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NONLINEAR OBSERVER DESIGN WITH APPLICATION TO THE SYNCHRONIZATION OF CHAOTIC SYSTEMS

A THESIS
SUBMITTED TO THE DEPARTMENT OF ELECTRICAL AND
ELECTRONICS ENGINEERING
AND THE INSTITUTE OF ENGINEERING AND SCIENCES
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
Ercan Solak
August 1996

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Ömer Morgül(Supervisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Erol Sezer

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Bülent Özgüler

Approved for the Institute of Engineering and Sciences:

Prof. Dr. Mehmet Baray
Director of Institute of Engineering and Sciences

ABSTRACT

NONLINEAR OBSERVER DESIGN WITH APPLICATION TO THE SYNCHRONIZATION OF CHAOTIC SYSTEMS

Ercan Solak
M.S. in Electrical and Electronics Engineering
Supervisor: Assoc. Prof. Dr. Ömer Morgül
August 1996

Observers are used to estimate the states of dynamical systems whenever they are not available through direct measurements. Although the design of linear observers is a well-developed branch of control theory, its counterpart for nonlinear systems is a relatively new field.

In this thesis, an observer construction methodology is proposed for a class of nonlinear systems satisfying certain conditions. Then, the problem of synchronizing chaotic systems, which has found recent applications in the area of secure message transmission, is addressed from the observer design point of view. In the design, we exploited one of the essential properties of the chaotic systems that the trajectories remain in a bounded region of the state space. It is also shown that, for certain well-known chaotic systems, the system structure enables one to use linear observer schemes in order to have global synchronization.

Keywords : Nonlinear observers, chaotic systems, chaos synchronization.

ÖZET

DOĞRUSAL OLMAYAN GÖZLEYİCİ TASARIMI VE KAOTİK SİSTEMLERİN EŞZAMANLAMASINA UYGULANMASI

Ercan Solak
Elektrik ve Elektronik Mühendisliği Bölümü Yüksek Lisans
Tez Yöneticisi: Doç. Dr. Ömer Morgül
Ağustos 1996

Gözleyiciler, dinamik sistemlerin durumları doğrudan ölçümlerle elde edilemediğinde, bu durumları tahmin etmekte kullanılırlar. Doğrusal gözleyici tasarımı, kontrol kuramının gelişmiş bir dalı olmasına rağmen, bunun doğrusal olmayan sistemlerdeki karşılığı göreceli olarak yeni bir alandır.

Bu tezde, doğrusal olmayan sistemlerin belli şartları sağlayan bir sınıfı için bir gözleyici tasarım yöntemi önerilmiştir. Daha sonra, son zamanlarda güvenli bilgi aktarımı konusunda uygulama alanı bulan, kaotik sistemlerin eşzamanlanması problemi, gözleyici tasarımı noktasından ele alınmıştır. Tasarımda, kaotik sistemlerin yörüngelerinin, durum uzayının sınırlı bir bölgesinde kalması özelliği, vurgulanarak kullanılmıştır. Ayrıca, bazı çok bilinen kaotik sistemler için, sistem yapısının global eşzamanlama amacıyla doğrusal gözleyici kullanımına olanak verdiği gösterilmiştir.

Anahtar Kelimeler : Doğrusal olmayan gözleyiciler, kaotik sistemler, kaos eşzamanlaması.

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And special thanks to my friends for their helpful discussions and suggestions.

humbly dedicated to Sherlock Holmes,
the most powerful observing machine ever imagined.

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Chapter 1

INTRODUCTION

In all control strategies, the state feedback gives more degrees of freedom to the designer than that the output feedback does, which is clearly evidenced by the fact that output is an algebraic combination of the states. Therefore it is natural for a system designer to seek to have the system states or their estimates available. While in some cases this can be achieved by a direct measurement, in general either the additional complexity required to perform a reliable measurement or the very nature of the system becomes a hindrance to such an approach.

A common solution to this problem is to incorporate into the design a new system called "state observer" or "state estimator" which gives an estimate of the true states using only the directly measurable variables of the system, namely, the output and the input. Under some mild conditions, any state feedback scheme performs as well even when the state variables in the formulation are replaced by those of the observer [1, page 251].

Other than a control objective, one can design an observer for the sole purpose of monitoring hard-to-measure variables of the system and using those estimation towards some other aim, such as system diagnostic [2].

Although the theory of observer design for linear systems is a well-developed field, its counterpart for nonlinear systems is a relatively new branch of control science, see [3]. Almost all of the existing research in this field focuses on some restricted classes of nonlinear systems satisfying certain conditions, see [4, 5, 6, 7, 8, 9, 10, 11].

Recently, independent of the ongoing research on nonlinear observer theory, there has been an increasing interest in the synchronization of chaotic systems through a set of common signals, see [12, 13, 14, 15]. The motivation underlying these attempts is the secure transmission by exploiting the non-periodicity of the chaotic signals. We show that this task can also be formulated as an observer design problem, where the original system and the observer are the two systems to be synchronized and the system output is the common signal.

The thesis is organized as follows; in the second chapter a survey on the existing nonlinear observer theory is presented with a brief reminder for the linear counterpart. For each method, the advantages and the drawbacks are highlighted. The discussion in the third chapter begins with the exposition of the limitations of linearization method. Then an explicit eigenvalue assignment procedure is given to improve the method based on the transformation of the system to observer canonical form, see [16]. Fourth chapter is an account of our application of the nonlinear observer design techniques to chaotic synchronization. We also indicate some special cases where the design is simplified due to the special form of the system. We also furnish the above approach with several examples of well-known chaotic systems.

The thesis is concluded with the description of an observation technique inspired by the gradient descent dynamics and a summarizing view of our work.

Chapter 2

BASICS AND OVERVIEW OF LITERATURE

2.1 Observability and Observer Notions

Observer design can be defined as the construction of an auxiliary dynamical system driven by the measurable variables of the original system such as its input and the output. Assuming that the state variables of the observer can easily be measured, we require those states to be a good estimation of the true states. Generally *a priori* knowledge of the system model is assumed. Namely, given a dynamical system described by,

$$\begin{aligned} \dot{x} &= f(x,u),\ x(0) = x_0, && (2.1) \\ y &= h(x), && (2.2) \end{aligned}$$

then the observer is a system described by,

$$\dot{\hat{x}} = F(\hat{x}, y, u), \tag{2.3}$$

which satisfies,

$$\lim_{t\to\infty}(x(t) - \hat{x(t)}) = 0, \tag{2.4}$$

where, $x \in \mathbf{R}^n$, $\hat{x} \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, $y \in \mathbf{R}^p$, $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$, $h : \mathbf{R}^n \to \mathbf{R}^p$ and $F : \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^p \to \mathbf{R}^n$.

When (2.4) is satisfied for every initial conditions $x(0) \in \mathbf{R}^n$ and $\hat{x}(0) \in \mathbf{R}^n$, (2.3) is a global observer for (2.1),(2.2). If convergence is guaranteed for $\hat{x}(0)$ in some neighborhood of $x(0)$, then we have a local observer.

Since the observer (2.3) estimates the set of all the system states, it is also called a "full order observer". When some of the states are available either through direct measurements or in the output, the set of states to be estimated can be reduced, yielding a "reduced order observer" [17, page 461]. In our work, we deal with full order observers.

For the above approach to work, observability is an important condition that has to be satisfied by the system.

Definition 2.1.1 [1] *Consider the system (2.1),(2.2). Two states x_0 and x_1 are said to be* **distinguishable** *if there exists an input function $u(\cdot)$ such that $y(\cdot, x_0, u) \neq y(\cdot, x_1, u)$, where $y(\cdot, x_i, u)$, $i = 1, 2$ is the output function of the system (2.1),(2.2) corresponding to the input function $u(\cdot)$ and the initial condition $x(0) = x_i$. The system is said to be* **locally observable** *at $x_0 \in \mathbf{R}^n$ if there exists a neighborhood N of x_0 such that every $x \in N$ other than x_0 is distinguishable from x_0. Finally, the system is said to be locally observable if it is locally observable at each $x_0 \in \mathbf{R}$. If the neighborhood extends to all the state space then we have* **global observability**.

Note that two states may be indistinguishable for some set of inputs but existence of any *one* distinguishing input is enough to guarantee local observability. In the next section we will see that analysis is quite simplified when the system is linear. For linear systems, notions of local and global observability are the same. Further, if a linear system is observable for an input function , so it is for *any* input function.

2.1.1 Linear Case

The following theorem summarizes the above mentioned properties of a linear time invariant system described by,

$$\dot{x} = Ax + Bu, \; x(0) = x_0, \tag{2.5}$$
$$y = Cx, \tag{2.6}$$

where $A \in \mathbf{R}^{n \times n}, B \in \mathbf{R}^{n \times m}, C \in \mathbf{R}^{p \times n}$.

Theorem 2.1.1 *For the system (2.5),(2.6) the following are equivalent,*

1. *The pair (C, A) is observable.*
2. *The following rank condition is satisfied;*

$$rank \begin{pmatrix} C \\ CA \\ \\ CA^{n-1} \end{pmatrix} = n. \tag{2.7}$$

3. *For any polynomial $p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n$, $a_i \in \mathbf{R}$, $i = 1, 2, \ldots, n$, there exists a constant matrix $K \in \mathbf{R}^{n \times m}$ such that $\det(\lambda I - A + KC) = p(\lambda)$.*

Proof : See [18]. □

One immediately realizes that the input has no effect on the observability of the linear system.

Hence for an observable LTI system we can construct the observer as,

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - \hat{y}), \; \hat{x}(0) = \hat{x}_0, \tag{2.8}$$
$$\hat{y} = C\hat{x}, \tag{2.9}$$

where A, B and C are the same as in (2.5) , (2.6) and $K \in \mathbf{R}^{n \times p}$ is the gain matrix. Let us define the state error as $\varepsilon \triangleq x - \hat{x}$. Then the error dynamics is given by,

$$\begin{aligned} \dot{\varepsilon} &= \dot{x} - \dot{\hat{x}}, \\ &= Ax - A\hat{x} - KC(x - \hat{x}), \\ &= (A - KC)\varepsilon. \end{aligned} \tag{2.10}$$

Thus, since (C, A) is observable, for a conjugate set of complex numbers $\{\lambda_1, \lambda_2, \ldots \lambda_n\}$ on the open left half plane, we can find a $K \in \mathbf{R}^{n \times p}$ such that the eigenvalues of $A - KC$ correspond exactly to this set, i.e.,

$$\det(\lambda I - A + KC) = \prod_{i=1}^{n} (\lambda - \lambda_i), \tag{2.11}$$

yielding a globally exponentially stable error system.

2.1.2 Generalization to Nonlinear Systems

There are some subtleties involved in the notion of observability for nonlinear systems. First, the distinguishability of any two states depends on the input function. There may exist some input function that yield the same output function for two different initial conditions although they are distinguishable. Another peculiarity is that in general, observability may only be satisfied locally. For examples of such cases, see [1, pages 415–416].

To give a sufficient condition for the local observability of an autonomous system, we successively differentiate the output and impose a rank condition to be able to extract the state information out of these quantities.

We consider a single input single output, (SISO), time-invariant nonlinear system,

$$\dot{x} = f(x), \quad x(0) = x_0, \tag{2.12}$$

$$y = h(x). \tag{2.13}$$

To see the pattern, let us successively differentiate the output with respect to time,

$$
\begin{aligned}
y &= h(x) \\
\dot{y} &= \nabla h(x) \cdot \dot{x} = \nabla h(x) \cdot f(x) \\
\ddot{y} &= \nabla(\nabla h(x) \cdot f(x)) \cdot f(x),
\end{aligned}
$$

where $\nabla h(x) = \left[\frac{\partial h(x)}{\partial x_1}, \frac{\partial h(x)}{\partial x_2}, \ldots, \frac{\partial h(x)}{\partial x_n}\right]$ is the gradient vector of $h(x)$. Remembering the definition of the Lie derivative $L_\varphi \theta$ of a C^∞ function $\theta(x)$ with respect to a C^∞ vector field $\varphi(x)$,

$$
\begin{aligned}
L_\varphi \theta(x) &= < \nabla\theta(x), \varphi(x) >, \\
L_\varphi^k \theta(x) &= L_\varphi(L_\varphi^{k-1}\theta(x)), \\
L_\varphi^0 \theta(x) &= \theta(x),
\end{aligned}
$$

we can express the time derivatives of the output as,

$$
\begin{aligned}
y &= h(x), \\
\dot{y} &= L_f h(x), \\
\ddot{y} &= L_f^2 h(x), \\
\\
y^{(n-1)} &= L_f^{n-1} h(x).
\end{aligned}
$$

Let us define the observability matrix $Q(x)$ as,

$$
Q(x) \triangleq \frac{d}{dx}\begin{bmatrix} h(x) \\ L_f h(x) \\ \\ L_f^{n-1} h(x) \end{bmatrix} = \frac{d\Phi}{dx}. \tag{2.14}
$$

Note that when the system is LTI, $Q(x)$ becomes the constant observability matrix introduced in (2.7).

Theorem 2.1.2 (Sufficient condition for local observability) *Consider the system (2.12),(2.13) and let $x_0 \in \mathbf{R}^n$ be given. If $Q(x_0)$ has rank n, then the system is locally observable at x_0.*

Proof : See [1, pages 418–421]. □

Remark 2.1.1 *This condition is also sufficient for the existence of a nonlinear diffeomorphic coordinate transformation $z = T(x)$, such that, in the new coordinates the system is linear up to output injection. Namely,*

$$\dot{z} = Az + g(y), \quad z(0) = z_0, \tag{2.15}$$

$$y = Cz, \tag{2.16}$$

with (C, A) observable. Obviously, the auxiliary system,

$$\dot{\hat{z}} = A\hat{z} + g(y) + K(y - \hat{y}), \quad z(0) = z_0, \tag{2.17}$$

$$\hat{y} = C\hat{z}, \tag{2.18}$$

is an exponential observer for (2.15),(2.16). For an in-depth discussion of the calculation of the nonlinear state transformation, see [19, page 244].

2.2 Methods of Observer Construction for Nonlinear Systems

Determination of the nonlinear state transformation mentioned above is quite difficult and to our knowledge, no systematic procedure has been proposed to explicitly solve this problem. Instead, many attempts have been made to deal with specific classes of nonlinear systems. In [20], a sufficient condition in terms of the gradients of the system function and the output function is given. Another approach is to impose a Lipschitz condition on the system nonlinearity, which would enable the linear error dynamics to suppress nonlinear effects [8]. [16] uses a similar constraint together with a nonlinear transformation.

Here, we first give a brief description of analysis method of [8]. Then a detailed exposition of the last technique proposed in [16] follows, since in our work we use this design strategy together with an eigenvalue assignment procedure of ours.

2.2.1 Linearization Method

As mentioned before this is an analysis approach rather than a constructive one. But the following discussion is useful in the sense that it exposes the limitations of the linearization approach. First, we state the following well-known lemma;

Lemma 2.2.1 (Bellman-Gronwall) *Let $u(\cdot)$, $\phi(\cdot)$ and $k(\cdot)$ be real valued piecewise continuous functions on $\mathbf{R}_+$. If $u(\cdot)$ satisfies*

$$u(t) \leq \phi(t) + \int_0^t k(\tau)u(\tau)d\tau, \qquad \forall t \geq 0, \tag{2.19}$$

then ,

$$u(t) \leq \phi(t) + \int_0^t \phi(\tau)k(\tau)e^{\int_\tau^t k(s)ds}d\tau, \qquad \forall t > 0. \tag{2.20}$$

Proof : See [21, page 476]. □

Consider the following autonomous system,

$$\begin{aligned} \dot{x} &= Ax + g(x), \quad x(0) = x_0, & (2.21)\\ y &= Cx, & (2.22) \end{aligned}$$

where the differentiable function $g : \mathbf{R}^n \to \mathbf{R}^n$ satisfies the following Lipschitz condition;

$$\|g(u_1) - g(u_2)\| \leq L\|u_1 - u_2\|, \quad \forall u_1, u_2 \in \mathbf{R}^n, \tag{2.23}$$

where $L > 0$ is a Lipschitz constant. In the following discussion $\|\cdot\|$ denotes either the standard Euclidean norm or the matrix norm induced by the 2-norm, unless otherwise stated. For a definition see [1, page 22].

Assume the pair (C, A) is observable. Hence we can choose a gain matrix $K \in \mathbf{R}^{n\times p}$ such that $A_c = A - KC$ is a stable matrix. Then for a symmetric and positive definite matrix $Q \in \mathbf{R}^{n\times n}$, there exists a symmetric and positive definite matrix $P \in \mathbf{R}^{n\times n}$ such that the following Lyapunov matrix equation is satisfied,

$$A_c^T P + PA_c = -Q. \tag{2.24}$$

For the system (2.21) , (2.22) we construct the following observer,

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + g(\hat{x}) + K(y - \hat{y}), \quad \hat{x}(0) = \hat{x}_0, & (2.25) \\ y &= C\hat{x}. & (2.26) \end{aligned}$$

Defining the error to be $\varepsilon = x - \hat{x}$, its dynamics becomes,

$$\dot{\varepsilon} = A_c\varepsilon + g(x) - g(\hat{x}). \tag{2.27}$$

To check the stability of (2.27) we use the Lyapunov function $V = \varepsilon^T P\varepsilon$. We note that for symmetric positive definite matrices P and Q, the following holds for $\forall u \in \mathbf{R}^n$,

$$\begin{aligned} \lambda_{min}(P)\|u\|^2 \leq u^T P u \leq \lambda_{max}(P)\|u\|^2, \\ \lambda_{min}(Q)\|u\|^2 \leq u^T Q u \leq \lambda_{max}(Q)\|u\|^2. \end{aligned} \tag{2.28}$$

By taking the time derivative of V along the error trajectory,

$$\begin{aligned} \dot{V} &= \varepsilon^T P\dot{\varepsilon} + \dot{\varepsilon}^T P\varepsilon, \\ &= \varepsilon^T(A_c^T P + PA_c)\varepsilon + 2\varepsilon^T P\left[g(x) - g(\hat{x})\right], \\ &\leq -\varepsilon^T Q\varepsilon + 2\|P\|\|g(x) - g(\hat{x})\|\|\varepsilon\|, \\ &\leq -(\lambda_{min}(Q) - 2L\lambda_{max}(P))\|\varepsilon\|^2, \\ &= -\left(\frac{\lambda_{min}(Q)}{2\lambda_{max}(P)} - L\right)2V. \end{aligned} \tag{2.29}$$

Hence we can have an exponentially decaying bound on the Lyapunov function. Namely,

$$V(t) \leq V(0)e^{-2\gamma t}, \tag{2.30}$$

where $\gamma = \frac{\lambda_{min}(Q)}{2\lambda_{max}(P)} - L$. Using (2.28), we have,

$$\|\varepsilon(t)\|^2 \leq \frac{V(t)}{\lambda_{min}(P)} \leq \frac{V(0)e^{-2\gamma t}}{\lambda_{min}(P)} \leq \frac{\lambda_{max}(P)}{\lambda_{min}(P)}e^{-2\gamma t}\|\varepsilon(0)\|^2, \tag{2.31}$$

or

$$\|\varepsilon(t)\| \leq \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}e^{-\gamma t}\|\varepsilon(0)\|. \tag{2.32}$$

Thus, for a given Lipschitz constant L if

$$\gamma = \frac{\lambda_{min}(Q)}{2\lambda_{max}(P)} - L > 0 \tag{2.33}$$

then the observer states converge to the actual states exponentially fast. Also note that the matrices P and Q are determined by the choice of the gain matrix K in the observer system.

An alternative way to see the same result, which could be related to our work in the sequel, is to use the solution of (2.27) as follows:

$$\varepsilon(t) = e^{A_c t}\varepsilon(0) + \int_0^t e^{A_c(t-\tau)}\left[g(x(\tau)) - g(\hat{x}(\tau))\right] d\tau. \qquad (2.34)$$

Since A_c is stable, the following holds for some $M > 0$ and $\alpha > 0$;

$$\|e^{A_c t}\| \leq M e^{-\alpha t}. \qquad (2.35)$$

By taking norms and using (2.35) in (2.34), we obtain

$$\|\varepsilon(t)\| \leq M e^{-\alpha t}\|\varepsilon(0)\| + \int_0^t M e^{-\alpha(t-\tau)}\|g(x(\tau)) - g(\hat{x}(\tau))\| d\tau. \qquad (2.36)$$

Using the Lipschitz condition (2.23), and multiplying by $e^{\alpha t}$

$$\|\varepsilon(t)e^{\alpha t}\| \leq M\|\varepsilon(0)\| + \int_0^t ML\|\varepsilon(\tau)e^{\alpha\tau}\| d\tau. \qquad (2.37)$$

Finally, applying Lemma 2.2.1 and multiplying by $e^{-\alpha t}$, we obtain

$$\|\varepsilon(t)\| \leq M e^{-(\alpha - ML)t}\|\varepsilon(0)\|. \qquad (2.38)$$

Hence if

$$\frac{\alpha}{M} > L, \qquad (2.39)$$

then the estimation error decays to zero exponentially fast.

This method relies on the suppression of the nonlinearity by linear dynamics. We will have more to say about the limits of this approach in the next chapter. For now, we state a lemma about the local performance of the above observer.

Lemma 2.2.2 *For the system (2.21),(2.22) assume that the pair (C, A) is observable, $g : \mathbf{R}^n \to \mathbf{R}^n$ is differentiable and that the following is satisfied,*

$$\lim_{x \to 0} \|Dg(x)\| = 0, \qquad (2.40)$$

where $Dg(\cdot)$ denotes the Jacobian of $g(\cdot)$. Then there exists a matrix $K \in \mathbf{R}^{n \times p}$ such that (2.33) holds if $\|\varepsilon(0)\| \leq r$ and $\|x(t)\| \leq r, \quad \forall t \geq 0$ for a sufficiently small real number $r > 0$.

Proof : Observability of (C, A) implies the existence of a $K \in \mathbf{R}^{n\times p}$ such that $A_c = A - KC$ is a stable matrix. Then we can find two symmetric and positive matrices P and Q which satisfy (2.24). In a ball of radius $R > 0$, a Lipschitz constant L can be chosen to be [22, page 199],

$$L = \sup\{\|Dg(x)\| \quad | \quad \|x\| \leq R\}. \tag{2.41}$$

Now choose $R > 0$ such that L given by (2.41) satisfies (2.33). Such an R can always be found since (2.40) holds.

Note that (2.25) can be written as

$$\dot{\hat{x}} = (A - KC)\hat{x} + g(\hat{x}) + KCx. \tag{2.42}$$

Since $A_c = A - KC$ is a stable matrix, it can be shown that the solutions of (2.42) remain bounded provided that $\|\hat{x}(0)\|$ and $\|x(t)\|$ are sufficiently small. To see this, we write the solution of (2.42) as

$$\hat{x}(t) = e^{A_c t}\hat{x}(0) + \int_0^t e^{A_c(t-\tau)}g(\hat{x}(\tau))d\tau + \int_0^t e^{A_c(t-\tau)}KCx(\tau)d\tau. \tag{2.43}$$

Stability of A_c implies the existence of the constants $M_1 > 0$ and $\delta > 0$ such that

$$\|e^{A_c t}\| \leq M_1 e^{-\delta t}. \tag{2.44}$$

By taking norms in (2.43) and using (2.44) we obtain,

$$\begin{aligned} \|\hat{x}(t)\| &\leq M_1 e^{-\delta t}\|\hat{x}(0)\| + \int_0^t M_1 e^{-\delta(t-\tau)}\|g(\hat{x}(\tau))\|d\tau \\ &+ \int_0^t M_1 e^{-\delta(t-\tau)}\|KC\|\|x(\tau)\|d\tau. \end{aligned} \tag{2.45}$$

Now assume that $\|\hat{x}(0)\| \leq r_1$ and $\|x(t)\| \leq r_2 \; \forall t \geq 0$. By using (2.23) in (2.45) one has

$$\|\hat{x}(t)\| \leq M_1 e^{-\delta t} r_1 + \int_0^t M_1 L e^{-\delta(t-\tau)}\|\hat{x}(\tau)\|d\tau + \frac{M_1\|KC\|r_2}{\delta}(1 - e^{-\delta t}). \tag{2.46}$$

By multiplying both sides by $e^{\delta t}$ and using Lemma 2.2.1,

$$\begin{aligned} \|e^{\delta t}\hat{x}(t)\| &\leq M_1 r_1 + \frac{M_1\|KC\|r_2}{\delta}(e^{\delta t} - 1) \\ &+ \int_0^t M_1 L\left[M_1 r_1 + \frac{M_1\|KC\|r_2}{\delta}(e^{\delta\tau} - 1)\right] e^{\int_\tau^t M_1 L ds} d\tau. \end{aligned} \tag{2.47}$$

By routine integration and then multiplying by $e^{-\delta t}$, (2.47) can be simplified as

$$\|\hat{x}(t)\| \leq A_1 r_2 + (A_2 r_1 - A_1 r_2) e^{(M_1 L - \delta)t}, \tag{2.48}$$

where

$$A_1 = \frac{M_1 \|KC\|}{\delta - M_1 L}, \quad A_2 = M_1. \tag{2.49}$$

Now, the constant $R > 0$ in (2.41) could be chosen sufficiently small so that L given by (2.41) satisfies $\delta - M_1 L > 0$. Then from (2.48) it follows that $\|\hat{x}\|$ is also bounded. Moreover, the existence of sufficiently small r_1 and r_2 guarantees that the Lipschitz constant given in (2.41) remains valid for $\forall t \geq 0$. Hence we can set $r = r_2$ so that, whenever $\|\varepsilon(0)\| \leq r$ and $\|u(t)\| \leq r$, (2.25) is an observer for the system (2.21),(2.22).

Remark 2.2.1 *The condition of (2.40) is always satisfied when the system description (2.21) is obtained by the linearization of a nonlinear system around an equilibrium point in which case g necessarily contains at least second order terms.*

2.2.2 Transformation to Observable Canonical Form

This section is devoted to the exposition of an observer design method for nonlinear systems proposed by [16]. Since we used an improved version of this method in our work, an elaborate treatment of this technique is given next. First we need to establish a lemma.

Lemma 2.2.3 *Let $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbf{R}$ and consider the Vandermonde matrix given below,*

$$V(\lambda) = \begin{bmatrix} \lambda_1^{n-1} & \lambda_1^{n-2} & & \lambda_1 & 1 \\ \lambda_2^{n-1} & \lambda_2^{n-2} & & \lambda_2 & 1 \\ \vdots & \vdots & & & \vdots \\ \lambda_n^{n-1} & \lambda_n^{n-2} & & \lambda_n & 1 \end{bmatrix}. \tag{2.50}$$

Then for any $\alpha > 0$ and $c > 0$, there exist $0 > \lambda_1 > \lambda_2 \ldots > \lambda_n$ such that the following is satisfied

$$\lambda_1 + \|V^{-1}(\lambda)\|c = -\alpha. \tag{2.51}$$

Proof : See [16]. $\square$

Now we give the description of the observer as a theorem in the lines of [16].

Theorem 2.2.1 *[16] Let $Q(x)$ be the observability matrix defined in (2.14) for the system (2.12) , (2.13). If*

- **H1** *$Q(x)$ has full rank for all $x \in \mathbf{R}^n$,*
- **H2** *$L_f^n h(\Phi^{-1}(u))$ is uniformly Lipschitz for all $u_1, u_2 \in \mathbf{R}^n$, i.e. the following holds for some $\gamma > 0$;*

$$\|L_f^n h(\Phi^{-1}(u_1)) - L_f^n h(\Phi^{-1}(u_2))\| \leq \gamma \|u_1 - u_2\|,$$

then there exists a finite gain vector $K \in \mathbf{R}^n$ such that the solution of the following system equation,

$$\dot{\hat{x}} = f(\hat{x}) + Q^{-1}(\hat{x})K(y - h(\hat{x})), \qquad \hat{x}(0) = \hat{x}_0 \tag{2.52}$$

converges exponentially to the solution of (2.12) , (2.13).

Proof : Let us define the nonlinear state transformation;

$$z = \Phi(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \\ L_f^{n-1} h(x) \end{bmatrix}, \qquad x \subset \mathbf{R}^n \tag{2.53}$$

which admits inverse because of the implicit function theorem and **H1**. In the new coordinates the system (2.12),(2.13) becomes;

$$\begin{aligned} \dot{z} &= Az + BL_f^n h(\Phi^{-1}(z)), && (2.54) \\ y &= Cz, && (2.55) \end{aligned}$$

where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times 1}$, $C \in \mathbf{R}^{1\times n}$ are given by the Brunowsky canonical form;

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & & 1 \\ 0 & 0 & 0 & & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C = [1 \; 0 \; \dots \; 0]\,. \tag{2.56}$$

In the same way, defining $\hat{z} = \Phi(\hat{x})$, the observer (2.52) assumes the form

$$\dot{\hat{z}} = A\hat{z} + BL_f^n h(\Phi^{-1}(\hat{z})) + K(y - C\hat{z}), \quad \hat{z} = \Phi(\hat{x}_0). \tag{2.57}$$

Then the dynamics of the error in the transformed domain is given by

$$\dot{\varepsilon} = (A - KC)\varepsilon + B\left[L_f^n h(\Phi^{-1}(z)) - L_f^n h(\Phi^{-1}(\hat{z}))\right]. \tag{2.58}$$

Since the pair (C, A) is observable, by an appropriate choice of the feedback gain matrix K, the eigenvalues of $A_c = A - KC$ can be assigned arbitrarily. Now, assume that the assigned eigenvalues, $\lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ are all real, negative and distinct such that $0 > \lambda_1 > \dots > \lambda_n$. Then the matrix A_c can be diagonalized by the Vandermonde matrix. Namely, we have

$$A_c = V^{-1}(\lambda)\Lambda V(\lambda), \tag{2.59}$$

where $\Lambda = diag\,[\lambda_1 \; \lambda_2 \dots \lambda_n]$. To see this, note that

$$A_c = \begin{bmatrix} -k_1 & 1 & 0 & \dots & 0 \\ -k_2 & 0 & 1 & \dots & 0 \\ & & & & \vdots \\ -k_{n-1} & & & 0 & 1 \\ -k_n & 0 & & \dots & 0 \end{bmatrix}. \tag{2.60}$$

The characteristic polynomial of A_c can easily be calculated to be

$$p(s) = \det(sI - A_c) = s^n + k_1 s^{n-1} + k_2 s^{n-2} + \ldots + k_{n-1}s + k_n. \tag{2.61}$$

We know that for an eigenvalue λ_i of A_c , $p(\lambda_i) = 0$ is satisfied. Namely,

$$\lambda_i^n = -k_1\lambda_i^{n-1} - k_2\lambda_i^{n-2} + \ldots - k_{n-1}\lambda_i - k_n, \quad i = 1, 2, \ldots, n. \tag{2.62}$$

Now rewrite (2.59) as

$$V(\lambda)A_c = \Lambda V(\lambda). \tag{2.63}$$

The RHS of the above equation is a matrix given as

$$\Lambda V(\lambda) = \begin{bmatrix} \lambda_1^n & \lambda_1^{n-1} & & \lambda_1^2 & \lambda_1 \\ \lambda_2^n & \lambda_2^{n-1} & & \lambda_2^2 & \lambda_2 \\ \vdots & \vdots & & \vdots & \\ \lambda_n^n & \lambda_n^{n-1} & & \lambda_n^2 & \lambda_n \end{bmatrix}. \tag{2.64}$$

Also, the LHS of (2.63) can be calculated as

$$V(\lambda)A_c = \begin{bmatrix} \sum_{j=1}^{n-1}(-k_{n-j}\lambda_1^j) & \lambda_1^{n-1} & & \lambda_1^2 & \lambda_1 \\ \sum_{j=1}^{n-1}(-k_{n-j}\lambda_2^j) & \lambda_2^{n-1} & & \lambda_2^2 & \lambda_2 \\ & & & & \\ \sum_{j=1}^{n-1}(-k_{n-j}\lambda_n^j) & \lambda_n^{n-1} & \ldots & \lambda_n^2 & \lambda_n \end{bmatrix}. \tag{2.65}$$

By using (2.62) in (2.65), we obtain (2.63).

Then we have

$$e^{A_c t} = V^{-1}(\lambda)e^{\Lambda t}V(\lambda). \tag{2.66}$$

The solution of (2.58) can be written as

$$\begin{aligned} V(\lambda)\varepsilon(t) &= e^{\Lambda t}V(\lambda)\varepsilon(0) \\ &+ \int_0^t e^{\Lambda(t-\tau)}V(\lambda)B\left[L_f^n h(\Phi^{-1}(z(\tau))) - L_f^n h(\Phi^{-1}(\hat{z}(\tau)))\right] d\tau. \end{aligned} \tag{2.67}$$

Taking the 2-norm of both sides and using **H2** and the fact that $\|V(\lambda)B\| = \sqrt{n}$, we have

$$\|V(\lambda)\varepsilon(t)\| \leq \|V(\lambda)\varepsilon(0)\|e^{\lambda_1 t} + \int_0^t e^{\lambda_1(t-\tau)}\sqrt{n}\gamma\|V^{-1}(\lambda)\|\|V(\lambda)\varepsilon(\tau)\|d\tau. \quad (2.68)$$

Now we multiply both sides by $e^{-\lambda_1 t}$ and use Lemma 2.2.1 to write

$$\|V(\lambda)\varepsilon(t)\| \leq e^{(\lambda_1 + \|V^{-1}(\lambda)\|\sqrt{n}\gamma)}\|V(\lambda)\varepsilon(0)\|. \quad (2.69)$$

By Lemma 2.2.3, we can choose the eigenvalues of $(A - KC)$ such that the exponent

$$\lambda_1 + \|V^{-1}(\lambda)\|\sqrt{n}\gamma = -\alpha \quad (2.70)$$

in (2.69) becomes negative. Hence we obtain

$$\|V(\lambda)\varepsilon(t)\| \leq e^{-\alpha t}\|V(\lambda)\varepsilon(0)\|, \qquad \alpha > 0. \quad (2.71)$$

Let $\sigma_1 > 0$ and $\sigma_n > 0$ be the maximum and the minimum singular values of $V(\lambda)$, respectively. Then we have

$$\sigma_1\|u\| \leq \|V(\lambda)u\| \leq \sigma_n\|u\|, \quad \forall u \in \mathbf{R}^n. \quad (2.72)$$

Thus, from (2.71) we obtain

$$\|\varepsilon(t)\| \leq \frac{\sigma_1}{\sigma_n}e^{-\alpha t}\|\varepsilon(0)\|, \quad (2.73)$$

showing the exponential decay of observation error. □

Chapter 3

LIMITATIONS AND IMPROVEMENTS

3.1 A Bound on the Linearization Method

It would be useful if we could give an explicit bound on the achievable performance by the linearization method. To do this we proceed, as in the previous section, by writing the solution of the error equation. First we state the following lemma relating the maximum eigenvalue of a stable matrix to the condition number of its diagonalizing matrix.

Lemma 3.1.1 *For a matrix $A \in \mathbf{R}^{n\times n}$ with real, distinct and negative eigenvalues $0 > \lambda_1 > \lambda_2 > \ldots > \lambda_n$, let $T \in \mathbf{R}^{n\times n}$ denote the matrix of eigenvectors of A, i.e., $A = T\Lambda T^{-1}$, where $\Lambda = diag[\ \lambda_1\ \lambda_2 \ldots \lambda_n\ ]$. Then the following inequality is satisfied,*

$$\frac{|\lambda_1|}{\|T\|\|T^{-1}\|} \leq \sigma_n(A), \tag{3.1}$$

where $\sigma_n(A)$ denotes the minimum singular value of A and $\|\cdot\|$ is the matrix norm induced by the 2-norm.

Proof : Let $v \in \mathbf{R}^n$ be a vector of unity norm and μ a real number that is different than any eigenvalue of A. Also let $r = Av - \mu v$. Then we have,

$$\begin{aligned} r &= T\Lambda T^{-1}v - \mu v, \\ &= (T\Lambda T^{-1} - \mu I)v, \\ &= T(\Lambda - \mu I)T^{-1}v, \end{aligned}$$

or

$$v = T(\Lambda - \mu I)^{-1}T^{-1}r. \tag{3.2}$$

Taking the norm,

$$1 \leq \|T\|\|(\Lambda - \mu I)^{-1}\|\|r\|\|T^{-1}\|, \tag{3.3}$$

where

$$\|(\Lambda - \mu I)^{-1}\| = \max_i(|\lambda_i - \mu|^{-1}) = (\min_i |\lambda_i - \mu|)^{-1}. \tag{3.4}$$

Thus, we have

$$\min_i |\lambda_i - \mu| \leq \|T\|\|T^{-1}\|\|r\|. \tag{3.5}$$

Now, let (v, μ) be an eigenvector-eigenvalue pair of the perturbed matrix $A + \delta A$ with $\|v\| = 1$. Expressing this as

$$(A + \delta A)v = \mu v, \tag{3.6}$$

or

$$-\delta Av = Av - \mu v, \tag{3.7}$$

and using (3.5), one has

$$\min_i |\lambda_i - \mu| \leq \|\delta A\|\|T\|\|T^{-1}\|. \tag{3.8}$$

Choose the perturbation matrix δA such that $A + \delta A$ becomes singular and choose $\mu = 0$. Then

$$\frac{\min_i |\lambda_i|}{\|T\|\|T^{-1}\|} \leq \|\delta A\|. \tag{3.9}$$

Since the minimum norm perturbation to make a matrix singular is the one whose norm is equal to the minimum singular value of the perturbed matrix[23, page 330], we finally get,

$$\frac{\min_i |\lambda_i|}{\|T\|\|T^{-1}\|} \leq \sigma_n(A). \qquad \square \tag{3.10}$$

Note that this ratio can be readily identified with the ratio $\frac{\alpha}{M}$ introduced in (2.39).

A shorter proof of the same result can be given as follows; we first write the SVD of A as

$$A = U\Sigma V^H, \tag{3.11}$$

and equate it to the modal decomposition of A to obtain

$$U\Sigma V^H = T\Lambda T^{-1}. \tag{3.12}$$

Taking the inverse of both sides, one has

$$V\Sigma^{-1}U^H = T\Lambda^{-1}T^{-1}, \tag{3.13}$$

or

$$\Sigma^{-1} = V^H T\Lambda^{-1}T^{-1}U. \tag{3.14}$$

By taking norm and using the unitarity of U and V, we obtain

$$\|\Sigma^{-1}\| \leq \|T\|\|\Lambda^{-1}\|\|T^{-1}\|, \tag{3.15}$$

or

$$\frac{1}{\sigma_n(A)} \leq \frac{\|T\|\|T^{-1}\|}{\min\limits_i |\lambda_i|}. \quad \square \tag{3.16}$$

Lemma 3.1.2 *For the system*

$$\dot{x} = Ax + g(x), \quad x(0) = x_0, \tag{3.17}$$

$$y = Cx, \tag{3.18}$$

let the eigenvalues of $A_c = A - KC$ be all real, negative and distinct. Then the observer (2.25),(2.26) is not guaranteed to work if $L > \sigma_n(A_c)$, where L is the Lipschitz constant of $g(\cdot)$ and $\sigma_n(A_c)$ is the smallest singular value of A_c.

Proof: Here we repeat the error equation;

$$\dot{\varepsilon} = A_c\varepsilon + g(x) - g(\hat{x}), \quad \varepsilon(0) = \varepsilon_0, \tag{3.19}$$

and write the solution of (3.19) as,

$$\varepsilon(t) = e^{A_c t}\varepsilon_0 + \int_0^t e^{A_c(t-\tau)}\left[g(x(\tau)) - g(\hat{x}(\tau))\right] d\tau. \tag{3.20}$$

Assuming the eigenvalues of A_c are all real, negative and distinct, we have the Jordan form of A_c as $A_c = T\Lambda T^{-1}$ where T is the matrix of eigenvectors of A_c. Then (3.20) becomes,

$$\varepsilon(t) = Te^{\Lambda t}T^{-1}\varepsilon_0 + \int_0^t Te^{\Lambda(t-\tau)}T^{-1}\left[g(x(\tau)) - g(\hat{x}(\tau))\right] d\tau. \tag{3.21}$$

Taking 2-norm of both sides and using the Lipschitz property of $g(x)$, one has

$$\|\varepsilon(t)\| \leq \|T\|\|T^{-1}\|\|\varepsilon_0\|e^{\lambda_1 t} + \|T\|\|T^{-1}\| \int_0^t e^{\lambda_1(t-\tau)}L\|\varepsilon(\tau)\|d\tau, \tag{3.22}$$

where λ_1 is the eigenvalue closest to the imaginary axis. Application of Lemma 2.2.1 to (3.22) yields,

$$\|\varepsilon(t)\| \leq cond(T)\|\varepsilon_0\|e^{cond(T)(L+\frac{\lambda_1}{cond(T)})t}, \tag{3.23}$$

where $cond(T) = \|T\|\|T^{-1}\|$ denotes the condition number of T. For the exponent in (3.23) to be negative, we require that

$$\frac{|\lambda_1|}{cond(T)} > L. \tag{3.24}$$

By Lemma 3.1.1, the quantity on the LHS of (3.24) cannot be larger than $\sigma(A_c)$, the minimum singular value of A_c. □

Note that the LHS of the inequality (3.24) can be adjusted by varying the feedback gain matrix K and this point may be exploited in the observer design. The inequality (3.24) gives a bound on the Lipschitz constant of the nonlinearity $g(\cdot)$ so that the observer given by (2.25), (2.26) is guaranteed to provide an estimate of the states of the system (2.21), (2.22). Hence in this approach, in order to tolerate a larger class of nonlinearities, the LHS of (3.24) may be maximized with an appropriate choice of the feedback gain K. Below such a maximization is given as an example.

Example 3.1.1 Suppose $A_c \in \mathbf{R}^{2\times 2}$ is stable and in companion form,

$$A_c = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix}, \tag{3.25}$$

which can be diagonalized by the Vandermonde matrix,

$$V = \begin{bmatrix} \lambda & 1 \\ a\lambda & 1 \end{bmatrix}, \tag{3.26}$$

where $a > 1$ and $\lambda < 0$ and λ , $a\lambda$ are the eigenvalues of A_c. Then, using 1-norm [1, page 22] we can write,

$$\|V\|_1 = \max\{2, |\lambda + a\lambda|\}, \qquad \|V^{-1}\|_1 = \frac{1 - a\lambda}{\lambda - a\lambda} \,. \tag{3.27}$$

The case where $|\lambda + a\lambda| \leq 2$ is easily ruled out by considering the ordering of the eigenvalues in the Vandermonde matrix. Thus for $|\lambda + a\lambda| > 2$, the quantity to be maximized is

$$C(\lambda, a) = \frac{-\lambda}{\|V\|_1 \|V^{-1}\|_1} = \frac{\lambda - a\lambda}{(1 - a\lambda)(1 + a)}. \tag{3.28}$$

Taking the partial derivatives of $C(\lambda, a)$ with respect to λ,

$$\frac{\partial C(\lambda, a)}{\partial \lambda} = \frac{1}{(1 - a\lambda)^2(1 + a)^2}, \tag{3.29}$$

we see that $C(\lambda, a)$ increases in the direction of decreasing λ. Also the solution of

$$\frac{\partial C(\lambda, a)}{\partial a} = 0, \tag{3.30}$$

is found to be

$$a = 1 + \sqrt{2 - \frac{1}{\lambda}}, \tag{3.31}$$

whose maximum value is $1 + \sqrt{2}$. The gains k_1 and k_2 are given by the formula

$$k_1 = 2\lambda + \sqrt{2\lambda^2 - \lambda}, \quad k_2 = \lambda^2 + \lambda\sqrt{2\lambda^2 - \lambda}. \tag{3.32}$$

We can calculate the maximum value of $C(\lambda, a)$ as

$$C(-\infty, 1 + \sqrt{2}) = \frac{\sqrt{2}}{(1 + \sqrt{2})(2 + \sqrt{2})} \approx 0.17 \tag{3.33}$$

This is illustrated below by a 3-D plot of $C(\lambda, a)$.

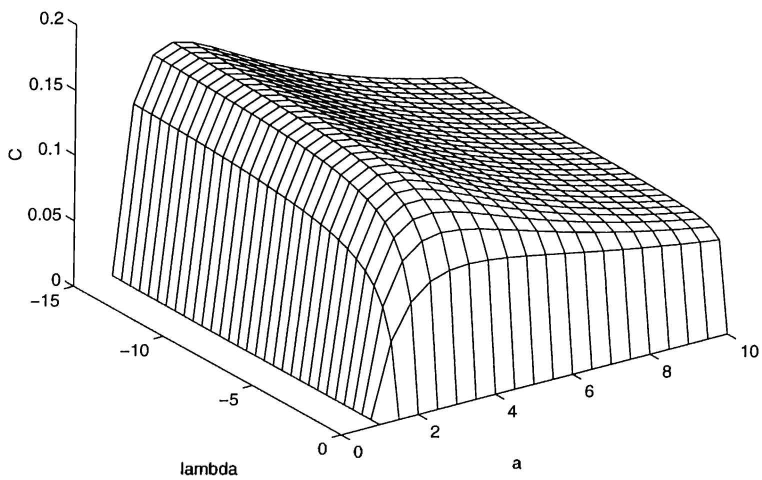


Figure 3.1: $C(\lambda, a)$

Hence for a suitable design we choose $a = \sqrt{2}+1$ and then choose λ as large as possible. This, of course, is restricted by the maximum obtainable gain in the implementation.

3.2 An Eigenvalue Assignment Procedure for Vandermonde Matrix

In the previous chapter, we intentionally skipped the discussion of the eigenvalue assignment scheme that is required to make the exponent of (2.69) negative. In [16] it is claimed that it would be enough if the maximum eigenvalue in the Vandermonde matrix is chosen to be larger in magnitude than the Lipschitz constant. This can easily be contradicted by an example. Instead we give an explicit eigenvalue assignment procedure for the error in (2.73) to converge exponentially to zero.

Lemma 3.2.1 *The determinant of the Vandermonde matrix is given by*

$$\det(V(\lambda)) = \det \begin{bmatrix} \lambda_1^{n-1} & \lambda_1^{n-2} & & \lambda_1 & 1 \\ \lambda_2^{n-1} & \lambda_2^{n-2} & & \lambda_2 & 1 \\ & & & & \vdots \\ \lambda_n^{n-1} & \lambda_n^{n-2} & \dots & \lambda_n & 1 \end{bmatrix} = \prod_{i>j}(\lambda_i - \lambda_j). \tag{3.34}$$

Proof : See [24, page 3]. □

Now we are ready to describe our eigenvalue assignment procedure by a theorem. Note that in [16] 2-norm was used. However, in the following discussion we will use ∞-norm, which is defined for $A \in \mathbf{R}^{n \times n}$ as $\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$, [1, page 22]. Since all the p-norms are topologically equivalent in $\mathbf{R}^n$ [25, page 258], this will make no major difference apart from changing the Lipschitz constant in Theorem 1 of [16]. Also the exponent given in (2.73) will be

$$-\alpha = \lambda_1 + \gamma \|V^{-1}(\lambda)\|_\infty. \tag{3.35}$$

Theorem 3.2.1 *Let $V(\lambda)$ denote the Vandermonde matrix constructed with the set $S_\lambda = \{\lambda_1,\ \lambda_2,\ \dots,\ \lambda_n\}$, and let $\lambda_i = a^{i-1}\lambda$ for some $\lambda < 0$ and $a > 1$. Then, if $|\lambda|$ and a are sufficiently large, we have $\|V^{-1}(\lambda)\|_\infty$ independent of λ and*

$$\lim_{a \to \infty} \|V^{-1}(\lambda)\|_\infty = 1. \tag{3.36}$$

Proof : We know that the inverse of a nonsingular matrix is found by dividing its adjoint matrix to its determinant. Let $M(\lambda)$ denote the inverse of the Vandermonde matrix. Then to find the i^{th} element in the last row of $M(\lambda)$ we delete the last column and i^{th} row of $V(\lambda)$ and calculate the corresponding cofactor. Namely,

$$m_{n,i} = \frac{(-1)^{i+n}}{\det(V(\lambda))} \det\left(\begin{bmatrix} \lambda_1^{n-1} & \lambda_1^{n-2} & & \lambda_1^2 & \lambda_1 \\ \vdots & & & \vdots & \\ \lambda_{i-1}^{n-1} & \lambda_{i-1}^{n-2} & & \lambda_{i-1}^2 & \lambda_{i-1} \\ \lambda_{i+1}^{n-1} & \lambda_{i+1}^{n-2} & & \lambda_{i+1}^2 & \lambda_{i+1} \\ \vdots & \vdots & & \vdots & \\ \lambda_n^{n-1} & \lambda_n^{n-2} & & \lambda_n^2 & \lambda_n \end{bmatrix}\right). \tag{3.37}$$

It is easy to see that, the matrix on the RHS of (3.37) can be scaled to get another Vandermonde structure. Then using Lemma 3.2.1, one has

$$m_{n,i} = (-1)^{i+n} \frac{\lambda_1 \ldots \lambda_{i-1}\lambda_{i+1} \ldots \lambda_n \prod\limits_{\substack{p>q \\ (p,q \neq i)}} (\lambda_p - \lambda_q)}{\prod\limits_{p>q} (\lambda_p - \lambda_q)}. \tag{3.38}$$

Canceling the common terms in the products we get,

$$m_{n,i} = (-1)^{i+n} \frac{\prod\limits_{p \neq i} \lambda_p}{\prod\limits_{\substack{p>q \\ (p=i)\vee(q=i)}} (\lambda_p - \lambda_q)}. \tag{3.39}$$

Now we assign $\lambda_j = a^{j-1}\lambda$, where $a > 1$ and $\lambda < 0$,

$$m_{n,i} = (-1)^{i+n} \frac{\prod\limits_{p \neq i} a^{p-1}\lambda}{\prod\limits_{\substack{p>q \\ (p=i)\vee(q=i)}} (a^{p-1} - a^{q-1})\lambda}. \tag{3.40}$$

Obviously the numbers of the factors in both of the products are $n-1$, thus cancelling λ's.

$$m_{n,i} = (-1)^{i+n} \frac{\prod\limits_{p \neq i} a^{p-1}}{\prod\limits_{\substack{p>q \\ (p=i)\vee(q=i)}} (a^{p-1} - a^{q-1})}, \tag{3.41}$$

or by calculating the product in the numerator, we obtain

$$m_{n,i} = (-1)^{i+n} \frac{a^{\frac{n^2-n+2-2i}{2}}}{\prod\limits_{\substack{p>q \\ (p=i)\vee(q=i)}} (a^{p-1} - a^{q-1})}. \tag{3.42}$$

The degree of the denominator can be calculated as the sum of the degrees of each product term. To see the calculation, let us write out the denominator as

$$\begin{aligned}\prod_{\substack{p>q \\ (p=i)\vee(q=i)}} (a^{p-1} - a^{q-1}) &= \underbrace{(a^{n-1} - a^{i-1})(a^{n-2} - a^{i-1})\dots(a^{i} - a^{i-1})}_{(n-i)\text{ times}} \\ &\quad + \underbrace{(a^{i-1} - a^{i-2})(a^{i-1} - a^{i-3})\dots(a^{i-1} - 1)}_{(i-1)\text{ times}}.\end{aligned}$$

Then we obtain,

$$\begin{aligned}\deg \prod_{\substack{p>q \\ (p=i)\vee(q=i)}} (a^{p-1} - a^{q-1}) &= \sum_{i+1}^{n}(j-1) + (i-1)(i-1), \\ &= \frac{n(n-1) + (i-1)(i-2)}{2}.\end{aligned}$$

For $i > 1$, the degree of the denominator is greater than that of the numerator, but for $i = 1$ they are equal. Thus one can conclude that, for the above assignment procedure, the last row of V^{-1} will uniformly converge to

$$\lim_{a\to\infty} m_n = \left[(-1)^{n+1} \;\; 0 \;\; 0 \;\; \dots \;\; 0\right]. \tag{3.43}$$

Note that the last row of V^{-1} is independent of λ.

Now, we will prove the dominance of this last row when calculating the ∞-norm of $V^{-1}(\lambda)$. To do this, let us consider an arbitrary cofactor $(-1)^{i+n-j}V_{i(n-j)}$ of V, where $j \neq 0$. By deleting the i^{th} row and $(n-j)^{th}$ column of V, one obtains

$V_{i(n-j)} =$

$$\begin{bmatrix} \lambda^{n-1} & & \lambda^{j+1} & \lambda^{j-1} & & \lambda & 1 \\ a^{n-1}\lambda^{n-1} & & a^{j+1}\lambda^{j+1} & a^{j-1}\lambda^{j-1} & & a\lambda & 1 \\ & & & & & & \vdots \\ a^{(i-1)(n-1)}\lambda^{n-1} & & a^{(i-1)(j+1)}\lambda^{j+1} & a^{(i-1)(j-1)}\lambda^{j-1} & & a^{i-1}\lambda & 1 \\ a^{i(n-1)}\lambda^{n-1} & & a^{i(j+1)}\lambda^{j+1} & a^{i(j-1)}\lambda^{j-1} & & a^{i}\lambda & 1 \\ & & & & & & \vdots \\ a^{(n-1)(n-1)}\lambda^{n-1} & \dots & a^{(n-1)(j+1)}\lambda^{j+1} & a^{(n-1)(j-1)}\lambda^{j-1} & & a^{n-1}\lambda & 1 \end{bmatrix}.$$

Considering the definition of the determinant of a matrix, to calculate the determinant of $V_{i(n-j)}$ we pick n entries no two of which lie in the same column or row and multiply those to get an element in the summation that we carry out over all possible permutations. A careful inspection reveals that the degree in λ of the determinant of $V_{i(n-j)}$ is at least one less than the degree in λ of the determinant of $V(\lambda)$ because a row and a column are missing in the multiplications. Hence,

$$\deg_\lambda(\det(V_{i(n-j)})) < \deg_\lambda(\det(V(\lambda))), \tag{3.44}$$

which implies that,

$$\lim_{\lambda\to-\infty} \frac{\det(V_{i(n-j)})}{\det(V(\lambda))} = 0. \tag{3.45}$$

Hence, as $|\lambda|$ gets larger, all the rows of $V^{-1}(\lambda)$ other than the last row uniformly converge to a zero-row vector. Then for $|\lambda|$ and a sufficiently large, the absolute sum of the last row of $V^{-1}(\lambda)$ will be dominant. In this case $\|V^{-1}(\lambda)\|_\infty$ is given by

$$\|V^{-1}(\lambda)\|_\infty = \sum_{i=1}^{n} \left| \frac{a^{\frac{n^2-n+2-2i}{2}}}{\prod\limits_{\substack{p>q \\ (p=i)\vee(q=i)}} (a^{p-1}-a^{q-1})} \right|. \quad \square \tag{3.46}$$

Although it is cumbersome to determine the exact relation between λ and a for (3.46) to be valid in the general case, here we give the results for $n = 2, 3$.

For n=2 :

$$V(\lambda) = \begin{bmatrix} \lambda & 1 \\ a\lambda & 1 \end{bmatrix},$$

$$V^{-1}(\lambda) = \frac{1}{a\lambda - \lambda} \begin{bmatrix} 1 & -1 \\ -a\lambda & \lambda \end{bmatrix},$$

and

$$\|V^{-1}\|_\infty = \frac{a+1}{a-1}, \qquad \text{for} \quad a > -\frac{2}{\lambda} - 1. \tag{3.47}$$

For n=3 :

$$V(\lambda) = \begin{bmatrix} \lambda^2 & \lambda & 1 \\ a^2\lambda^2 & a\lambda & 1 \\ a^4\lambda^2 & a^2\lambda & 1 \end{bmatrix},$$

$$V^{-1}(\lambda) = \frac{1}{\lambda^3(a^2-1)(a^2-a)(a-1)} \begin{bmatrix} (a-a^2)\lambda & (a^2-1)\lambda & (1-a)\lambda \\ (a^4-a^2)\lambda^2 & (1-a^4)\lambda^2 & (a^2-1)\lambda^2 \\ (a^4-a^5)\lambda^3 & (a^4-a^2)\lambda^3 & (a-a^2)\lambda^3 \end{bmatrix},$$

and

$$\|V^{-1}\|_\infty = \frac{a^2+1}{(a-1)^2} \qquad \text{for} \quad a > -\frac{2}{\lambda}. \tag{3.48}$$

Now we give a step by step outline of the observer design procedure;

1. Given a nonlinear single-input single-output TI system

$$\begin{aligned} \dot{x} &= f(x), & (3.49) \\ y &= h(x), & (3.50) \end{aligned}$$

find $Q(x)$ by using (2.14).

2. Determine if $Q(x)$ has rank n for all $x \in \mathbf{R}^n$.

3. If so, this means that the nonlinear state transformation $z = \Phi(x)$ is invertible, where $\Phi(\cdot)$ is defined in (2.14). Then, using ∞-norm, find a global Lipschitz bound γ on the function $L_f^n h(\Phi^{-1}(z))$.

4. Choose $a > 1$ and by using (3.46), calculate

$$w = \gamma \|V^{-1}(\lambda)\|_\infty. \tag{3.51}$$

5. Assign the first eigenvalue λ such that $\|V^{-1}(\lambda)\|_\infty$ depends only on a and

$$\lambda < -w. \tag{3.52}$$

   The number $\alpha = \lambda + w$ determines the lower bound on the exponential decay rate of observation error.

6. Assign the remaining eigenvalues as

$$\lambda_i = a^{i-1}\lambda, \quad i = 1, \ldots, n. \tag{3.53}$$

7. Determine the gain vector K by using

$$s^n + k_1 s^{n-1} + \ldots + k_{n-1}s + k_n = (s - \lambda_1)(s - \lambda_2)\ldots(s - \lambda_n). \tag{3.54}$$

   and $K = [k_1 \; k_2 \; \ldots \; k_n]^T$.

8. Construct the observer as

$$\dot{\hat{x}} = f(\hat{x}) + [Q(\hat{x})]^{-1} K(y - h(\hat{x})). \tag{3.55}$$

We conclude this chapter by mentioning a disadvantage of the above eigenvalue assignment procedure. Usually the observer gains are so high that the transient oscillations in the convergence may be quite damaging in practical applications, apart from the fact that such high gains are not so easy to implement.

Chapter 4

APPLICATION TO SYNCHRONIZATION OF CHAOTIC SYSTEMS

Recently there has been a great deal of interest toward the synchronization of nonlinear systems operating in a chaotic regime. Its major application area is the secure transmission of information imposed on chaotic signals. The non-periodicity of the chaotic signal makes it almost impossible to tap into the channel by classical methods. On the other hand, the intented receiver of the information has to possess means, which, assumedly no eavesdropper has, of extracting the information out of the chaotically modulated signal. Apart from a robust synchronizing scheme, the receiver end of the communication has the extra knowledge of the system model that has produced the chaotic signals.

In this chapter we show that synchronization of chaotic systems can be achieved by using state observers in the receiver end. For some chaotic systems the results of the linear observer theory can easily be applied while for some other cases one needs to employ the observer construction methods described in the previous chapter.

The fact that the system operates in chaotic regime can be exploited to facilitate the observer design. Loosely speaking, the trajectories of a chaotic system passes through almost all points in a bounded region of the state space. This peculiarity enables one to define global Lipschitz bounds on the nonlinearities involved.

In Figure 4.1, a communication system using chaotic signals in modulation is depicted. Here, $s(t)$ and $y(t)$ can be viewed as two states of a chaotic system. We modulate $s(t)$ additively by the message signal $m(t)$ and send the resulting signal $f(t)$ to the receiver end. Also another state $y(t)$ of the chaotic system is directly sent to the receiver end through a different channel. This signal is used to reconstruct $s(t)$ which, in turn, is subtracted from $f(t)$ to get a copy of the original message.

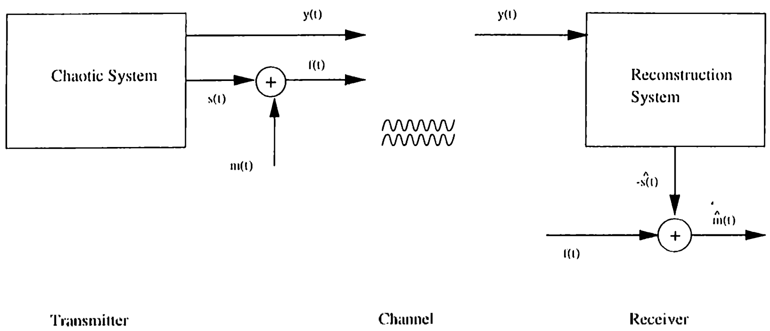


Figure 4.1: A communication system using chaotic modulation

4.1 Synchronization by Exploiting the System Structure

A common approach to the synchronization problem is the one that has been proposed in [12, 13]. In this section we will briefly outline their method with an example. This method is based on the separation of the system into two subsystems, i.e.,

$$\dot{u} = f(u, w), \tag{4.1}$$

$$\dot{w} = g(u, w). \tag{4.2}$$

(4.3)

where $u \in \mathbf{R}^n$, $w \in \mathbf{R}^m$, $f : \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}^n$ and $g : \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}^m$. At the receiver end, we replicate the second subsystem and call it the "response system".

$$\dot{\hat{w}} = g(u, \hat{w}). \tag{4.4}$$

We assume that the state u of the first subsystem is known and is directly sent to the receiver end. Thus this scheme can be viewed as some of the original state variables driving the response system, for that reason our original system is called the "drive system". The two systems synchronize if the error $\varepsilon_w = w - \hat{w}$ goes asymptotically to zero,

$$\lim_{t \to \infty} \varepsilon_w(t) = 0. \tag{4.5}$$

Although the scheme may seem simple at first glance, there does not exist an explicit procedure to choose response subsystem to guarantee the stability of error system. Moreover there may not exist any plausible choice at all, see [13].

Example 4.1.1 [13] Consider the Lorentz chaotic attractor system

$$\begin{aligned} \dot{x} &= \sigma(y - z), & (4.6)\\ \dot{y} &= -xz + rx - y, & (4.7)\\ \dot{z} &= xy - bz. & (4.8) \end{aligned}$$

We choose the parameters $\sigma = 16$, $b = 4$, and $r = 45.92$ so that the system operates in chaotic regime. Then dubbing y and z as the response variables, we replicate the response system at the receiver end. Note that the first state variable $x(t)$ is sent as the synchronizing signal.

$$\begin{aligned} \dot{\hat{y}} &= -x\hat{z} + rx - \hat{y}, & (4.9)\\ \dot{\hat{z}} &= x\hat{y} - b\hat{z}. & (4.10) \end{aligned}$$

Below is shown the convergence of the response variable y to the true state of the system for a typical initial condition and for the above choice of parameters.

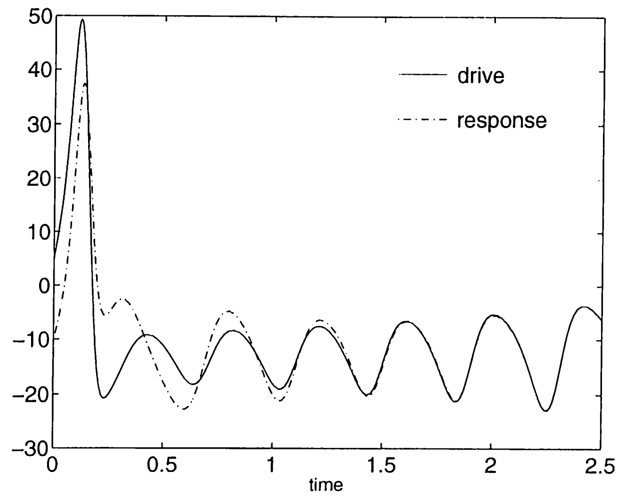


Figure 4.2: Convergence of $\hat{y}(t)$ of the response system to $y(t)$ of the drive system for the initial conditions $x(0) = 5$, $y(0) = 5$, $z(0) = -4$, $\hat{y}(0) = -10$, and $\hat{z}(0) = 15$.

4.2 Observer Based Synchronization

The synchronization problem described so far can also be addressed from the observation point of view, see [26, 27]. We can take the common synchronizing signal to be the system output of the drive system and the response system can be chosen as a full order observer. Then it becomes possible to use existing observer design strategies for the purpose of synchronization. Besides, the system output is not *a priori* defined for chaotic systems. Hence one can tailor an output such that the observer design is facilitated.

Another peculiarity of the chaotic systems is that, since the trajectories always remain in a compact region, we can always find a global Lipschitz bound on the nonlinearities involved. Thus the observer proposed in the previous chapter works globally whenever a diffeomorphic transformation to Brunowsky canonical form can be found.

In the following discussion, we first expose the simplification of the observer design for two classes of nonlinear systems, then the nonlinear state transformation method is applied to Rössler and Lorentz systems and the Chua oscillator.

4.2.1 Systems in Lur'e Form

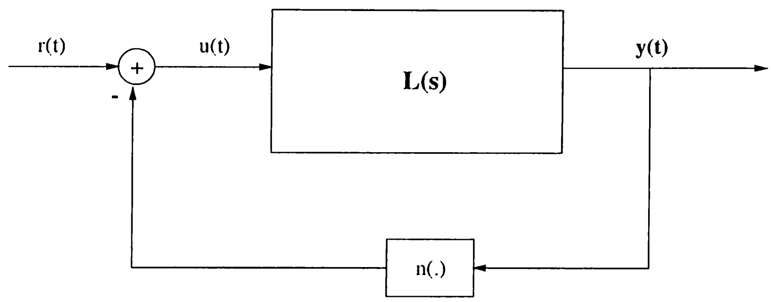


Figure 4.3: Lur'e Form

We consider the class of systems having the structure shown in Figure 4.3. Here $\mathbf{L}(s)$ represents the transfer function of a single-input single-output LTI system and $n(\cdot) : \mathbf{R} \to \mathbf{R}$ is a memoryless nonlinearity. This is the well-known Lur'e form which have been heavily investigated [1] and is known to exhibit chaotic behavior for certain cases. Such a system can always be synchronized using a global observer. We assume that $\mathbf{L}(s)$ is a strictly proper transfer function, then we can find an observable realization (A, B, C) of $\mathbf{L}(s)$ such that $\mathbf{L}(s) = C(sI - A)^{-1}B$. Rewriting the system description in state space,

$$\dot{x} = Ax - Bn(y), \tag{4.11}$$

$$y = Cx, \tag{4.12}$$

we choose the observer as

$$\dot{\hat{x}} = A\hat{x} - Bn(y) + K(y - \hat{y}), \tag{4.13}$$

$$\hat{y} = C\hat{x}. \tag{4.14}$$

Hence the error dynamics is given by

$$\dot{\varepsilon} = A_c \varepsilon, \tag{4.15}$$

where, by the observability of (C, A) , $A_c = A - KC$ can be chosen to be a stable matrix with an appropriate choice of K. The idea is just the design of observer for a system that is linear up to output injection.

Example 4.2.1 Let $\mathbf{L}(s)$ and $n(\cdot)$ be

$$\mathbf{L}(s) = \frac{1}{s^3 + s^2 + 1.25s}, \qquad n(y) = \begin{cases} -ky & |y| \leq 1 \\ 2ky - 2k\ \mathrm{sgn}(y) & 1 \leq |y| \leq 3 \\ 3k\ \mathrm{sgn}(y) & |y| \geq 3 \end{cases} \tag{4.16}$$

with $k = 1.8$. This system is known to exhibit chaotic behavior for this set of parameters [28]. Realization is given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1.25 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = [\,1 \ \ 0 \ \ 0\,]. \tag{4.17}$$

Below is shown the system's chaotic behavior and the exponential convergence of the observer states to the system states when the observer is as described above. The feedback gain is chosen as $K = [\frac{29}{10}, \frac{11}{20}, \frac{-19}{8}]^T$.

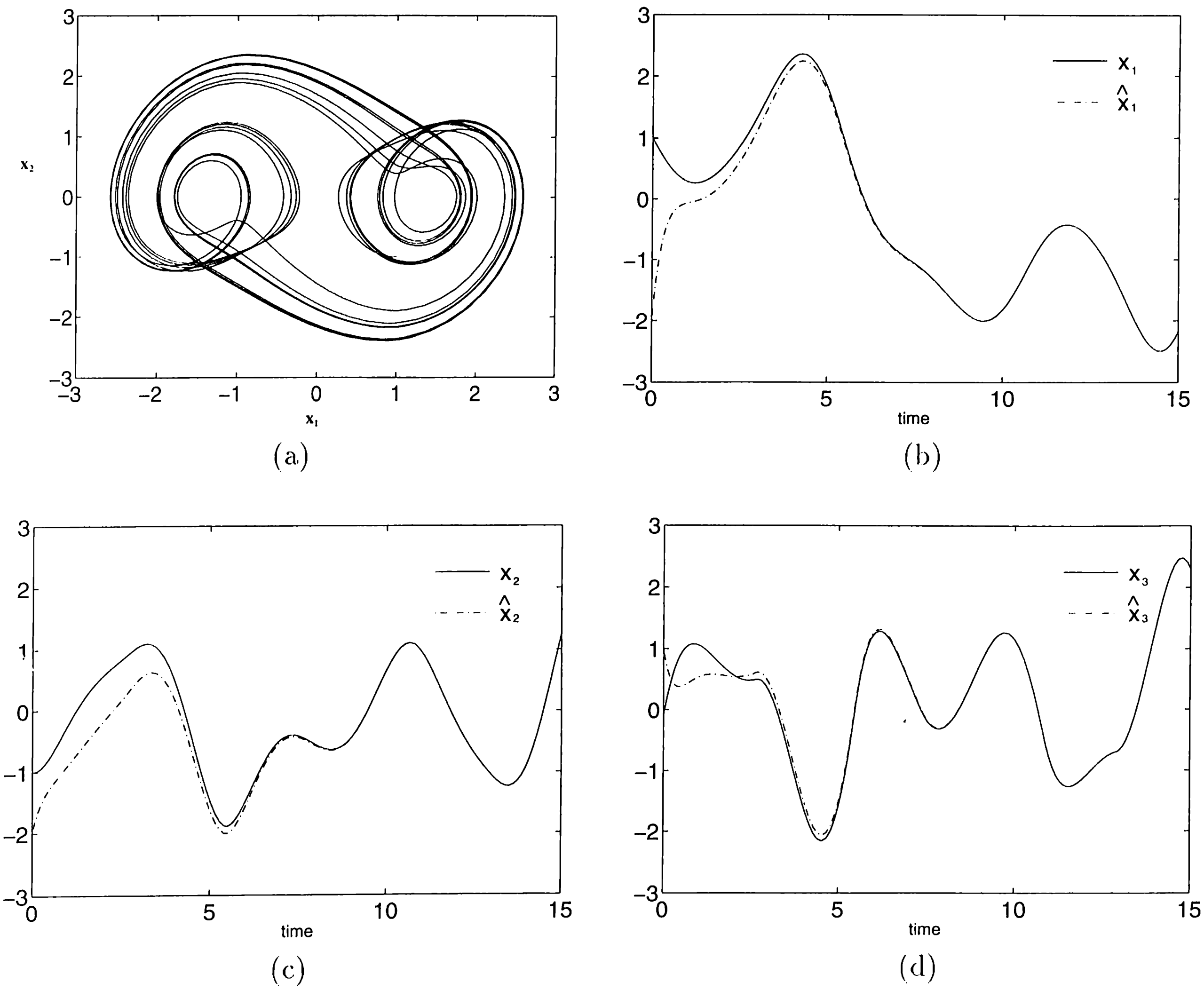


Figure 4.4: (a) The chaotic behavior of the system. (b)-(c)-(d) System and the observer states for $x(0) = [1,\ -1,\ -0.1]^T$ and $\hat{x}(0) = [-2,\ -2,\ 1]^T$

Example 4.2.2 The well-known Chua's Oscillator circuit [29] can also be represented in Lur'e form. The state equations are,

$$\dot{x}_1 = -\frac{R_0}{L}x_1 - \frac{1}{L}x_2, \tag{4.18}$$

$$\dot{x}_2 = -\frac{1}{C_2}x_1 - \frac{G}{C_2}x_2 + \frac{G}{C_2}x_3, \tag{4.19}$$

$$\dot{x}_3 = \frac{G}{C_1}x_2 - \frac{G}{C_1}x_3 - \frac{1}{C_1}f(x_3), \tag{4.20}$$

where $x_1 = i_L$, $x_2 = v_2$, $x_3 = v_1$, and $G = \frac{1}{R}$. The nonlinear resistor is described by $i_R = f(v_R)$, where $f : \mathbf{R} \to \mathbf{R}$ is a three segment piecewise linear

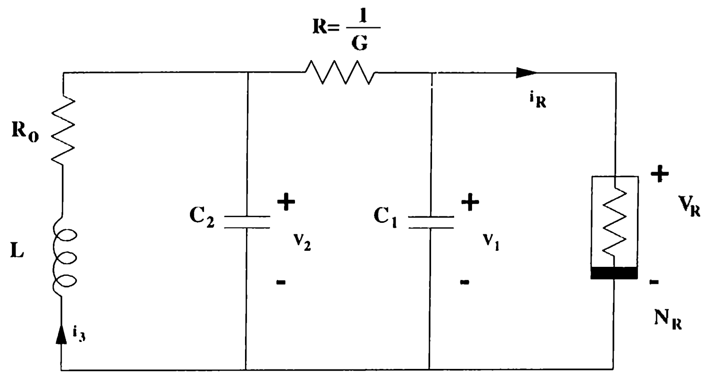


Figure 4.5: Chua Oscillator

function given as

$$f(x_3) = G_2 x_3 + 0.5(G_1 - G_2)(|x_3 + E| - |x_3 - E|), \tag{4.21}$$

and $G_1 < 0$, $G_2 < 0$, $E > 0$ are some constants.

If the output is chosen as $y = x_3$, the system is already a realization of Lur'e form with

$$A = \begin{bmatrix} -\frac{R_0}{L} & -\frac{1}{L}x_2 & 0 \\ -\frac{1}{C_2} & -\frac{G}{C_2} & \frac{G}{C_2} \\ 0 & \frac{G}{C_1} & -\frac{G}{C_1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{C_1} \end{bmatrix}, \quad C = [0 \ \ 0 \ \ 1], \tag{4.22}$$

and

$$n(y) = G_2 y + 0.5(G_1 - G_2)(|y + E| - |y - E|). \tag{4.23}$$

For the simulations, to facilitate the numerical integration, we define a new independent variable $\tau = \frac{G}{C_2}t$ and scale x_1 by $\frac{1}{G}$. After these changes the system is rewritten as ($R_0 = 0$)

$$\dot{x_1} = -\beta x_2, \tag{4.24}$$

$$\dot{x_2} = x_1 - x_2 + x_3, \tag{4.25}$$

$$\dot{x_3} = \alpha x_2 - \alpha x_3 - \frac{\alpha}{G} f(x_3), \tag{4.26}$$

where $\alpha = \frac{C_2}{C_1}$ and $\beta = \frac{C_2}{LG_2}$. The linear part of this system is observable if $\alpha \neq 0$. We choose the parameters as $G_1 = 0.8$, $G_2 = 0.5$, $\alpha = 8$, $\beta = 11$, $E = 1$, and $G = 0.7$. When we give examples of observer design by nonlinear state transformation, we will see that the Chua's oscillator also satisfies the necessary conditions for such a diffeomorphic transformation to exist, see Example 4.2.7. Now, we give the simulation results for the above design with $K = [-\frac{65}{9},\ \frac{9}{8},\ -1]^T$.

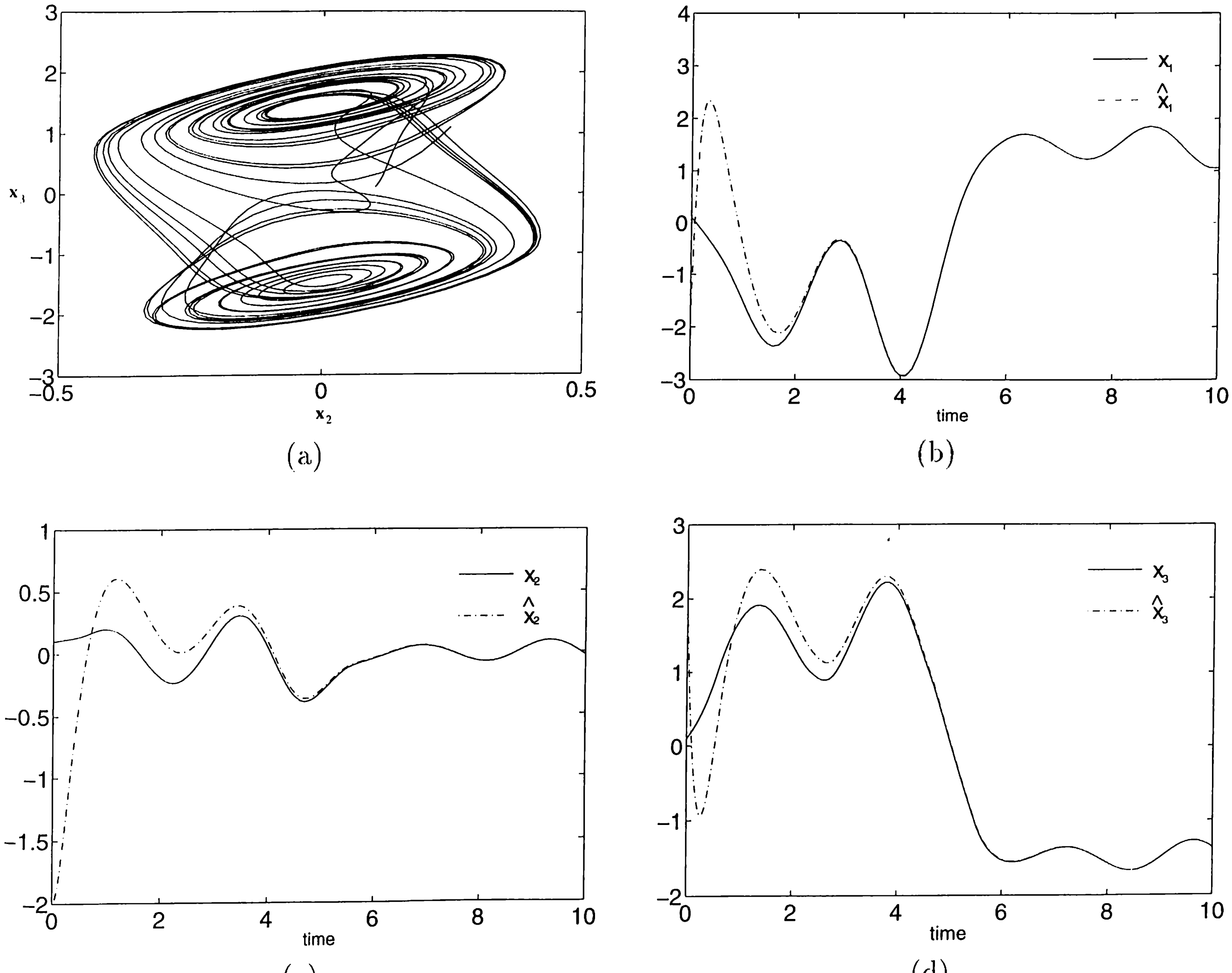


Figure 4.6: (a) The chaotic behavior of the Chua Oscillator. (b)-(c)-(d) System and the observer states for $x(0) = [0.1,\ 0.1,\ 0.1]^T$ and $\hat{x}(0) = [-2,\ -2,\ 2]^T$

4.2.2 Forced Oscillators

Consider an n^{th} order differential equation

$$w^{(n)} + F(w, \dot{w}, \ldots, w^{(n-1)}) = h(t), \tag{4.27}$$

where F is a differentiable function of its arguments. Van der Pol and Duffing systems are two examples of the above type displaying chaotic behavior. Choosing $x_1 = w$ and $x_i = \dot{x}_{i-1}$, $i = 1, 2, \ldots, n$, we write the state space representation of (4.27) as

$$\dot{x} = Ax + Bg(x) + Bu, \tag{4.28}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & & & 0 & 1 \\ 0 & & & & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \; g(x) = -F(x), \; u = h(t).$$

With the choice of output as $y = x_1$, the above representation is in the Brunowsky canonical form of previous chapter. Hence the eigenvalue assignment procedure described therein can directly be applied to design the observer as

$$\dot{\hat{x}} = A\hat{x} + Bg(\hat{x}) + Bu + K(x_1 - \hat{x}_1). \tag{4.29}$$

This observer has been used in [30] to design controllers for the purpose of driving following two systems to stable limit cycles.

Example 4.2.3 As a first example of this type, consider the following forced Van der Pol oscillator;

$$\ddot{x} + d(x^2 - 1)\dot{x} + x = a\cos wt + r(t). \tag{4.30}$$

It was shown in [31] that for various values of d , a and w, this oscillator exhibits a large variety of nonlinear phenomena, including chaos. This system is in the form given by (4.27) with

$$F(x, \dot{x}) = d(x^2 - 1)\dot{x} + x. \tag{4.31}$$

Transforming to state space coordinates and choosing $y = x_1$, we obtain

$$\begin{aligned} \dot{x}_1 &= x_2, & (4.32) \\ \dot{x}_2 &= -d(x_1^2 - 1)x_2 - x_1 + a\cos wt + r(t). & (4.33) \end{aligned}$$

We note that, although the nonlinearity is not globally Lipschitz, the solutions which are of interest to us remain in a bounded convex region Ω, of the state space. Thus a Lipschitz constant can be found by

$$L = \sup_{x\in\Omega} \|\nabla f(x)\|_\infty, \tag{4.34}$$

where, $f(x) = -d(x_1^2 - 1)x_2 - x_1$. Proceeding this way,

$$\|\nabla f(x)\|_\infty = \| \begin{bmatrix} -2dx_1x_2 - 1 \\ -d(x_1^2 - 1) \end{bmatrix} \|_\infty \tag{4.35}$$

and

$$L = \sup_{x\in\Omega} \max\{|2dx_1x_2 + 1|, |d(x_1^2 - 1)|\}. \tag{4.36}$$

For a particular trajectory with $d = 6$, $a = 2.5$, $w = 3$, we inspect the phase portrait of (4.32),(4.33) to set $L = 241$, then we arbitrarily pick a ratio $r > 1$. The first eigenvalue is chosen such that

$$\lambda_1 = \lambda < -\frac{r+1}{r-1}L \tag{4.37}$$

and the second eigenvalue is given by $\lambda_2 = r\lambda$. Finally the gains are found by equating

$$s^2 + k_1 s + k_2 = (s - \lambda_1)(s - \lambda_2). \tag{4.38}$$

So, choosing $r = 4$ and $\lambda = -403$, we find $\lambda_2 = -1612$, $k1 = 2015$, $k2 = 649636$.

One can use this observer to design a state feedback for the purpose of eliminating chaos. Using the bifurcation diagram given in [31], it can be seen that for certain different ranges of the parameter d , the system exhibits chaos or limit cycle. Hence effectively changing the value of this parameter with state feedback of the form $r(t) = d_f(x_1^2 - 1)x_2$, the system behavior can be changed from chaos to limit cycle, with the new parameter being $d_n = d - d_f$. For the present example, the bifurcation diagram reveals that when $d = 6$, the system operates in chaotic regime, and for $d = 0.5$, we have a limit cycle. Thus, choosing $d_f = 5.5$ we achieve the desired change. When the feedback states are taken from the observer, a nonlinear feedback of the form $r(t) = 5.5(\hat{x}_1^2 - 1)\hat{x}_2$, still drives the system to a stable limit cycle, see [30]. In the simulations both the chaotic regime and the limit cycle are shown. Note that the convergence of the observer states to the system states is quite fast while we have a large overshoot in $\hat{x}_2$.

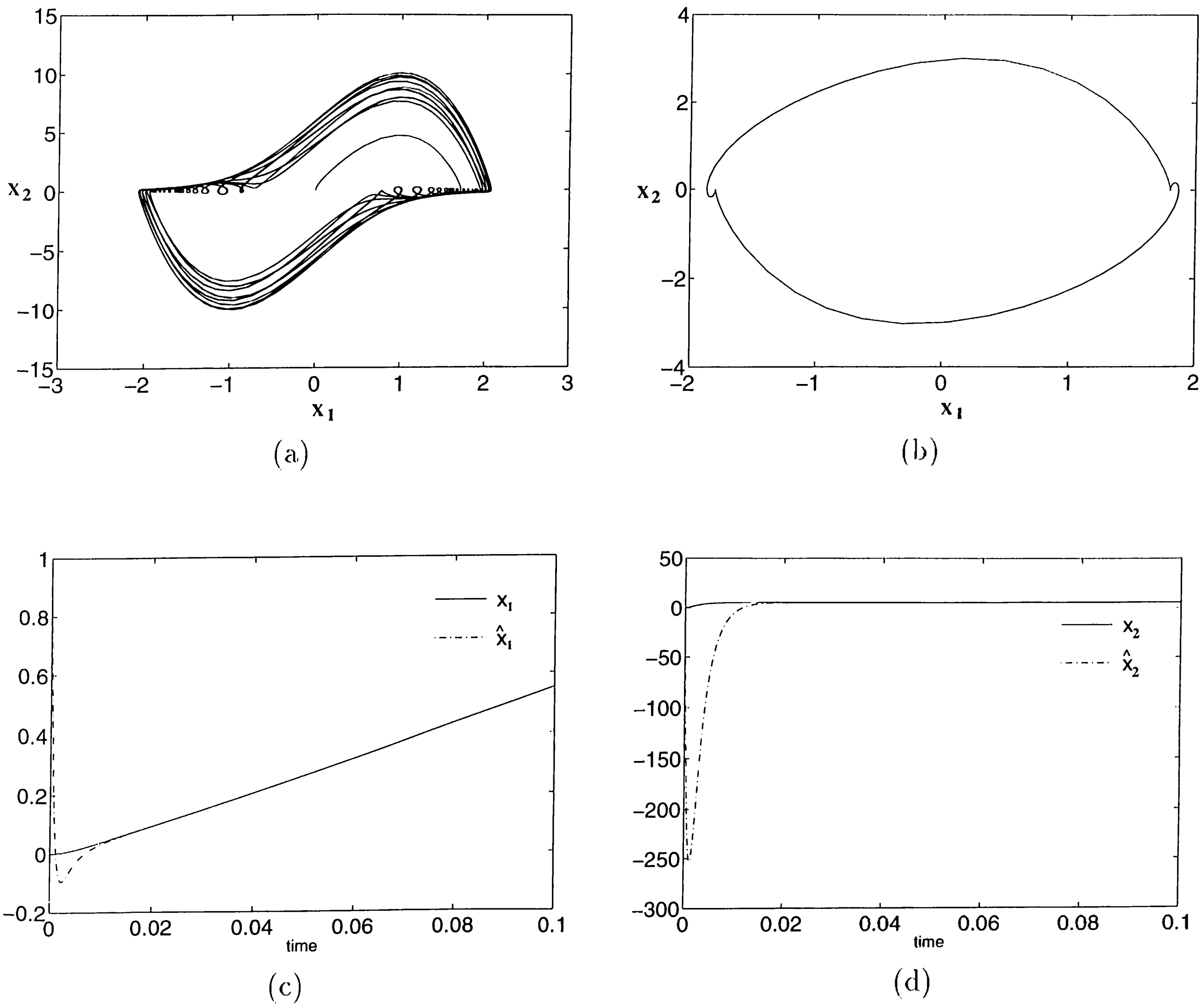


Figure 4.7: (a) The chaotic behavior of the Van der Pol forced Oscillator. (b) The limit cycle when the observer-state control is applied (c)-(d) System and the observer states for $x(0) = [0,\ 0]^T$ and $\hat{x}(0) = [1,\ 1]^T$.

Example 4.2.4 Our second example for forced oscillators is Duffing Equation which is used to model different natural phenomena, see [32]. It is described by the differential equation

$$\ddot{x} + a_0 x + a_1 \dot{x} + a_2 x^3 = q \cos wt + r(t). \tag{4.39}$$

The bifurcation structure of this system with respect to parameters a_0 , a_1 , a_2 can be found in [32].

The state space description of Duffing equation is

$$\begin{aligned} \dot{x}_1 &= x_2, & (4.40) \\ \dot{x}_2 &= -a_0 x_1 - a_1 x_2 - a_2 x_1^3 + q \cos wt + r(t). & (4.41) \end{aligned}$$

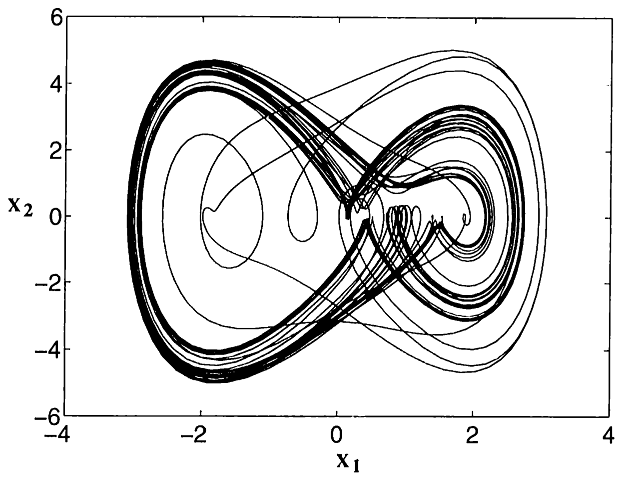
Although the observer design scheme of the previous example is quite applicable here, we note that the system becomes a realization of Lur'e form when x_1 is chosen as output. Then the observer given by

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + k_1(y - \hat{x}_1), & (4.42) \\ \dot{\hat{x}}_2 &= -a_0 \hat{x}_1 - a_1 \hat{x}_2 + k_2(y - \hat{x}_1) - a_2 y^3 + q \cos wt + r(t) & (4.43) \end{aligned}$$

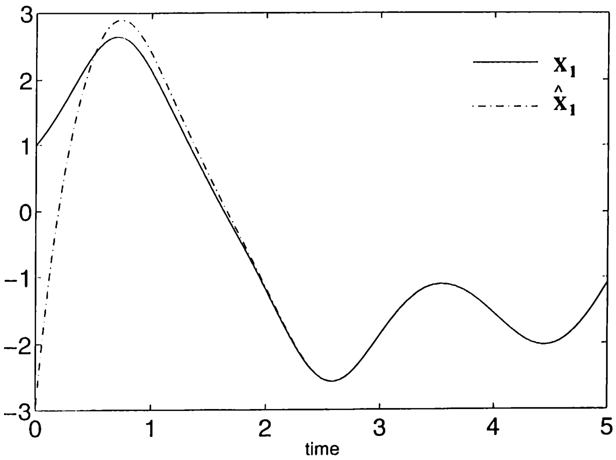
works globally and converges exponentially to the true states. With appropriate choice of k_1 and k_2, we can place all the eigenvalues of $A - KC$ on the open left half plane. This is always possible since the pair (C, A) with

$$A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}, \quad C = [\, 1 \;\; 0 \,] \tag{4.44}$$

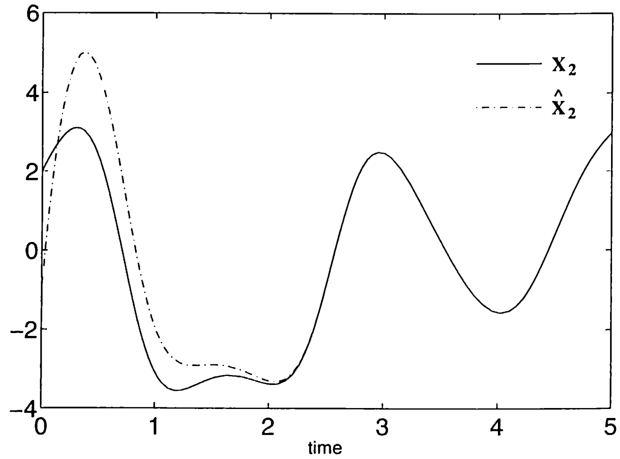
is observable whenever $a_0 \neq 0$. The parameters are chosen such that the system operates in chaotic regime, $a_0 = 0.25$, $a_1 = 0.2$, $a_2 = 1$, $q = 7.5$ and $w = 1$. Then by choosing the gain vector K such that $A - KC$ is stable, we construct the observer (4.42),(4.43). Here we give the simulation results for $K = [\frac{26}{5}, \frac{729}{100}]^T$ and $r(t) = 0$.



(a)



(b)



(c)

Figure 4.8: (a) The chaotic behavior of the Duffing system. (b)-(c) System and the observer states for $x(0) = [2,\ 2]^T$ and $\hat{x}(0) = [-3,\ -1]^T$.

4.2.3 Rössler and Lorentz Systems

Example 4.2.5 A common test system for the performance of synchronization schemes is the Rössler system,

$$\dot{x}_1 = -x_2 - x_3, \tag{4.45}$$
$$\dot{x}_2 = x_1 + ax_2, \tag{4.46}$$
$$\dot{x}_3 = -cx_3 + x_1x_3 + b. \tag{4.47}$$

This system exhibits chaotic motion for certain range of parameters $a > 0$, $b > 0$, $c > 0$, [13].

We first try $y = x_1$. Using (2.14),

$$\begin{aligned} Q(x) &= \frac{d\Phi}{dx} = \frac{d}{dx}\begin{bmatrix} x_1 \\ -x_2 - x_3 \\ -x_1 - ax_2 + cx_3 - x_1x_3 - b \end{bmatrix}, \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ -1 - x_3 & -a & c - x_1 \end{bmatrix}, \end{aligned}$$

which is singular for $x_1 = a + c$. Thus, the sufficiency condition of global observability is not met for this choice of output since $x = \Phi(x)$ is not a globally invertible transformation.

This time, we choose $y = x_2$. Then $\Phi(x)$ becomes (the constant input b can be ignored)

$$\Phi(x) = \begin{bmatrix} x_2 \\ x_1 + ax_2 \\ -x_2 - x_3 + ax_1 + a^2x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & a & 0 \\ a & -1 + a^2 & -1 \end{bmatrix} x = Tx, \quad (4.48)$$

and

$$Q(x) = \frac{d\Phi}{dx} = T, \quad (4.49)$$

which is always nonsingular, enabling us to define a global diffeomorphic state transformation by $x = \Phi(x)$. In the new domain the state equations become

$$\dot{z}_1 = z_2, \quad (4.50)$$

$$\dot{z}_2 = z_3, \quad (4.51)$$

$$\dot{z}_3 = f(z), \quad (4.52)$$

where

$$f(z) = -cz_1 + (ca - 1)z_2 + (a - c)z_3 - az_1^2 - az_2^2 + (a^2 + 1)z_1z_2 + z_2z_3 - az_1z_3.$$

A Lipschitz bound on $f(z)$ can be found by

$$L = \sup_{z \in \Omega} \|\nabla f(z)\|_\infty, \tag{4.53}$$

where Ω is the compact domain confining the chaotic trajectories of the system.

$$\begin{aligned}
\nabla f(z) &= \begin{bmatrix} -2az_1 + (a^2+1)z_2 - az_3 - c \\ -2az_1 + (a^2+1)z_1 - 2az_2 + z_3 + (ca-1) \\ -az_1 + z_2 + (a-c) \end{bmatrix}, \\
&= \begin{bmatrix} -2a & a^2+1 & -a \\ a^2+1 & -2a & 1 \\ -a & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} -c-2a \\ ca-1 \\ a-c \end{bmatrix}, \\
&= Dz + E.
\end{aligned}$$

Using (4.48) we can write

$$\begin{aligned}
\|\nabla f(z)\|_\infty &\le \|D\|_\infty \|z\|_\infty + \|E\|_\infty, \\
&\le \|D\|_\infty \|T\|_\infty \|x\|_\infty + \|E\|_\infty
\end{aligned} \tag{4.54}$$

Thus for a parameter choice of $a = 0.2$, $b = 0.2$, $c = 5$, (4.54) becomes

$$\begin{aligned}
\|\nabla f(z)\|_\infty &\le (1.64)(2.16)\|x\|_\infty + (5.4) \\
&= (3.5424)\|x\|_\infty + (5.4)
\end{aligned} \tag{4.55}$$

Hence the Lipschitz constant L can be found by replacing $\|x\|_\infty$ in (4.55) by the maximum absolute coordinate of the trajectories. Inspecting a typical trajectory we find a bound on the maximum absolute coordinate as $\|x\|_\infty \le 18$. Thus the Lipschitz constant can be assigned $L = 70$. Picking a ratio $a = 3$ and using (3.48) we obtain

$$\|V^{-1}\| = \frac{3^2+1}{(3-1)^2} = 2.5 \tag{4.56}$$

Hence the largest eigenvalue λ_1 is assigned such that

$$\lambda_1 = \lambda \le -(70)(2.5) = -175. \tag{4.57}$$

Let us choose $\lambda = -180$. Then we obtain $\lambda_2 = a\lambda = -540$ and $\lambda_3 = a^2\lambda = -1620$. The feedback gain matrix is calculated using (3.54) as

$$k_1 = 2340, \quad k_2 = 1263600, \quad k_3 = 157464000. \tag{4.58}$$

Below are shown the simulation results for this choice of gains.

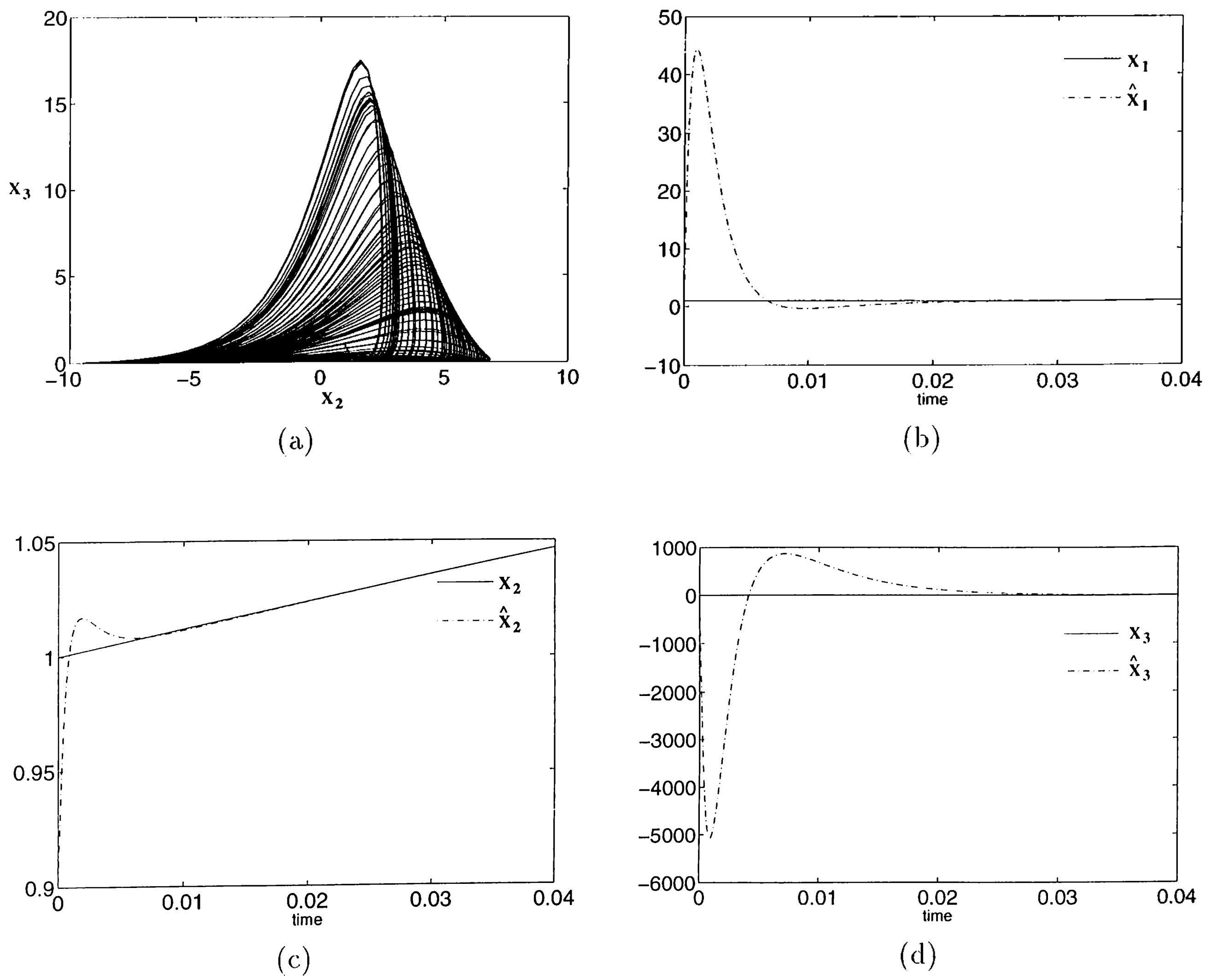


Figure 4.9: (a) The chaotic behavior of the Rössler system (b)-(c)-(d) System and the observer states for $x(0) = [1,\ 1,\ 1]^T$ and $\hat{x}(0) = [1.1,\ 0.9,\ 0.99]^T$.

Example 4.2.6 Consider the system described by

$$\dot{x}_1 = -\beta x_1 + x_3 x_2, \tag{4.59}$$

$$\dot{x}_2 = -\sigma x_2 + \sigma x_3, \tag{4.60}$$

$$\dot{x}_3 = -\rho x_2 - x_3 - x_1 x_2, \tag{4.61}$$

also known as the Lorentz chaotic attractor. When the output is chosen as one of the states, $y = x_i,\ \ i = 1, 2, 3$, it can be shown by some lengthy but routine calculations that the state transformation defined in (2.53) is not invertible.

But here, we would like to stress an interesting observation that we obtained through simulations. For all of the chaotic systems that are investigated in our work, we noted that the observers designed by considering only the linear part of the system perform globally converging state estimation. Namely, given a chaotic system of the form

$$\begin{aligned} \dot{x} &= Ax + g(x), && (4.62) \\ y &= Cx, && (4.63) \end{aligned}$$

where $g(\cdot)$ contains only nonlinear terms, the observer

$$\dot{\hat{x}} = A\hat{x} + g(\hat{x}) + K(y - C\hat{x}) \quad (4.64)$$

works globally for $A_c = A - KC$ stable. Hence we are led to conjecture that such an observation scheme is always valid for chaotic systems. Although the behavior of the chaotic systems is not well-understood for the present [33], there have been reports that chaotic systems may have superior properties that can be employed in control applications [34, 35]. Further developments in this field may reveal the underlying paradigm which, as it seems, enabled us to simplify the observer design for chaotic systems. If our conjecture turns out to be true, then such an observer design would be quite a simplification, considering the pathologically high feedback gains that is obtained in the design using the nonlinear state transformation to observer canonical form [36].

For now, we give the simulation results for the Lorentz system. The system is separated into linear and nonlinear parts as

$$\begin{aligned} \dot{x} &= Ax + g(x), && (4.65) \\ y &= Cx, && (4.66) \end{aligned}$$

where

$$A = \begin{bmatrix} -\beta & 0 & 0 \\ 0 & -\sigma & \sigma \\ 0 & \rho & -1 \end{bmatrix}, \quad g(x) = \begin{bmatrix} x_3 x_2 \\ 0 \\ -x_1 x_2 \end{bmatrix}, \quad C = [0\ 1\ 0]. \quad (4.67)$$

Note that with this choice of output $y = x_2$, the pair (C, A) is observable. For the parameter set $\sigma = 10$, $\beta = \frac{8}{3}$ and $\rho = 28$, we assigned the three eigenvalues of $A_c = A - KC$, real, negative, and distinct by choosing $K = [0, \frac{-11}{3}, \frac{253}{9}]$. Below are shown the simulation results for this choice of gains.

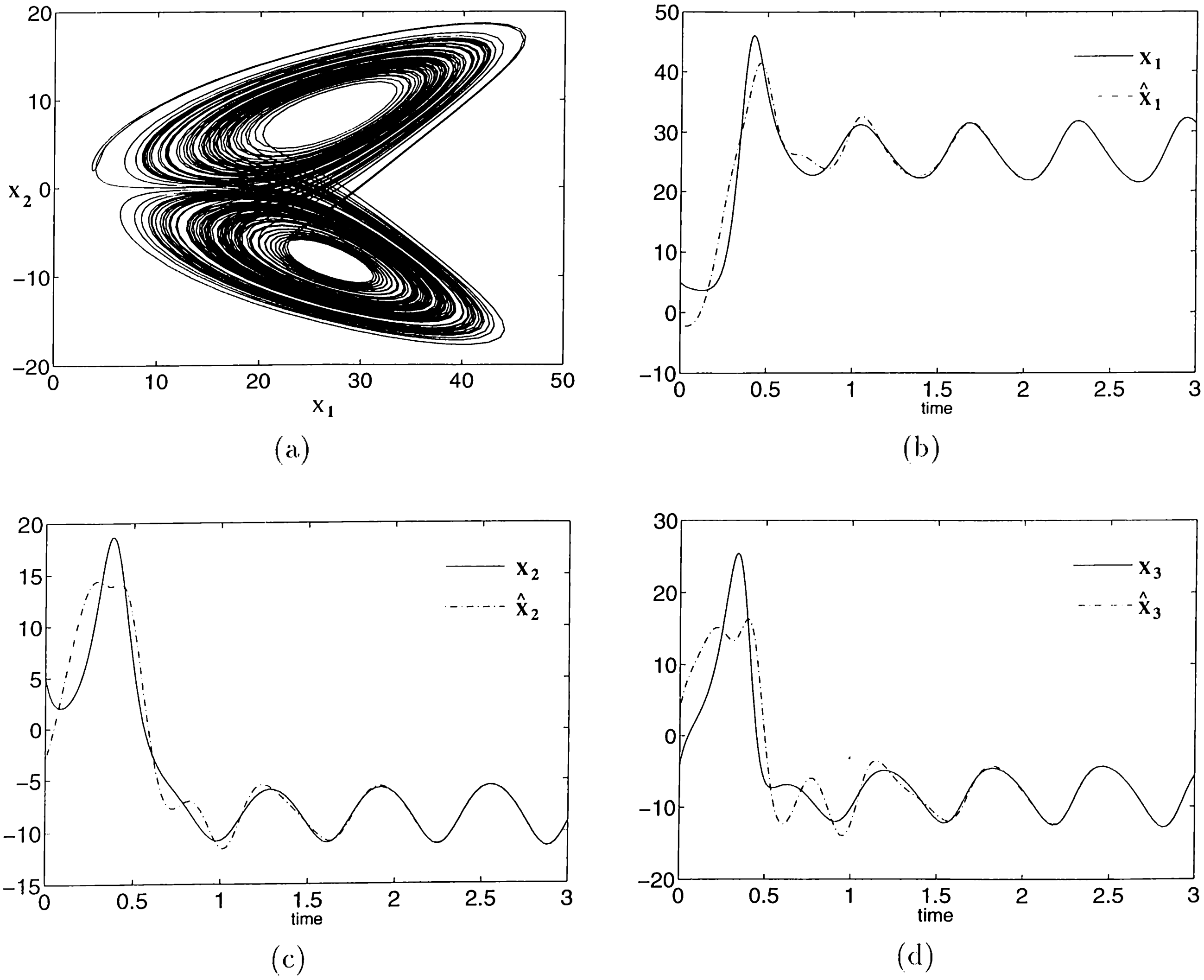


Figure 4.10: (a) The chaotic behavior of the Lorentz attractor (b)-(c)-(d) System and the observer states for $x(0) = [5,\ 5,\ -4]^T$ and $\hat{x}(0) = [-2,\ -3,\ 4]^T$.

Example 4.2.7 (*Chua Oscillator revisited*) We have already seen that with the appropriate choice of the system output, Chua oscillator can be represented as a realization of a system in Lur'e form. This example shows that it is also possible to transform the system by a diffeomorphic change of coordinates to observer canonical form. Let us rewrite the system equations

$$\dot{x_1} = -\beta x_2, \tag{4.68}$$

$$\dot{x_2} = x_1 - x_2 + x_3, \tag{4.69}$$

$$\dot{x_3} = \alpha x_2 - \alpha x_3 - \frac{\alpha}{G} f(x_3), \tag{4.70}$$

and choose the output $y = x_1$. Then using (2.53), the coordinate transformation is given by

$$z_1 = x_1, \tag{4.71}$$
$$z_2 = -\beta x_2, \tag{4.72}$$
$$z_3 = -\beta x_1 + \beta x_2 - \beta x_3, \tag{4.73}$$

or

$$z = Tx, \tag{4.74}$$

where

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\beta & 0 \\ -\beta & \beta & -\beta \end{bmatrix}. \tag{4.75}$$

Hence

$$Q(x) = \frac{d\Phi(x)}{dx} = T \tag{4.76}$$

is nonsingular whenever $\beta \neq 0$. This corresponds to the condition that $\frac{C_2}{LG_2} \neq 0$ which is trivially satisfied.

After some routine but tedious algebraic manipulations the system equations in the new coordinates can be obtained as

$$\dot{z}_1 = z_2, \tag{4.77}$$
$$\dot{z}_2 = z_3, \tag{4.78}$$
$$\dot{z}_3 = g(z), \tag{4.79}$$

where

$$g(z) = -\alpha\beta z_1 - \beta z_2 - (\alpha+1)z_3 + \frac{\beta\alpha}{G} f(-z_1 - \frac{1}{\beta}z_2 - \frac{1}{\beta}z_3). \tag{4.80}$$

A Lipschitz bound on $g(z)$ can be found as

$$L = \sup_{z \in \mathbf{R}^n} \|\nabla g(z)\|_\infty = \| \begin{bmatrix} -\alpha\beta - \frac{\alpha\beta}{G} f'(\cdot) \\ -\beta - \frac{\alpha}{G} f'(\cdot) \\ -(\alpha+1) - \frac{\alpha}{G} f'(\cdot) \end{bmatrix} \|_\infty. \tag{4.81}$$

Using the fact that $|f'(\cdot)| \leq G_1$, for the set of parameters $G_1 = 0.8$, $G_2 = 0.5$, $\alpha = 8$, $\beta = 11$, $E = 1$, and $G = 0.7$, the Lipschitz constant can be assigned as $L = 189$. Note that, although the nonlinearity is not differentiable at two points, this poses no problem in the assignment of the Lipschitz constant. Picking a ratio $a = 3$ and using (3.48) we obtain

$$\|V^{-1}\|_\infty = \frac{3^2+1}{(3-1)^2} = 2.5 \tag{4.82}$$

Hence the largest eigenvalue λ_1 is assigned such that

$$\lambda_1 = \lambda \leq -(189)(2.5) = -472.5. \tag{4.83}$$

Let us choose $\lambda = -475$. Then we obtain $\lambda_2 = a\lambda = -1425$ and $\lambda_3 = a^2\lambda = -4275$. The feedback gain matrix is calculated using (3.54) as

$$k_1 = 6175, \quad k_2 = 8799375, \quad k_3 = 2.89 \times 10^9. \tag{4.84}$$

Below is the simulation results for these unusually huge gains.

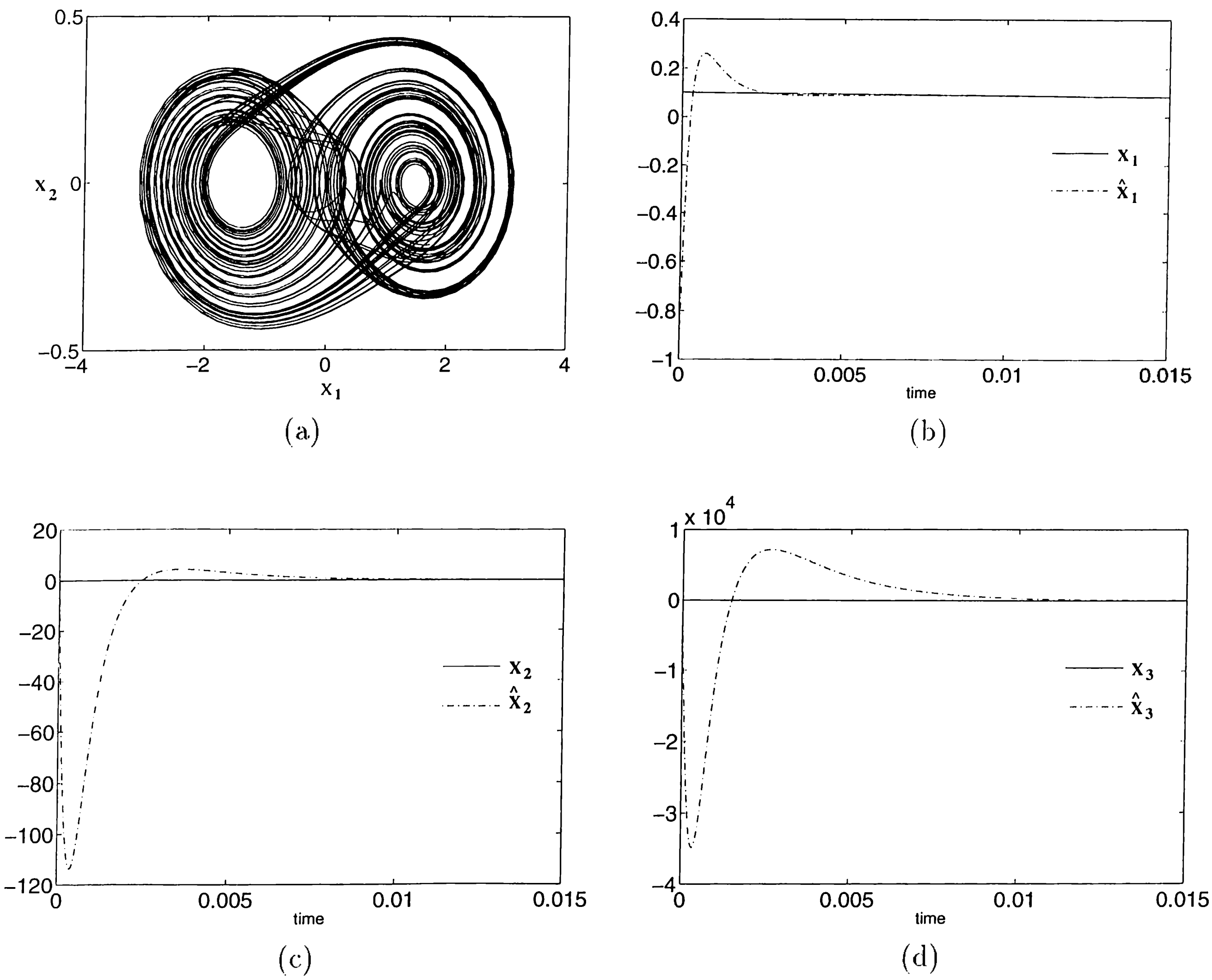


Figure 4.11: (a) The chaotic behavior of the Chua oscillator (b)-(c)-(d) System and the observer states for $x(0) = [0.1,\ 0.1,\ 0.1]^T$ and $\hat{x}(0) = [-1,\ 1,\ -1]^T$.

Chapter 5

AN OBSERVER WITH GRADIENT UPDATE

In this chapter we examine the possibility of adapting the gradient descent algorithm to observer design. The observer discussed in the previous chapter, like many existing observers, is a replica of the original system with the additive injection term calculated using the error between the outputs of the system and the observer. Namely, given a single-input single-output TI nonlinear system

$$\begin{aligned} \dot{x} &= f(x), && (5.1) \\ y &= h(x), && (5.2) \end{aligned}$$

the observer is constructed as

$$\dot{\hat{x}} = f(\hat{x}) + \underbrace{G(\hat{x})(y - h(\hat{x}))}_{\text{output error injection}} \tag{5.3}$$

Note that $x = \hat{x}$ is an equilibrium point of the error system since the injection term in (5.3) diminishes when $x = \hat{x}$. This is equivalent to assert that if

$$\hat{x}(t_0) = x(t_0) \quad \text{for} \;\; t_0 \in \mathbf{R}_+ \tag{5.4}$$

then

$$\hat{x}(t) = x(t) \quad \text{for} \;\; \forall t \geq t_0. \tag{5.5}$$

Another observer form that satisfies this condition might be

$$\dot{\hat{x}} = \frac{y(x) - g(\hat{x})}{y(\hat{x}) - g(\hat{x})} f(\hat{x}) \tag{5.6}$$

where $g : \mathbf{R}^n \to \mathbf{R}$ and $K \in \mathbf{R}^{n \times n}$ are the parameters to be chosen in the design such that the error system is stable. In order to overcome a possible singularity at $x = \hat{x} = 0$, $g(\cdot)$ should be chosen such that $g(0) \neq 0$. Note that while we have an additive injection in (5.3), in (5.6) dynamics are modified by a multiplicative term.

Now consider the system given in (5.1),(5.2). Choose the observer as

$$\begin{aligned} \dot{\hat{x}} &= F(\hat{x}, y), && (5.7) \\ \hat{y} &= g(\hat{x}), && (5.8) \end{aligned}$$

where $F : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ and $g : \mathbf{R}^n \to \mathbf{R}$ are to be determined later in the analysis. Let us define the output error as

$$\varepsilon_o = \frac{1}{2}(y - \hat{y})^2. \tag{5.9}$$

We would like to update the observer state in the opposite direction of its contribution to the output error defined above. To do this, we calculate the directional derivative of the output error with respect to the observer states,

$$\begin{aligned} \frac{\partial \varepsilon_o}{\partial \hat{x}} &= -\frac{\partial \hat{y}}{\partial \hat{x}}(y - \hat{y}), && (5.10) \\ &= -\left[h(x) - g(\hat{x})\right] \nabla g(\hat{x}). && (5.11) \end{aligned}$$

Then, we choose the observer dynamics as

$$\dot{\hat{x}} = \Sigma \left[h(x) - g(\hat{x})\right] \nabla g(\hat{x}) \tag{5.12}$$

where $\Sigma = diag[\sigma_1,\ \sigma_2, \ldots, \sigma_n]$ and $\sigma_i > 0$ are constant scalars.

To satisfy the equilibrium condition we should have

$$F(x, h(x)) = f(x) \tag{5.13}$$

or

$$\Sigma \left[h(x) - g(x)\right] \nabla g(x) = f(x). \tag{5.14}$$

We see that (5.14) is a system of nonlinear partial differential equations with $f(\cdot)$ and $h(\cdot)$ known and $g(\cdot)$ to be solved. Also note that Σ brings n free parameters.

Assuming (5.14) has a solution $g_0(x)$, the observer becomes

$$\dot{\hat{x}} = \frac{h(x) - g_0(\hat{x})}{h(\hat{x}) - g_0(\hat{x})} f(\hat{x}). \tag{5.15}$$

Note that this observer has the form given in (5.6).

Although the existence of a solution for (5.14) seems unlikely for all $f(\cdot)$ and $h(\cdot)$, it may still be possible to obtain plausible simplifications by narrowing down the class of nonlinearities involved. This issue together with the stability problem of the error system will be investigated further in a future work.

Chapter 6

CONCLUSION

In this work, we addressed the problem of synchronizing chaotic systems from the nonlinear observer design point of view. A solution to this problem is provided for some classes of nonlinear systems satisfying some mild requirements. In doing this, properties of chaotic systems is exploited in order to simplify the observer design.

Observer design is achieved by first transforming the nonlinear system to observer canonical form and then choosing the output error injection gains in this transformed coordinates. An explicit eigenvalue assignment procedure is given in order to choose the gain vector. This procedure is incorporated in a step-by-step design scheme.

We provided a restriction on the class of nonlinearities for which the linear observer design paradigm is guaranteed to yield exponentially decaying errors. For this, we derived a bound on the ratio of the minimum eigenvalue of a matrix to the condition number of its matrix of eigenvectors.

In applying the above nonlinear observer design techniques to the synchronization of chaotic circuits, we noted that for certain widely known chaotic systems (e.g. Chua oscillator) the system output can be chosen such that the system becomes linear up to output injection, which admits simple global observer. Besides, based upon the simulations we carried out, we conjectured that linear observer design schemes are directly applicable to nonlinear systems operating in chaotic regime. Such a technique does not suffer from the shortcomings of the observer based on the nonlinear state transformation to observer canonical form.

In the latter design, observer gains are unusually large which may cause serious problems in the actual implementations. Moreover, the system structure should satisfy certain requirements in order for the transformation to exist. However, the former design does not involve any state transformation.

Finally we proposed a gradient descent observer for nonlinear systems. Although restrictions imposed in this design seem rather stringent, this topic can be further investigated by narrowing down the class of nonlinear systems involved.

Another topic of further research is the issue of finding the underlying behavior of chaotic systems that led us to conjecture that the linear observer techniques can be applied for this class of nonlinear systems.

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