.2 Topology of the Real Part

Notation: In what follows, we use the notation S_g and V_p to stand, respectively, for the connected sum of g copies of a 2-torus and the connected sum of p copies of a real projective plane. The 2-sphere S belongs to both families, $S = S_0 = V_0$.

Types of the real part: Let X be a nonsingular compact complex surface with a real structure. Then, since the real part $X_{\mathbb{R}}$ is a closed 2-dimensional manifold, it has a well defined $\mathbb{Z}/2$ -homology fundamental class $\left[X_{\mathbb{R}}\right]$. We say that $X_{\mathbb{R}}$ is of type I_{abs} if $X_{\mathbb{R}}$ is homologous to zero in $H_2(X)$ and of type I_{rel} if $X_{\mathbb{R}}$ is homologous to $w_2(X)$. The surface is said to be of type I if it is of type I_{abs} or I_{rel} ; otherwise it is said to be of type II. In the case of an Enriques surface E and its double covering X the notion of type obviously extends to the halves $E_{\mathbb{R}}^{(i)}$ and $X_{\mathbb{R}}^{(i)}$. For the covering and its halves the types I_{abs} and I_{rel} coincide.

The real part of an Enriques surface is a closed 2-manifold with finitely many components, each being either $S_g = \sharp_g(S^1 \times S^1)$ or $V_p = \sharp_p RP^2$ ($\sharp_i$ denotes the connected sum of i copies).

Definition 3.3 *A* Morse simplification *is a Morse transformation which decreases the total Betti number. There are two types of such simplifications:*

i) removing a spherical component ($S \to \emptyset$), *and*

ii)contracting a handle ($S_{g+1} \to S_g$ *or* $V_{p+2} \to V_p$).

By *topological type* we mean a class of surfaces with homeomorphic real parts. A topological type of an Enriques surface is called *extremal* if it cannot be obtained from the topological type of another Enriques surface by a Morse simplification.

Theorem 3.3 *(see [4]) There are 87 topological types of real Enriques surfaces. Each of them can be obtained by a sequence of Morse simplifications from one of the 22 extremal types listed below. Conversely, with the exception of the two types* $6S$ *and* $S_1 \sqcup 5S$*, any topological type obtained in this way is realized by a real Enriques surface.*

The 22 extremal types are:

1. M-surfaces:

(a) $\chi(E_{\mathbb{R}}) = 8$	(b) $\chi(E_{\mathbb{R}}) = -8$
$4V_1 \sqcup 2S$,	$V_{11} \sqcup V_1$,
$V_2 \sqcup 2V_1 \sqcup 3S$,	$V_{10} \sqcup V_2$,
$V_3 \sqcup V_1 \sqcup 4S$,	$V_9 \sqcup V_3$,
$2V_2 \sqcup 4S$,	$V_8 \sqcup V_4$,
$V_4 \sqcup 5S$,	$V_7 \sqcup V_5$,
$V_2 \sqcup S_1 \sqcup 4S$,	$2V_6$,
	$V_{10} \sqcup S_1$;

2. $(M-2)$-surfaces with $\chi(E_{\mathbb{R}}) = 0$:

$V_4 \sqcup 2V_1$,	$V_5 \sqcup V_1 \sqcup S$,
$V_3 \sqcup V_2 \sqcup V_1$,	$V_4 \sqcup V_2 \sqcup S$,
$V_6 \sqcup 2S$,	$2V_3 \sqcup S$,
$V_4 \sqcup S_1 \sqcup S$,	$2V_2 \sqcup S_1$;

3. Pair of tori: $2S_1$.

Theorem 3.4 *(see [5])Each half of a real Enriques surface may be either* S_1*, or* $2V_2$*, or* $\alpha V_g \sqcup aV_1 \sqcup bS$*,* $g \geq 1$*,* $a \geq 0$*,* $b \geq 0$*,* $\alpha = 0, 1$*. With the exception of the types* kS *and* $V_{2r} \sqcup kS$ *any decomposition into halves satisfying the above condition is realizable.*

3.3 Kalinin's Spectral Sequences

The original construction of this sequence is due to I.Kalinin see [13].

3.3.1 Kalinin's Homology Spectral Sequence

Let X be a smooth compact manifold with an involution $c : X \to X$. There exists a filtration:

$$0 = F^{n+1} \subset F^n \subset ... \subset F^0 = H_*(\text{Fixc})$$

a Z graded spectral sequence (H^r_*, d^r_*), where

$$d^r_q : H^r_q \to H^r_{q+r-1} \ , \ d^r_{q+r-1} \circ d^r_q = 0 \ ,$$

(H^0_*, d^0_*) is the chain complex of X, and H^{r+1}_q =Ker d^r_q / Im d^r_{q-r+1} ,

and homomorphisms $\text{bv}_r : F^r \to H^\infty_r$ such that

(1) $H^1_* = H_*(X)$ and $d^1_* = 1 + c_*$;

(2) a cycle $x_p \in H^0_p$ survives to H^r_p if and only if there are some chains $y_p = x_p, y_{p+1}, ..., y_{p+r-1}$ in X so that $\partial y_{i+1} = (1+c_*)y_i$ (∂ denotes the boundary operator). In this case $d^r_p x_p = (1 + c_*)y_{p+r-1}$;

(3) bv_q annihilates F^{q+1} and maps F^q/F^{q+1} isomorphically onto H^∞_q;

(4) the filtration, spectral sequence, and homomorphisms are all natural with respect to equivariant mappings.

Definition 3.4 *If a cycle admits a representation by an equivariant chain, it survives to $H^\infty_*(X)$. Thus there exist homomorphisms $H_p(Fixc) \to H^\infty_p(X)$, which we will call the inclusion homomorphisms.*

Viro Homomorphisms:

The homomorphisms bv_* appearing in Kalinin's spectral sequence were discovered, in an equivalent form, by O. Viro before Kalinin's work. The following is the geometrical description of Viro homomorphisms, given in terms of Kalinin's spectral sequence.

(1) $\text{bv}_0 : H^*(Fixc) \to H^\infty_0(X)$ is zero on $H_{\geq 1}(Fixc)$; its restriction to $H_0(Fixc) \to H^\infty_0(X) = H_0(X)$ coincides with the inclusion homomorphism ;

(2) a (nonhomogeneous) element $x \in H_*(Fixc)$ represented by a cycle $\sum x_i$ belongs to F_p =Ker bv_{p-1} if and only if there exists some chains y_i, $1 \leq i \leq p$,

so that $\partial y_1 = x_0$ and $\partial y_{i+1} = x_i + (1+c_*)y_i$ for $i \geq 1$; the class of $x_p + (1+c_*)y_p$ in $H_p^\infty(X)$ represents then $\mathrm{bv}_p x$.

3.3.2 Kalinin's Cohomology Spectral Sequence

There exists a filtration

$$H^*(\mathrm{Fixc}) = F_n \supset F_{n-1} \supset \ldots \supset F_{-1} = 0,$$

a Z graded spectral sequence (H_r^*, d_r^*), where

$$d_r^q : H_r^q \rightarrow H_r^{q-r+1},\ d_r^{q-r+1} \circ d_r^q = 0\ ,$$

(H_0^*, d_0^*) is the cochain complex of X , and $H_{r+1}^q = \mathrm{Ker} d_r^q /\ \mathrm{im}\ d_r^{q+r-1}$, and homomorphisms $\mathrm{bv}^r : H_\infty^r \rightarrow H^*(Fixc)/F_{r-1}$ such that

(1) bv^q maps H_∞^q isomorphically onto F_q/F_{q-1};

(2) the spectral sequence , homomorphisms, and filtration are all natural with respect to equivariant mappings;

(3) the spectral sequence is multiplicative, the multiplication being induced by the cup-product in H_0^*; the filtration and homomorphisms bv^q preserve the multiplication;

(4) $H_*^r(X)$ is a graded differential module over H_r^* (via the cup product); the homology filtration and homomorphisms bv_q preserve the module structure.

This spectral sequence is dual to the homology one in the following sense : $H_r^q = Hom(H_q^r; \mathbb{Z}/2)$, $F_{r-1} = \mathrm{Ker}\ [H^*(Fixc) \rightarrow \mathrm{Hom}(F^r; \mathbb{Z}/2)]$, and d_r^q and bv^q are dual to d_{q-r+1}^r and bv_q respectively.

Let ${}^r\mathrm{B}_p \subset {}^r\mathrm{Z}_p \subset H_p(X)$ be the pull-backs of $\mathrm{Im} d_p^{r-1}$ and $\mathrm{Ker} d_p^{r-1}$, respectively, so that ${}^rH_p = {}^r\mathrm{Z}_p / {}^rB_p$. There are obvious cohomology analogues ${}^rB^p \subset {}^rZ^p \subset H^p(X)$, and ${}^rH^p = {}^rZ^p/{}^rB^p$ mod F_{p-1} .

Proposition 3.1 *(see [9]): We have* ${}^\infty Z_p = Ker[pr_* : H_p(X) \rightarrow H_p(\overline{X}, Fixc)$ *and* ${}^\infty B^p = Im[pr^* : H^p(\overline{X}, Fixc) \rightarrow H^p(X)]$, *where* $\overline{X} = X/c$ *is the orbit space.*

3.4 Pontrjagin-Viro Form on a Real Enriques Surface

3.4.1 Quadratic Extensions and Brown Invariant

Definition 3.5 *Let V be a $\mathbb{Z}/2$-vector space and $\circ : V \otimes V \to \mathbb{Z}/2$ a symmetric bilinear form. A function $q : V \to \mathbb{Z}/4$ is called a quadratic extension of $\circ$ if $q(x+y) = q(x) + q(y) + 2(x \circ y)$.*

Pair (V, q) is called a *quadratic space* ($\circ$ is recovered from q). A quadratic space is called nonsingular if the bilinear form is nonsingular; it is called *informative* if $q|_{V^\perp} = 0$.

The Brown invariant $\mathrm{Br}(V, q)$ (or just Br q) of a nonsingular quadratic space is the (mod 8)-residue defined by

$$exp(\tfrac{1}{4} i\pi Br\mathrm{q}) = 2^{\frac{1}{2} dim V} \textstyle\sum_{x \in V} exp(\tfrac{1}{2} i\pi q(x)).$$

This can be extended to informative spaces: since q vanishes on $V^\perp$, it descends to a quadratic form $q' : V/V^\perp \to \mathbb{Z}/4$, and we have Br q =Br q' .

A subspace W of an informative quadratic space (V, q) is called informative if $W^\perp \subset W$ and $q\,|_{W^\perp} = 0$.

Proposition 3.2 *If W is an informative subspace of an informative quadratic space (V, q), then $Br(W, q\,|_W) = Br(V, q)$.*

The proposition above can be interpreted as follows: the Brown invariant of any extension of q to a quadratic form on V equals Br q.

Proposition 3.3 *(see [9]) For any informative quadratic space (V, q) we have:*

(1) *Br* $q \equiv \dim(V/V^\perp)$ *mod2*;

(2) Br $q \equiv q(u)$ mod 4 for any characteristic element $u \in V$;

(3) Br $(q+v) \equiv$ Br $q - 2q(v)$ for any $v \in V$, where $q+v$ is the quadratic form $x \to q(x) + 2(x \circ v)$;

(4) Br $q = 0$ if and only if (V, q) is null cobordant, i.e., there is a subspace $H \subset V$ such that $H^{\perp} = H$ and $q\mid_H = 0$.

Rokhlin-Guillou-Marin form (see [9]): Let Y be an oriented closed smooth 4-manifold and U a characteristic surface in Y, i.e., a smooth closed 2-submanifold (not necessarily orientable) with $[U] = u_2(Y)$ in $H_2(Y)$. (u_2 is the Wu class)

Let $i : U \hookrightarrow Y$ be the inclusion and $K = Ker[i_* : H_1(U) \to H_1(Y)]$. Then there exists a function $q : K \to Z/4$, which is a quadratic extension of the intersection index form on $H_1(U)$, called the *Rokhlin-Guillou-Marin* form of (Y, U).

Theorem 3.5 *(see [9]) Let Y, U and (K, q) be as above. Then (K, q) is an informative subspace of $H_1(U)$ and $2Brq = \sigma(Y) - U \circ U mod 16$, where $U \circ U$ stands for the normal Euler number of U in Y, and $\sigma(Y)$ is the signature of Y.*

3.4.2 Definition of the Pontrjagin-Viro Form

The *Pontrjagin square* is the cohomology operation $P^{2n} : H^{2n}(X) \to H^{4n}(X;\ Z/4)$ uniquely defined by the following properties (see [9]):

(1) $P^{2n}(x + y) = P^{2n}(x) + P^{2n}(y) + 2(x \cup y)$ for any $x, y \in H^{2n}(X)$;

(2) $P^{2n}(x) \equiv x^2 \text{mod } 2$ for any $x \in H^{2n}(X)$;

(3) $P^{2n}(\overline{x} \text{ mod } 2\ ) = \overline{x}^2$ for any $\overline{x} \in H^{2n}(X; \mathbb{Z}/4)$.

Let X be a closed 4n-manifold,denote by $P_{2n} : H_{2n}(X) \to \mathbb{Z}/4$ the composition

$$H_{2n}(X) \to H^{2n}(X) \to H^{4n}(X; \mathbb{Z}/4) \to \mathbb{Z}/4,$$

where the first arrow is the Poincare duality.

If X is a closed n-manifold and $Fixc \neq \emptyset$, then the Poincare duality D induces isomorphisms $D :^r H^p_r \to H^r_{n-p}$ and in the usual way one can define intersection pairings $* : H^r_p \otimes H^r_q \to H^r_{p+q-n}$. The induced pairing (via bv_*) on the graded group $Gr^*_F H_*(Fixc)$ is called *Kalinin's intersection pairing.*

Definition 3.6 *If $P_{2n}(^\infty B_{2n}) = 0$, then P_{2n} descends to a well-defined quadratic function $^\infty H_{2n} \to \mathbb{Z}/4$. The composition of this function and the Viro homomorphism $bv_{2n} : F^{2n} \to^\infty H_{2n}$ is denoted by P and called the Pontrjagin-Viro form. It is a quadratic extension of Kalinin's intersection form*

$$* : F^{2n} \otimes F^{2n} \to Z/2, \text{ i.e.,} P(x+y)=P(x)+P(y)+2(x*y) \text{ for any } x,y \in F^{2n}.$$

Lemma 3.1 *(see [9]) The following are sufficient conditions for the existence of the Pontrjagin-Viro form $P : F^2 \to \mathbb{Z}/4$ on a real Enriques Surface E:*

(1) E is an M surface, i.e.,it has maximal total $\mathbb{Z}/2$-Betti number $\beta_*(E_{\mathbb{R}}) =$ 16.

(2) E is of type I_{rel} and either $E_{\mathbb{R}}$ is nonorientable or both $E^{(1)}_{\mathbb{R}}$ and $E^{(2)}_{\mathbb{R}}$ are nonemtpy;

(3) E is of type I, $E_{\mathbb{R}}$ is nonorientable, and either both $E^{(1)}_{\mathbb{R}}$ and $E^{(2)}_{\mathbb{R}}$ are nonempty or $E_{\mathbb{R}}$ contains a nonorientable component of odd genus.

Lemma 3.2 *(see [6]) Let F_1, F_2 be two components of $E_{\mathbb{R}}$. Then $bv_1\langle F_1 - F_2\rangle = 0$ if and only if these two components belong to the same half of $E_{\mathbb{R}}$.*

Pontrjagin-Viro form and Rohklin-Guillou-Marin forms: Assume that X is an oriented closed smooth 4-manifold, c is smooth and orientation preserving, and $Fixc \neq \emptyset$ has pure dimension 2 and P is well defined. Denote by $F^p_{[i]}$ and $\widehat{F}^p_{[i]}$, respectively, the intersection $F^p \cap H_i(Fixc)$ and the projection of F^p to $H_i(Fixc)$.

Proposition 3.4 *(see [9])Let $F' \subset Fixc$ be a union of components of $Fixc$ such that $P(x) = 2([F'] \circ x)$ mod 4 for all $x \in F^2_{[0]}$. Let $H' = H_1(F^1) \cap \widehat{F}^2_{[1]}$ and define a quadratic function $P' : H' \to \mathbb{Z}/4$ via $x_1 \to P(x_1 + x_0) + 2([F'] \circ x_0)$, where $x_0 \in H_0(Fixc)$ is any element such that $x_1 + x_0 \in F^2$. Then P' coincides with the Rohklin-Guillou-Marin form q' of the characteristic surface F' in $\overline{X}$. In particular, (H', P') is an informative subspace of $H_1(F')$.*

Fix a real Enriques surface $E_{\mathbb{R}}$. Recall that $E_{\mathbb{R}}$ decomposes into two halves $E_{\mathbb{R}}^{(1)}$, $E_{\mathbb{R}}^{(2)}$, where $E_{\mathbb{R}} = E_{\mathbb{R}}^{(1)} \cup E_{\mathbb{R}}^{(2)}$. Assume that $E_{\mathbb{R}}$ has well-defined Pontrjagin-Viro form P.

Since P is linear on $F_{[0]}^{2}$, each half $E_{\mathbb{R}}^{(i)}$ splits into two quarters, which consist of whole components of $E_{\mathbb{R}}^{(i)}$. Denote this splitting by $E_{\mathbb{R}}^{(i)}$ =(quarter 1) $\sqcup$ (quarter 2) and call it *complex seperation.* Geometrically this means that a subsurface $F' \subset E_{\mathbb{R}}$ is characteristic in $E/conj$ if and only if it is the union of two quarters which belong to distinct halves.

Let $Q_i^{(j)}$ denote the i-th quarter in the j-th half, and $E_{\mathbb{R}} = \{(Q_1^{(1)}) \sqcup (Q_2^{(1)})\} \sqcup \{(Q_1^{(2)}) \sqcup (Q_2^{(2)})\}$ be the decomposition of $E_{\mathbb{R}}$ into four quarters. If both halves are non-empty, let $q_{ij}^{(1)}$ and $q_{ji}^{(2)}$ denote the restriction to $H_1(Q_i^{(1)})$ and $H_1(Q_j^{(2)})$ of the Rokhlin-Guillou-Marin form of the characteristic surface $Q_i^{(1)} \cup Q_j^{(2)}$. We have

(3.1) Br $q_{i1}^{(1)} = -$Br $q_{i2}^{(2)}$, Br $q_{j1}^{(1)} = -$Br $q_{j2}^{(2)}$, which follows from Proposition 3.3.

Proposition 3.5 *(see [9]) If both the halves are nonempty , then for i,j =1,2*

$$\chi(Q_i^{(1)}) + \chi(Q_j^{(2)}) \equiv 2 + \tfrac{1}{4}\chi(E_{\mathbb{R}}) + Br\, q_{ij}^{(1)} + Br\, q_{ji}^{(2)} \quad mod\ 8$$

If $E_{\mathbb{R}}^{(2)} = \emptyset$ *, then for* $i = 1,2$

$$\chi(Q_i^{(1)}) \equiv 2 + \tfrac{1}{4}\chi(E_{\mathbb{R}}) + Br\, q_i^{(1)} \ mod\ 8\ .$$

($q_i^{(1)}$ *denote the restriction of* P *to* $H_1(Q_i^{(1)})$*).*

3.5 Main Result

Table 1: M-surfaces of parabolic type

Case $E_{\mathbb{R}} = S_1 \sqcup V_2 \sqcup 4S$		
$^*(V_2 \sqcup 2S) \sqcup (2S)$	$(S_1) \sqcup ()$	0
Case $E_{\mathbb{R}} = 2V_2 \sqcup 4S$		
$^*(V_2) \sqcup (V_2)$	$(2S) \sqcup (2S)$	0
$^*(V_2) \sqcup (V_2)$	$(3S) \sqcup (S)$	2

* $(2V_2) \sqcup ()$	$(2S) \sqcup (2S)$	$0 or 2$
$(V_2 \sqcup 2S) \sqcup (2S)$	$(V_2) \sqcup ()$	0
$(V_2 \sqcup S) \sqcup (2S)$	$(V_2 \sqcup S) \sqcup ()$	2
$(V_2 \sqcup 2S) \sqcup (S)$	$(V_2) \sqcup (S)$	2
$(V_2 \sqcup S) \sqcup (S)$	$(V_2 \sqcup S) \sqcup (S)$	0

Case $E_{\mathbb{R}} = V_2 \sqcup 2V_1 \sqcup 3S$

* $(V_2 \sqcup 2S) \sqcup (2V_1 \sqcup S)$		0
* $(V_2 \sqcup 2V_1 \sqcup S) \sqcup (2S)$		$0 or 2$
$(V_2 \sqcup V_1 \sqcup S) \sqcup (V_1 \sqcup S)$	$(S) \sqcup ()$	$0 or 2$
$(V_2 \sqcup S) \sqcup (2V_1)$	$(S) \sqcup (S)$	0
$(V_2 \sqcup S) \sqcup (2V_1)$	$(2S) \sqcup ()$	2
$(V_2 \sqcup 2V_1) \sqcup (S)$	$(S) \sqcup (S)$	$0 or 2$
$(V_2 \sqcup V_1) \sqcup (V_1)$	$(2S) \sqcup (S)$	$0 or 2$
$(V_2 \sqcup 2S) \sqcup (V_1 \sqcup S)$	$(V_1) \sqcup ()$	0
$(V_2 \sqcup V_1 \sqcup S) \sqcup (2S)$	$(V_1) \sqcup ()$	$0 or 2$
$(V_2 \sqcup S) \sqcup (V_1 \sqcup S)$	$(V_1) \sqcup (S)$	0
$(V_2 \sqcup S) \sqcup (V_1 \sqcup S)$	$(V_1 \sqcup S) \sqcup ()$	2
$(V_2 \sqcup V_1 \sqcup S) \sqcup (S)$	$(V_1) \sqcup (S)$	$0 or 2$
$(V_2 \sqcup S) \sqcup (V_1)$	$(V_1 \sqcup S) \sqcup (S)$	0
$(V_2 \sqcup S) \sqcup (V_1)$	$(V_1) \sqcup (2S)$	2
$(V_2 \sqcup V_1) \sqcup (S)$	$(V_1 \sqcup S) \sqcup (S)$	$0 or 2$
$(V_2) \sqcup (V_1)$	$(V_1 \sqcup S) \sqcup (2S)$	0
$(V_2) \sqcup (V_1)$	$(V_1 \sqcup 2S) \sqcup (S)$	2
$(V_2 \sqcup V_1) \sqcup ()$	$(V_1 \sqcup S) \sqcup (2S)$	$0 or 2$
$(V_2 \sqcup S) \sqcup (2S)$	$(V_1) \sqcup (V_1)$	0
$(V_2 \sqcup S) \sqcup (2S)$	$(2V_1) \sqcup ()$	2
$(V_2 \sqcup 2S) \sqcup (S)$	$(V_1) \sqcup (V_1)$	0
$(V_2 \sqcup S) \sqcup (S)$	$(2V_1) \sqcup (S)$	0
$(V_2 \sqcup S) \sqcup (S)$	$(V_1 \sqcup S) \sqcup (V_1)$	2
$(V_2) \sqcup (S)$	$(V_1 \sqcup S) \sqcup (V_1 \sqcup S)$	0
$(V_2) \sqcup (S)$	$(2V_1 \sqcup S) \sqcup (S)$	2
$(V_2 \sqcup S) \sqcup ()$	$(V_1 \sqcup S) \sqcup (V_1 \sqcup S)$	0
$(V_2 \sqcup S) \sqcup ()$	$(2V_1) \sqcup (2S)$	2
$(V_2) \sqcup ()$	$(2V_1 \sqcup S) \sqcup (2S)$	0
$(V_2) \sqcup ()$	$(V_1 \sqcup 2S) \sqcup (V_1 \sqcup S)$	2

Table 2: M-surfaces of elliptic type ($E_{\mathbb{R}} = 4V_1 \sqcup 2S$)

$(2V_1 \sqcup S) \sqcup (2V_1 \sqcup S)$		$(V_1 \sqcup S) \sqcup (S)$	$(2V_1) \sqcup (V_1)$
$(4V_1) \sqcup (2S)$		$(V_1) \sqcup (2S)$	$(2V_1) \sqcup (V_1)$
$(2V_1 \sqcup S) \sqcup (V_1 \sqcup S)$	(V_1)	$(3V_1) \sqcup (S)$	$(V_1) \sqcup (S)$
$(3V_1) \sqcup (2S)$	(V_1)	$(2V_1) \sqcup (V_1 \sqcup S)$	$(V_1) \sqcup (S)$
$(2V_1 \sqcup S) \sqcup (S)$	$(V_1) \sqcup (V_1)$	$(2V_1) \sqcup (V_1 \sqcup S)$	$(V_1 \sqcup S)$
$(V_1 \sqcup S) \sqcup (V_1 \sqcup S)$	$(V_1) \sqcup (V_1)$	$(S) \sqcup (S)$	$(2V_1) \sqcup (2V_1)$
$(2V_1) \sqcup (2S)$	$(V_1) \sqcup (V_1)$	$(2S)$	$(2V_1) \sqcup (2V_1)$
$(V_1 \sqcup S) \sqcup (V_1 \sqcup S)$	$(2V_1)$	$(2V_1) \sqcup (S)$	$(V_1 \sqcup S) \sqcup (V_1)$
$(2V_1) \sqcup (2S)$	$(2V_1)$	$(V_1 \sqcup S) \sqcup (V_1)$	$(V_1 \sqcup S) \sqcup (V_1)$
$(3V_1) \sqcup (V_1 \sqcup S)$	(S)	$(2V_1) \sqcup (S)$	$(2V_1) \sqcup (S)$

Theorem 3.6 *Given a decomposition $E_{\mathbb{R}} = E_{\mathbb{R}}^{(1)} \sqcup E_{\mathbb{R}}^{(2)}$ from tables 1 and 2, the Pontrjagin-Viro form can take any value satisfying Proposition* 3.5. *Furthermore, in all cases listed in the tables the Pontrjagin-Viro form is uniquely recovered (up to autohomemorphism of $E_{\mathbb{R}}$ preserving the complex seperation) from the complex seperation and $P(w_1)$ via Proposition* 3.5. *Here $P(w_1)$ is the value of P on the first Stiefel-Whitney class of (any of) components V_2 see table* 1.

Proof: Recall that V_1, V_2, S stand for real projective plane, Klein bottle and the 2-sphere, respectively.

The proof consists in classification of quadratic forms on $\mathrm{H}_*(E_{\mathbb{R}})$ satisfying Proposition 3.5 and comparing the result with the known deformation classification of real Enriques surfaces. The restriction of a form to a component C of $E_{\mathbb{R}}$ is determined up to isomorphism by its Brown invariant, which may take the following values :

$C = S : Br \equiv 0 \ mod 8,$

$C = V_1 : Br \equiv \pm 1 \ mod 8,$

$C = V_2 : Br \equiv 0, \pm 2 \ mod 8,$

$C = S_1 : Br \equiv 0, 4 \ mod 8$

Thus, it remains to enumerate the collections of componentwise Brown invariants such that the resulting values of Br $\mathrm{q}_{ij}^{(k)}$ satisfy Proposition 3.5.

As an example let us consider the decomposition $E_{\mathbb{R}} = [V_1 \sqcup (V_2 \sqcup V_1)] \sqcup [(S \sqcup S) \sqcup S]$,

$E_{\mathbb{R}}^{(1)} = [V_1 \sqcup (V_2 \sqcup V_1)]$, $E_{\mathbb{R}}^{(2)} = [(S \sqcup S) \sqcup S]$

For any pair $Q_i^{(1)}$, $Q_j^{(2)}$ of quorters, one from each half, a quadratic extension $q_{ij} : H_1(Q_i^{(1)} \cup Q_j^{(2)}) \to \mathbb{Z}/4$ is defined :

1) $q_{11} : H_1(V_1 \sqcup (S \sqcup S)) \to \mathbb{Z}/4$, 2) $q_{12} : H_1(V_1 \sqcup S) \to \mathbb{Z}/4$,

3) $q_{21} : H_1((V_2 \sqcup V_1) \sqcup (S \sqcup S)) \to \mathbb{Z}/4$, 4) $q_{22} : H_1((V_2 \sqcup V_1) \sqcup S) \to \mathbb{Z}/4$,

Let $q_{ij}^{(1)}$ and $q_{ji}^{(2)}$ denote the restriction to $H_1(Q_i^{(1)})$ and $H_1(Q_j^{(2)})$ of the Rokhlin-Guillou-Marin form of the characteristic surface $Q_i^{(1)} \cup Q_j^{(2)}$. Then the values of their Brown invariants must satisfy the following congruences:

1) $\chi(V_1) + \chi(S \sqcup S) \equiv 2 + \frac{1}{4}\chi(E_{\mathbb{R}}) + \text{Br } q_{11}^{(1)} + \text{Br } q_{11}^{(2)} \mod 8$

$1+4 \equiv 2 + 2 + \{1, -1\} + 0$; the congruence is satisfied if and only if we choose 1 for Br $q_{11}^{(1)}$

2) $\chi(V_1) + \chi(S) \equiv 2 + \frac{1}{4}\chi(E_{\mathbb{R}}) + \text{Br } q_{12}^{(1)} + \text{Br } q_{21}^{(2)} \mod 8$

$1 + 2 \equiv 2 + 2 + \{1, -1\} + 0$; choose 1 for Br $q_{12}^{(1)}$

3) $\chi(V_2 \sqcup V_1) + \chi(S \sqcup S) \equiv 2 + \frac{1}{4}\chi(E_{\mathbb{R}}) + \text{Br } q_{21}^{(1)} + \text{Br } q_{12}^{(2)} \mod 8$

$0 + 1 + 4 \equiv 2 + 2 + \{0, \pm 2\} + \{1, -1\} + 0$; choose 0,-1 for Br $q_{21}^{(1)}$

4) $\chi(V_2 \sqcup V_1) + \chi(S) \equiv 2 + \frac{1}{4}\chi(E_{\mathbb{R}}) + \text{Br } q_{22}^{(1)} + \text{Br } q_{22}^{(2)} \mod 8$

$0 + 1 + 2 \equiv 2 + 2 + \{0, \pm 2\} + \{1, -1\} + 0$; choose $0, -1$ for Br $q_{22}^{(1)}$.

Note that due to 3.1 the sign of Br changes when we change quarter.

Thus, given a decomposition we classified octuples of forms $q_{ij}^{(k)}$ satisfying Proposition 3.5 and restricted Brown invariants. After the classification is done we compared it with known classification of deformation types of surfaces.

Remark: In fact the Pontrjagin-Viro form determines a real Enriques M-surface up to deformation.

3.6 Possible Applications

An application of our main result is the study of the fundamental group of the moduli space of real Enriques surfaces. More precisely, given a deformation type, there is an obvious representation of the fundamental group of the corresponding component of the moduli space in the mapping class group of the real part. (The latter is, by definition, the group of autodiffeomorphisms of the real part considered modulo diffeotopies.) The image of the above representation is of particular interest; we refer to it as the monodromy group of the deformation type. In other words, we are interested in the autodiffeomorphisms of the real part which can be realized by changing the surface continuously in the class of real Enriques surfaces.

Since the Pontrjagin-Viro form is a topological invariant, it must be preserved by any element of the monodromy group. In particular, the complex separation and the Brown invariants of the restrictions of $\mathrm{q}_{ij}^{(k)}$ to the components of E_R must be preserved. This gives certain restrictions to possible autodiffeomorphisms.

Consider, for example, the deformation type with the seperation $[S \sqcup (V_2 \sqcup S)] \sqcup [(V_1 \sqcup V_1) \sqcup V_2]$. Let f be an element of the monodromy group. We claim that f acts identically on the set of components of E_R. Indeed, since the complex separation must be preserved, the only possible permutation is the transposition of the two components V_1. However, from our calculation it follows that the restrictions of, say, $q_{11}^{(2)}$ to the two components have Brown invariants of opposite signs. Hence, the components cannot be transposed.

Similar prohibitions can be found for other deformation types. As usual, after certain diffeomorphisms have been prohibited, the rest should be constructed. We hope that Pontrjagin-Viro form does describe the monodromy groups (at least, for M-surfaces), and the diffeomorphisms preserving it can be found in algebraic families of real Enriques surfaces.

Chapter 4

Conclusion

We showed that the Pontrjagin-Viro form of a real Enriques surface when defined satisfies the congruence relation stated in Proposition 3.5 and besides any quadratic form $P : H_*((E_{\mathbb{R}}^{(1)}) \oplus H_*(E_{\mathbb{R}}^{(2)})) \to \mathbb{Z}/4$ of a triad $(E_{\mathbb{R}}, E_{\mathbb{R}}^{(1)}, E_{\mathbb{R}}^{(2)})$ satisfying Proposition 3.5 can be realized as the Pontrjagin-Viro form of a real Enriques surface.

Our results give useful information about the fundamental group of the moduli space of real Enriques surfaces considered by the autodiffeomorphisms of the real part. We found several restrictions for possible autodiffeomorphisms of a fixed deformation type and similar prohibitions can be found for other deformation types. We hope that the Pontrjagin-Viro form does describe the monodromy groups (at least, for M -surfaces), and the diffeomorphisms preserving it can be found in algebraic families of real Enriques surfaces.

Our approach is topological,not algebraic, and thus gives stronger results, can be applied to flexible real Enriques surfaces. (A flexible Enriques surface is a closed smooth 4-manifold with involution which possesses certain topological properties of real Enriques surfaces, see [4].) We conjecture that the monodromy groups of true real Enriques surfaces coincide with those of flexible ones.

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