ASYMPTOTIC THEORY OF CHARACTERS OF THE SYMMETRIC GROUPS

A THESIS

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS

AND THE INSTITUTE OF ENGINEERING AND SCIENCES

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

By

Elif Kurtaran

August, 1996

ASYMPTOTIC THEORY OF CHARACTERS OF THE SYMMETRIC GROUPS

A THESIS

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS

AND THE INSTITUTE OF ENGINEERING AND SCIENCES

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF
MASTER OF SCIENCE

By

Elif Kurtaran

August, 1996

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof.Dr. Alexander Klyachko(Principal Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof.Dr. İsmail Güloğlu

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Mahmut Kuzucuoğlu

Approved for the Institute of Engineering and Sciences:

Prof. Dr. Mehmet Baray
Director of Institute of Engineering and Sciences

ABSTRACT

ASYMPTOTIC THEORY OF CHARACTERS OF THE SYMMETRIC GROUPS

Elif Kurtaran
M.S. in Mathematics
Advisor: Prof.Dr. Alexander Klyachko
August, 1996

In this work, we studied the connection between ramified coverings of Riemann surfaces $\pi : X \to Y$ of degree n and characters of symmetric group S_n. We considered asymptotics of characters of S_n as $n \to \infty$ and normalized characters $\frac{\chi(g)}{\chi(1)}$ of S_n under some restrictions.

Keywords : Coverings, Riemann surfaces, triangulations, symmetric group, characters.

ÖZET

SİMETRİK GRUPLARIN KARAKTERLERİNİN ASİMTOTİK TEORİSİ

Elif Kurtaran
Matematik Bölümü Yüksek Lisans
Danışman: Prof.Dr. Alexander Klyachko
Ağustos, 1996

Bu çalışmada Riemann yüzeyleri arasındaki n.dereceden $\pi : X \to Y$ dallanmış örtüleri ile S_n simetrik grubu arasındaki bağıntıyı inceledik. Ayrıca, n sonsuza giderken S_n simetrik grubunun karakterlerinin asimtotikleri ile normalize edilmiş $\frac{\chi(g)}{\chi(1)}$ karakterleri bazı kısıtlamalar altında ele aldık.

Anahtar Kelimeler : Örtüler, Riemann yüzeyleri, üçgenleştirme, simetrik grup, karakterler.

ACKNOWLEDGMENTS

I am greatful to Prof. Alexander Klyachko who expertly and patiently guided my research up to this point and without whom this thesis wouldn't exist.

I would like to thank my family for their unfailing support and influence in my life.

Finally, I would like to thank to my friends for all they have done for me, and especially to Özgül for her patience and support on my sleepless nights and to Ferruh for his helps.

TABLE OF CONTENTS

Chapter 1

Introduction

1.1 Review of known results

In this thesis we will consider the asmyptotic behaviour of characters of symmetric group S_n as n tends to ∞. There are at least two reasons of interest of this problem.

i) The first one, which is not our interest of study, is its connection with representations of the infinite symmetric group S_∞. S_∞ is a nontrivial experimental model in the theory of locally finite groups and has been studied by Zalesskii [1], Vershik and Kerov[2]. In the theory of representations of symmetric group, each irreducible representation of S_n with character χ_λ corresponds to a Young diagram λ. Vershik and Kerov[3], in their paper studied the limit form of Young diagrams with respect to the Plancherel measure, given as $\frac{dim^2\lambda}{n!}$, for an irreducible representation λ. They obtained that, with respect to Placherel measure almost all diagrams have the same shape, given by the function

$$\Omega(X) = \begin{cases} \frac{2}{\pi}(x\ arc\ sinX + \sqrt{1-X^2}) & for\ \ |X| \leq 1 \\ |X| & for\ \ |X| \geq 1 \end{cases}$$

In the same paper, two sided bounds of the largest dimension (w.r.t Plancherel measure)of irreducible representations of S_n as $n \to \infty$ is found.

Thoma, in his paper [4], considered the problem of finding limit for the ratio $\frac{\chi_\lambda(g)}{\chi_\lambda(1)}$ as $n \to \infty$ (called the *normalized character of representation* λ)

for $g\epsilon S_n \subset S_\infty$, n is fixed and $|\lambda| \to \infty$. He gave an explicit formula for all normalized characters of S_∞ as

$$\prod_{m\geq 2} (\sum_{i=1}^{\infty} \alpha_i^m + (-1)^{m+1} \sum_{i=1}^{\infty} \beta_i^m)^{\rho_m(g)},$$

where

$$\begin{aligned} \alpha_k &= lim_{n\to\infty} \frac{f_k(\lambda)}{n}, \\ \beta_k &= lim_{n\to\infty} \frac{g_k(\lambda)}{n}, \\ f_k(\lambda) &= max\{i:\ (i,k) \in \lambda\} \ - \ k \ + \ (\frac{1}{2}), \qquad (1.1) \\ g_k(\lambda) &= max\{i:\ (k,i) \in \lambda\} \ - \ k \ + \ (\frac{1}{2}), \qquad (1.2) \end{aligned}$$

$\alpha_1 \geq \alpha_2 \ldots \geq 0$, $\beta_1 \geq \beta_2 \geq \ldots \geq 0$, $\sum \alpha_i + \sum \beta_i \leq 1$ and ρ_m is the number of cycles of length m in the permutation σ.

ii) The second reason of interest of the problem is its connection with triangulations of surfaces. We will focus our attention to this case (section 1.2)

1.2 Triangulations and ramified coverings

Our approach to the problem is motivated by its connection with triangulations of Riemann surfaces and ramified coverings. As to give an idea, observe the following.

Let $\sum$ be a triangulation of compact Riemann surface X, and $\sum'$ be its barycentric subdivision. Let

f_i : barycenters of triangles,

e_i : centers of edges,

v_i : vertices of $\sum$ (see fig. 1.1).

The triangulation $\sum'$ gives us a ramified covering $\pi : X \to \mathbb{P}^1$, $\mathbb{P}^1$: Riemann sphere. π maps black triangles onto upper hemisphere, white triangles onto lower hemisphere, and barycenters e_i, f_i, v_i to 0,1 and ∞ respectively.

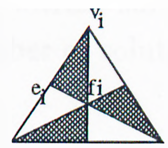

Figure 1.1:

Hence, we get a ramified covering with the following properties:

i) $deg\ \pi\ =\ 3\ \#(triangles\ in\ \sum)\ =\ 2\ \#(edges\ in\ \sum)$,

ii) Barycenters of triangles have ramification index 3,

iii) Centers of edges have ramification index 2.

It is easy to see that we have a one-to-one correspondance between triangulations and coverings of sphere ramified only over $0,1,\infty$ with ramification indices

2 over 0,

3 over 1,

arbitrary indices over ∞.

It is worth to mention the special attractiveness of triangulations of a Riemann surface X for physicists since they are used as a model for random metric on X (see papers 5,6).

1.3 Connection with characters

As we have seen in previous item, the problem of counting the triangulations is particular case of counting ramified coverings of given degree and prescribed ramification indices. It turns out that the last problem is closely related with characters of symmetric group. This connection follows from two classical results.

i) The first is Hurwitz theorem which gives a one-to-one correspondance between ramified coverings $\pi : X \rightarrow \mathbb{P}^1$ of given degree and ramification indices, and solutions of the equation

$$g_1 g_2 \dots g_k\ =\ 1\ \ ; g_i \in C_i \subset S_n \tag{1.3}$$

up to conjugacy, where cycle lengths of g_i is equal to ramification indices of points in fibers.

If instead of $\mathbb{P}^1$, we consider an arbitrary surface Y of genus g, then the number of coverings is equal to the number of solutions of the equation below, up to conjugacy.

$$g_1 g_2 \dots g_k [f_1, h_1] \dots [f_g, h_g] = 1 \; ; \; f_i, h_i \in S_n \; , \; g_i \in C_i \subset S_n \tag{1.4}$$

where $[f, g] = fgf^{-1}g^{-1}$ is a commutator.

ii) The second is Burnside theorem which gives the number of solutions of the equations (1.3) and (1.4) for an arbitrary group G in terms of the characters.

$$\#\{g_1 g_2 \dots g_k = 1 : g_i \in C_i \in G\} = \frac{|C_1||C_2| \dots |C_k|}{|G|} \sum_\chi \frac{\chi(g_1)\chi(g_2)\dots\chi(g_k)}{\chi(1)^{k-2}} \tag{1.5}$$

$$\#\{g_1 g_2 \dots g_k [f_1, h_1] \dots [f_g, h_g] = 1\} = \frac{|C_1||C_2| \dots |C_k|}{|G|^{1-2g}} \sum_\chi \frac{\chi(g_1)\chi(g_2)\dots\chi(g_k)}{\chi(1)^{k+2g-2}} \tag{1.6}$$

where the summation is over all irreducible characters χ of G, $g_i \in C_i$ are elements from fixed conjugacy classes C_i.

The theorems of Hurwitz and Burnside leads to the following formula for the number of coverings, which is the starting point of our approach. Before, let us remark that the number $\sum_{\pi:X\to Y} \frac{1}{|Aut\pi|}$ is called as "Eisenstein number of coverings".

Theorem 1.1 *The Eisenstein number of ramified coverings $\pi : X \to Y$ of degree n with given ramification indices, of the surface Y of genus g_Y, ramified over k points $y_1, \dots, y_k$ in Y is given by*

$$\sum_{\pi:X\to Y} \frac{1}{|Aut\pi|} = \frac{|C_1||C_2| \dots |C_k|}{(n!)^{2-2g_Y}} \sum_\chi \frac{\chi(g_1)\chi(g_2)\dots\chi(g_k)}{(\chi(1))^{k-(2-2g_Y)}} \tag{1.7}$$

where $g_i \in C_i$ are elements from fixed conjugacy classes C_i, cycle lenghts of g_i are ramification indices in fiber $\pi^{-1}(y_i)$, and the summation is over all irreducible characters χ of S_n.

In the case Y=$\mathbb{P}^1$, we get

Corollary 1.1 *Under the hypothesis of theorem (1.1) with Y replaced by* $\mathbb{P}^1$ *we have*

$$\sum_{\pi:X\to\mathbb{P}^1} \frac{1}{|Aut\pi|} = \frac{|C_1||C_2|\dots|C_k|}{(n!)^2}\sum_{\chi}\frac{\chi(g_1)\chi(g_2)\dots\chi(g_k)}{(\chi(1))^{k-2}} \tag{1.8}$$

Hence, using the argument in section 1.2, we can write the following formula:

The Eisenstein number of triangulations on X is equal to

$$\frac{1}{(\frac{n}{2})!\, 2^{\frac{n}{2}}\, (\frac{n}{3})!\, 3^{\frac{n}{3}}}\, |C(g)| \sum_{\chi} \frac{\chi(\delta_2)\,\chi(\delta_3)\,\chi(g)}{\chi(1)}$$

where $g \epsilon S_n$, C(g) denotes its conjugacy class, δ_2 consists of 2 cycles and δ_3 consists of 3 cycles. By (1.24) with $g_Y = 0\,, k = 3\,, d_1 = 2\,d_2 = 3$

$$g_X = 1 + \frac{n}{2}(\frac{1}{6} - \frac{1}{d}) \quad \text{, d:mean.value of cycle lengths of g.}$$

1.4 Main results

Theorem (1.1) can be used in both directions, i.e. information on coverings may be transferred in information on characters and vice versa. When the structure of the covering is known, it is easier to carry information on coverings to characters. Let us begin from coverings.

1.4.1 Explicit formulae

There exists several cases in which the number of coverings can be evaluated explicitly. In each of these cases, ramification indices in the fibers are the same.

Let $\pi : X \to \mathbb{P}^1$ be a ramified covering of degree n ramified over k points $y_1, y_2, \dots, y_k$ with ramification indices m_i, equal in each fiber. In the case

$$\frac{1}{m_1} + \frac{1}{m_2} + \dots + \frac{1}{m_k} \geq k - 2 \tag{1.9}$$

all coverings may be explicitely described in terms of finite groups of Möbius transformations or plane Coxeter groups. Since we know the structure of these groups, we can get explicit formulae for (1.8).

1. Elliptic Case

If

$$\frac{1}{m_1}+\frac{1}{m_1}+\frac{1}{m_2}+\ldots+\frac{1}{m_k}>k-2 \tag{1.10}$$

the possible solutions are

ai) Cyclic case : k=2 , $m_1 = m_2 = m$,

aii) Dihedral case : k=3 ,$m_1 = m_2 = 2$, $m_3 = m$,

bi) Tetrahedral case : $m_1 = 2$, $m_2 = m_3 = 3$,

bii) Cubic case : $m_1 = 2$, $m_2 = 4$, $m_3 = 3$,

biii) Icosahedral case : $m_1 = 2$, $m_2 = 3$, $m_3 = 5$.

In this case, all coverings may be described using finite groups of Möbius transformations.

Finite groups of Möbius transformations: The transformations

$$T(z)=\frac{az+b}{cz+d} \quad a,b,c,d\in\mathbb{C} \;:\; ad-bc\neq 0 \tag{1.11}$$

are known as Möbius transformations and they form a group under composition. Finite groups of Möbius transformations are:

ai) Cyclic group of rotations of order m by multiples of $\frac{2\pi}{m}$.

aii) Dihedral group of symmetries of order 2m of a regular m-gon.

bi) Tetrahedral group of 12 rotations carrying a regular tetrahedron to itself.

bii) The group of rotations of cube of order 24.

biii) The icosahedral group of 60 rotations of a regular icosahedron.

Extended complex plane $\mathbb{C}\bigcup\{\infty\}$ and sphere S^2 may be identified via steographic projection. Under this corresonpondance, finite groups of Möbius transformations correspond to finite group of rotations of sphere. They are in fact subgroups of finite Coxeter groups, as will be seen in chapter 4.

It turns out that when G is a finite group of Möbius transformations the map $\pi:\mathbb{P}^1\to\mathbb{P}^1/G$ is a ramified covering with equal ramification indices in each fiber, say m_i, m_i's satisfying (1.10). We get the following formula for Eisenstein number of coverings.

Theorem 1.2 *The Eisenstein number of ramified coverings $\pi : X \to \mathbb{P}^1$ of degree n, ramified over k points $y_1,\ldots,y_k$, with equal ramification indices m_i*

in each fiber $\pi^{-1}(y_i)$, m_i's satisfying (1.10), is given by

$$\sum_{\pi} \frac{1}{|Aut\pi|} = \frac{1}{(\frac{n}{|G|})!\ |G|^{\frac{n}{|G|}}} \tag{1.12}$$

where G is finite group of Möbius transformations corresponding to solution of (1.10).

Combining the above theorem with theorem (1.1) , we get the following .

Theorem 1.3 *The following equalities holds*

$$\sum_{\chi} \chi(\sigma_m)^2 = (\frac{n}{m})!\ m^{\frac{n}{m}} \tag{1.13}$$

$$\sum_{\chi} \frac{\chi(\sigma_2)^2\ \chi(\sigma_m)}{\chi(1)} = \frac{\left[(\frac{n}{2})!\ 2^{\frac{n}{2}}\right]^2\ (\frac{n}{m})!\ m^{\frac{n}{m}}}{n!\ (2m)^{\frac{n}{2m}}\ (\frac{n}{2m})!} \tag{1.14}$$

$$\sum_{\chi} \frac{\chi(\sigma_3)^2\ \chi(\sigma_2)}{\chi(1)} = \frac{\left[(\frac{n}{3})!\ 3^{\frac{n}{3}}\right]^2\ (\frac{n}{2})!\ 2^{\frac{n}{2}}}{n!\ (12)^{\frac{n}{12}}\ (\frac{n}{12})!} \tag{1.15}$$

$$\sum_{\chi} \frac{\chi(\sigma_2)\ \chi(\sigma_3)\chi(\sigma_4)}{\chi(1)} = \frac{(\frac{n}{2})!\ 2^{\frac{n}{2}}\ (\frac{n}{3})!\ 3^{\frac{n}{3}}\ (\frac{n}{4})!\ 4^{\frac{n}{4}}}{n!\ (24)^{\frac{n}{24}}\ (\frac{n}{24})!} \tag{1.16}$$

$$\sum_{\chi} \frac{\chi(\sigma_2)\ \chi(\sigma_3)\chi(\sigma_5)}{\chi(1)} = \frac{(\frac{n}{2})!\ 2^{\frac{n}{2}}\ (\frac{n}{3})!\ 3^{\frac{n}{3}}\ (\frac{n}{5})!\ 5^{\frac{n}{5}}}{n!\ (60)^{\frac{n}{60}}\ (\frac{n}{60})!} \tag{1.17}$$

where the summations are taken over all irreducible characters χ of S_n and σ_m denotes the permutation consisting of $\frac{n}{m}$ cycles of length m.

2.Parabolic Case

If

$$\frac{1}{m_1} + \frac{1}{m_2} + \ldots + \frac{1}{m_k} = k - 2 \tag{1.18}$$

the possible solutions are

a) $m_1 = m_2 = m_3 = m_4 = 2$,

bi) $m_1 = 2$, $m_2 = m_3 = 4$,

bii) $m_1 = 2$, $m_2 = 3$, $m_3 = 6$,

biii) $m_1 = m_2 = m_3 = 3$.

In this case, all coverings can be explicitely described in terms of affine Coxeter groups.

Affine Coxeter groups Affine Coxeter group G is generated by reflections in sides of a k-gon $\Delta \subset \mathbb{R}^2$. More generally, any k-gon with angles $\frac{\pi}{m_1}, \frac{\pi}{m_2}, \ldots, \frac{\pi}{m_k}$ satisfying $\sum \frac{1}{m_i} = k-2$, can be repeated by successive reflections in sides to cover the Euclidean plane. For m_i's satisfying (1.18), the corresponding affine Coxeter groups are as follows:

a) Group generated by reflections in sides of quadrangle (see figure (1.2)).

bi) Group generated by reflections in sides of triangle with angles $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}$ (see figure (1.3)).

bii) Group generated by reflections in sides of triangle with angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}$ (see figure (1.4).

biii) Group generated by reflections in sides of equilateral triangle (see figure (1.5).

Similar to elliptic case, we can evaluate Eisenstein number of coverings using affine Coxeter groups.

Theorem 1.4 *The Eisenstein number of ramified coverings $\pi : X \longrightarrow \mathbb{P}^1$ of degree $n\mu$, ramified over k points $y_1, \ldots, y_k$, with equal ramification indices m_i in each fiber $\pi^{-1}(y_i)$, m_i's satisfying (1.18), is given by*

$$\sum_{\pi} \frac{1}{|Aut\pi|} = \text{coefficient at } q^n \text{ in } [\prod_{k=1}^{\infty} (1-q^k)]^{\frac{-1}{\mu}} \tag{1.19}$$

where $\mu \in \mathbb{N}$ depends on the affine Coxeter group corresponding to the solution of (1.18) more explicitely, for m_i's satisfying the case

a) $\mu = 2$,

bi) $\mu = 4$,

bii) $\mu = 6$,

biii) $\mu = 3$.

Unexpectedly, we see that right side of the equality (1.19) contains a function close to Dedekind η function.

Dedekind η function

The function $\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n)$, $q = e^{2\pi i z}$ is called as Dedekind η

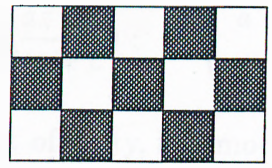

Figure 1.2:

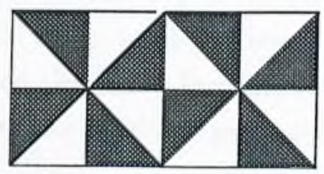

Figure 1.3:

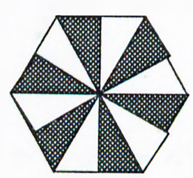

Figure 1.4:

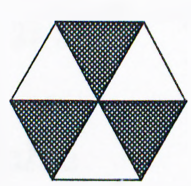

Figure 1.5:

function. $\eta(z)$ being holomorphic everywhere and verifying the relation

$$\eta(z) = \epsilon(cz+d)^{\frac{-1}{2}}\, \eta(\frac{az+b}{cz+d}) : \ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \epsilon \ SL_2(\mathbb{Z})$$

where $\epsilon(a,b,c,d)$ is a 24'th root of unity, is a modular form of weight $\frac{1}{2}$.

Relation with p(n)

The number p(n) of partitions of n is an important object in number theory.

$$\frac{1}{\prod(1-q^n)} = \sum p(n)\, q^n$$

In 1917, Hardy and Ramanujan developed a method which yields an asymptotic formula for p(n). After some modification of Hardy and Ramanujan's method, Rademacher obtained the exact formula for p(n). Proof of this exact formula is based essentially on the modular properties of the function $\frac{1}{\prod(1-q^n)}$.

H.Rademacher and H.Zuckerman, in their paper[8], have found the Fourier coefficients of the modular form $\eta(z)^{-2r}$. Using these, we get the next theorem.

Theorem 1.5 *The following asymptotic formulae holds.*

$$\sum_{\chi \epsilon S_{2n}} \frac{\chi(\sigma_2)^4}{\chi(1)^2} \sim \frac{\pi}{2^2\, 3^{\frac{3}{8}}}\, n^{\frac{1}{8}}\, \exp \pi\sqrt{\frac{n}{3}} \tag{1.20}$$

$$\sum_{\chi \epsilon S_{4n}} \frac{\chi(\sigma_2)\chi(\sigma_4)^2}{\chi(1)} \sim \frac{\pi}{2^{\frac{25}{16}}\, 3^{\frac{5}{16}}}\, n^{\frac{3}{16}}\, \exp \pi\sqrt{\frac{n}{6}} \tag{1.21}$$

$$\sum_{\chi \epsilon S_{6n}} \frac{\chi(\sigma_2)\chi(\sigma_3)\chi(\sigma_6)}{\chi(1)} \sim \frac{\pi}{2^{\frac{2}{3}}\, 3^{\frac{7}{12}}}\, n^{\frac{5}{24}}\, \exp \frac{\pi}{3}\sqrt{n} \tag{1.22}$$

$$\sum_{\chi \epsilon S_{3n}} \frac{\chi(\sigma_3)^3}{\chi(1)} \sim \frac{\pi}{2^{\frac{1}{2}}\, 3^{\frac{7}{6}}}\, n^{\frac{1}{6}}\, \exp(\frac{\pi}{3})\sqrt{2n} \tag{1.23}$$

1.4.2 Asymptotic formulae

In the previous item, we deduced results on characters using theorem (1.1) and known structure of ramified coverings. Now, let us consider the other direction, i.e. getting information on coverings from that of characters.

The problem of estimating the number of coverings with given ramification indices in some extend can be reduced to estimation of the ratio $\frac{\chi(g)}{\chi(1)^{\frac{1}{d}}}$, where d is mean value of cycle lengths of $g \in S_n$.To see this let us write Riemann-Hurwitz formula in the following form.

Let $\pi : X \to Y$ be a ramified covering of degree n, of surface Y of genus g_Y by surface X with genus g_X, ramified over k points $y_1, \ldots, y_k$ in Y. Riemann-Hurwitz formula connecting genus of X and Y may be written in the form:

$$2g_X - 2 = n\left[2g_Y - 2 + k - \sum_i \frac{1}{d_i}\right] \tag{1.24}$$

where $d_i = \frac{n}{\#(cycles\ in\ g_i)}$ is mean value of cycle lengths of g_i, for g_i:monodromy permutation with cycle lengths equal to ramification indices of points in fiber $\pi^{-1}(y_i)$.

Hence, using equality (1.7) and Riemann-Hurwitz formula we get

$$\sum_{\pi:X\to Y} \frac{1}{|Aut\pi|} = \frac{|C_1|\ldots|C_k|}{(n!)^{2-2g_Y}} \sum_\chi \frac{\chi(g_1)}{(\chi(1))^{\frac{1}{d_1}}} \cdots \frac{\chi(g_k)}{(\chi(1))^{\frac{1}{d_k}}} \frac{1}{\chi(1)^{\frac{2g_X-2}{n}}} \tag{1.25}$$

Since $\chi(1) < \sqrt{n!}\ \ \forall\ \chi$, $1 > \frac{1}{\chi(1)^{\frac{2g_X-2}{n}}} \geq \frac{c}{n^{g_X-1}}$ for some constant c. Hence, decreases polynomially for $g_X > 0$. On the other hand, $\frac{\chi(g)}{\chi(1)^{\frac{1}{d}}}$ grows exponentially, as will be seen in corollary (1.3). So, estimation of right side in above equality is mainly reduced to that of the ratio $\frac{\chi(g)}{\chi(1)^{\frac{1}{d}}}$.

Description of problem In our study, we considered the asymptotics of characters χ_β of S_n corresponding to Young diagrams β under the following conditions.

Let diagram β be given by $b_1 \geq b_2 \geq \ldots \geq b_m$ and the cycle structure of g given by $1^{a_1}2^{a_2}\ldots n^{a_n}$. Suppose that

i) Diagram β has fixed number of rows and g has fixed number of cycles.

ii) Number of cells in each row increases as $n \to \infty$ with fixed frequency, i.e. $\frac{b_i}{n} = \beta_i$, β_i is fixed as $n \to \infty$.

iii) Lengths of all cyles in $g \in S_n$ are coprime.

iv) Multiplicity of cycles in $g \in S_n$ increases as $n \to \infty$ with fixed frequency, i.e $\frac{a_k}{n} = \alpha_k$, α_k is fixed as $n \to \infty$.

It turns out that the estimation of the value of $\chi_\beta(g)$as $n \to \infty$ depends on the solution of the following system of non-linear algebraic equation.

$$\sum_k k\ \alpha_k \frac{x_i^k}{x_1^k + \ldots + x_m^k} = \beta_i \ ; \ i = 1, \ldots, m \tag{1.26}$$

where

$$\beta_i = \frac{b_i}{n},$$
$$\alpha_k = \frac{a_k}{n}.$$

We prove the followings:

Theorem 1.6 *The system (1.26) has, up to proportionality , unique positive solution* $x = (x_1, \ldots, x_m)$, $x_1 \geq x_2 \geq \ldots \geq x_m \geq 0$..

This theorem is crucial in proving the following asymptotic formulae.

Theorem 1.7 *Let us consider a sequence of diagrams* β *such that* $b_1 \geq b_2 \geq \ldots \geq b_m$, $\beta_i = \frac{b_i}{n}$ *fixed, and a sequence of permutations* $g \in S_n$ *with cycle structure* $1^{a_1}2^{a_2}\ldots n^{a_n}$ *such that* $\alpha_k = \frac{a_k}{n}$ *is fixed. If lengths of all cycles involved in* $g \epsilon S_n$ *are coprime, and* $\beta_i \neq \beta_j$, $i \neq j$, *asymptotics for* $\chi_\beta(g)$ *as* $n \to \infty$ *is given as:*

$$\chi_\beta(g) \sim \frac{e^{nw(x)}\sqrt{m}}{(2\pi n)^{\frac{m-1}{2}}} \prod_{i<j} (1 - \frac{x_j}{x_i}) \frac{1}{\sqrt{\sum_{i=1}^m H_{ii}}} \tag{1.27}$$

where x_i*'s are positive roots of the system (1.26).*

$$w(x) = \sum_k \alpha_k \log(x_1^k + \ldots + x_m^k) - \sum_{i=1}^m \beta_i \log x_i \tag{1.28}$$

and H_{ii} *is the principal minor of order* $m-1$ *of the quadratic form in variable* dt_i

$$Hess(w) = \sum_{k=1}^n k^2\ \alpha_k \left(\frac{\sum_{i=1}^m x_i^k dt_i^2}{\sum_i x_i^k} - (\frac{\sum_{i=1}^m x_i^k dt_i}{\sum_{i=1}^m x_i^k})^2 \right). \tag{1.29}$$

Taking $\alpha_i = 0$, $i > 0$, we get $x_i = \beta_i$, $w(x) = H(\beta)$, and get the following.

Corollary 1.2 *The asymptotics for the dimension of the irreducible representation corresponding to diagram β with different lengths of rows is*

$$\chi_\beta(1) \sim \frac{e^{nH(\beta)} \prod_{i<j}(1-\frac{\beta_j}{\beta_i})}{(2\pi n)^{\frac{m-1}{2}} \sqrt{\beta_1\beta_2\ldots\beta_m}} \tag{1.30}$$

where $H(\beta) = -\sum_i \beta_i log(\beta_i)$ is the entropy function.

We observe that the above asymptotic is no more valid if $x_i = x_j$ for some i,j. When diagram is rectangular, $x_i = x_j \; \forall \; i,j$. In this case, we evaluated the asymptotic for $\chi_\beta(g)$ using Selberg integral.

Theorem 1.8 *Under the assumptions of theorem (1.7), if lengths of cycles involved in g are coprime and the diagram β is rectangular , i.e. all rows are of the same length, then*

$$\chi_\beta(g) \sim \frac{m^{\frac{n}{d}}}{(2\pi)^{\frac{m-1}{2}}(n\,\bar{d})^{\frac{m^2-1}{2}}}\,(m)^{\frac{m^2}{2}} \prod_{j=1}^{m-1} j! \tag{1.31}$$

where $\sum_k \alpha_k = \frac{1}{d}$, $\sum_k k^2\alpha_k = \bar{d}$.

From theorem (1.7), it follows that main term in asymptotic is the expression $e^{nw(x)}$, hence it is essential to estimate $w(x)$.

Theorem 1.9 *Let x be the unique positive root of (1.26), $w(x)$ as in theorem (1.28) and β be diagram described as in theorem (1.7). Then*

$$w(x) \geq \frac{1}{d}\, H(\beta)$$

The equality is only if all cycles are of the same length or diagram is rectangular.

Corollary 1.3 *If the diagram β is not rectangular and if all cycles involved in g are of different length, then $\frac{\chi_\beta(g)}{\chi_\beta(1)^{\frac{1}{d}}}$ exponentially increases to ∞ as $n \to \infty$.*

In addition to these, in this thesis we proved the following theorem, which solve the problem proposed by Zalesskii.

Theorem 1.10 *If*
i) g_n *is any sequence of elements of* S_n *with fixed number of cycles,*
ii) $\chi_{\lambda(n)}$ *is any sequence of faithful characters of* S_n *labelled by partitions* $\lambda(n)$*,*
then

$$\frac{\chi_{\lambda(n)}(g_n)}{\chi_{\lambda(n)}(1)} \to 0 \quad as\ n \to \infty. \tag{1.32}$$

Chapter 2

Preliminaries

This chapter contains basic definitions and theorems needed for the rest of chapters.

An important part of the theory of functions of a complex variable is devoted to the study of algebraic functions. An analytic function $w = w(z)$ is called an algebraic function if it satisfies a functional equation

$$A(z,w) = a_0(z)w^n + a_1(z)w^{n-1} + ... + a_n(z) = 0, a_0(z) \neq 0 \tag{2.1}$$

in which the $a_i(z)$ are polynomials in z with complex numbers as coefficients. From this algebraic equation in w , we note that each value of z determines several values of w, so that w is a multiple-valued function of z.

Starting from a single function element of an algebraic function $w(z)$,we could use analytic continuation to piece together the whole function and study in this way its multiple-valuedness. Riemann's approach to this situation is to look for a new surface(instead of the z-plane) on which to consider the algebraic function defined, and on which it is an ordinary single-valued function. This surface is called a *Riemann surface.*

It can be shown that the Riemann surface for any algebraic function is topologically a sphere with g handles and the algebraic function is a single-valued function on this surface(For interested , refer [2]).

This number g is called as *genus* of the surface. The genus can be calculated by using polygonal subdivision.

Definition 2.1 *A polygonal subdivision $\mathcal{M}$ of a surface S consists of a finite* *set of points of $\mathcal{S}$, called vertices, and a finite set of simple points on $\mathcal{S}$, called edges, such that*

i) every edge has two end-points ,these points being vertices,

ii) edges can only intersect at their end-points,

iii) the union of edges is connected,

iv) the components of the complement $S \setminus M$ are homeomorphic to open discs. These components are called faces.

It can be shown that every compact, connected surface $\mathcal{S}$ has a polygonal subdivision. This was first proved (for Riemann surfaces) by T.Rado in 1925.

The *Euler characteristic* of a surface $\mathcal{S}$ is $\chi(S) = \chi(\mathcal{M}) = V - E + F$ where $\mathcal{M}$ is a polygonal subdivision of $\mathcal{S}$ with V vertices, E edges and F faces. Homeomorpic surfaces have the same Euler characteristic. [17]

Theorem 2.1 *The Euler characteristic of a compact, connected, orientable surface $\mathcal{S}$ of genus g is given by*

$\chi(S) = 2 - 2g.$

Now , we can introduce covering surfaces of Riemann surfaces.

Definition 2.2 *A continuous surjection $p : \tilde{S} \rightarrow S$,where S and $\tilde{S}$ are Riemann surfaces, is a ramified covering map of S if each $s \epsilon S$ has an open neighborhood U and a homeomorphism $\phi : U \rightarrow D$(open unit disc) such that for each connected component V of $p^{-1}(U)$ there is a homeomorphism $\psi : V \rightarrow D$ satisfying $\phi o p = \pi_n o \psi$ for some integer $n \geq 1$ $(\pi_n : D \rightarrow D, z \rightarrow z^n)$.*

We have $n = 1$ iff p is a homeomorphism $V \rightarrow U$, in this case p is called an unramified covering map.

If $n > 1$ for some V then we say that the unique element $\tilde{s}$ of $V \bigcap p^{-1}(s)$ is a branch point of order $n-1$ (Since p is like π_n, locally n-to-one near $\tilde{s}$). The points of S over which there exists branch points are called ramified points and the integer n is called ramification index of the ramified point.

In case when $\tilde{S}$ is simply connected, p is called a universal covering map.

Theorem 2.2 (Riemann-Hurwitz) *Let $p : \tilde{S} \rightarrow S$ be a ramified covering of degree n. The following formula is valid*

$$2(g_{\tilde{s}} - 1) = 2n(g_s - 1) + \sum_x (m_x - 1) \ , \qquad (2.2)$$

where the summation is taken over all ramified points x in $\tilde{S}$ with ramification index m_x.

We can classify Riemann surfaces according to their universal coverings.

Theorem 2.3 *Every compact Riemann surface has a universal covering.*

In case of genus 0 it is its own universal covering.

In case of genus 1, its universal covering is the complex plane $\mathbb{C}$.

In case of higher genus its universal covering is the upper halfplane (Lobachevsky plane).

For any surface R, we can select a point z on R and consider the class $C(z)$ of all closed curves from z. Identifying curves in $C(z)$ which are homotopic to each other and introducing a product on the homotopy classes, we can construct a group which, for the moment we denote by $\pi_1(R, z)$. It is easy to see that any two such groups $\pi_1(R, z)$ and $\pi_1(R, w)$ are isomorphic for R path connected. Hence we can refer to both as the *fundamental group* $\pi_1(R)$ of R. For a simply connected R, the group $\pi_1(R)$ is the trivial group since any closed curve from z is automatically homotopic to the point curve z.

Definition 2.3 *The degree of a covering space $(\tilde{X}, \pi)$ of X is the cardinal of a fiber. If the degree is m, one also says that $(\tilde{X}, \pi)$ is an m-sheeted covering of X, or an m-fold covering (It can be proved that all fibers in a covering space have the same cardinal) .*

Theorem 2.4 *Let (X, x_o) be a pointed space, let $(\tilde{X}, \pi)$ be a covering space of X, and let $Y = \pi^{-1}(x_o)$. Let the ordering of the points in the fiber over the base point x_o be : $\pi^{-1}(x_o) = z_1, z_2 ... z_n$. Path lifting defines a homomorphism (called the characteristic homomorphism for π):*

$$\chi(\pi) : \pi_1(X) \rightarrow S_n$$

of the fundamental group $\pi_1(X)$ into the symmetric group of n elements. Image of $\chi(\pi)$ is called monodromy group of $(\tilde{X}, \pi)$.

Main idea in this theorem is as follows: Given a loop $l : [0, 1] \rightarrow X$ in X based at $x_0 \in X$, i.e. $l(0) = l(1) = x_o$, there exists unique lifting of l to a path

$l_i : [0,1] \to \tilde{X}$ in $\tilde{X}$ with $l_i(0) = z_i$ $\forall\, i = 1,..n$ (So the fundamental group acts transitively on fibers). Since $l_i(1)$ is again a point in the fibre$\pi^{-1}(x_o)$, these liftings define a permutation $\tau \epsilon S_n$ such that $l_i(1) = z_{\tau(i)}$. τ depends only on the homotopy class of the loop l , and the assignement of τ to the homotopy class of l , defines a homomorphism $\chi(\pi)$. If we change the base point or change the ordering in the fibre over the base point, this will change $\chi(\pi)$ by a conjugation in S_n.

Now , let us define what is meant by equivalent(or isomorphic) coverings.

Definition 2.4 *Two covering spaces $(\tilde{Y}, q)$ and $(\tilde{X}, p)$ of a space X are equivalent if there is a homeomorphism $\varphi : \tilde{Y} \to \tilde{X}$ such that $q = \varphi p$.*

Theorem 2.5 *Two n-fold coverings are equivalent iff their characteristic homomorphisms are conjugate homomorphisms.*

2.1 Covering Transformations and Galois Correspondance For Coverings

Definition 2.5 *If $(\tilde{X}, \pi)$ is a covering space of X, then a covering transformation is a homeomorphism $h : \tilde{X} \to \tilde{X}$ with $\pi h = \pi$. Define $Aut(\pi)$ as the set of all covering transformations of $\tilde{X}$. Under composition of functions , $Aut(\pi)$ forms a group.*

By theorem (2.4), a covering $\pi : \tilde{X} \to X$ of degree n, gives an action of the fundamental group of X , on the set of n elements, i.e. on the general fibre $\pi^{-1}(x_o)$, $x_o \epsilon X$. The stabilizer π_* of a point $y \epsilon \pi^{-1}(x_o)$ is a subgroup of $\pi_1(X)$ and corresponding to another point y' in the fibre , a conjugate subgroup of π_* appears.

We have the following theorem about subgroups of $\pi_1(X)$ and coverings of X [for proofs, see [10]].

Theorem 2.6 *The following correspondances hold*

1) $\exists$ a one-to-one correspondance between conjugacy classes of subgroups H of $G = \pi_1(X)$ and equivalence classes of coverings π of X , where $deg\pi = [G :$

H].

2) Connected coverings corresponds to normal subgroups of $\pi_1(X)$.

3) $\exists$ *a one-to-one correspondance between coverings of X , of degree n, and the actions of* $\pi_1(X)$ *on n element set Y, where conjugate actions correponds to equivalent coverings by theorem (2.4). In case when the covering is connected, this action is transitive.*

4) $Aut\pi \cong Aut_G Y = \{\sigma : Y \to Y : \sigma g = g\sigma \ \forall\ g \in G\}$ *where G is the fundamental group of X. Otherwise stated*
$Aut\pi \cong C_{\{g_1,...,g_k\}}$ *where* $\{g_1,..,g_k\}$ *is the set of generators of* $\pi_1(X)$ *, and* $C_{\{g_1,...,g_k\}}$ *denotes its centralizer in* S_n.

We can summarize the Galois Correspondance considered in this theorem as below: ($\leftrightarrow$ denotes one-to-one correspondance)

$\pi : \tilde{X} \to X$ covering of degree n up to isomorphism $\leftrightarrow$ Subgroups H of $\pi_1(X)$ up to conjugacy $\leftrightarrow$ Actions of $\pi_1(X)$ on an n-element set.

Remark 2.1 *These notions are valid for non-ramified coverings but since removing ramified points leads to non-ramified coverings , we can use them in our study dealing with ramified coverings of sphere.*

Disconnected Coverings Now, let us consider the case when $\pi : \tilde{X} \to X$ is non-connected covering. i.e. $\tilde{X}$ is union of connected components X_i, $\tilde{X} = \bigcup_i X_i$, where the restriction $\pi|_{X_i} = \pi_i$ gives connected covering of X. By collecting the isomorphic components, we can write $\tilde{X} = \bigcup_j m_j X_j$, where X_j's are pairwise non-isomorphic components and

$$m_j X_j = \underbrace{X_j \bigcup X_j \bigcup \ldots \bigcup X_j}_{m_j \ times}$$

Symmetric group S_m acts on the isomorphic components mX_j in the following way : Labelling m isomorphic components X_j by $X_1, X_2, .., X_m$ for $\sigma \epsilon S_m$, $\sigma(X_1 \bigcup ... \bigcup X_m) = X_{\sigma(1)} \bigcup ... \bigcup X_{\sigma(m)}$. In this way , group of permutations of isomorphic components, namely $\prod S_{m_j}$ becomes a subgroup of $Aut\pi$. Moreover it can be shown that:

i) $\prod_i Aut\pi_i$ is a normal subgroup of $Aut\pi$.

ii)$\prod_i Aut\pi_i \bigcap \prod S_{m_j} = 1$.

Hence, $Aut\pi$ is semidirect product of $\prod_i Aut\pi_i$ and the group of permutations of isomorphic components. And

$$|Aut\pi| = \prod_{i=1}^{k} |Aut\pi_i| \prod_{j=1}^{l} m_j! \tag{2.3}$$

where k is the number of connected components, m_j is the number of isomorphic components, $\sum_{j=1}^{l} m_j = k$.

Chapter 3

Connection between coverings and characters

In this chapter we will prove the following theorem which gives the connection between coverings and characters of S_n.

Theorem 3.1 *The Eisenstein number of ramified coverings $\pi : X \to \mathbb{P}^1$ of degree n ramified over k points $y_1, \ldots, y_k$ in $\mathbb{P}^1$ with given ramification indices is given by*

$$\sum_{\pi:X\to\mathbb{P}^1} \frac{1}{|Aut\pi|} = \frac{|C_1||C_2|\ldots|C_k|}{(n!)^2} \sum_{\chi} \frac{\chi(g_1)\chi(g_2)\ldots\chi(g_k)}{(\chi(1))^{k-2}} \tag{3.1}$$

where the summation is over all irreducible characters χ of S_n, $g_i \in C_i \subset S_n$ are elements from fixed conjugacy classes C_i and cycle lenghts of g_i are ramification indices in fiber $\pi^{-1}(y_i)$.

This connection is due to two classical results of Hurwitz and Burnside.

3.1 Hurwitz interpretation of solutions

For a better understanding of Hurwitz interpretation of solutions, let us first consider what is really meant by the fundamental group of a surface R of genus g , by giving some examples.

The torus, and manifold of genus g can be described as in figures (3.1) and (3.2). One can calculate the fundamental group from these polygons. The fundamental group $\pi(R)$ is generated by the loops $a_1, ..., a_g, b_1, .., b_g$ with the

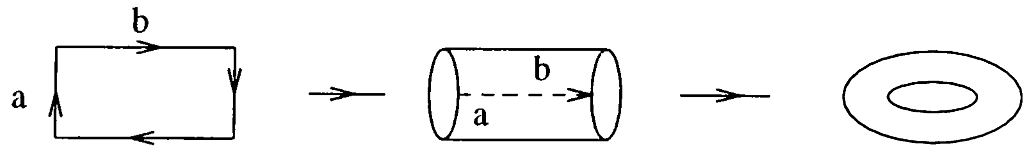

Figure 3.1: torus

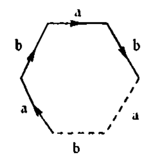

Figure 3.2: manifold of genus g

relation $\prod_i a_i b_i a_i^{-1} b_i^{-1} = 1$ i.e.

$$\pi(R) = \{a_1, a_2, ..., a_g, b_1, ..., b_g : [a_1 b_1]...[a_g b_g] = 1\} , \qquad (3.2)$$

where $[a_i b_i] = a_i b_i a_i^{-1} b_i^{-1}]$ is a commutator.
The fundamental group of surface Y with the points $y_1, y_2, ..., y_k$ removed is given by

$$\pi_1(Y) = \{a_1, .., a_g, b_1, .., b_g, c_1, .., c_k; [a_1 b_1][a_2 b_2]...[a_g b_g] c_1 ... c_k = 1\} . \qquad (3.3)$$

Where c_i's corresponds to loops aroud the removed points.
If Y is $\mathbb{P}^1$: Riemann sphere with the points $y_1, y_2, ..., y_k$ removed, its fundamental group is given by

$$\pi_1(Y) = \{c_1, .., c_k; c_1 ... c_k = 1\} . \qquad (3.4)$$

Remark: When $\pi : X \to Y$ is a ramified covering map of degree n ramified over k points, lifting c_i leads to monodromy permutation $g_i \in S_n$, lifting $2g$ non-contractible cycles $a_1, b_1, .., a_g, b_g$ leads to permutations $f_1, h_1, .., f_g, h_g \in S_n$.
We can now state Hurwitz interpretation of solutions.

Theorem 3.2 (Hurwitz) $\exists$ *a one-to-one correspondance between the solutions of the equation*

$$g_1 g_2 \dots g_k \ = \ 1 \ : \ g_i \ \in \ C_i \ \subset \ S_n \qquad (3.5)$$

up to conjugacy, and ramified coverings $\pi : X \to \mathbb{P}^1$ of degree n up to isomorphism, ramified over k points $y_1, \dots, y_k$ with prescribed ramification indices. Cycle lengths of g_i are ramification indices in fiber $\pi^{-1}(y_i)$.

Let us outline the proof.

Proof:

i) Let $S = \mathbb{P}^1 \backslash \{y_1, \ldots, y_k\}$, $\widetilde{\pi}$: $X \backslash \pi^{-1}(y_1, \ldots, y_k) \rightarrow S$. Then, $\pi_1(S) = \{c_1, c_2, \ldots, c_k : c_1 c_2 \ldots c_k = 1\}$. It is easily seen that there exists a one-to-one correspondance between solutions of the equation

$$g_1 g_2 \ldots g_k = 1 : g_i \in S_n \tag{3.6}$$

and action of $\pi_1(S)$ on an n-element set, where conjugate actions corresponds to conjugate solutions. Hence by theorem (2.6), $\exists$ a one-to-one correspondance between coverings $\pi : X \rightarrow \mathbb{P}^1$ of degree n up to isomorphism and solutions of above equation up to conjugacy.

ii) The set $g_1, \ldots, g_k$ are monodromy permutations due to liftings of loops generating $\pi_1(S)$. By definition, at each branch point in X of ramification index m, ramified covering π looks locally like $\pi_m : z \rightarrow z^m$. Hence, monodromy cyclically permutes $z^{\frac{1}{m}}$ to $e^{\frac{2\pi i}{m}} z^{\frac{1}{m}}$. Therefore, cycle lengths of g_i are ramification indices in fiber $\pi^{-1}(y_i)$.

Combining i and ii proves the theorem.

The following theorem may be proved in much the same way as theorem(3.2).

Theorem 3.3 (Generalized Hurwitz theorem) $\exists$ *a one-to-one correspondance between the solutions of the equation*

$$g_1 g_2 \ldots g_k [f_1, f_2] \ldots [f_g, h_g] = 1 : g_i \in C_i \in S_n,\ f_i, g_i \in S_n \tag{3.7}$$

up to conjugacy and coverings $\pi : X \rightarrow Y$ up to isomorphism, where π is as described in above theorem, with $\mathbb{P}^1$ replaced by an arbitrary surface Y of genus g.

3.2 Burnside's interpretation of solutions

Burnside theorem gives the number of solutions of the equations (3.5) and (3.7) for an arbitrary group G in terms of characters as follows [9].

$$\#\{g_1 g_2 \ldots g_k = 1 : g_i \in C_i \subset G\} = \frac{|C_1||C_2| \ldots |C_k|}{|G|} \sum_{\chi} \frac{\chi(g_1)\chi(g_2) \ldots \chi(g_k)}{\chi(1)^{k-2}} \tag{3.8}$$

$$\#\{g_1g_2\dots g_k[f_1,h_1]\dots[f_g,h_g] = 1\ g_i \in C_i\ h_g, f_g \in S_n\} = \frac{|C_1||C_2|\dots|C_k|}{|G|^{1-2g}} \sum_{\chi} \frac{\chi(g_1)\chi(g_2)\dots\chi(g_k)}{\chi(1)^{k+2g-2}} \tag{3.9}$$

where the summation is over all irreducible characters χ of G, $g_i \in C_i$ are elements from fixed conjugacy classes C_i.

3.3 Main theorem

Combining Hurwitz's and Burnside's results, we get the following theorem, which is the starting point of our approach.

Theorem 3.4 *The Eisenstein number of ramified coverings $\pi : X \to \mathbb{P}^1$ of degree n ramified over k points $y_1, \dots, y_k$ in $\mathbb{P}^1$ with given ramification indices is given by*

$$\sum_{\pi:X\to\mathbb{P}^1} \frac{1}{|Aut\pi|} = \frac{|C_1||C_2|\dots|C_k|}{(n!)^2} \sum_{\chi} \frac{\chi(g_1)\chi(g_2)\dots\chi(g_k)}{(\chi(1))^{k-2}} \tag{3.10}$$

where the summation is over all irreducible characters χ of S_n, $g_i \in C_i$ are elements from fixed conjugacy classes C_i and cycle lenghts of g_i are ramification indices in fiber $\pi^{-1}(y_i)$.

Proof:
i) It suffices first to show

$$\#\{g_1g_2\dots g_k = 1 : g_i \in C_i \subset S_n\} = \sum_{\pi:X\to\mathbb{P}^1} \frac{n!}{|Aut\pi|}, \tag{3.11}$$

where the summation is taken over ramified coverings $\pi : X \to \mathbb{P}^1$ described as in the statement of theorem. By Hurwitz theorem, $\exists$ a one-to-one correspondance between solutions of $g_1g_2\dots g_k = 1$: $g_i \in C_i$ up to conjugacy and ramified coverings $\pi : X \to \mathbb{P}^1$ with given degree and ramification indices. Let $\{g_1, g_2, \dots, g_k\}$ be a solution of $g_1g_2\dots g_k = 1$: $g_i \in C_i$. We have

$$\#(\text{solutions conjugate to })\{g_1, g_2, \dots, g_k\} = [S_n : C_{\{g_1,g_2,\dots,g_k\}}], \tag{3.12}$$

where $C_{\{g_1,g_2,\dots,g_k\}}$ is the centralizer of the set $\{g_1, g_2, \dots, g_k\}$.
By theorem (2.6)

$$C_{\{g_1,g_2,\dots,g_k\}} \cong Aut\pi \tag{3.13}$$

and the result follows.
ii) Since by Hurwitz interpretation of solutions g_i's are monodromy permutations in fiber $\pi^{-1}(y_i)$, cycle lengths of g_i is equal to ramification indices of points in fiber. Combining (3.11) with equation (3.8) in Burnside theorem for $G = S_n$, implies the desired result.

Using similar ideas, the folowing theorem may be proved.

Theorem 3.5 *The Eisenstein number of ramified coverings $\pi : X \to Y$ of degree n with given ramification indices, of the surface Y of genus g_Y, ramified over k points $y_1, \ldots, y_k$ in Y is given by*

$$\sum_{\pi:X\to Y} \frac{1}{|Aut\pi|} = \frac{|C_1||C_2|\ldots|C_k|}{(n!)^{\chi(Y)}} \sum_{\chi} \frac{\chi(g_1)\chi(g_2)\ldots\chi(g_k)}{(\chi(1))^{k-\chi(Y)}} \tag{3.14}$$

where $\chi(Y) = 2 - 2g_Y$ is the Euler characteristic of the surface Y, $g_i \in C_i$, cycle lenghts of g_i are ramification indices in fiber $\pi^{-1}(y_i)$, and the summation is over all irreducible characters of S_n.

Chapter 4

Explicit Formulae

In this chapter we will give a detailed exposition of carrying information on coverings to that of characters of symmetric group and give our results.

Classification of ramified coverings with the same ramification indices.

Given a ramified covering $\pi : X \to Y$ with the same ramification indices in each fiber, we will determine the type of components of X using ramification indices.

We will use the following Riemann-Hurwitz formula.

Let $\pi : X \to Y$ be a ramified connected covering of degree n, of the surface Y of genus g_Y by surface X of genus g_X. Then

$$2(g_X - 1) \;=\; 2n(g_Y - 1) \;+\; \sum_{x \in X}(m_x - 1) \tag{4.1}$$

where the summation is over all $x \in X$ with ramification indices m_x.

In the case of $Y \;=\; \mathbb{P}^1$ and of equal ramification indices m_i in each fiber $\pi^{-1}(y_i)$, the Riemann-Hurwitz formula can be written in the following form:

$$\chi(X) \;=\; n\left[\sum_{i=1}^{k}\frac{1}{m_i} \;-\; (k-2)\right] , \tag{4.2}$$

for $\pi : X \to \mathbb{P}^1$, ramified over k points $y_1, \ldots, y_k$.

So, if all ramification indices in a fiber are equal, the above equality gives us a tool for determining the components of X. Namely, we have three cases.

i) Elliptic Case: If $\sum_{i=1}^{k}\frac{1}{m_i} \;>\; k-2$, then all components of X are Riemann

sphere.
ii) Parabolic Case : If $\sum_{i=1}^{k} \frac{1}{m_i} = k-2$, then all components of X are torus.
iii) Hyperbolic Case : If $\sum_{i=1}^{k} \frac{1}{m_i} < k-2$, then all components of X have genus greater than 1.
In cases i) and ii), we will get an explicit formulae for the number of such coverings. There exists no explicit formulae in hyperbolic case. Asymptotics of the number of coverings in hyperbolic case is closely connected with asymptotics of characters of symmetric group, and will be studied in the next chapter.

Let us first concentrate on elliptic case.

4.1 Elliptic case

In elliptic case we will deal with ramified coverings $\pi : X \to \mathbb{P}^1$ ramified over k points with equal ramification indices, say m_i, in each fiber and m_i's satisfying

$$\sum_{i=1}^{k} \frac{1}{m_i} > k-2 . \tag{4.3}$$

There exists finitely many solutions for m_i's, these are

ai) Cyclic case : k=2 , $m_1 = m_2 = m$,
aii) Dihedral case : k=3 ,$m_1 = m_2 = 2$, $m_3 = m$,
bi) Tetrahedral case : $m_1 = 2$, $m_2 = m_3 = 3$,
bii) Cubic case : $m_1 = 2$, $m_2 = 4$, $m_3 = 3$,
biii) Icosahedral case : $m_1 = 2$, $m_2 = 3$, $m_3 = 5$.

4.2 Description of coverings in terms of triangulations

We will explicitely describe ramified coverings $\pi : \mathbb{P}^1 \to \mathbb{P}^1$ in terms of triangulations of $\mathbb{P}^1$.

Definition 4.1 *A bicolored triangulation on a surface is the decomposition of the surface into triangles such that each edge has a neighborhood colored black and white.*

Remark 4.1 *Since each edge has a neighbourhood colored black and white, just three indices can be used to label each vertex of triangles.*

Bicolored triangulation of a surface X defines a ramified covering of $\mathbb{P}^1$.

Example 1: Let $\sum$ be a bicolored triangulation on X, with vertices of triangles labelled by a, b and c. Let $\pi : X \rightarrow \mathbb{P}^1$ sending

i) black triangles to north hemisphere,

ii) white triangles to south hemisphere,

iii) vertices a,b, and c to 0,1 and ∞ respectively.

Then, π is a ramified covering map ramified over 0,1 and ∞. Ramification points in X are the vertices of triangles in $\sum$ with ramification index at a vertex equal to $\frac{1}{2}$ (# of triangles meeting at the vertex).

Proposition 4.1 *Let $\pi : X \rightarrow \mathbb{P}^1$ be a ramified covering of degree n, ramified over three points a,b,c $\in \mathbb{P}^1$. Then π induces a bicolored triangulation $\sum$ on X. It has the following properties:*

i) #(triangles in)$\sum$ = 2 degπ,

ii)#(triangles meeting at a vertex j) = $2m_j$, m_j : ramification index of j.

Proof: Joining the points a,b and c partition $\mathbb{P}^1$ consisting of two triangles, triangle abc(colored black) and its complement (colored white). Topologically, it can be assumed that a,b,c are on the equator of $\mathbb{P}^1$ hence dividing the sphere into two hemispheres north (colored black) and south (colored white) hemispheres. π is continous, hence $\pi^{-1}(abc)$ is simply connected. Labelling preimages of vertices a,b,c by the same letters, a bicolored triangulation of X is obtained.

i) Let $z_o \in \mathbb{P}^1$: unramified point inside triangle abc. deg π = n implies $\pi^{-1}(z_o)$ lies in n triangles in X colored black. Similarly, for $z_1 \in \mathbb{P}^1$ unramified point in the complement of abc $\pi^{-1}(z_1)$ lies in n black triangles. Hence giving 2n triangles in X, n of them are inverse images of north, n of them of south hemisphere.

ii) Follows from the fact that at a point with ramification index m_j, π looks locally like $\pi_j \; : \; z \rightarrow z^{m_j}$. Combining the above example and above proposition we get the following theorem.

Theorem 4.1 *There exists a one-to-one correspondance between bicolored triangulations $\sum$ on surface X and ramified coverings $\pi : X \rightarrow \mathbb{P}^1$ with given degree and ramification indices, ramified over 3 points, such that*

i) #(triangles in)$\sum$ = 2 degπ ,

ii)#(triangles meeting at a vertex j) = $2m_j$, m_j : ramification index of j.

As we will see, in elliptic case all coverings can be explicitely described using finite groups of Möbius transformations.

Finite groups of Möbius transformations: As explained in chapter 1, finite groups of Möbius transformations are:

ai) Cyclic group of rotations of order m by multiples of $\frac{2\pi}{m}$.

aii) Group of rotations of regular m-gon (dihedral group) of order 2m.

bi) Tetrahedral group of 12 rotations carrying a regular tetrahedron to itself.

bii) The group of rotations of cube of order 24.

biii) The icosahedral group of 60 rotations of a regular icosahedron.

and they correspond to finite groups of rotations of sphere [for details,see 7]. In fact, these are highly related with finite Coxeter groups.

Finite Coxeter groups are generated by reflections in planes $\Delta \in \mathbb{R}^3$ all passing through the origin. Finite groups of Möbius transformations correspond to subgroups of finite Coxeter group of index 2. More explicitely, each group listed above is the subgroup of

ai) Finite Coxeter group of order 2m generated by reflections in planes of symmetry of a regular m-gon.

aii) Finite Coxeter group of order 4m generated by reflections in planes of symmetry of dihedron.

bi) Finite Coxeter group of order 24 generated by reflections in planes of symmetry of regular tetrahedron.

bii) Finite Coxeter group of order 48 generated by reflections in planes of symmetry of cube.

biii) Finite Coxeter group of order 120 generated by reflections in planes of symmetry of regular icosahedron.

Regular polytopes Polytopes are geometrical figures bounded by portions of lines, planes or hyperplanes. In two dimensional geometry, they are known as polygones and comprise figures as triangles, squares e.t.c. In three dimensional geometry, they are known as polyhedra and include figures as tetrahedra, cubes e.t.c.

Remark A plane p-gon is said to be regular if it is both equilateral and equiangular, and denoted by $\{p\}$. A polyhedron is said to be regular if its faces are regular and equal, while its vertices are all surrounded alike. If its faces are $\{p\}$'s, q surrounding each vertex, the polyhedron is denoted by $\{p, q\}$.

There are 5 regular polyhedra:

1){3, 3} Tetrahedron,
2){3, 4} Octahedron,
3){4, 3} Cube,
4){3, 5} Icosahedron,
5){5, 3} Dodecahedron.

Rotation groups of regular polyhedra Two reciprocal polyhedra $\{p, q\}$ and $\{q, p\}$ have the same rotation group. The center of $\{p, q\}$ is joined to vertices, mid-edge points and centers of faces and rotations of polyhedron consists of rotations through angles $\frac{2k\pi}{q}$, π, $\frac{2j\pi}{p}$, about these respective lines [For details, see 18].

The following example is crucial for describing ramified coverings $\pi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ using finite groups of Möbius transformations.

Example 2: Let G be a finite group of Möbius transformations. Consider the orbit space $\mathbb{P}^1/G$ and the natural projection $\pi : \mathbb{P}^1 \rightarrow \mathbb{P}^1/G$. π is given by $\pi(z) = [z]$ where $[z]$ denotes the G-orbit of z. In fact, the quotient space has genus 0 and hence is just Riemann sphere due to Riemann-Hurwitz formula. Hence, π is a covering of sphere by sphere, with the following properties

i) $\deg \pi = |G|$,

ii) $z \in \mathbb{P}^1$ is ramified iff $|[z]| < |G|$ iff $C_z = \{g \in G : gz = z\} \neq 1$. Hence, ramification index of z is equal to $|C_z|$.

iii) If G is one of rotation groups of regular polyhedra, ramification points are f: center of faces, e: mid-edge points and v: vertices. π is ramified over three points a,b,c , in $\mathbb{P}^1$, $\pi^{-1}(a) = f, \pi^{-1}(b) = e, \pi^{-1}(c) = v$, with equal ramification indices m_1, m_2, m_3 in each fiber. $m_1 = |C_{face}|, m_2 = |C_{vertex}|, m_3 = |C_{edge}|$.

Using rotation groups of regular polyhedra, we can give the following table.

G	$\lvert G\rvert$	$\lvert C_{face}\rvert$	$\lvert C_{vertex}\rvert$	$\lvert C_{edge}\rvert$
tetrahedral	12	3	3	2
cube	24	3	4	2
icosahedral	60	5	3	2

iv) If G is cyclic group of rotations by angle $\frac{2\pi}{m}$ about a line, π is ramified over two points with equal ramification indices $m_1 = m_2 = m$ in each fiber. v) If G is rotation group of order 2m of dihedron, π is ramified over three points a,b, and c, with equal ramification indices in each fiber. $\pi^{-1}(a) = m$ summits, $\pi^{-1}(b) =$ mid-edge points, $\pi^{-1}(c) =$ *poles* of dihedron and ramification indices are : $m_1 = m_2 = 2$, $m_3 = m$ respectively.

The corresponding bicolored triangulation for example 2: By proposition (4.1) for $\pi : \mathbb{P}^1 \to \mathbb{P}^1/G$ with $deg\pi = |G|$ each finite group G of Möbius transformations partition the sphere into $2|G|$ symmetric triangles which meet in sets of $2m_j$ at vertices with ramification index m_j. The corresponding triangulation for each G is as follows:

i) G:cyclic group of order m.

$2m$ lunes.

ii) G:dihedral group of order 2m.

Decomposition of sphere into $4m$ triangles. 4, 4 and 2n triangles meeting at each vertex respectively.

iii) G:tetrahedral group.

Decomposition of sphere into 24 triangles. 4, 6 and 6 triangles meeting at each vertex respectively.

iv) G:Group of rotations of cube.

Decomposition of sphere into 48 triangles. 4, 6 and 8 triangles meeting at each vertex respectively.

v) G:icosahedral group.

Decomposition of sphere into 120 triangles. 4, 6 and 10 triangles meeting at each vertex respectively.

Theorem 4.2 *Let $\pi : \mathbb{P}^1 \to \mathbb{P}^1$ be a ramified covering ramified over $y_1, \ldots, y_k$ with equal ramification indices m_i in each fiber $\pi^{-1}(y_i)$, m_i's satisfying (4.3). Then π is isomorphic to $\pi_G : \mathbb{P}^1 \to \mathbb{P}^1/G$, i.e. factorization by one of finite group of Möbius transformations.*

Proof: By theorem (4.1) it suffices to show that π and π_G induce the same triangulation on $\mathbb{P}^1$. Degree of π = n may be found by Riemann-Hurwitz formula

$$2 = n \left[\sum_{i=1}^{k} \frac{1}{m_i} - (k-2) \right] .$$

Triangulation correponding to π_G for each G was analyzed in example 2. If we investigate each case corresponding to possible values of m_i's, we observe that the bicolored triangulation corresponding to π is the same as that of π_G, for some finite group of Möbius transformations. Namely,

i) If $m_1 = m_2 = m$, $deg\, \pi = m$, $\pi \sim \pi_G$ for G: cyclic group.

ii) If $m_1 = m_2 = 2$, $m_3 = m$, $deg\pi = 2m$, $\pi \sim \pi_G$ for G: dihedral group.

iii) If $m_1 = 2, m_2 = 3, m_3 = 3$, $deg\ \pi = 12$, $\pi \sim \pi_G$ for G: tetrahedral group.
iv) If $m_1 = 2, m_2 = 3, m_3 = 4$, $deg\ \pi = 24$, $\pi \sim \pi_G$ for G: group of rotations of cube.
v) If $m_1 = 2, m_2 = 3, m_3 = 5$, $deg\ \pi = 60$, $\pi \sim \pi_G$ for G: icosahedral group.

Proposition 4.2 *Let $\pi : \mathbb{P}^1 \to \mathbb{P}^1/G$ be a connected covering map, G: finite group of Möbius transformation. Then, $Aut\pi = G$.*

Proof: Clearly $G \subset Aut\pi$. Conversely, let $h \in Aut\pi$, $h \neq identity$. Then h has no fixed points. i.e. $hz_1 = z_2$ for z_1, z_2 in the same fiber $\pi^{-1}(z)$. G acts transitively on fibers, so $\exists g \in G$ such that $gz_1 = z_2$. We have

$$gz_1 = z_2 \Rightarrow hg^{-1}(z_2) = z_2 . \tag{4.4}$$

Since $hg^{-1} \in Aut\pi$ and hence has no fixed points unless it is identity map , last equality implies that $h = g$. So $Aut\pi \subset G$. And the result follows.

As we have seen in theorem (4.2), all connected coverings of $\mathbb{P}^1$ ramified over k points $y_1, \ldots, y_k$ with equal ramification indices, m_i, in a fiber $\pi^{-1}(y_i)$, m_i's satifying

$$\sum_{i=1}^{k} \frac{1}{m_i} > k-2$$

are isomorphic. Hence, we can give the following theorem for non-connected coverings $\pi : X \to \mathbb{P}^1$.

Theorem 4.3 *The Eisenstein number of ramified coverings $\pi : X \to \mathbb{P}^1$ of degree n, ramified over k points $y_1, \ldots, y_k$, with equal ramification indices m_i in each fiber $\pi^{-1}(y_i)$, m_i's satisfying*

$$\sum_{i=1}^{k} \frac{1}{m_i} > k-2 ,$$

is given by

$$\sum_{\pi} \frac{1}{|Aut\pi|} = \frac{1}{(\frac{n}{|G|})!\ |G|^{\frac{n}{|G|}}} , \tag{4.5}$$

where G: finite group of Möbius transformation corresponding to solution of m_i's.

Proof: We have $X = \bigcup_i X_i$. $X_i \sim \mathbb{P}^1$, and $\forall i\ \pi|_{X_i} = \pi_i$ a connected covering of sphere ramified over k points $y_1, \ldots, y_k$, with equal ramification indices m_i in each fiber $\pi^{-1}(y_i)$, m_i's satisfying

$$\sum_{i=1}^{k} \frac{1}{m_i} > k - 2 .$$

By theorem (4.2) $\pi_i \sim \pi_G$ for G : finite group of Möbius transformations determined by solutions for m_i's. Hence, each connected component is isomorphic. Since by example 2, $deg\pi_i = |G| \forall i$, we have

$$m = \frac{n}{|G|} , \tag{4.6}$$

where m is the number of connected components in X. And, by proposition (4.2)

$$|Aut\pi_i| = |G| . \tag{4.7}$$

Using (2.3) in chapter 2 with $k = \frac{n}{|G|}$, $l = 1$ we get

$$|Aut(\pi)| = (\frac{n}{|G|})!\ |G|^{\frac{n}{|G|}} , \tag{4.8}$$

and the result follows.

Combining the above theorem with main theorem in chapter 3 , we get the following.

Theorem 4.4 *The following equalities holds*

$$\sum_{\chi} \chi(\sigma_m)^2 = (\frac{n}{m})!\ m^{\frac{n}{m}} \tag{4.9}$$

$$\sum_{\chi} \frac{\chi(\sigma_2)^2\ \chi(\sigma_m)}{\chi(1)} = \frac{\left[(\frac{n}{2})!\ 2^{\frac{n}{2}}\right]^2\ (\frac{n}{m})!\ m^{\frac{n}{m}}}{n!\ (2m)^{\frac{n}{2m}}\ (\frac{n}{2m})!} \tag{4.10}$$

$$\sum_{\chi} \frac{\chi(\sigma_3)^2\ \chi(\sigma_2)}{\chi(1)} = \frac{\left[(\frac{n}{3})!\ 3^{\frac{n}{3}}\right]^2\ (\frac{n}{2})!\ 2^{\frac{n}{2}}}{n!\ (12)^{\frac{n}{12}}\ (\frac{n}{12})!} \tag{4.11}$$

$$\sum_{\chi} \frac{\chi(\sigma_2)\ \chi(\sigma_3)\chi(\sigma_4)}{\chi(1)} = \frac{(\frac{n}{2})!\ 2^{\frac{n}{2}}\ (\frac{n}{3})!\ 3^{\frac{n}{3}}\ (\frac{n}{4})!\ 4^{\frac{n}{4}}}{n!\ (24)^{\frac{n}{24}}\ (\frac{n}{24})!} \tag{4.12}$$

$$\sum_{\chi} \frac{\chi(\sigma_2)\,\chi(\sigma_3)\chi(\sigma_5)}{\chi(1)} = \frac{(\frac{n}{2})!\,2^{\frac{n}{2}}\,(\frac{n}{3})!\,3^{\frac{n}{3}}\,(\frac{n}{5})!\,5^{\frac{n}{5}}}{n!\,(60)^{\frac{n}{60}}\,(\frac{n}{60})!} \tag{4.13}$$

where the summations are taken over all irreducible characters χ of S_n and σ_m denotes the permutation consisting of $\frac{n}{m}$ cycles of length m in conjugacy class $C_i \subset S_n$.

Proof: Each equation can be proved similarly. Let us prove one of them.
Proof of (4.12):
$m_1 = 2$, $m_2 = 3$, $m_3 = 4$ case.
i) By theorem (4.3) and theorem (4.2)

$$\sum_{\pi} \frac{1}{|Aut\pi|} = \frac{1}{(\frac{n}{24})!\,24^{\frac{n}{24}}}, \tag{4.14}$$

where the summation is over all ramified coverings $\pi : X \to \mathbb{P}^1$ of degree n, ramified over 3 points y_1, y_2, y_3 with equal ramification indices, m_i, in each fiber $\pi^{-1}(y_i)$, $m_1 = 2$, $m_2 = 3$, $m_3 = 4$. From theorem (4.2) G is the group of rotations of cube, $|G| = 24$ and combining with theorem (4.3) statement follows.
ii) By theorem (3.4) in chapter 3

$$\sum_{\pi} \frac{1}{|Aut\pi|} = \frac{|C_1||C_2||C_3|}{(n!)^2} \sum_{\chi} \frac{\chi(\sigma_2)\chi(\sigma_3)\chi(\sigma_4)}{\chi(1)}, \tag{4.15}$$

where the summation is over all coverings $\pi : X \to \mathbb{P}^1$ as described in i) and

$$|C_1| = \frac{n!}{(\frac{n}{2})!\,2^{\frac{n}{2}}}, \tag{4.16}$$

$$|C_2| = \frac{n!}{(\frac{n}{3})!\,3^{\frac{n}{3}}}, \tag{4.17}$$

$$|C_3| = \frac{n!}{(\frac{n}{4})!\,4^{\frac{n}{4}}}. \tag{4.18}$$

(4.19)

The result follows combining i) and ii).

Now, let us consider Parabolic case.

4.3 Parabolic case

In parabolic case, we will deal with ramified coverings $\pi : X \to \mathbb{P}^1$ ramified over k points $y_1, \ldots, y_k$, with equal ramification indices m_i in each fiber $\pi^{-1}(y_i)$, and m_i's satisfying

$$\sum_{i=1}^{k} \frac{1}{m_i} = k - 2 \,. \tag{4.20}$$

Hence, by Riemann-Hurwitz formula, X turns out to be union of tori. The possible solutions for m_i's are:

a) $m_1 = m_2 = m_3 = m_4 = 2$,

bi) $m_1 = 2 \,,\ m_2 = m_3 = 4$,

bii) $m_1 = 2 \,,\ m_2 = 3 \,,\ m_3 = 6$,

biii) $m_1 = m_2 = m_3 = 3$.

Main difference between elliptic and parabolic case is the following:
Let $\pi : X \to \mathbb{P}^1$ be a ramified connected covering of degree n, ramified over k points $y_1, \ldots, y_k$ with equal ramification indices m_i in each fiber $\pi^{-1}(y_i)$. In elliptic case, $\sum_{i=1}^{k} \frac{1}{m_i} > k-2$, hence degree π can be found by Riemann-Hurwitz formula

$$2 - 2g_X = n \left[\sum_{i=1}^{k} \frac{1}{m_i} - (k-2) \right] \,.$$

Whereas in parabolic case two sides of the equation vanishes. So, it is not possible to find degree π using this formula.

4.3.1 Description of connected coverings

We have seen that Galois correspondance for coverings gives a one-to-one correspondance between coverings $\pi : \tilde{X} \to (\mathbb{P}^1 \backslash \{y_1, \ldots, y_k)$ and subgroups $H \subset \pi_1(\mathbb{P}^1 \backslash \{y_1, \ldots, y_k)$ of index n. If the ramification in fibers $\pi^{-1}(y_i)$ are equal to m_i, it turns out that there exists a one-to-one correspondance between such coverings and subgroups of index n of the group with generators $\delta_1, \ldots, \delta_k$ and relations

$$\delta_1{}^{m_1} = \delta_2{}^{m_3} = \ldots = \delta_k{}^{m_k} = \delta_1 \delta_2 \ldots \delta_k = 1 \,.$$

Denote this group by $\Gamma^+(m_1, \ldots, m_k)$. We will see in the next section that Γ^+ is a subgroup of index 2 in affine Coxeter group $\Gamma(m_1, \ldots, m_k)$. So, to understand the structure of coverings, we have to focus our attention to affine Coxeter groups.

Affine Coxeter groups

Affine Coxeter groups are generated by reflections in sides of a k-gon $\Delta \in \mathbb{R}^2$. More generally, any k-gon with angles $\frac{\pi}{m_1}, \ldots, \frac{\pi}{m_k}$ satisfying $\sum_{i=1}^{k} \frac{1}{m_i} = k-2$ can be repeated, by successive reflections in sides to cover the Euclidean plane. Affine Coxeter group $\Gamma(m_1, \ldots, m_k)$ can be described as an abstract group with generetors $R_1, R_2, \ldots, R_k$ satisfying

$$R_1{}^2 = R_2{}^2 = \ldots = R_k{}^2 = (R_1R_2)^{m_1} = (R_2R_3)^{m_2} = \ldots = (R_kR_1)^{m_k} = 1.$$

where R_i is the reflection in the i'th side of the polygon. The elements of the group either preserves or reverses orientation according as the number of reflections in the product is even or odd. There is a subgroup of index 2 consisting of rotations and translations alone, these being the only orientation-preserving transformations. We call this subgroup rotation subgroup of affine Coxeter group.

Rotation subgroup $\Gamma^+(m_1, \ldots, m_k)$ of affine Coxeter group $\Gamma(m_1, \ldots, m_k)$ can be described as an abstract group with generators $\delta_1, \ldots, \delta_k$ satisfying

$$\delta_1{}^{m_1} = \delta_2{}^{m_2} = \ldots = \delta_k{}^{m_k} = \delta_1\delta_2\ldots\delta_k = 1 \ , \qquad (4.21)$$

where $\delta_i = R_iR_{i+1}$. Hence, δ_i correspond to rotation around the corresponding vertex of polygon by angle $\frac{2\pi}{m_i}$.

In chapter1 (1.4.2), the affine Coxeter groups corresponding to solutions of $\sum_{i=1}^{k} \frac{1}{m_i} = k-2$ are given. In the corresponding figures, regions with the same color represent orbits of regions under the action of rotation subgroup of affine Coxeter group.

Index $[\Gamma^+ : L]$

The translation group L is generated by two translations and the transforms of any point by such a group make a 2 dimensional lattice. For the following part of the section, the index $[\Gamma^+ : L]$, where L is the translation subgroup of Γ^+ will be of use. So, let us find these indices for each affine Coxeter group as listed in chapter1 (1.4.2).

$[\Gamma : \Gamma^+][\Gamma^+ : L] = [\Gamma : L]$. Since $[\Gamma : \Gamma^+] = 2$, $2[\Gamma^+ : L] = [\Gamma : L]$. $[\Gamma : L]$ can be found by comparing number of fundamental polygons for Γ and L. The fundamental polygons for Γ and L corresponding to each Coxeter group Γ given in chapter 1, are given in figures (4.1)-(4.4). First figure shows fundamental polygon for Γ and second that of L.

Hence, we have

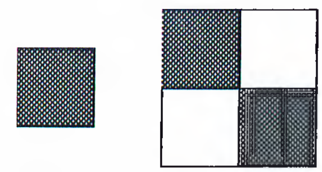

Figure 4.1:

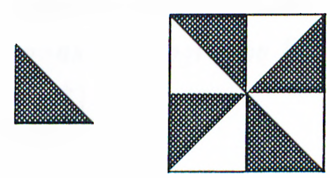

Figure 4.2:

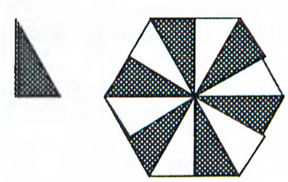

Figure 4.3:

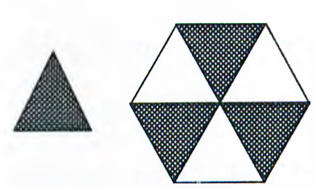

Figure 4.4:

Table for index $[\Gamma^+ : L]$

a) $[\Gamma : L] = 4, [\Gamma^+ : L] = 2$ for $\Gamma(2,2,2,2)$: group generated by reflections in sides of quadrangle.

bi) $[\Gamma : L] = 8, [\Gamma^+ : L] = 4$ for $\Gamma(2,4,4)$: group generated by reflections in sides of triangle with angles $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}$

bii) $[\Gamma : L] = 12, [\Gamma^+ : L] = 6$ for $\Gamma(2,3,6)$: group generated by reflections in sides of triangle with angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}$

biii) $[\Gamma : L] = 6, [\Gamma^+ : L] = 3$ for $\Gamma(3,3,3)$: group generated by reflections in sides of equilateral triangle.

Proposition 4.3 *Connected coverings* $\pi : X \to \mathbb{P}^1$ *of degree* n*, ramified over* k *points* $y_1, \ldots, y_k$ *in* $\mathbb{P}^1$ *with equal ramification indices* m_i *in each fiber* $\pi^{-1}(y_i)$*,* m_i*'s satisfying (4.20), corresponds to subgroups* Γ_1^+ *of translation group* $L \subset \Gamma^+(m_1, \ldots, m_k)$ *such that* $[\Gamma^+ : \Gamma_1^+] = n$.

Proof: By Galois correspondance for coverings, $\exists$ a one-to-one correspondance between such coverings $\pi : X \to \mathbb{P}^1$ and subgroups Γ_1^+ of $\Gamma^+(m_1, \ldots, m_k)$ such that $[\Gamma^+ : \Gamma_1^+] = n$. Let

$$G = \{g_1, \ldots, g_k \; : \; g_1 \ldots g_k = g_1^{m_1} = \ldots = g_k^{m_k} = 1, g_i \in C_i \subset S_n\} \; ,$$

g_i:monodromomy permutations due to liftings of loops around ramified points, product of cycles of length m_i. We have a surjective homomorphism

$$\begin{aligned} \Gamma^+ &\to G \subset S_n \; , \\ \delta_i &\to g_i . \end{aligned} \tag{4.22}$$

For connected covering G acts transitively on n element set and

$$\Gamma^+ / \Gamma_1^+ = G/G_1 \simeq \{1, 2, \ldots, n\},$$

where $G_1 = \{g \in G : g1 = 1\}$.

Γ^+ acts transitively on residue classes $\Gamma^+/\Gamma_1^+ \simeq \{1, 2, \ldots, n\}$.

$$\begin{aligned} g_i \in S_n \text{ decompose in cycles of length } m_i \iff & \sigma_i^k \gamma \Gamma_1^+ \neq \gamma \Gamma_1^+ \;\; , k \not\equiv 0 (m_i) \; , \; \forall \gamma \in \Gamma^+ \\ \iff & \gamma^{-1} \sigma_i^k \gamma \notin \Gamma_1^+ . \end{aligned}$$

i.e. Γ_1^+ contains no elements conjugate to σ_i^k. In Γ^+ all elements of finite order are conjugate to some σ_i^k. Hence, Γ_1^+ is torsion free. The result now follows since, in Γ all elements of infinite order are translations.

The above proposition implies the following corollary.

Corollary 4.1 *Let $\pi : X \longrightarrow \mathbb{P}^1$ be as in above proposition. Then, deg π is divisible by* $[\Gamma^+ : L]$.

Proof: By the above proposition, *deg* $\pi = [\Gamma^+ : \Gamma_1^+]$ and $\Gamma_1^+ \subset L \subset \Gamma^+$. Since

$$[\Gamma^+ : \Gamma_1^+] = [\Gamma^+ : L][L : \Gamma_1^+] \tag{4.23}$$

the result follows.

In the following discussions we will use the following known results.
1) For $\pi : X \longrightarrow Y$ a connected covering of degree n, $G = \pi_1(Y)$ acts transitively on an n-element set N. To this covering, there corresponds the subgroup $H \subset G$ of index n. $H = G_y$, where G_y is the stabilizer of an element $y \in N$ [10].
2) If G acts transitively on a set Y, then

$$Aut_G(Y) \simeq N_G(G_y)/G_y \tag{4.24}$$

where $N_G(G_y)$ is the normalizer of G_y, and G_y denotes the stabilizer of $y \in Y$ [10].
3) Let L_o denote a sublattice of lattice L of index n. Then

$$\#\{L_o \subset L \ : \ [L : L_o] = n\} = \sum_{d|n} d = \delta(d) \tag{4.25}$$

[see, 12].

The following proposition gives the number of Eisenstein coverings in parabolic case.

Proposition 4.4 *The Eisenstein number of ramified connected coverings $\pi : X \longrightarrow \mathbb{P}^1$ of degree $n\mu$, ramified over k points $y_1, \ldots, y_k$ with equal ramification indices m_i in each fiber $\pi^{-1}(y_i)$, m_i's satisfying (4.20) is given by*

$$\sum_{\pi : X \to \mathbb{P}^1} \frac{1}{|Aut\pi|} = \frac{1}{n\mu} \sum_{d|n} d \,, \tag{4.26}$$

where $\mu = [\Gamma^+(m_1, \ldots, m_k) : L]$, L:translation subgroup of $\Gamma^+(m_1, \ldots, m_k)$.

Proof: Using proposition (4.3) and equality (4.24) we can write

$$\sum_{\pi : X \to \mathbb{P}^1} \frac{1}{|Aut\pi|} = \sum_{\Gamma_1^+ \subset \Gamma^+} \frac{1}{[N_{\Gamma^+}(\Gamma_1^+) : \Gamma_1^+]}, \tag{4.27}$$

where the summation on right side is over all torsion free subgroups $\Gamma_1^+ \subset \Gamma^+$ up to conjugacy of index $n\mu$.
Using $n\mu = [\Gamma^+ : \Gamma_1^+] = [\Gamma^+ : L][L : \Gamma_1^+], [\Gamma^+ : L] = \mu, [L : \Gamma_1^+] = n$ (4.27) can be written as

$$\frac{1}{n\mu} \sum_{\Gamma_1^+ \subset \Gamma^+} \frac{[\Gamma^+ : \Gamma_1^+]}{[N_\Gamma^+(\Gamma_1^+) : \Gamma_1^+]}$$
$$= \frac{1}{n\mu} \sum_{\Gamma_1^+ \subset \Gamma^+} [\Gamma^+ : N_{\Gamma^+}(\Gamma_1{}^+)]. \tag{4.28}$$

We have

$$\sum_{\Gamma_1^+ \subset \Gamma^+} [\Gamma^+ : N_{\Gamma^+}(\Gamma_1{}^+)] = \#(\text{torsion free subgroups} \Gamma_1{}^+ \subset \Gamma^+ \ni [\Gamma^+ : \Gamma_1{}^+] = n\mu \tag{4.29}$$

Since $n\mu = [\Gamma^+ : \Gamma_1^+] = [\Gamma^+ : L][L : \Gamma_1^+], [\Gamma^+ : L] = \mu, [L : \Gamma_1^+] = n$ (4.29) is equal to

$$\#(L \subset \Gamma_1^+ \ : \ [L : \Gamma_1{}^+]) = n. \tag{4.30}$$

By (4.25), (4.30) is equal to $\sum_{d|n} d = \delta(d)$. And the result follows.

4.3.2 Disconnected coverings

We will explicitely describe the Eisenstein number of ramified coverings in parabolic case. Hence we will give the proof of theorem 4 stated in chapter 1.

The following proposition shows the connection between connected and disconnected coverings.

Lemma 4.1 *The following equality holds*

$$\sum_{\pi : X \to \mathbb{P}^1} \frac{q^{deg \pi}}{|Aut \pi|} = exp\left(\sum_{\pi : X \to \mathbb{P}^1} \frac{q^{deg \pi}}{|Aut \pi|} \right) \tag{4.31}$$

where the summation on LHS is over all coverings $\pi : X \to \mathbb{P}^1$, the summation on RHS is over all connected coverings $\pi : X \to \mathbb{P}^1$.

Proof: Let $X = \bigcup_i d_i X_i$, X_i: pairwise non-isomorphic connected components. $\pi|_{X_i} = \pi_i$, $\pi_i : X_i \to \mathbb{P}^1$ connected covering. Then LHS of the equality is equal to

$$\sum_{n\geq 1} q^n \sum_{\substack{\pi:X\to\mathbb{P}^1 \\ deg\pi=n}} \frac{1}{|Aut\pi|} \,. \tag{4.32}$$

Since $|Aut\pi| = \prod_i |Aut\pi_i|^{d_i}(d_i)!$, and $deg\pi = \sum_i d_i deg\pi_i$ (4.32) can be written as

$$\sum_{\pi_i:X\to\mathbb{P}^1} \prod_i \frac{q^{d_i deg\pi_i}}{|Aut\pi_i|^{d_i} d_i!} \tag{4.33}$$

$$= \prod_{\pi:X\to\mathbb{P}^1} \sum_{d\geq 0} \frac{q^{d deg\pi}}{|Aut\pi|^d d!} \,, \tag{4.34}$$

where last summation is over all connected coverings π. Since

$$\sum_{d\geq 0} \frac{q^{d deg\pi}}{|Aut\pi|^d d!} = \exp\left(\frac{q^{deg\pi}}{|Aut\pi|}\right) \,, \tag{4.35}$$

the result follows.

Lemma 4.2 *Let $\pi : X \to \mathbb{P}^1$ denote connected ramified covering ramified over k points $y_1, \ldots, y_k$ in $\mathbb{P}^1$ with equal ramification indices m_i in each fiber $\pi^{-1}(y_i)$, m_i's satisfying (4.20). Then the following holds*

$$\sum_{\pi:X\to\mathbb{P}^1} \frac{q^{deg\pi}}{|Aut\pi|} = -\frac{1}{\mu}\sum_{n=1}^{\infty} log(1-q^{n\mu}) \tag{4.36}$$

where $\mu = [\Gamma^+(m_1, \ldots, m_k) : L]$, L : translation subgroup of $\Gamma^+(m_1, \ldots, m_k)$.

Proof: Let $N = n\mu$, $n \geq 1$. Then

$$\sum_{\pi:X\to\mathbb{P}^1} \frac{q^{deg\pi}}{|Aut\pi|} = \sum_N q^N \sum_{\pi:X\to\mathbb{P}^1} \frac{1}{|Aut\pi|} \,, \tag{4.37}$$

where the summation is over all connected ramified coverings π as described in statement, of degree N.
Using proposition (4.4), (4.37) can be written as

$$\sum_N q^N \frac{1}{N} \sum_{d|n} d \,. \tag{4.38}$$

Writing $n = md$, $N = \mu n$, (4.38) is equal to

$$\frac{1}{\mu}\sum_{m\geq 1}\sum_{d\geq 1} \frac{q^{\mu m d}}{d} = -\frac{1}{\mu}\sum_{m\geq 1} \log(1-q^{\mu m}) \,. \tag{4.39}$$

Hence the result follows.

From lemma (4.1) and lemma (4.2) there follows immediately

Proposition 4.5 *Let $\pi : X \to \mathbb{P}^1$ denote ramified coverings as described in lemma (4.2). Then the following holds*

$$\sum_{\pi : X \to \mathbb{P}^1} \frac{q^{deg \pi}}{|Aut \pi|} = \frac{1}{[\prod_{n=1}^{\infty}(1 - q^{\mu n})]^{\frac{1}{\mu}}} \tag{4.40}$$

where μ is as in lemma (4.2).

Hence, the following theorem is an immediate consequence of above proposition using table for index $[\Gamma^+ : L]$.

Theorem 4.5 *The Eisenstein number of ramified coverings $\pi : X \to \mathbb{P}^1$ of degree $n\mu$, ramified over k points $y_1, \ldots, y_k$, with equal ramification indices m_i in each fiber $\pi^{-1}(y_i)$, m_i's satisfying (4.20), is given by*

$$\sum_{\pi : X \to \mathbb{P}^1} \frac{1}{|Aut \pi|} = \textit{coefficient at } q^n \textit{ in } [\prod_{k=1}^{\infty} (1 - q^k)]^{\frac{-1}{\mu}} \tag{4.41}$$

where $\mu \in \mathbb{N}$ depends on the affine Coxeter group corresponding to the solution of (4.20) more explicitely, for m_i's satisfying the case
a) $\mu = 2$,
bi) $\mu = 4$,
bii) $\mu = 6$,
biii) $\mu = 3$.

4.3.3 Estimation of coefficients

The above theorem gains interest if the coefficients on RHS can be evaluated. It turns out that the function $[\prod_{k=1}^{\infty} (1 - q^k)]^{\frac{-1}{\mu}}$ is closely related to Dedekind η function which was considered in Chapter 1. H.Rademacher and H.Zuckerman found the Fourier coefficients of the modular form $\eta(z)^{-2r}$ and gave the following asymptotic formula [for details see, 8].

The modular form $F(z) = \eta(z)^{-2r}$ admits for $Im\ z > 0$, the Fourier expansion

$$F(z) = q^{1-\frac{2r}{24}}\{q^{-1} + \sum_{m=0}^{\infty} a_m q^m\} , \tag{4.42}$$

where the asymptotics for coefficients can be given as

$$a_m \sim \frac{(\frac{r}{12})^{\frac{2r+1}{4}}}{\sqrt{2}(m+1)^{\frac{2r+3}{4}}} \exp\left(4\pi\sqrt{\frac{r(m+1)}{12}}\right) . \tag{4.43}$$

Putting $n = m+1$ corollary immediatly follows

Corollary 4.2 *Asymptotics for the coefficient at* q^n *as* $n \longrightarrow \infty$ *in* $[\prod_{n=1}^{\infty}(1-q^n)]^{-2r}$ *is given as*

$$\frac{(\frac{r}{12})^{\frac{2r+1}{4}}}{\sqrt{2}n^{\frac{2r+3}{4}}} \exp(\pi\sqrt{\frac{4rn}{3}}) . \tag{4.44}$$

Example: The number $p(n)$ of partitions of n has the generating function

$$\frac{1}{\prod_n(1-q^n)} = \sum_n p(n)q^n . \tag{4.45}$$

The above corollary for $r = \frac{1}{2}$ gives the asymptotics for $p(n)$ as

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp(\pi\sqrt{\frac{2n}{3}}) . \tag{4.46}$$

Letting $r = \frac{1}{2\mu}$ in above corollary and combining with theorem (4.5) we get the following corollary.

Corollary 4.3 *The Eisenstein number of ramified coverings as described in theorem (4.5) has the following asymptotic*

$$\sum_{\pi:X\rightarrow\mathbb{P}^1} \frac{1}{|Aut\pi|} \sim \frac{(\frac{1}{24\mu})^{\frac{1+\mu}{4\mu}}}{\sqrt{2}n^{\frac{3\mu+1}{4\mu}}} \exp\left(\pi\sqrt{\frac{2n}{3\mu}}\right) , \tag{4.47}$$

where μ *is as described in theorem (4.5).*

Example: The Eisenstein number of ramified coverings $\pi : X \rightarrow \mathbb{P}^1$ of degree $6n$, ramified over 3 points y_1, y_2, y_3 with ramification indices 2,3 and 6 , equal in each fiber has the following asymptotic

$$\sum_{\pi:X\rightarrow\mathbb{P}^1} \frac{1}{|Aut\pi|} \sim \frac{1}{2^{\frac{5}{3}}\, 3^{\frac{7}{12}} n^{\frac{19}{24}}} \exp(\frac{\pi}{3}\sqrt{n}) . \tag{4.48}$$

Proof: $m_1 = 2$, $m_2 = 3$, $m_3 = 6$ case. So $\mu = 6$. Applying the above corollary gives the result.

Combining corollary (4.3) with theorem 1 in chapter 1, we get the following theorem.

Theorem 4.6 *The following asymptotic formulae holds.*

$$\sum_{\chi \epsilon S_{2n}} \frac{\chi(\sigma_2)^4}{\chi(1)^2} \sim \frac{\pi}{2^2\, 3^{\frac{3}{8}}} n^{\frac{1}{8}} \exp \pi \sqrt{\frac{n}{3}} \tag{4.49}$$

$$\sum_{\chi \epsilon S_{4n}} \frac{\chi(\sigma_2)\chi(\sigma_4)^2}{\chi(1)} \sim \frac{\pi}{2^{\frac{25}{16}}\, 3^{\frac{5}{16}}} n^{\frac{3}{16}} \exp \pi \sqrt{\frac{n}{6}} \tag{4.50}$$

$$\sum_{\chi \epsilon S_{6n}} \frac{\chi(\sigma_2)\chi(\sigma_3)\chi(\sigma_6)}{\chi(1)} \sim \frac{\pi}{2^{\frac{2}{3}}\, 3^{\frac{7}{12}}} n^{\frac{5}{24}} \exp \frac{\pi}{3}\sqrt{n} \tag{4.51}$$

$$\sum_{\chi \epsilon S_{3n}} \frac{\chi(\sigma_3)^3}{\chi(1)} \sim \frac{\pi}{2^{\frac{1}{2}}\, 3^{\frac{7}{6}}} n^{\frac{1}{6}} \exp (\frac{\pi}{3})\sqrt{2n} \tag{4.52}$$

where σ_m is product of $\frac{n}{m}$ cycles of length m.

Proof: The idea in each case is the same. Hence, we will give the detailed proof of one case.

Proof of (4.51):

1) First, observe that applying theorem 1 in Chapter 1 for

i) $Y = \mathbb{P}^1$,

ii) Covering π of degree $6n$, ramified over 3 points,

iii) Equal ramification indices in each fiber, with values 2,3 and 6,

gives

$$\sum_{\pi} \frac{1}{|Aut\pi|} = \frac{|C_1||C_2||C_3|}{(6n!)^2} \sum_{\chi \epsilon S_{6n}} \frac{\chi(\sigma_2)\chi(\sigma_3)\chi(\sigma_6)}{\chi(1)}. \tag{4.53}$$

2) By the above example we have

$$\sum_{\pi : X \to \mathbb{P}^1} \frac{1}{|Aut\pi|} \sim \frac{1}{2^{\frac{5}{3}}\, 3^{\frac{7}{12}} n^{\frac{19}{24}}} \exp(\frac{\pi}{3}\sqrt{n}) . \tag{4.54}$$

3) Using

$$|C_1| = \frac{(6n)!}{2^{3n}(3n)!}, \tag{4.55}$$

$$|C_2| = \frac{(6n)!}{3^{2n}(2n)!}, \tag{4.56}$$

$$|C_3| = \frac{(6n)!}{6^n n!}, \tag{4.57}$$

$$\tag{4.58}$$

and Stirlings formula $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$ and combining with 1) and 2) the result follows.

Chapter 5

Asymptotic Formulae

As was explained in chapter 1 (1.4.2), the problem of estimating the number of coverings with given ramification indices in some extend can be reduced to estimation of characters.

In this chapter, we will consider asymptotics of characters $\chi_\beta(g)$ of S_n under the following conditions.

Let diagram β be given by $b_1 \geq b_2 \geq \ldots \geq b_m$ and the cycle structure of g given by $1^{a_1}2^{a_2}\ldots n^{a_n}$ with fixed length of cycles. Suppose that
i) Diagram β has fixed number of rows.
ii) Number of cells in each row increases as $n \to \infty$ with fixed frequency, i.e. $\frac{b_i}{n} = \beta_i$, β_i is fixed as $n \to \infty$.
iii) Lengths of all cyles in $g \in S_n$ are coprime.
iv) Multiplicity of cycles in $g \in S_n$ increases as $n \to \infty$ with fixed frequency, i.e $\frac{a_k}{n} = \alpha_k$, α_k is fixed as $n \to \infty$.

5.1 Frobenius formula

Frobenius formula is used to compute the value of χ_β, an irreducible character of S_n corresponding to a Young diagram β as follows:

Introduce independent variables $z_1, \ldots, z_m$ with m: number of rows in β. Define

$$s_j = z_1^j + \ldots + z_m^j \ , \ 1 \leq j \leq n, \tag{5.1}$$

$$\Delta(z_1,\ldots,z_m) = \prod_{i<j}(z_i - z_j), \tag{5.2}$$

$$\delta = (m-1, m-2, \ldots, 0)\ , \tag{5.3}$$

$$z^{\beta+\delta} = z_1^{b_1+m-1} z_2^{b_2+m-2} \ldots z_m^{b_m}\ . \tag{5.4}$$

Then

$$\chi_\beta(g) = \text{ coefficient at } z^{\beta+\delta} \text{ in } \Delta \prod_j s_j^{a_j}\ . \tag{5.5}$$

5.2 Reduction to contour integral

Evaluation of the value of $\chi_\beta(g)$ can be reduced to evaluation of some contour integral.

By Frobenius formula $\chi_\beta(g)$ is the coefficient of some multivariable polynomial. Hence, using generalization of Cauchy's formula to multidimensional case, (5.5) can be written as

$$\chi_\beta(g) = \frac{1}{(2\pi i)^m} \int_{|z_1|=1} \cdots \int_{|z_m|=1} \frac{\Delta(z_1,\ldots,z_m)\prod_j s_j^{a_j}}{{z_1}^{b_1+m}{z_2}^{b_2+m-1}\ldots{z_m}^{b_m+1}} dz_1 \ldots dz_m\ . \tag{5.6}$$

Under the conditions for g and β mentioned in the beginning of chapter, (5.6) can be written in the form

$$\chi_\beta(g) = \frac{1}{(2\pi i)^m} \int_{|z_1|=1} \cdots \int_{|z_m|=1} \frac{\Delta(z_1,\ldots,z_m)}{z_1^m z_2^{m-1} \ldots z_m} \left[\frac{s_1^{\alpha_1} s_2^{\alpha_2} \ldots s_n^{\alpha_n}}{z_1^{\beta_1} z_2^{\beta_2} \ldots z_m^{\beta_m}}\right]^n \tag{5.7}$$

where $a_k = \alpha_k n$, $b_i = \beta_i n$, α_k, β_i fixed as $n \to \infty$.

5.3 Asymptotic analysis of contour integral

Let

$$F(\lambda) = \int_{\gamma^n} g(z) exp(\lambda w(z)) dz \tag{5.8}$$

where $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $dz = dz_1 \ldots dz_n$ and γ^n is an n-dimensional smooth compact manifold. By the many dimensional method of steepest descent, the asymptotics for $F(\lambda)$ as $\lambda \to \infty$ is determined by the critical values of $w(z)$ i.e. by z_o's such that $\nabla w(z_o) = 0$. The idea is to take integral in a small neighborhood of critical point responsible of asymptotics. We have:

Let $max_{z\in\gamma^n} Rew(z)$ be attained only at a point $z_o \in \gamma^n$ such that $\bigtriangledown w(z_o) \neq 0$ and $\det Hess(w)|_{z=z_o} = \det \frac{\partial^2 w}{\partial z_i \partial z_j}|_{z=z_o} \neq 0$. If $g(z_o) \neq 0$, as $\lambda \to \infty$ there is the asymptotic formula

$$F(\lambda) \sim (\frac{2\pi}{\lambda})^{\frac{n}{2}} \frac{1}{\sqrt{\det Hess(w)|_{z=z_o}}} \exp(\lambda w(z_o)) g(z_o) , \tag{5.9}$$

where

$$Hess(w) = \left(\frac{\partial^2 w}{\partial z_i \partial z_j} \right) .$$

[For details, see 13].

In order to use this argument for $\chi_\beta(g)$ write χ_β in the form

$$F(n) = \chi_\beta(g) = \frac{1}{(2\pi i)^m} \int_{|z_1|=1} \cdots \int_{|z_m|=1} g(z) exp(nw(z)) dz \tag{5.10}$$

where

$$z = (z_1, \dots, z_m) \in \mathbb{C}^m, \tag{5.11}$$

$$g(z) = \frac{\Delta(z_1, \dots, z_m)}{z_1^m z_2^{m-1} \dots z_m} = \prod_{i<j} (1 - \frac{z_j}{z_i}), \tag{5.12}$$

$$w(z) = \sum_{k=1}^{n} \alpha_k log(s_k) - \sum_{i=1}^{m} \beta_i \log z_i . \tag{5.13}$$

The equation for the critical points of $w(z)$ is of the form

$$\sum_k k\alpha_k \frac{z_i^k}{z_1^k + \dots + z_m^k} = \beta_i \ , \ i = 1, \dots, m . \tag{5.14}$$

This equation has a lot of complex roots. But it turns out that the asymptotics of integral is determined by positive real roots of this equation.

5.4 Asymptotic formulae

Let us state and prove our results.

The following theorem is crucial for the asymptotic formulae.

Theorem 5.1 *The system*

$$\sum_k k\alpha_k \frac{z_i^k}{z_1^k + \dots + z_m^k} = \beta_i \ , \ i = 1, \dots, m \ ; \ \beta_1 \geq \ \beta_2 \geq \dots \geq \beta_m \geq 0 \tag{5.15}$$

has up to proportionality unique positive solution $x = (x_1, x_2, \dots, x_m)$; $x_1 \geq x_2 \geq \dots \geq x_m \geq 0$.

Proof: Let $z_i = e^{t_i}$, $t_i \in \mathbb{R}$, $p_k = k\alpha_k$, $\sum_k p_k = 1$. Then (5.15) can be written as

$$\sum_k p_k \frac{e^{kt_i}}{e^{kt_1} + \ldots + e^{kt_m}} = \beta_i \; ; \; i = 1, 2, \ldots, m \tag{5.16}$$

and (5.13)

$$w(e^{t_i}) = \sum_{k=1}^{n} \alpha_k \log(e^{kt_1} + \ldots + e^{kt_m}) - \sum_{i=1}^{m} \beta_i t_i \, . \tag{5.17}$$

The proof may be dividen in two steps.
step 1: *w is a convex function of* $t = (t_1, \ldots, t_m)$,
i.e.

$$Hess(w) = \sum_{i,j} \frac{\partial^2(w)}{\partial t_i \partial t_j} X_j X_i \geq 0 \; , \; \forall X_i, X_j \in \mathbb{R} \, . \tag{5.18}$$

For $z_i = e^{t_i}$

$$Hess(w) = \sum_{k=1}^{n} \alpha_k \sum_{i,j} \frac{\partial^2}{\partial t_i \partial t_j} \log(e^{kt_1} + \ldots + e^{kt_m}) X_j X_i \, . \tag{5.19}$$

It suffices to show that

$$h(t) = \sum_{i,j} \frac{\partial^2}{\partial t_i \partial t_j} \log(e^{t_1} + \ldots + e^{t_m}) X_j X_i \geq 0 \, . \tag{5.20}$$

After some computations we get

$$h(t) = \frac{\sum_i e^{t_i} X_i^2}{\sum_i e^{t_i}} - \left(\frac{\sum_i e^{t_i} X_i}{\sum_i e^{t_i}} \right)^2 \tag{5.21}$$

Let $\bar{X}$ denote mean value of X_i's w.r.t. $q_i = \frac{e^{t_i}}{\sum_i e^{t_i}}$. Then

$$h(t) = \sum q_i (X_i - \bar{X})^2 \geq 0 \tag{5.22}$$

and equality only when $X_1 = X_2 = \ldots = X_m = \bar{X}$.
Hence $Hess(w) \geq 0$ is proved. We can conclude from this the followings.
1) Restriction of $Hess(w)$ on the hyperplane $\sum_i X_i = 0$ is positive.
2) The mapping

$$\Omega : (t_1, \ldots, t_m) \to \left(\frac{\partial f}{\partial t_1}, \ldots, \frac{\partial f}{\partial t_m} \right) = (\beta_1, \ldots, \beta_m)$$

is locally invertible for $\sum t_i = 0$, $f(t) = \sum_k \alpha_k \log(e^{kt_1} + \ldots + e^{kt_m})$.
step 2: *The mapping*

$$\Omega : (z_1, \ldots, z_m) \to (\beta_1, \beta_2, \ldots, \beta_m) \ , \ z_i \geq 0 \ , \ \sum_i z_i = 1 \tag{5.23}$$

is a homeomorphism between simplexes

$$\Delta_z = \{z_i \ : \ \sum_i z_i = 1 \ , \ z_i \geq 0\} \ ,$$
$$\Delta_\beta = \{\beta_i \ : \ \sum_i \beta_i = 1 \ , \ \beta_i \geq 0\} \ .$$

The proof is by induction on m. By induction hypothesis

$$\Omega_i : (z_1, \ldots, z_{i-1}, 0, z_{i+1}, \ldots, z_m) \to (\beta_1, \ldots, \beta_{i-1}, 0, \beta_{i+1}, \ldots, \beta_m)$$

is a homeomorphism $\forall i$. Hence the restriction of Ω to the boundary δ

$$\Omega|_\delta : \delta\Delta_z \to \delta\Delta_\beta$$

is a homeorphism between boundaries of simplexes. Hence, by Brauwer theorem the map Ω is surjective. By part 2 of step1 Ω is a local homeomorphism. It turns out that the homeomorphism is global, sinceΔ_β is simply connected. And the result follows.

The above theorem is central in proving the following theorem.

Theorem 5.2 *Let us consider a sequence of diagrams β such that $b_1 \geq b_2 \geq \ldots \geq b_m$, $\beta_i = \frac{b_i}{n}$ fixed, and a sequence of permutations $g \in S_n$ with cycle structure $1^{a_1}2^{a_2}\ldots n^{a_n}$ such that $\alpha_k = \frac{a_k}{n}$ is fixed. If lengths of all cycles involved in $g \epsilon S_n$ are coprime, and $\beta_i \neq \beta_j$, $i \neq j$, asymptotics for $\chi_\beta(g)$ as $n \to \infty$ is given as:*

$$\chi_\beta(g) \sim \frac{e^{nw(x)}\sqrt{m}}{(2\pi n)^{\frac{m-1}{2}}} \prod_{i<j} (1 \ - \ \frac{x_j}{x_i}) \frac{1}{\sqrt{\sum_{i=1}^m H_{ii}}} \tag{5.24}$$

where $x = (x_1, \ldots, x_m)'$ is positive solution of the system (5.14)

$$w(x) \ = \ \sum_k \alpha_k \log(x_1^k + \ldots + x_m^k) \ - \ \sum_{i=1}^m \beta_i \log x_i \tag{5.25}$$

and H_{ii} is diagonal minor of order $m-1$ of the quadratic form in variable dt_i

$$Hess(w) \ = \ \sum_{k=1}^n k^2 \alpha_k \left(\frac{\sum_{i=1}^m \ x_i^k dt_i^2}{\sum_i x_i^k} \ - \ (\frac{\sum_{i=1}^m x_i^k dt_i}{\sum_{i=1}^m \ x_i^k})^2 \right) \ . \tag{5.26}$$

Proof: The proof may be dividen in several steps.
First, recall that in section 5.3 we write $\chi_\beta(g)$ in the form

$$\chi_\beta(g) = \frac{1}{(2\pi i)^m} \int_{|z_1|=1} \cdots \int_{|z_m|=1} g(z) exp(nw(z)) dz \tag{5.27}$$

where

$$z = (z_1, \ldots, z_m) \in \mathbb{C}^m , \tag{5.28}$$

$$g(z) = \frac{\Delta(z_1, \ldots, z_m)}{z_1^m z_2^{m-1} \ldots z_m} = \prod_{i<j} (1 - \frac{z_j}{z_i}) , \tag{5.29}$$

$$w(z) = \sum_{k=1}^{n} \alpha_k log(s_k) - \sum_{i=1}^{m} \beta_i \log z_i . \tag{5.30}$$

1) *Deformation of surface of integration.*
Let $(e^{\tau_1}, e^{\tau_2}, \ldots, e^{\tau_m}) = (x_1, x_2, \ldots, x_m)$ denote the positive solution of (5.15). Deforming the contour of integration, we get

$$\chi_\beta(g) = \frac{1}{(2\pi i)^m} \int_{|z_1|=e^{\tau_1}} \cdots \int_{|z_m|=e^{\tau_m}} g(z) e^{nw(z)} dz . \tag{5.31}$$

2) *Asymptotics of integral depends on the positive solution.*
By section 5.3 it suffices to show that for coprime lengths of cycles $MaxRe(w(z))$ on the contour $|z_i| = e^{\tau_i}$ is attained only at the positive solution of (5.14).

$$Re(w(z)) = \sum_k \alpha_k \log |{z_1}^k + \ldots + {z_m}^k| - \sum_i \beta_i \log |z_i| . \tag{5.32}$$

By triangle inequality,

$$|{z_1}^k + \ldots + {z_m}^k| \le |z_1|^k + \ldots + |z_m|^k = e^{k\tau_1} + \ldots + e^{k\tau_m} . \tag{5.33}$$

Equality is only for collinear ${z_i}^k$'s. i.e $z_i = \epsilon_i e^{\tau_i} \lambda$, $|\lambda| = 1$, ${\epsilon_i}^k = 1 \; \forall k \ni \alpha_k \neq 0$. But since k's are coprime, equality is only when $\epsilon_i = 1$. For all other z_i's on the contour strict inequality occurs. Hence, only the positive solution of (5.15) gives the maximum.

3) *Passing to real varibles.*
Let $z_j = e^{\tau_j + it_j}$, $-\pi \le t_j \le \pi$. $t = (t_1, \ldots, t_m)$. Then

$$\chi_\beta(g) = \frac{1}{(2\pi)^m} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{i<j} \left(1 - \frac{e^{\tau_j + it_j}}{e^{\tau_i + it_i}}\right) e^{nw(t)} dt . \tag{5.34}$$

Let

$$F(t) = \prod_{i<j} \left(1 - \frac{e^{\tau_j + it_j}}{e^{\tau_i + it_i}}\right) e^{nw(t)} dt . \tag{5.35}$$

Observe that

$$w(\lambda z) = w(z) \ ,$$
$$g(\lambda z) = g(z) \ .$$

In terms of t_i, this means invariance under shifts $t_i \to t_i + a$. Hence,

$$F(t_1, \ldots, t_m) = F(t_1 + a, \ldots, t_m + a) \ ; \ \forall a \in \mathbb{R} \ , \tag{5.36}$$

so that $F(t)$ is constant on lines parallel to main diagonal $t_1 = t_2 = \ldots = t_m$; $-\pi < t_i < \pi$.

4) *Asymptotics of the integral depends only of diagonal(set of critical points of w).* Hence, in a small neighborhood of main diagonal, asymptotically the integral is equal to

$$\frac{1}{(2\pi)^m} \int_H \int_L F(t) dl dt \sim \frac{2\pi\sqrt{m}}{(2\pi)^m} \int_H F(t) dt \tag{5.37}$$

where H is the hyperplane $\sum_i t_i = 0$ and L : lines orthogonal to H i.e. lines parallel to main diagonal, $2\pi\sqrt{m} =$ length of main diagonal.

5) *Prefinal formula.*

We can apply (5.9) to $\int_H F(t)dt$. Denoting the positive solution of (5.14) which determines the asymptotics of integral by $x = (x_1, \ldots, x_m)$, (5.9) gives

$$\int_H F(t)dt \sim \left(\frac{2\pi}{n}\right)^{\frac{m-1}{2}} e^{nw(x)} \prod_{i<j} \left(1 - \frac{x_j}{x_i}\right) \frac{1}{\sqrt{\det(w_{ij}|_H(x))}} \ . \tag{5.38}$$

Hence ,

$$\chi_\beta(g) \sim \frac{e^{nw(x)}\sqrt{m}}{(2\pi n)^{\frac{m-1}{2}}} \prod_{i<j} \left(1 - \frac{x_j}{x_i}\right) \frac{1}{\sqrt{\det w_{ij}|_H(x)}} \tag{5.39}$$

where $w_{ij} = \frac{\partial^2 w}{\partial t_i \partial t_j}$.

6) *Evaluation of* $\det w_{ij}|_H(x)$.

Consider the quadratic form in variables dt_i

$$Hess(w) = \sum_{i,j} \frac{\partial^2 w(x)}{\partial t_i \partial t_j} dt_i dt_j \ . \tag{5.40}$$

Using (5.21),

$$Hess(w) = \sum_k k^2 \alpha_k \left[\frac{{x_1}^k {dt_1}^2 + \ldots + {x_m}^k {dt_m}^2}{{x_1}^k + \ldots + {x_m}^k} - \left(\frac{\sum_i {x_i}^k dt_i}{\sum_i {x_i}^k} \right)^2 \right] \tag{5.41}$$

$$= \sum_{ij} h_{ij} dt_i dt_j \ . \tag{5.42}$$

Let

$$H = (h_{ij}), (h_{ij}) = \left(\frac{\partial^2 w(x)}{\partial t_i \partial t_j}\right) \tag{5.43}$$

denote the Hessian matrix with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_m$. Since $Hess(w) \geq 0$ iff $\lambda_i \geq 0$, $\forall i = 1, \ldots, m$, by the first part in the proof of theorem (5.1) $\lambda_m = 0$. Hence $\det Hess(w)|_{dt_1+\ldots+dt_m=0}$ is quadratic form with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_{m-1}$ and

$$\det Hess(w)|_{dt_1+\ldots+dt_m=0} = \lambda_1 \ldots \lambda_{m-1} \ . \tag{5.44}$$

$\lambda_1 \lambda_2 \ldots \lambda_{m-1} = \sum_i H_{ii}$ where H_{ii} : principal minor of order $m-1$ of H obtained by deleting i'th row and i'th column of H. The result follows combining with the preceding step.

Using the above theorem, we can get an asymptotic formula for dimensions of irreducible characters of S_n.

Corollary 5.1 *The asymptotics for the dimension of the irreducible representation corresponding to diagram β with different lengths of rows is*

$$\chi_\beta(1) \sim \frac{e^{nH(\beta)} \prod_{i<j}(1 - \frac{\beta_j}{\beta_i})}{(2\pi n)^{\frac{m-1}{2}} \sqrt{\beta_1 \beta_2 \ldots \beta_m}} \tag{5.45}$$

where $H(\beta) = -\sum_i \beta_i log(\beta_i)$ is the entropy function, $\beta_i = \frac{b_i}{n}$ for diagram $\beta : b_1 \geq \ldots \geq b_m$.

Proof: Notice that $g = 1$ iff $\alpha_1 = 1$, $\alpha_k = 0 \ \forall k > 1$. Hence, applying the above theorem for $g = 1$ (5.14) can be written as

$$\frac{x_i}{\sum_{i=1}^m x_i} = \beta_i \ , \ i = 1, \ldots, m \ . \tag{5.46}$$

Hence we may take $x_i = \beta_i$ and the positive root of system is $(x_1, \ldots, x_m) = (\beta_i, \ldots, \beta_m)$. Equation for $w(x)$ may be written as

$$w(x) = \log(\beta_1 + \ldots, \beta_m) - \sum_{i=1}^m \beta_i \log(\beta_i) \ . \tag{5.47}$$

Since $\sum_{i=1}^m \beta_i = 1$,

$$w(x) = -\sum_{i=1}^m \beta_i \log(\beta_i) := H(\beta) \tag{5.48}$$

where $H(\beta)$ is the entropy function. The quadratic form (5.26) is now equal to

$$\sum_{i=1}^{m} \beta_i dt_i^2 - (\sum_{i=1}^{m} \beta_i dt_i)^2 \tag{5.49}$$

$$= \sum_i \beta_i dt_i^2 - \sum_{i,j} \beta_i \beta_j dt_i dt_j \ . \tag{5.50}$$

It can be seen that all diagonal minors of the form are equal to $\beta_1 \ldots \beta_m$. Hence

$$\sqrt{\sum_i H_{ii}} = \sqrt{m\beta_1 \ldots \beta_m} \ . \tag{5.51}$$

And the result immediately follows.

Observe that the above theorem fails when $x_i = x_j$ for some i, j. Another way of stating this is to say that it fails when diagram β has some equal rows. In case of rectangular diagrams we could evaluate the asymptotics for $\chi_\beta(g)$ using Selberg integral.

Remark 5.1 *The* Selberg integral *[14] is given as*

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |\Delta(x)|^{2\gamma} e^{-a(x_1^2+\ldots+x_m^2)} dx = (2\pi)^{\frac{m}{2}} (2a)^{\frac{-m(\gamma(m-1)+1)}{2}} \prod_{j=1}^{m} \frac{\Gamma(1+j\gamma)}{\Gamma(1+\gamma)} \tag{5.52}$$

where

$$x = (x_1, \ldots, x_m) \ , \tag{5.53}$$

$$dx = dx_1 \ldots dx_m \ , \tag{5.54}$$

$$\Delta(x) = \prod_{i<j} (x_i - x_j) \ , \tag{5.55}$$

$$\Gamma : \ \textit{Gamma function} \ . \tag{5.56}$$

Theorem 5.3 *Under the assumptions of theorem (5.2) for β and g, if lengths of cycles involved in g are coprime and the diagram β is rectangular , i.e. all rows are of the same length, then*

$$\chi_\beta(g) \sim \frac{m^{\frac{n}{d}}}{(2\pi)^{\frac{m-1}{2}} (n \bar{d})^{\frac{m^2-1}{2}}} (m)^{\frac{m^2}{2}} \prod_{j=1}^{m-1} j! \tag{5.57}$$

where $\sum_k \alpha_k = \frac{1}{d}$, $\sum_k k^2 \alpha_k = \bar{d}$.

Proof: In the proof we will use some steps of the proof of theorem (5.2).

1) *Critical point.*

Diagram β is rectangular iff $\beta_1 = \beta_2 = \ldots = \beta_m$. Hence the solutions of (5.14) can be taken as

$$x_1 = x_2 = \ldots = x_m = 1 \ .$$

Hence for $x = (1, 1 \ldots, 1)$ (5.25) is equal to

$$w(x) = \frac{1}{d} \log m \ , \ \frac{1}{d} = \sum_k \alpha_k \ . \tag{5.58}$$

2) *Reduction to integral.*

Using steps (1-3) of theorem (5.2) for $\tau_1 = \ldots = \tau_m = 0$

$$\chi_\beta(g) = \frac{1}{(2\pi)^m} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{\Delta(z_1, \ldots, z_m)}{z_1^{m-1} \ldots z_{m-1}} e^{nw(t)} dt \ , \tag{5.59}$$

$$z_j = e^{it_j} \ . \tag{5.60}$$

Approximating $w(t)$ by its Taylor expansion about x

$$w(t) \sim w(x) - \frac{1}{2} H(t) \tag{5.61}$$

where $H(t)$ is given by (5.42) for $x_1 = x_2 = \ldots = x_m = 1$ as

$$H(t) = \sum_k k^2 \alpha_k \left[\frac{t_1^2 + \ldots + t_m^2}{m} - \left(\frac{t_1 + \ldots + t_m}{m} \right)^2 \right] \ . \tag{5.62}$$

Combining with step 4 of theorem (5.2) we have

$$\chi_\beta(g) \sim \frac{\sqrt{m} e^{nw(x)}}{(2\pi)^{m-1}} \int_H \frac{\Delta(z_1, \ldots, z_m)}{z_1^{m-1} \ldots z_{m-1}} e^{-\frac{n}{2} H(t)} dt \tag{5.63}$$

where H is the hyperplane $t_1 + t_2 + \ldots + t_m = 0$.

3) Observe that $\Delta(z_1, \ldots, z_m)$ is skew-symmetric w.r.t. z_i's and $H(t)$ is symmetric w.r.t t_i's. i.e. for $I = (i_1, \ldots, i_m) \in S_m$

$$\Delta(z_{i_1}, \ldots, z_{i_m}) = sgn\ I \Delta(z_1, \ldots, z_m) \ , \tag{5.64}$$

$$H(t_{i_1}, \ldots, t_{i_m}) = H(t_1, \ldots, t_m) \ . \tag{5.65}$$

Hence (5.63) can be written as

$$\chi_\beta(g) \sim$$

$$\frac{\sqrt{m} e^{nw(x)}}{(2\pi)^{m-1} m!} \int_H \Delta(z_1, \ldots, z_m) \left[\sum_{I \in S_m} sgn I \bar{z}_{i_1}^{\,m-1} \bar{z}_{i_2}^{\,m-2} \ldots \bar{z}_{i_{m-1}} \right] e^{\frac{-n}{2} H(t)} dt \tag{5.66}$$

where the summation is over all permutations $I \in S_m$. Noticing that

$$\sum_{I \in S_m} sgnI \bar{z}_{i_1}^{m-1} \bar{z}_{i_2}^{m-2} \ldots \bar{z}_{i_{m-1}} = \Delta(\bar{z}_1, \ldots, \bar{z}_m) ,$$

(5.66) for $z_j = e^{it_j}$ can be written as

$$\chi_\beta(g) \sim \frac{\sqrt{m} e^{nw(x)}}{(2\pi)^{m-1} m!} \int_H |\Delta(t)|^2 e^{\frac{-n}{2} H(t)} dt . \tag{5.67}$$

4) *Computation using Selberg integral.*
For $t_1 + \ldots + t_m = 0$ we have,

$$H(t) = \frac{\bar{d}}{m}(t_1^2 + \ldots + t_m^2) , \tag{5.68}$$

$$\text{where } \bar{d} = \sum_k k^2 \alpha_k . \tag{5.69}$$

Hence

$$\int_H |\Delta(t)|^2 e^{\frac{-n}{2} H(t)} dt = \int_H |\Delta(t)|^2 e^{\frac{-n\bar{d}}{2m}(t_1^2 + \ldots + t_m^2)} dt . \tag{5.70}$$

Using Selberg's formula in remark (5.1) for $\gamma = 1$, $a = \frac{n\bar{d}}{2m}$,

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |\Delta(t)|^2 e^{\frac{-n\bar{d}}{2m}(t_1^2 + \ldots + t_m^2)} dt = \frac{(2\pi)^{\frac{m}{2}}}{(n\bar{d})^{\frac{m^2}{2}}} m^{\frac{m^2}{2}} \prod_{j=1}^{m} j! . \tag{5.71}$$

The integral

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |\Delta(t)|^2 e^{\frac{-n}{2} H(t)} dt \tag{5.72}$$

can be written by making change of coordinates as

$$\int_{-\infty}^{\infty} \int_{H'} |\Delta(t)|^2 e^{\frac{-n\bar{d}}{2m}(t_1^2 + \ldots + t_m^2)} dt da , \tag{5.73}$$

where H' is the hyperplane $\frac{t_1 + \ldots + t_m}{\sqrt{m}} = a$. Putting $t_i = \tilde{t}_i + \frac{a}{\sqrt{m}}$ (5.73) reduces to

$$\int_{-\infty}^{\infty} \int_{\tilde{t}_1 + \ldots + \tilde{t}_m = 0} |\Delta(\tilde{t})|^2 e^{\frac{-n\bar{d}}{2m}(\tilde{t}_1^2 + \ldots + \tilde{t}_m^2 + a^2)} d\tilde{t} \, da , \tag{5.74}$$

which is equal to

$$\sqrt{\frac{2\pi m}{n\bar{d}}} \int_{\tilde{t}_1 + \ldots + \tilde{t}_m = 0} |\Delta(\tilde{t})|^2 e^{\frac{-n\bar{d}}{2m}(\tilde{t}_1^2 + \ldots + \tilde{t}_m^2)} d\tilde{t} . \tag{5.75}$$

Combining (5.75) and (5.71),

$$\int_H |\Delta(t)|^2 e^{\frac{-n}{2}H(t)}dt = \sqrt{\frac{n\bar{d}}{2\pi m}} \frac{(2\pi)^{\frac{m}{2}}}{(n\bar{d})^{\frac{m^2}{2}}} m^{\frac{m^2}{2}} \prod_{j=1}^{m} j! \; . \tag{5.76}$$

Combining (5.67) and (5.76) with (5.58) we get the final result.

Theorem 5.4 *Let x be the unique positive root of (5.14), $w(x)$ as in (5.25) and β be diagram described as in theorem (5.2). Then*

$$w(x) \geq \frac{1}{\bar{d}} H(\beta) \; .$$

The equality is only if all cycles are of the same length or diagram is rectangular.

Proof: We have

$$w(x) = \sum_k \alpha_k \log(x_1^k + \ldots + x_m^k) - \sum_i \beta_i \log x_i \; . \tag{5.77}$$

Using $\sum_k k\alpha_k = 1$, we can write

$$w(x) = \sum_k \alpha_k [\log(x_1^k + \ldots + x_m^k) - \sum_i \beta_i \log x_i^k] \; . \tag{5.78}$$

1) We will first prove the following

$$f(y_1, y_2, \ldots, y_m) = \log(y_1 + y_2 + \ldots + y_m) - \sum_i \beta_i \log y_i \geq H(\beta) \; . \tag{5.79}$$

Let $y_i = e^{t_i}$. Then

$$f(t_1, \ldots, t_m) = \log(e^{t_1} + \ldots + e^{t_m}) - \sum_i \beta_i t_i \; . \tag{5.80}$$

Extremum condition for f: $\frac{\partial f}{\partial t_i} = 0$, $\forall i$. i.e.

$$\frac{e^{t_i}}{e^{t_1} + \ldots + e^{t_m}} = \beta_i \; , \; \forall i \tag{5.81}$$

therefore, f has an extremum at $e^{t_i} = \lambda\beta_i$, $\lambda \geq 0$. For $y_i = \lambda\beta_i$

$$f(y_1, \ldots, y_m) = -\sum_i \beta_i \log \beta_i = H(\beta) \; . \tag{5.82}$$

And result follows.

2) Using first step, we get

$$w(x) \geq \sum_k \alpha_k H(\beta) \; . \tag{5.83}$$

Since $\sum_k \alpha_k = \frac{1}{d}$, we get

$$w(x) \geq \frac{H(\beta)}{d} \; . \tag{5.84}$$

3) In inequality (5.84) the equality occurs iff

$$x_i^k = \lambda_k \; \beta_i \; , \; \forall k \; s.t. \; \alpha_k \neq 0 \; .$$

a) If we have at least two different cycle lengths, i.e. $\alpha_k \neq 0$ and $\alpha_r \neq 0$ for $k \neq r$, then

$$x_i^k = \lambda_k \beta_i \; ,$$
$$x_i^r = \lambda_r \beta_i \; ,$$

implies

$$x_1 = x_2 = \ldots = x_m \; .$$

So

$$\beta_1 = \beta_2 = \ldots = \beta_m \; ,$$

i.e. the diagram β is rectangular. Conversely, if diagram is rectangular equality directly follows.

b) If all cycles have the same length then $k\alpha_k = 1$, $\alpha_r = 0$ for $r \neq k$. Then (5.14) implies that x_i^k is proportional to β_i, $\forall k$ such that $\alpha_k \neq 0$ and we have equality. And the theorem is proved.

Combining the above theorem with corollary (5.1) the below corollary is immediate.

Corollary 5.2 *Under the assumptions of theorem (5.2) for g and β, if the diagram β is not rectangular and if all cycles involved in g are of different length, then $\frac{\chi_\beta(g)}{\chi_\beta(1)^{\frac{1}{d}}}$ exponentially increases to ∞ as $n \to \infty$.*

Chapter 6

Vanishing of normalized characters

Definition 6.1 *The ratio* $\frac{\chi}{\chi(1)}$ *is called normalized character of character* χ.

We are interested in limits of normalized characters of S_n as $n \longrightarrow \infty$. We suppose that
i) number of cycles in $g_n \in S_n$ is fixed.
ii) $\chi_{\lambda(n)}$ is any sequence of faithful characters of S_n labelled by partitions $\lambda(n)$.

In this chapter we will give the proof of the following theorem.

Theorem 6.1 *Under the conditions i) and ii)*

$$\frac{\chi_{\lambda(n)}(g_n)}{\chi_{\lambda(n)}(1)} \longrightarrow 0 \tag{6.1}$$

as $n \longrightarrow \infty$.

As will be seen, case of exterior powers and two row diagrams is essential in the proof. Let us begin with exterior powers case.

6.1 Vanishing of normalized characters for exterior powers

In this section we will give the proof of theorem for representations of S_n given by exterior powers of standart representation.

Notations

1) V_1 denotes the standart irreducible representation of S_n of dimension $n-1$.
2) χ_k denotes the character of $\Lambda^k V_1$.
3) For a representation $G : V$ $det_V(\lambda - g)$ is $det(\lambda I - A)$ where A is the matrix of linear transformation gV.

Recall: Standart Representation of S_n Natural representation of S_n arises with the action of S_n on C^n by permuting the coordinates , which is not irreducible as the line spanned by the sum of the basis vectors , i.e. $<(1,1,..,1)>$, is invariant with complementary subspace

$$V_1 = \{(x_1, x_2, ..., x_n) \in \mathbb{C}^n : \sum x_i = 0\}$$

hence $V = V_1 \oplus <(1,1...,1)>$. This n-1 dimensional representation V_1 is called the standart representation of S_n.

Definition 6.2 *The diagram below , denoted as $[n-k, 1^k]$, is called hook diagram.*

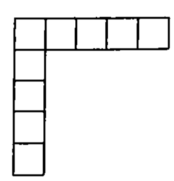

Figure 6.1: Hook diagram $[n-k, 1^k]$

Remark 6.1 $\Lambda^k V_1$ *corresponds to hook diagram $[n-k, 1^k]$ [15].*

6.1.1 Character formula for exterior powers

By combinatorial way, we will give the formula for characters of exterior powers.

Lemma 6.1 *For $g \epsilon S_n$, if $\varepsilon_1, \varepsilon_2, ..., \varepsilon_{n-1}$ are eigenvalues of g in V_1, then $\chi_k(g) = \sigma_k(\varepsilon_1, \varepsilon_2, ..., \varepsilon_{n-1})$, where χ_k is the character of g in $\bigwedge^k V_1$ and σ_k denotes k'th elementary symmetric function in variables $\varepsilon_1, \varepsilon_2, ..., \varepsilon_{n-1}$.*

Proof: Follows immediately from the following:
i) The eigenvalues of g in $\bigwedge^k V_1$ are $\varepsilon_{i_1}\varepsilon_{i_2}...\varepsilon_{i_k}$; $\{i_1, i_2, ..., i_k\} \subset \{1, 2, ..., n-1\}$,
ii) The character of an element g in a representation V just means the sum of eigenvalues of g in V.

Corollary 6.1 *For $g \in S_n$, V_1: standart representation of S_n the following holds*

$$det_{V_1}(\lambda - g) = \sum_{k=0}^{n-1} (-1)^k \chi_k(g) \lambda^{n-1-k} . \tag{6.2}$$

Proof: Let g have eigenvalues $\varepsilon_1, \varepsilon_2, ..., \varepsilon_{n-1}$ in V_1. Then

$$det_{V_1}(\lambda - g) \quad = \quad \prod_{j=1}^{n-1} (\lambda - \varepsilon_j) \tag{6.3}$$

which is equal to

$$\sum_{k=0}^{n-1} \lambda^{n-1-k}(-1)^k \sigma_k(\varepsilon_1, \ldots, \varepsilon_{n-1}) . \tag{6.4}$$

By lemma (6.1) $\chi_k(g) = \sigma_k(\varepsilon_1, \ldots, \varepsilon_{n-1})$. Hence we get

$$det_{V_1}(\lambda - g) = \sum_{k=0}^{n-1} (-1)^k \; \chi_k(g) \lambda^{n-1-k} . \tag{6.5}$$

And the statement is proved.

Proposition 6.1 *Let $g \in S_n$ with cycle structure $(1^{a_1} 2^{a_2} \ldots n^{a_n})$. Then*

$$\chi_k(g) \;=\; (-1)^k \;\; \textit{coefficient of} \;\; \lambda^{n-1-k} \; \textit{in} \; \frac{\prod_{i=1}^n (\lambda^i - 1)^{a_i}}{\lambda - 1} . \tag{6.6}$$

Proof: Proof may be divided in two steps.
i) For V: natural representation of S_n, g: full cycle in S_n it can easily be seen that

$$det_V(\lambda - g) = \lambda^n - 1 . \tag{6.7}$$

Hence for g having cycle structure $(1^{a_1}, 2^{a_2}, \ldots, n^{a_n})$

$$det_V(\lambda - g) = (\lambda - 1)^{a_1} (\lambda^2 - 1)^{a_2} \ldots (\lambda^n - 1)^{a_n} . \tag{6.8}$$

ii) Since $V = V_1 \oplus < (1,1,\ldots,1) >$, eigenvalues of g in V_1 are those in V except 1. Hence

$$det_{V_1}(\lambda - g) = \frac{(\lambda-1)^{a_1}(\lambda^2-1)^{a_2}\ldots(\lambda^n-1)^{a_n}}{(\lambda-1)} . \tag{6.9}$$

The proof now follows combining (6.9) with corollary (6.1)

The above proposition for $g = 1$, $a_1 = n$, $a_k = 0$, $\forall k > 1$ immediatley gives the following corollary.

Corollary 6.2 *Dimension of $\Lambda^k V_1$ is given as*

$$\chi_k(1) = \binom{n-1}{k} . \tag{6.10}$$

Now, we can show that normalized characters of exterior powers vanishes as $n \to \infty$.

Theorem 6.2 *For χ_m, character of m'th exterior power of standart representation, $m \neq 0$, $m \neq n-1$*

$$\frac{\chi_m(g)}{\chi_m(1)} \to 0 \ as \ n \to \infty \tag{6.11}$$

where $g \in S_n$ with fixed number of cycles.

Proof: Let$g \in S_n$ with cycle structure $d_1 d_2 \ldots d_k$, k: fixed. By proposition (6.1)

$$\chi_m(g) = (-1)^m \text{coeff. of } z^{n-1-m} \ in \ \frac{(z^{d_1}-1)(z^{d_2}-2)\ldots(z^{d_k}-1)}{(z-1)}$$

Observe that for any coefficient, say z^p

$$\text{coeff. at } z^p \text{ in } \frac{(z^{d_1}-1)(z^{d_2}-2)\ldots(z^{d_k}-1)}{(z-1)} \leq \text{coff. at } z^p \text{ in } \frac{(z^{d_1}+1)(z^{d_2}+1)\ldots(z^{d_k}+1)}{(z-1)}$$
$$\leq d_1 2^{k-1}$$

Hence,

$$\chi_m(g) \leq d_1 2^{k-1} . \tag{6.13}$$

Since $d_i \leq n$, $\forall i$

$$\chi_m(g) \leq n2^{k-1} . \tag{6.14}$$

By corollary (6.2) we have

$$\chi_m(1) = \begin{pmatrix} n-1 \\ m \end{pmatrix} . \tag{6.15}$$

Hence

$$\frac{\chi_m(g)}{\chi_m(1)} \leq \frac{n2^{k-1}}{\begin{pmatrix} n-1 \\ m \end{pmatrix}} . \tag{6.16}$$

$\begin{pmatrix} n-1 \\ m \end{pmatrix}$ is a polynomial of degree m of n.
For $n-3 \geq m \geq 2$ from (6.16) we get

$$\frac{\chi_m(g)}{\chi_m(1)} \longrightarrow 0 \tag{6.17}$$

as $n \longrightarrow \infty$.
For $m = 1$;
χ_m is the character of standart representation of S_n of dimension $n-1$. Hence

$$\chi_m(g) = (\# \text{ trivial cycles in g}) - 1 . \tag{6.18}$$

Since # of cycles in g is k: fixed

$$\chi_m(g) \leq k-1 . \tag{6.19}$$

Hence

$$\frac{\chi_m(g)}{\chi_m(1)} \leq \frac{k-1}{n-1} , \tag{6.20}$$

implying

$$\frac{\chi_m(g)}{\chi_m(1)} \longrightarrow 0 , \ as \ n \longrightarrow \infty . \tag{6.21}$$

For $m = n-2$, χ_{n-2} is the character of S_n corresponding to conjugate diagram of χ_1. So, $\chi_{n-2}(g) = (sign\ g)\chi_1(g)\ g \in S_n$ [16]. Hence, limit of normalized character vanishes in this case also. And the theorem is proved.

6.2 Vanishing of normalized characters for two row representations

In this section, we will consider normalized characters of irreducible representations of S_n which corresponds to two row diagrams. For convenience, we call such representations as two row representations.

Notations

1) $\chi_{a,b}$ denotes the character of irreducible representation (a, b) of S_n corresponding to two row diagram $a \geq b, a + b = n$.

2) $[a][b]$ denotes the permutation representation of S_n induced by trivial representation of $S_a \times S_b$, with character $\chi_{[a][b]}$.

Recall

1) $\chi_{[a][b]}(g)$: character of permutation representation of S_n on the set $I \subset \{1, 2, \ldots, n\}$ such that $\#I = a$.

And it is equal to

$$\chi_{[a][b]}(g) = \#\{I \subset \{1, 2, \ldots, n\} : \#I = a \ , \ gI = I\} \ . \tag{6.22}$$

2) *Weyl's determinant formula*

Let (a, b) denote the two row representation corresponding to two row diagram $a \geq b \ , \ a + b = n$. Then

$$(a, b) = [a][b] - [a + 1][b - 1] \ , \tag{6.23}$$

with character relation

$$\chi_{a,b} = \chi_{[a][b]} - \chi_{[a+1][b-1]} \ . \tag{6.24}$$

For details, we refer the reader to [16].

6.2.1 Character formula for two row representations

As in exterior power case, we will give the formula for characters of two row representations by combinatorial way.

The following lemma is central.

Lemma 6.2 *Let $g \in S_n$ with cycle structure $d_1^{a_1} d_2^{a_2} \ldots d_n^{a_n}$. Then, the following formula holds*

$$\chi_{[a][b]}(g) = \textit{ coefficient at } z^a \textit{ in } (1 + z^{d_1})^{a_1}(1 + z^{d_2})^{a_2} \ldots (1 + z^{d_n})^{a_n} \ . \tag{6.25}$$

Proof: The proof is based on equality (6.22)

$$\chi_{[a][b]}(g) = \#\{I \subset \{1,2,\ldots,n\} : \#I = a \ , \ gI = I\} \tag{6.26}$$

Since $gI = I$ iff I is a union of cycles of g, it follows that

$$\chi_{[a][b]}(g) = \#\{k_1 d_1 + k_2 d_2 + \ldots + k_n d_n = a : \ 0 \le k_i \le a_i\} \ , \tag{6.27}$$

which can be expressed as

$$\chi_{[a][b]}(g) = \text{coeff. at } z^a \text{ in } (1+z^{d_1})^{a_1} \ldots (1+z^{d_n})^{a_n} \ . \tag{6.28}$$

And the proof is completed.

Using Weyl's determinant formula, we get the following proposition.

Proposition 6.2 *Let $g \in S_n$ as in lemma (6.2). Character $\chi_{a,b}(g)$ of two row representation (a,b) of S_n, $a \ge b$, $a+b=n$ is given by*

$$\chi_{a,b}(g) = \ \textit{coeff. at } z^{a+1} \textit{ in } (z-1)(1+z^{d_1})^{a_1} \ldots (1+z^{d_n})^{a_n} \ . \tag{6.29}$$

Proof: By Weyl's determinant formula, we have

$$\chi_{a,b}(g) = \chi_{[a][b]}(g) - \chi_{[a+1][b-1]}(g) \ . \tag{6.30}$$

By lemma (6.2) this is equal to

$$\text{coeff. at } z^a \text{ in } (1+z^{d_1})^{a_1} \ldots (1+z^{d_n})^{a_n} - \ \text{coeff. at } z^{a+1} \text{ in } (1+z^{d_1})^{a_1} \ldots (1+z^{d_n}) \tag{6.31}$$

And the result follows.

For $g = 1$, $d_1 = 1$, $a_1 = n$, $a_k = 0 \ \forall k > 1$ the corollary immediately follows from the above proposition.

Corollary 6.3 *Dimension of two row representation (a,b) , $a \ge b, a+b=n$ is given by*

$$\chi_{a,b}(1) = \begin{pmatrix} n \\ a \end{pmatrix} - \begin{pmatrix} n \\ a+1 \end{pmatrix} \ . \tag{6.32}$$

Similar to exterior power case, we can show that limit of normalized characters of two row representations vanishes as $n \to \infty$.

Theorem 6.3 *The normalized character of two row representation (a,b), $a \geq b$, $a+b=n$, of S_n vanishes as $n \to \infty$. i.e.*

$$\frac{\chi_{a,b}(g)}{\chi_{a,b}(1)} \to 0 \ , \ as \ n \to \infty \tag{6.33}$$

where $g \in S_n$ with fixed number of cycles.

Proof: Let $g \in S_n$ with cycle structure $d_1 d_2 \ldots d_k$, k: fixed. By proposition (6.2) we have

$$\chi_{a,b}(g) = \text{ coeff. at } z^{a+1} \text{ in } (z-1)(1+z^{d_1})\ldots(1+z^{d_k}) \ . \tag{6.34}$$

Observe that

$$\text{any coeff. of } (z-1)(1+z^{d_1})\ldots(1+z^{d_k}) \leq 2^{k+1} \ . \tag{6.35}$$

Hence,

$$\chi_{a,b}(g) \leq 2^{k+1} \ . \tag{6.36}$$

By corollary (6.3)

$$\chi_{a,b}(1) = \begin{pmatrix} n \\ a \end{pmatrix} - \begin{pmatrix} n \\ a+1 \end{pmatrix} , \tag{6.37}$$

which is a polynomial of n. Hence

$$\frac{\chi_{a,b}(g)}{\chi_{a,b}(1)} \to 0 \ , as \ n \to \infty \ . \tag{6.38}$$

Remark 6.2 *If λ' is conjugate diagram of λ obtained by changing rows and columns of λ, then $\chi_{\lambda'}(g) = sign g \chi_{\lambda}(g)$ [15]. Hence since two row diagrams and two column diagrams are conjugate, the theorem above is still valid for two column representations.*

6.3 Vanishing of normalized characters for general representations

We can now prove the theorem (6.1) for any irreducible representation of S_n.

Theorem 6.4 *Let $\chi_{\lambda(n)}$ be an irreducible character of S_n correponding to diagram $\lambda(n)$. Then*

$$\frac{\chi_{\lambda(n)}(g_n)}{\chi_{\lambda(n)}(1)} \to 0\ ,\quad n \to \infty \tag{6.39}$$

if $\chi_{\lambda(n)}(1) \to \infty$ and $g_n \in S_n$ such that $|g_n| = k$ i.e. # of cycles in g_n is equal to k, k: fixed.

Proof: Proof is by induction on k, i.e. on the number of cycles in g_n.
1) For $k = 1$:
Then g_n is a long cycle in S_n. Hence

$$|\chi_{\lambda(n)}(g_n)| = \left\{ \begin{array}{ll} 0 & \text{if } \lambda(n) \text{ is not a hook diagram} \\ 1 & \text{if } \lambda(n) \text{ is hook diagram} \end{array} \right\} . \tag{6.40}$$

Hence

$$\frac{|\chi_{\lambda(n)}(g)|}{\chi_{\lambda(n)}(1)} \leq \frac{1}{\begin{pmatrix} n-1 \\ m \end{pmatrix}} , \tag{6.41}$$

where $\begin{pmatrix} n-1 \\ m \end{pmatrix} = \chi_{\lambda(n)}(1)$ for $\lambda(n) = [n-m, 1^m]$.

$$\frac{1}{\begin{pmatrix} n-1 \\ m \end{pmatrix}} \to 0\ ,\ as\ n \to \infty \tag{6.42}$$

unless $m = 0$ or $m = n-1$, i.e. unless $\chi_{\lambda(n)}$ is trivial or sign character, but this is impossible by assertion. Hence, for $k = 1$ the statement is true.
2) Assume the statement is true for $|g_n| < k$.
i) If g_n has more than one cycle with length increasing to ∞ as $n \to \infty$:
Let

$$g_n = g_p g_q \in S_p \times S_q \tag{6.43}$$

$$p + q = n\ ,\ |g_p| < k\ ,\ |g_q| < k,\ \ p, q \to \infty\ as\ n \to \infty \tag{6.44}$$

By Littlewood-Richardson rule

$$\chi_{\lambda(n)}(g_n) = \sum_{i,j} m_{i,j} \chi_i(g_p) \chi_j(g_q) \tag{6.45}$$

where χ_i and χ_j are irreducible characters of S_p and S_q respectively. Hence

$$\frac{|\chi_{\lambda(n)}(g_n)|}{\chi_{\lambda(n)}(1)} \leq \frac{\sum_{i,j} m_{ij}|\chi_i(g_p)||\chi_j(g_q)|}{\sum_{i,j} m_{ij}\chi_i(1)\chi_j(1)} \tag{6.46}$$

Generalizing the fact that for $a, b, c, d > 0$, $\frac{a+b}{c+d}$ is between $\frac{a}{c}$ and $\frac{b}{d}$ we get

$$\frac{|\chi_{\lambda(n)}(g_n)|}{\chi_{\lambda(n)}(1)} \leq max_{i,j} \frac{|\chi_i(g_p)||\chi_j(g_q)|}{\chi_i(1)\chi_j(1)}\ , \tag{6.47}$$

Since $|g_p|, |g_q| < k$ using induction hypothesis we get

$$\frac{|\chi_{\lambda(n)}(g_n)|}{\chi_{\lambda(n)}(1)} \to 0\ as\ n \to \infty\ , \tag{6.48}$$

implying

$$\frac{\chi_{\lambda(n)}(g_n)}{\chi_{\lambda(n)}(1)} \to 0\ as\ n \to \infty \tag{6.49}$$

unless $\chi_i(1) = \chi_j(1) = 1$.
By Frobenious reciprocity theorem $\chi_{\lambda(n)}$ is a component of $Ind_{S_p \times S_q}^{S_n}(\chi_i \times \chi_j)$. Hence,
a) If χ_i and χ_j are trivial characters then $Ind_{S_p \times S_q}^{S_n}(\chi_i \times \chi_j)$ contains only two row characters. So $\chi_{\lambda(n)}$ is a two row character.
b) If χ_i is trivial character and χ_j is sign character, $Ind_{S_p \times S_q}^{S_n}(\chi_i \times \chi_j)$ contains only exterior power characters. Hence $\chi_{\lambda(n)}$ is exterior power character.
c) If χ_i and χ_j are both sign characters, $Ind_{S_p \times S_q}^{S_n}(\chi_i \times \chi_j)$ contains only two column characters. Hence, $\chi_{\lambda(n)}$ is two column character.
By the previous section, in each case the statement is true. And the assertion is proved.

ii) If g_n has only one cycle with length increasing to ∞ as $n \to \infty$:
Let $g_n = g_{n-k}g_k \in S_{n-k} \times S_k$ where g_{n-k} is a long cycle in S_{n-k} and g_k be the rest, k: fixed. By second orthogonality relations if $\chi_1, \ldots, \chi_p$ denotes irreducible characters of S_n then

$$\sum_i |\chi_i(g_n)|^2 = |C_{S_n}(g_n)| \tag{6.50}$$

where $|C_{S_n}(g_n)|$ is the centralizer of $g_n \in S_n$.
Hence it follows that

$$|\chi_{\lambda(n)}(g_n)| \leq \sqrt{|C_{S_n}(g_n)|}\ . \tag{6.51}$$

Since $C_{S_n}(g_n) \subset C_{n-k} \times S_k$ where C_{n-k} : cyclic group of order $n-k$, we have

$$|C_{S_n}(g)| \leq (n-k)k! \, . \tag{6.52}$$

Combining with (6.51)

$$|\chi_{\lambda(n)}(g_n)| \leq c\sqrt{n} \tag{6.53}$$

for some constant c.
For $n \geq 5$ the minimum dimension for any non-trivial character is $n-1$. So

$$\chi_{\lambda(n)}(1) \geq n-1 \tag{6.54}$$

Combining (6.53) and (6.54) we get

$$\frac{|\chi_{\lambda(n)} g_n|}{\chi_{\lambda(n)}(1)} \to 0 \; as \; n \to \infty \tag{6.55}$$

which implies

$$\frac{\chi_{\lambda(n)}(g_n)}{\chi_{\lambda(n)}(1)} \to 0 \; as \; n \to \infty \tag{6.56}$$

And the theorem is proved.

Chapter 7

Conclusion

We considered the connection between ramified coverings $\pi : X \to Y$ of Riemann surfaces and characters of S_n. In cases when the structure of covering is known, we carried information on coverings to that of characters of S_n and get some explicit formulae for sums over characters of S_n, involving Dedekind η function. For carrying information on characters to coverings, we developed asymptotic theory for characters of S_n as $n \to \infty$ under some restrictions. We restricted ourselves to diagrams with bounded number of rows. Hence, our results on asymptotics is not sufficient to get an estimation on the number of coverings. So, asymptotic theory for characters of S_n may be developed more. Finally, we have shown that under certain conditions, normalized characters $\frac{\chi(g)}{\chi(1)}$ of S_n vanishes as $n \to \infty$.

REFERENCES

[1] A. E. Zalesskii, "Group Rings of Simple Locally Finite Groups," preprint, 1995

[2] A. M. Vershik, S. V. Kerov, " Asymptotic theory of characters of the symmetric group," Funkts. Anal. Prilozhen., Vol. 15, No.4, 15-17, 1981

[3] A. M. Vershik, S. V. Kerov, " Asymptotic of the largest and the typical dimensions of irreducible representations of a symmetric group," Funkts. Anal. Prilozhen., Vol. 19, 25-36, 1985

[4] E. Thoma, "Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe," Math. Z., 85, 40-61, 1964

[5] M. Bauer, C. Itzykson, "Triangulations," London Mathematical Society Lecture Note Series 200, Cambridge University Press, 1994

[6] A. A. Migda, "Quantum gravity as dynamical triangulation," Two Dimensional Quantum Gravity and Random Surfaces, Vol. 8, Singapore, 1992

[7] F. Klein, "Lectures on Icosahedron, " Dover Publications, Inc., New York, 1956

[8] H. Rademacher, H. S. Zuckerman, "On the Fourier coefficients of certain modular forms of positive dimension," Annals of Mathematics, Vol. 39, 433-462, 1938

[9] Z. Arad, M. Herzog, "Powers and Products of Conjugacy Classes in Groups," Lecture Notes in Mathematics 1112, Springer-Verlag, 1985

[10] J. J. Rotman, "An Introduction to Algebraic Topology," Springer-Verlag, 1988

[11] H. S. M. Coxeter, W. O. J. Moser, "Generators and Relations for Discrete Groups," Springer-Verlag, 1965

[12] J. P. Serre, "A Course in Arithmetic," Springer-Verlag, 1973

[13] R. V. Gamkrelidze, "Analysis I," Encyclopedia of Mathematical Sciences Vol.13, Springer-Verlag, 1989

[14] Selberg, A. Bemerkninger OM et multiplet integral. Norsk Metematisk Tidsskrift, 26, 71-78, 1944

[15] W. Fulton, J. Harris "Representation Theory: a first course," Springer-Verlag, 1991

[16] G. D. James, "The Representation Theory of the Symmetric Groups," Springer-Verlag, 1978

[17] W. S. Massey, "Algebraic Topology," Springer-Verlag, 1977

[18] H. S. M. Coxeter, "Regular Polytopes," Dover Publications, Inc., 1973