TWO-DIMENSIONAL FRACTIONAL FOURIER TRANSFORM AND ITS OPTICAL IMPLEMENTATION

A THESIS
SUBMITTED TO THE DEPARTMENT OF ELECTRICAL AND ELECTRONICS ENGINEERING
AND THE INSTITUTE OF ENGINEERING AND SCIENCES
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
Ayşegül Şahin
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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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M.S. in Electrical and Electronics Engineering
Supervisor: Assoc. Prof. Dr. Haldun M. Özaktaş
August 1996

The fractional Fourier transform of order a is defined in a manner such that the common Fourier transform is a special case with order $a = 1$. The definition is easily extended to two dimensions by just repeating the transform in x and y directions independently. The properties of the separable two dimensional fractional Fourier transform defined in this manner are derived and several optical implementations are given. However, this definition, for certain purposes, motivated us to look for a new, non-separable definition. We present such a definition of the two dimensional fractional Fourier transform with its optical implementation. The usefulness of the new definition is justified with a noise filtering example.

ÖZET

İKİ BOYUTLU KESİRLİ FOURIER DÖNÜŞÜMÜ VE OPTİK GERÇEKLEMESİ

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Elektrik ve Elektronik Mühendisliği Bölümü Yüksek Lisans
Tez Yöneticisi: Doç. Dr. Haldun M. Özaktaş
Ağustos 1996

Derecesi a olan kesirli Fourier dönüşümü, bilinen Fourier dönüşümü bu dönüşümün $a = 1$ için özel bir hali olacak şekilde tanımlanır. Bu tanım, iki boyuta dönüşüm x ve y yönlerinlerinde bağımsız olarak tekrar edilerek genellenebilir. Çalışmamızda, bu şekilde tanımlanan iki boyutlu ayrıştırılabilir kesirli Fourier dönüşümünün özellikleri çıkartıldı ve birçok optik gerçeklemesi sunuldu. Fakat bu tanım belli amaçlar için bizi yeni, ayrıştırılamaz bir tanım aramaya teşvik etti. İki boyutlu kesirli Fourier dönüşümünün bu yeni tanımı optik gerçeklemesiyle birlikte sunuldu. Tanımın kullanılışlılığı bir gürültü filtreleme örneğiyle doğrulandı.

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To my parents . . .

Chapter 1

Introduction

The fractional Fourier transform of order a is defined in a manner such that the common Fourier transform is a special case with order $a = 1$. The one-dimensional fractional Fourier transform of order a can be defined for $0 < |a| < 2$ as

$$\mathcal{F}^a[f(x)](x) = \int_{-\infty}^{\infty} B_a(x, x') f(x') dx' \qquad (1.1)$$

$$B_a(x, x') = \frac{e^{-i(\pi\hat{\phi}/4 - \phi/2)}}{\sqrt{|\sin\phi|}} \exp[i\pi(x^2 \cot\phi - 2xx' \csc\phi + x'^2 \cot\phi)] \qquad (1.2)$$

where $\phi = a\pi/2$ and $\hat{\phi} = \mathrm{sgn}(\sin\phi)$. The kernel is defined separately for $a = 0$ and $a = \pm 2$ as $B_0(x, x') = \delta(x - x')$ and $B_{\pm 2}(x, x') = \delta(x + x')$. The definition can easily be extended outside the interval [-2,2] by noting that $\mathcal{F}^{4j+a}(x) = \mathcal{F}^a(x)$ [4].

The fractional Fourier transform was first discovered by mathematicians. In 1937, Condon introduced the concept of fractional Fourier transform in mathematics literature [1]. Later in 1961, Bargmann gave two definitions of fractional Fourier transform, one based on Hermite polynomials and the other one as the integral transformation [2]. Namias reinvented the transform in 1980 and solved several types of Schrödinger equation by using the fractional Fourier transform [3]. In 1987, McBride and Kerr extended the work of Namias and developed an operational calculus for the fractional Fourier transform [4].

Up to 1993, fractional Fourier transform was a purely mathematical transform. However, in 1993, Ozaktas and Mendlovic introduced the concept of fractional Fourier transform in optics and used graded-index (GRIN) media as a basis for defining fractional Fourier transform. In retrospect, they saw that their definition was fully consistent with the former mathematical definition of fractional Fourier transform [5-7]. Lohmann gave another definition of the fractional Fourier transform through its effect on Wigner distribution function and suggested two optical systems consisting of thin lenses separated by free space to implement fractional Fourier transform optically [8]. The equivalence of graded-index based definition and Wigner distribution based definition is also demonstrated in [9].

Fractional Fourier transform is widely used to explain optical phenomena. The process of propagation of light can be interpreted as a continuous fractional Fourier transformation. The common Fourier transform and imaging are special cases that occur when $a = 1$ and $a = 2$ respectively. There exists a fractional Fourier transform relation between amplitude distributions of light on two spherical surfaces of given radii and separation. Thus, fractional Fourier transform is presented as a tool for analyzing optical systems composed of thin lenses and sections of free space [10]. The relation between Fraunhofer diffraction phenomena at far field and common Fourier transform is generalized to Fresnel diffraction and fractional Fourier transform [11, 12]. Some optical transforms like Fourier transform, imaging systems and correlators, can be implemented by cascading fractional Fourier transform units [13-15]. Propagation in graded-index media and Gaussian beam propagation and spherical mirror resonators are also studied in terms of fractional Fourier transform [5-7,16,17]. The parameters of the fractional Fourier transform can be determined in terms of ray optical parameters. The relation between fractional Fourier transform and ray optics provides a more intuitive way of understanding the concept of fractional Fourier transform [18]. The success of fractional Fourier transform in explaining optical phenomena led to a generalization from 'Fourier Optics' to 'Fractional Fourier Optics' [10].

Fractional Fourier transform can be optically realized like the common

Fourier transform [8,19-21]. Thus, it has many applications in optical signal processing [5-8,10-12,14,15,17,20-27]. Optical phase retrieval problem is solved by the fractional Fourier transform approach in [28-30] and a lens design problem is given in [31].

The fractional Fourier definition is also extended to two dimensions. The first generalization [8] assumed identical transform orders in both directions while the others [20, 21] used different transform orders in x and y directions.

Fractional Fourier transform is closely related to Wigner distribution. Performing the fractional Fourier transform with order a corresponds to rotating the Wigner distribution by an angle $\phi = a\pi/2$ [8, 32]. The relationship between fractional Fourier transform, Wigner distribution, ambiguity function and other time-frequency representations is also examined [33-35].

The fractional Fourier transform is a special quadratic-phase system (linear canonical transform). Hence, like all the quadratic-phase systems, it can be characterized by a transformation matrix. Use of transformation matrices makes the analysis of systems easier, especially when two or more dimensional analysis is considered [8,10,19,36-40]. The fractional Fourier transform has a continuous parameter a. As a increases from 0 to 1, the function evolves smoothly from the original function to its common Fourier transform. Since a is a continuous parameter, there is a continuum of domains and the function has its corresponding representation in each domain, leading to alternative representations for the signal other than the conventional time and frequency domain representations [32, 41, 42].

The discrete-time implementation of fractional Fourier transform also exists. In [43], a fast algorithm that calculates fractional Fourier transform in $O(N \log N)$ time is presented. Being a generalization of common Fourier transform, fractional Fourier transform is expected to yield improvements in signal processing applications in which Fourier transform is widely used. Some of the applications are space-variant filtering and signal detection [32,44-47], time- or space-variant multiplexing and data compression [32], correlation, matched filtering, and pattern recognition [13, 48], signal synthesis [35] and radar [46]. The theory of optimal Wiener filtering is generalized to fractional Fourier domains and improvement is achieved. Since the transform can be implemented in

$O(N \log N)$ time, the improvement is achieved with no additional cost [45, 46]. Alternative definitions of fractional Fourier transform and its generalizations also exist [26,49-51].

The fractional Fourier transform has also applications in quantum optics [28, 42, 52, 53] and statistical optics [54]. The recent work on fractional Fourier transform is collected in [22].

This study focuses on the two-dimensional fractional Fourier transform. In Chapter 2, the properties of two-dimensional fractional Fourier transform are given. While some properties like additivity, linearity follow from one dimensional case, some properties are specific to two dimensions. Some of these properties are derived and some of them are directly generalized from one-dimensional properties. However, two-dimensional fractional Fourier transform fails to satisfy some of the desired properties. That is the reason why present a new definition in Chapter 4. Besides fractional Fourier transform, we mention quadratic phase systems, which will be the initial point of our study in Chapter 3. In Chapter 3, we propose various optical implementations for two-dimensional fractional Fourier transform by using two different approaches. The first approach depends on the optical implementation of the quadratic-phase systems. Since fractional Fourier transform belongs to the family of quadratic-phase systems, once the optical implementations of quadratic-phase systems are found, the same systems can also be used as fractional Fourier transformers. The second approach is specific to fractional Fourier transform. Several practical optical systems with different complexity are proposed. In Chapter 4, a new definition is suggested for two-dimensional fractional Fourier transform. The development of the definition is discussed in detail and its properties are derived. Chapter 5, consists of the optical implementation of the new fractional Fourier transform definition. The last chapter provides an application of the new definition to a filtering problem. It is shown that the new definition is remarkably better than the former one in the separation of additive chirp noise under certain circumstances.

To summarize, we derive the properties of two-dimensional fractional Fourier transform and present many optical systems that realize this transform optically. We also suggest a new, non-separable definition for two-dimensional fractional Fourier transform. Both the optical and discrete-time implementations of the new definition are given. The usefulness of our definition is justified by using a noise filtering example.

Chapter 2

Two-dimensional fractional Fourier transform

2.1 Definition of two-dimensional fractional Fourier transform

The definition of the two-dimensional fractional Fourier transform was previously made by using the same orders in both directions. But in [21], we defined the two-dimensional fractional Fourier transform with different orders in the two dimensions. The kernel for this transform is nothing but the product of two one-dimensional kernels. The two-dimensional fractional Fourier transform with order a_x along the x axis and a_y along the y axis is defined as

$$\mathcal{F}^{a_x,a_y}[f(x,y)](x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_{a_x,a_y}(x,y;x',y')\, f(x',y')\, dx'\, dy', \tag{2.1}$$

where

$$B_{a_x,a_y}(x,y;x',y') = B_{a_x}(x,x')B_{a_y}(y,y'). \tag{2.2}$$

Hence the two-dimensional kernel can be written as

$$B_{a_x,a_y}(x,y;x',y') = A_{\phi_x}\exp[i\pi(x^2\cot\phi_x - 2xx'\csc\phi_x + x'^2\cot\phi_x)] \times A_{\phi_y}\exp[i\pi(y^2\cot\phi_y - 2yy'\csc\phi_y + y'^2\cot\phi_y)]. \quad (2.3)$$

where

$$A_{\phi_x} = \frac{e^{-i(\pi\hat{\phi_x}/4-\phi_x/2)}}{\sqrt{|\sin\phi_x|}} \qquad A_{\phi_y} = \frac{e^{-i(\pi\hat{\phi_y}/4-\phi_y/2)}}{\sqrt{|\sin\phi_y|}} \quad (2.4)$$

and $\phi_x = a_x\pi/2$, $\phi_y = a_y\pi/2$, $\hat{\phi_x} = \mathrm{sgn}(\sin\phi_x)$, $\hat{\phi_y} = \mathrm{sgn}(\sin\phi_y)$.

As the above equation suggests, the kernel B_{a_x,a_y} is a separable kernel. Throughout this study we will refer to this definition as the two-dimensional separable fractional Fourier transform. The kernel for two-dimensional transform can be obtained by multiplying two one-dimensional fractional Fourier transform kernels and letting the orders change independent from each other. Thus the kernel has two parameters a_x and a_y. The definition may be simplified by using vector-matrix notation:

$$\mathcal{F}[f(\mathbf{r})](\mathbf{r}) = \int_{-\infty}^{\infty} A_{\phi_\mathbf{r}}\exp[i\pi(\mathbf{r}^\mathrm{T}\mathbf{C_t}\mathbf{r} - 2\mathbf{r}^\mathrm{T}\mathbf{C_s}\mathbf{r}' + \mathbf{r}'^\mathrm{T}\mathbf{C_t}\mathbf{r}')]f(\mathbf{r}')d\mathbf{r}', \quad (2.5)$$

where

$$A_{\phi_\mathbf{r}} = A_{\phi_x}A_{\phi_y}, \qquad \mathbf{r} = \begin{bmatrix} x & y \end{bmatrix}^\mathrm{T}, \quad \mathbf{r}' = \begin{bmatrix} x' & y' \end{bmatrix}^\mathrm{T},$$

$$\mathbf{C_t} = \begin{bmatrix} \cot\phi_x & 0 \\ 0 & \cot\phi_y \end{bmatrix}, \quad \mathbf{C_s} = \begin{bmatrix} \csc\phi_x & 0 \\ 0 & \csc\phi_y \end{bmatrix}.$$

For the two-dimensional case, the kernel B_{a_x,a_y} is a separable function of x and y. It is also possible to use this definition as the n-dimensional separable fractional Fourier transform definition. The constant $A_{\phi_\mathbf{r}}$, vectors $\mathbf{r}$,$\mathbf{r}'$, and matrices $\mathbf{C_t}, \mathbf{C_s}$ should have the following generalized expressions:

$$A_{\phi_\mathbf{r}} = A_{\phi_{x_1}}A_{\phi_{x_2}}\dots A_{\phi_{x_n}}, \qquad \mathbf{r} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^\mathrm{T}, \quad \mathbf{r}' = \begin{bmatrix} x_1' & \dots & x_n' \end{bmatrix}^\mathrm{T},$$

$$\mathbf{C_t} = \begin{bmatrix} \cot\phi_{x_1} & & 0 \\ & & \\ 0 & \dots & \cot\phi_{x_n} \end{bmatrix}, \quad \mathbf{C_s} = \begin{bmatrix} \csc\phi_{x_1} & & 0 \\ & & \\ 0 & \dots & \csc\phi_{x_n} \end{bmatrix}.$$

2.2 Properties of two-dimensional fractional Fourier transform

1. *Additivity*

   The two-dimensional fractional Fourier transform kernel is additive in the index; i.e.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_{a_x,a_y}(x,y;x'',y'')B_{a'_x,a'_y}(x'',y'';x',y')\,dx''\,dy'' \tag{2.6}$$

$$= B_{a_x+a'_x,a_y+a'_y}(x,y;x',y').$$

   This property may be rewritten as

$$\mathcal{F}^{a_x,a_y}[\mathcal{F}^{a'_x,a'_y}f(x,y)] = \mathcal{F}^{a_x+a'_x,a_y+a'_y}[f(x,y)] \tag{2.7}$$

   allowing us to add the orders of successive fractional Fourier transforms. Fractional Fourier transforms of different orders commute with each other, thus their orders can be changed freely.

   By substituting 2.2 in 2.6, the following equation is obtained

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_{a_x,a_y}(x,y;x'',y'')B_{a'_x,a'_y}(x'',y'';x',y')\,dx''\,dy'' \tag{2.8}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_{a_x}(x,x'')B_{a_y}(y,y'')B_{a'_x}(x'',x')B_{a'_y}(y'',y')\,dx''\,dy'' \tag{2.9}$$

   The proof follows directly by using the additivity property of the one-dimensional property [4] which is

$$\int_{-\infty}^{\infty} B_{a_x}(x,x'')B_{a'_x}(x'',x')dx'' = B_{a_x+a'_x}(x,x'). \tag{2.10}$$

2. *Inverse Transform*

   The kernel of the inverse transform is given as

$$B^{-1}_{a_x,a_y}(x,y;x',y') = B_{-a_x,-a_y}(x,y;x',y'). \tag{2.11}$$

   Letting $a_x + a'_x = a_y + a'_y = 0$ in the additivity property and noting that transform of order $a = 0$ corresponds to the function itself, the result follows.

3. *Linearity*

The fractional Fourier transform is linear. For arbitrary real constants a_k,

$$\sum_k a_k f(x,y) \overset{\mathcal{F}^{a_x,a_y}}{\longrightarrow} \sum_k a_k \mathcal{F}^{a_x,a_y}[f(x,y)]. \tag{2.12}$$

Since the fractional Fourier transform is a linear integral transform, it satisfies the linearity property.

4. *Separability*

If $f(x,y) = f(x)f(y)$ then,

$$\mathcal{F}^{a_x,a_y}[f(x,y)] = \mathcal{F}^{a_x}[f(x)]\mathcal{F}^{a_y}[f(y)]. \tag{2.13}$$

The two-dimensional fractional Fourier transform is separable by definition, hence the property is evident.

5. *Unitarity*

The two-dimensional kernel is unitary, i.e.

$$B^*_{a_x,a_y}(x,y;x',y') = B^{-1}_{a_x,a_y}(x',y';x,y) = B_{-a_x,-a_y}(x',y';x,y). \tag{2.14}$$

By using the kernel of the transform in 2.3 and the kernel of the inverse transform in 2.11, the property can be verified.

6. *Parseval relation*

The Parseval relation for two-dimensional fractional Fourier transform is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{f}(\mathbf{r}')^*\mathrm{g}(\mathbf{r}')\mathrm{d}\mathbf{r}' = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{\mathcal{F}^{\mathrm{a_x,a_y}}[\mathrm{f}](\mathbf{r}')\}^*\{\mathcal{F}^{\mathrm{a_x,a_y}}[\mathrm{g}](\mathbf{r}')\}\mathrm{d}\mathbf{r}' \tag{2.15}$$

where

$$\mathbf{r}' = \begin{bmatrix} x' & y' \end{bmatrix}^{\mathrm{T}}$$

A direct consequence of this equality is the energy-preserving property of fractional Fourier transform

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(\mathbf{r}')|^2 d\mathbf{r}' = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\mathcal{F}^{a_x,a_y}[f](\mathbf{r}')|^2 d\mathbf{r}'. \tag{2.16}$$

This property follows from the unitarity property of the fractional Fourier transform.

7. *The effect of shift*
The fractional Fourier transform of $f(x - s_x, y - s_y)$ can be expressed in terms of the fractional Fourier transform of $f(x, y)$ as

$$\mathcal{F}^{a_x,a_y}[f(\mathbf{r}-\mathbf{s})](\mathbf{r}) = e^{-i2\pi[\mathbf{b}^{\mathrm{T}}(\mathbf{r}-\frac{1}{2}\mathbf{a})]}\mathcal{F}^{a_x,a_y}[f(\mathbf{r})](\mathbf{r}-\mathbf{a}) \tag{2.17}$$

where

$$\mathbf{r} = \begin{bmatrix} x & y \end{bmatrix}^{\mathrm{T}}, \quad \mathbf{s} = \begin{bmatrix} s_x & s_y \end{bmatrix}^{\mathrm{T}},$$

$$\mathbf{a} = \begin{bmatrix} s_x \cos\phi_x & s_y \cos\phi_y \end{bmatrix}^{\mathrm{T}}, \quad \mathbf{b} = \begin{bmatrix} s_x \sin\phi_x & s_y \sin\phi_y \end{bmatrix}^{\mathrm{T}}.$$

8. *Effect of multiplication by a complex exponential*
If a function $f(x, y)$ is multiplied by an exponential $e^{i2\pi(m_x x + m_y y)}$, then the resulting fractional Fourier transform becomes

$$\mathcal{F}^{a_x,a_y}[e^{i2\pi\mathbf{m}^{\mathrm{T}}\mathbf{r}}f(\mathbf{r})] = e^{i2\pi[\mathbf{m}^{\mathrm{T}}(\mathbf{r}-\frac{1}{2\pi}\mathbf{c})]}\mathcal{F}^{a_x,a_y}[f(\mathbf{r})](\mathbf{r}-\frac{1}{2\pi}\mathbf{a}). \tag{2.18}$$

where

$$\mathbf{r} = \begin{bmatrix} x & y \end{bmatrix}^{\mathrm{T}}, \quad \mathbf{m} = \begin{bmatrix} m_x & m_y \end{bmatrix}^{\mathrm{T}},$$

$$\mathbf{c} = \begin{bmatrix} m_x \sin\phi_x & m_y \sin\phi_y \end{bmatrix}^{\mathrm{T}}, \quad \mathbf{d} = \begin{bmatrix} m_x \sin\phi_x & m_y \sin\phi_y \end{bmatrix}^{\mathrm{T}},$$

This property is easily derived by using the definition of fractional Fourier transform.

9. *Multiplication by powers of coordinate variables* The fractional Fourier transform of $x^m y^n f(x, y)$ for $m, n \geq 0$ is

$$\begin{aligned} &\mathcal{F}^{a_x,a_y}[x^m y^n f(x,y)] \\ &= [x\cos\phi_x + \tfrac{i}{\pi}\sin\phi_x \tfrac{\partial}{\partial x}]^m [y\cos\phi_y + \tfrac{i}{\pi}\sin\phi_y \tfrac{\partial}{\partial y}]^n \mathcal{F}^{a_x,a_y}[f(x,y)]. \end{aligned} \tag{2.19}$$

When $m = 0$ or $n = 0$, the property reduces to the one-dimensional transform's property.

10. *Derivative of* $f(x,y)$

The dual of the multiplication property is the derivative property. The fractional Fourier transform of $\frac{\partial^m}{\partial^m x}\frac{\partial^n}{\partial^n y}f(x,y)$ is

$$\begin{aligned}&\mathcal{F}^{a_x,a_y}[\frac{\partial^m}{\partial x^m}\frac{\partial^n}{\partial y^n}f(x,y)]\\&=[i2\pi x\sin\phi_x+\cos\phi_x\frac{\partial}{\partial x}]^m[i2\pi y\sin\phi_y+\cos\phi_y\frac{\partial}{\partial y}]^n\mathcal{F}^{a_x,a_y}[f(x,y)].\end{aligned} \tag{2.20}$$

Property 9 and 10 are general forms of the corresponding properties of one-dimensional transform, which will be recovered when $m=0$ or $n=0$.

11. *Scaling*

The fractional Fourier transform is not scale-invariant like the common Fourier transform. However, the fractional Fourier transform of a scaled function with orders a_x and a_y can be represented in terms of the fractional Fourier transform of the original function but with different orders a'_x and a'_y. The fractional Fourier transform of $f(k_x x, k_y y)$ can be represented in terms of the fractional Fourier transform of $f(x,y)$ as

$$\mathcal{F}^{a_x,a_y}[f(\mathbf{kr})](\mathbf{r})=C\exp[i\pi\mathbf{r}^{\mathrm{T}}\mathbf{Pr}]\mathcal{F}^{a_{x'},a_{y'}}\{f(\mathbf{r})\}(\mathbf{Sr}) \tag{2.21}$$

where

$$C=\frac{A_{\phi_x}A_{\phi_y}}{|k_x||k_y|A_{\phi'_x}A_{\phi'_y}},$$

$$\mathbf{k}=\begin{bmatrix}k_x & 0\\ 0 & k_y\end{bmatrix},\qquad \mathbf{r}=\begin{bmatrix}x & y\end{bmatrix},$$

$$\phi'_x=\arctan(k_x^2\tan\phi_x),\qquad a'_x=\frac{2a_{\phi'_x}}{\pi},$$

$$\phi'_y=\arctan(k_y^2\tan\phi_y),\qquad a'_y=\frac{2a_{\phi'_y}}{\pi},$$

$$\mathbf{P}=\begin{bmatrix}\cot\phi_x\frac{k_x^4-1}{k_x^4+\cot^2\phi_x} & 0\\ 0 & \cot\phi_y\frac{k_y^4-1}{k_y^4+\cot^2\phi_y}\end{bmatrix},\qquad \mathbf{S}=\begin{bmatrix}\frac{\sin\phi'_x}{k_x\sin\phi_x} & 0\\ 0 & \frac{\sin\phi'_y}{k_y\sin\phi_y}\end{bmatrix}.$$

12. *Rotation*

Let

$$\mathbf{A}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix},$$

then $f(\mathbf{Ar}) = f(\cos\theta x + \sin\theta y, -\sin\theta x + \cos\theta y)$ represents the rotated function with angle θ. For $\phi_x \neq \phi_y$, we cannot represent the two-dimensional fractional Fourier transform of $f(\mathbf{Ar})$ in terms of the two-dimensional fractional Fourier transform of $f(\mathbf{r})$. But for $\phi_x = \phi_y$ For $\phi_x = \phi_y$

$$\mathcal{F}^a[f(\mathbf{Ar})](\mathbf{r}) = \mathcal{F}^a(\mathbf{r})(\mathbf{Ar}) \tag{2.22}$$

which means that when the function is rotated by an angle θ, its fractional Fourier transform is also rotated by the same angle. But this is valid only when the transform orders are equal in both directions.

13. *Arbitrary affine transform*
Let us try to generalize the rotation property to general affine transformation by setting

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

In this case, it is not possible to represent $\mathcal{F}^{a_x,a_y}[f(\mathbf{Ar})](\mathbf{r})$ in terms of a scaled version of fractional Fourier transform of $f(\mathbf{r})$ with a similar relation to 2.22. It is disturbing that, the fractional Fourier transform fails to satisfy this property. In Chapter 4, the same property will be discussed again, treating an alternative definition.

14. *Wigner Distribution and fractional Fourier transform*
Let $W_f(x, y; \mu_x, \mu_y)$ be the Wigner distribution of $f(x, y)$. If $g(x, y)$ is the fractional Fourier transform of $f(x, y)$, then Wigner distribution of $g(x, y)$ is related to that of $f(x, y)$ through the following equation

$$W_g(\mathbf{r}, \mathbf{s}) = W_f(\mathbf{Ar} + \mathbf{Bs}, \mathbf{Cr} + \mathbf{Ds}), \tag{2.23}$$

where

$$\mathbf{r} = \begin{bmatrix} x & y \end{bmatrix}^{\mathrm{T}}, \quad \mathbf{s} = \begin{bmatrix} \mu_x & \mu_y \end{bmatrix}^{\mathrm{T}} \tag{2.24}$$

and

$$\mathbf{A} = \begin{bmatrix} \cos\phi_x & 0 \\ 0 & \cos\phi_y \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -\sin\phi_x & 0 \\ 0 & -\sin\phi_y \end{bmatrix}, \tag{2.25}$$

$$\mathbf{C} = \begin{bmatrix} \sin\phi_x & 0 \\ 0 & \sin\phi_y \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \cos\phi_x & 0 \\ 0 & \cos\phi_y \end{bmatrix}. \tag{2.26}$$

As the above equation suggests, the effect of fractional Fourier transform on the Wigner distribution is a counterclockwise rotation with angle ϕ_x in the x-μ_x plane and ϕ_y in the y-μ_y plane. In the following section, this property will be discussed again as a special case.

15. *Projection*
The projection property of one-dimensional kernel [32, 34] states that the projection of the Wigner distribution function on an axis making angle ϕ with the x axis, is the absolute square of fractional Fourier transform of the function with order $a(\phi = a\pi/2)$. This effect can be represented in terms of the Radon transform as

$$\mathcal{R}_\phi[W(x,\mu)] = |\mathcal{F}^a[f(x)]|^2, \tag{2.27}$$

where the Radon transform of a two-dimensional function is its projection on an axis making angle ϕ with the x axis. The separability of the two-dimensional kernel may be used to derive the corresponding property for two-dimensional case. If the Radon transform is applied successively to the Wigner distribution $W(x,y;\mu_x,\mu_y)$, then the property becomes

$$\mathcal{R}_{\phi_y}[\mathcal{R}_{\phi_x}[W(x,y;\mu_x,\mu_y)]] = |\mathcal{F}^{a_x,a_y}[f(x,y)]|^2. \tag{2.28}$$

Thus, the projection of the Wigner distribution $W(x,y;\mu_x,\mu_y)$ of any function $f(x,y)$ on the plane determined by the two lines, first making an angle ϕ_x with the x axis and second making an angle ϕ_y with the y axis, is the absolute square of its two-dimensional fractional Fourier transform with orders a_x and a_y.

16. *Eigenvalues and eigenfunctions*
Two-dimensional Hermite-Gaussian functions are eigenfunctions of the two-dimensional fractional Fourier transform, i.e.,

$$\int_{-\infty}^{\infty} B_{a_x,a_y}(x,y;x'y')\Psi_{nm}(xx,y')\,dx'\,dy' = \lambda_{nm}\Psi_{nm}(x,y) \tag{2.29}$$

where the eigenfunctions are determined by

$$\Psi_{nm}(x,y) = \frac{2^{1/2}}{\sqrt{2^n 2^m n!\,m!}} H_n(\sqrt{2\pi}x)H_m(\sqrt{2\pi}y)\exp[-\pi(x^2+y^2)] \tag{2.30}$$

with the corresponding eigenvalue

$$\lambda_{nm} = \exp(-i\pi a_x n/2)\exp(-i\pi a_y m/2). \tag{2.31}$$

By using the separability of the two-dimensional fractional Fourier transform and the corresponding property in one-dimension [4, 33, 32], this property may easily be derived. For $a_x = a_y = 1$, the eigenvalues and eigenfunctions corresponding to the common Fourier transform can be recovered.

2.3 Quadratic-phase systems

Fractional Fourier transforms, Fresnel transforms, chirp multiplication and scaling operations are widely used in optics to analyze systems composed of sections of free space and thin lenses. These linear integral transforms belong to the class of quadratic-phase systems. The one-dimensional quadratic-phase system with parameters α, β, γ is defined as [55]

$$g(x) = \int_{-\infty}^{\infty} h(x, x') f(x') dx',$$

$$h(x, x') = \beta^{1/2} e^{-i\pi/4} \exp[i\pi(\alpha x^2 - 2\beta x x' + \gamma x'^2)]. \tag{2.32}$$

Quadratic-phase systems have 3 parameters whereas fractional Fourier transform has only one. Eqn. 2.32 reduces to the definition of fractional Fourier transform if the parameters α, β and γ are chosen as

$$\alpha = \gamma = \cot\phi \quad \text{and} \quad \beta = \csc\phi.$$

Any quadratic-phase system can be completely specified by its parameters α, β, γ as 2.32 suggests. However, an alternative way of specifying quadratic-phase systems is using a transformation matrix. The transformation matrix of such a system specified by the parameters α, β, γ is

$$\mathbf{T} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \alpha/\beta & 1/\beta \\ -\beta + \alpha\gamma/\beta & \alpha/\beta \end{bmatrix} \tag{2.33}$$

with $AD - BC = 1$. The transformation matrix approach is practical in the analysis of quadratic-phase systems. First of all, if several systems are cascaded, the overall system matrix can be found by multiplying the corresponding transformation matrices. Second, the transformation matrix corresponds to the ray-matrix in optics [56]. Third, the effect of the system on the Wigner distribution of the input function can be expressed in terms of this transformation matrix. This topic is extensively discussed in [36-40].

It is possible to generalize one-dimensional quadratic-phase system to two dimensions. A straightforward generalization is to multiply two one-dimensional kernel and form the definition for two-dimensional quadratic-phase system.

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y;x',y')f(x',y')dx'dy',$$

$$\begin{aligned} h(x,y;x',y') &= e^{-i\pi/4}\beta_x^{1/2}\exp[i\pi(\alpha_x x^2 - 2\beta_x xx' + \gamma_x x'^2)] \\ &\times e^{-i\pi/4}\beta_y^{1/2}\exp[i\pi(\alpha_y y^2 - 2\beta_y yy' + \gamma_y y'^2)]. \end{aligned} \tag{2.34}$$

It is also possible to completely specify this two-dimensional transform through its transformation matrix.

$$\mathbf{T} \equiv \begin{bmatrix} A_x & 0 & B_x & 0 \\ 0 & A_y & 0 & B_y \\ C_x & 0 & D_x & 0 \\ 0 & C_y & 0 & D_y \end{bmatrix} \equiv \begin{bmatrix} \gamma_x/\beta_x & 0 & 1/\beta_x & 0 \\ 0 & \gamma_y/\beta_y & 0 & 1/\beta_y \\ -\beta_x + \alpha_x\gamma_x/\beta_x & 0 & \alpha_x/\beta_x & 0 \\ 0 & -\beta_y + \alpha_y\gamma_y/\beta_y & 0 & \alpha_y/\beta_y \end{bmatrix}$$

with $A_x D_x - B_x C_x = 1$ and $A_y D_y - B_y C_y = 1$. By noting that

$$\alpha_x = \gamma_x = \cot\phi_x \quad \text{and} \quad \beta_x = \csc\phi_x \tag{2.35}$$

and

$$\alpha_y = \gamma_y = \cot\phi_y \quad \text{and} \quad \beta_y = \csc\phi_y, \tag{2.36}$$

the transformation matrix for the two-dimensional fractional Fourier transform turn out to be

$$\mathbf{T} \equiv \begin{bmatrix} \cos\phi_x & 0 & \sin\phi_x & 0 \\ 0 & \cos\phi_y & 0 & \sin\phi_y \\ -\sin\phi_x & 0 & \cos\phi_x & 0 \\ 0 & -\sin\phi_y & 0 & \cos\phi_y \end{bmatrix}. \tag{2.37}$$

After deriving the transformation matrix for the two-dimensional quadratic-phase systems, let us examine property 14 which describes the effect of fractional Fourier transform on the Wigner distribution of the input function. The inverse of the transformation matrix characterizes the effect of any quadratic-phase system on the Wigner distribution of the input [57]. The Wigner distribution of the input function W_f and the Wigner distribution of the output W_g are related to each other by the following relation

$$W_g(\mathbf{u}) = W_f(\mathbf{T}^{-1}\mathbf{u}), \tag{2.38}$$

where

$$\mathbf{u} = \begin{bmatrix} x & y & \mu_x & \mu_y \end{bmatrix}^{\mathrm{T}} \tag{2.39}$$

We have already found the transformation matrix $\mathbf{T}$ of the system. When the inverse of the matrix is substituted in 2.38, the result in property 14 is verified.

Chapter 3

Optical implementation of the two-dimensional fractional Fourier transform

In this chapter, various optical implementations of two-dimensional fractional Fourier transform will be presented. Two approaches are used for this purpose. The first approach is based on the canonical type-1 and type-2 decompositions. The second approach classifies the systems according to the number of lenses and then shows the advantages and limitations of each system.

In Chapter 2, it was shown that the fractional Fourier transform is not scale-invariant. In some physical applications, it is necessary to introduce input and output scale parameters. It is possible to modify our definition by including the scale parameters and also the additional phase factors that may occur at the output,

$$\begin{aligned} B_{a_x,a_y}(x,y;x',y') &= A_{\phi_x}\exp[i\pi x^2 p_x]\exp[i\pi(\tfrac{x^2}{s_2^2}\cot\phi_x - \tfrac{2xx'}{s_1 s_2}\csc\phi_x + \tfrac{x'^2}{s_1^2}\cot\phi_x)] \\ &\quad \times A_{\phi_y}\exp[i\pi y^2 p_y]\exp[i\pi(\tfrac{y^2}{s_2^2}\cot\phi_y - \tfrac{2yy'}{s_1 s_2}\csc\phi_y + \tfrac{y'^2}{s_1^2}\cot\phi_y)]. \end{aligned} \tag{3.1}$$

In this definition, s_1 stands for the input scale parameter and s_2 stands for the output scale parameters. In the previous chapter, we derived the transformation matrix for the fractional Fourier transform. But allowing phase factors p_x, p_y and scaling factors s_1 and s_2, the transformation matrix of the fractional Fourier transform can be modified as

$$\mathbf{T} \equiv \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \tag{3.2}$$

where

$$\mathbf{A} = \begin{bmatrix} \frac{s_2}{s_1} \cos\phi_x & 0 \\ 0 & \frac{s_2}{s_1} \cos\phi_y \end{bmatrix} \tag{3.3}$$

$$\mathbf{B} = \begin{bmatrix} s_1 s_2 \sin\phi_x & 0 \\ 0 & s_1 s_2 \sin\phi_y \end{bmatrix} \tag{3.4}$$

$$\mathbf{C} = \begin{bmatrix} \frac{1}{s_1 s_2}[p_x \cos\phi_x - \sin\phi_x] & 0 \\ 0 & \frac{1}{s_1 s_2}[p_y \cos\phi_y - \sin\phi_y] \end{bmatrix} \tag{3.5}$$

$$\mathbf{D} = \begin{bmatrix} \frac{s_1}{s_2} \sin\phi_x (p_x + \cot\phi_x) & 0 \\ 0 & \frac{s_1}{s_2} \sin\phi_y (p_y + \cot\phi_y) \end{bmatrix} \tag{3.6}$$

In our optical set-ups, we will try to control as many parameters as we can. Here is a list of parameters that we would like to control:

Order parameters a_x and a_y: The main objective of designing optical set-ups is to control the orders of the fractional Fourier transform. Control on the order parameters is our primary interest.

Scale parameters, s_1 and s_2: It is desirable to specify both the input and output scale parameters to provide practical set-ups.

Additional phase factors p_x and p_y : In our designs, we try to obtain $p_x = p_y = 0$ in order to remove the additional phase factors at the output plane and observe the fractional Fourier transform on a flat surface.

Before going through the optical systems in detail, the characterization of optical components will be given.

3.1 Characterization of optical components

In Chapter 2, the concept of transformation matrix is introduced. Here both the kernels and the transformation matrices of the optical components will be given. The transformation kernel for a free-space propagation of length d is expressed as

$$h_f(x,y,x',y') = K_f \exp\left(i\pi\left[\frac{(x-x')^2}{\lambda d} + \frac{(y-y')^2}{\lambda d}\right]\right), \tag{3.7}$$

and its corresponding transformation matrix is

$$\mathbf{T}_f(d) = \begin{bmatrix} 1 & 0 & \lambda d & 0 \\ 0 & 1 & 0 & \lambda d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.8}$$

Similarly, the kernel for a cylindrical lens with focal length f_x along the x direction is

$$h_{xl}(x,y,x',y') = K_{xl}\,\delta(x-x')\,\exp(-i\pi x^2/\lambda f_x) \tag{3.9}$$

with its transformation matrix

$$\mathbf{T}_{xl}(f_x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-1}{\lambda f_x} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.10}$$

and the kernel for a cylindrical lens with focal length f_y along the y direction is

$$h_{yl}(x,y,x',y') = K_{yl}\,\delta(y-y')\,\exp(-i\pi y^2/\lambda f_y) \tag{3.11}$$

with its transformation matrix

$$\mathbf{T}_{yl}(f_y) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{-1}{\lambda f_y} & 0 & 1 \end{bmatrix}. \tag{3.12}$$

When we consider an anamorphic lens with focal length f_x in the x direction, f_y in the y direction and f_{xy} in the xy direction, the kernel is

$$h_{xyl}(x,y,x',y') = K_{xyl}\,\delta(x-x',y-y')\,\exp\left(-i\pi\left[\frac{x^2}{\lambda f_x}+\frac{y^2}{\lambda f_y}+\frac{xy}{\lambda f_{xy}}\right]\right) \quad (3.13)$$

with the transformation matrix

$$\mathbf{T}_{xyl}(f_y) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-1}{\lambda f_x} & \frac{-1}{2\lambda f_{xy}} & 1 & 0 \\ \frac{-1}{2\lambda f_{xy}} & \frac{-1}{\lambda f_y} & 0 & 1 \end{bmatrix}. \quad (3.14)$$

3.2 Optical implementation using canonical decompositions

We will begin our discussion with the canonical type-1 and type-2 systems [58] which can be used to implement one-dimensional quadratic-phase systems. Then the canonical systems will be generalized to two dimensions. Since fractional Fourier transform belongs to the family of quadratic-phase systems, once the optical implementations of quadratic-phase systems are found, the results may be specialized to fractional Fourier transform.

3.2.1 Optical implementation of one-dimensional quadratic-phase systems

It is possible to use type-1 and type-2 realizations to implement any quadratic-phase system with desired parameters α, β and γ optically.

TYPE-1:

Both the optical system in 3.1 and the quadratic-phase system have three parameters. In order to determine the system parameters the relation between

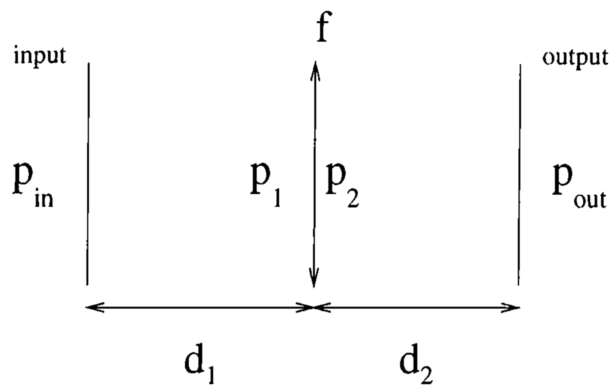


Figure 3.1: Type-1 system that realizes one-dimensional quadratic-phase system

the light distribution $p_{\rm in}(x)$ at the input and light distribution at the output $p_{\rm out}(x)$ should be found. Assuming propagation from left to right, $p_1(x)$ (the light distribution just before the lens) is related to $f(x)$ by a Fresnel integral:

$$p_1(x) = \frac{\exp(i2\pi d_1/\lambda)}{\sqrt{i\lambda d_1}} \int_{-\infty}^{\infty} \exp[i\pi(x-x')^2/\lambda d_1] p_{\rm in}(x')\, dx' \tag{3.15}$$

The light distribution at the right of the lens is

$$p_2(x) = p_1(x,y) \exp\left[\frac{-i\pi x^2}{\lambda f}\right] \tag{3.16}$$

Propagation in the second section of free space results in another convolution. The light distribution at the output is

$$p_{\rm out}(x) = \frac{\exp(i2\pi d_2/\lambda)}{\sqrt{i\lambda d_2}} \int_{-\infty}^{\infty} \exp[i\pi(x-x'')^2/\lambda d_2] p_2(x'')\, dx'' \tag{3.17}$$

When the terms are rearranged and the integral on x'' is carried out, the resulting relation becomes

$$p_{\rm out}(x) = K \int_{-\infty}^{\infty} \exp[i\pi(Ax^2 - 2Bxx' + Cx^2)] p_{\rm in}(x')\, dx', \tag{3.18}$$

where

$$K = \frac{\exp(i2\pi(d_1+d_2)/\lambda)}{\sqrt{i\lambda(d_1+d_2)}}$$

$$A = \frac{f-d_1}{\lambda(d_1 f + d_2 f - d_1 d_2)}$$

$$B = \frac{f}{\lambda(d_1 f + d_2 f - d_1 d_2)}$$
$$C = \frac{f - d_2}{\lambda(d_1 f + d_2 f - d_1 d_2)}$$

If we wish 3.18 to represent a quadratic-phase system with parameters α, β and γ, the following necessary and sufficient conditions should be satisfied:

$$A = \frac{f - d_1}{\lambda(d_1 f + d_2 f - d_1 d_2)} = \alpha \tag{3.19}$$

$$B = \frac{f}{\lambda(d_1 f + d_2 f - d_1 d_2)} = \beta \tag{3.20}$$

$$C = \frac{f - d_2}{\lambda(d_1 f + d_2 f - d_1 d_2)} = \gamma \tag{3.21}$$

It is possible to define the system parameters uniquely by solving the above equations. The equations for d_1, d_2 and f in terms of α, β and γ are

$$d_1 = \frac{\beta - \alpha}{\lambda(\beta^2 - \gamma\alpha)}, \quad d_2 = \frac{\beta - \gamma}{\lambda(\beta^2 - \gamma\alpha)}, \quad f = \frac{\beta}{\lambda(\beta^2 - \gamma\alpha)}. \tag{3.22}$$

By using this set-up, it is possible to implement one-dimensional fractional Fourier transform of the desired order. The scale parameters s_1 and s_2 may be specified by the designer and the additional phase factors p_x and p_y may be made equal to zero. Letting $\alpha = \cot\phi/s_2^2$, $\gamma = \cot\phi/s_1^2$ and $\beta = \csc\phi/s_1 s_2$, one recovers Lohmann's type-1 system that performs fractional Fourier transform. In this case, the system parameters are

$$d_1 = \frac{(s_1 s_2 - s_1^2 \cos\phi)}{\lambda \sin\phi}, \quad d_2 = \frac{(s_1 s_2 - s_2^2 \cos\phi)}{\lambda \sin\phi}, \quad f = \frac{s_1 s_2}{\lambda \sin\phi}. \tag{3.23}$$

Since the additional phase factors are set to zero, they do not appear in the equations. However, if one wishes to set p_x and p_y to a value other than zero, it is again possible by setting $\alpha = p_x \cot\phi/s_2^2$ and substituting it in Eqn. 3.22.

TYPE-2:

Instead of one lens and two sections of free space, we have two lenses separated by a single section of free space. For this system, the parameters d, f_1 and f_2 are given by the following equations:

$$d = \frac{1}{\lambda\beta}, \quad f_1 = \frac{1}{\lambda(\beta - \gamma)}, \quad f_2 = \frac{1}{\lambda(\beta - \alpha)}. \tag{3.24}$$

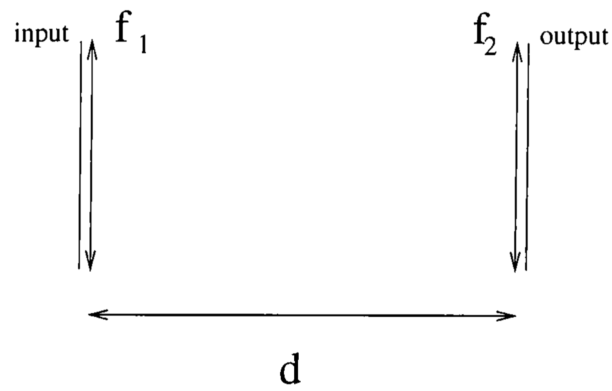


Figure 3.2: Type-2 system that realizes one-dimensional quadratic-phase system

If $\alpha = \cot\phi/s_2^2$, $\gamma = \cot\phi/s_1^2$ and $\beta = \csc\phi/s_1 s_2$ is substituted in these equations the expressions for fractional Fourier transform can be found. The designer can again specify the scale parameters and there is no additional phase factor at the output. The system parameters are

$$d = \frac{s_1 s_2 \sin\phi}{\lambda}, \quad f_1 = \frac{s_1^2 s_2 \sin\phi}{s_1 - s_2\cos\phi}, \quad f_2 = \frac{s_1 s_2^2 \sin\phi}{s_2 - s_1\cos\phi}. \tag{3.25}$$

Equations 3.22 and 3.24 give the expressions for the system parameters of type-1 and type-2 systems. But for some values of α, β and γ, the lengths of free space sections may turn out to be negative. But in our optical systems, we must require that the lengths of free space sections be positive. However, this constraint will restrict the range quadratic-phase systems that can be realized with the suggested set-ups. In section 3.2.3, we will solve this problem by designing an optical set-up that simulates anamorphic free space. This system is designed in such a way that its effect is equivalent to propagation in free space with different (and possibly negative) distances along the two dimensions.

3.2.2 Optical implementation of two-dimensional quadratic-phase systems

In order to find an optical realization of the two-dimensional fractional Fourier transform, a two-dimensional analysis is needed. Hence, we will have to deal with two-dimensional kernels or 4×4 matrices. But the following theorem allows us to analyze multi-dimensional systems as many one-dimensional systems, which makes the analysis remarkably easier.

Theorem 3.1 *Let*

$$g(\mathbf{r}) = \int_{-\infty}^{\infty} h(\mathbf{r}, \mathbf{r}') f(\mathbf{r}')\, d\mathbf{r}',$$

where

$$\mathbf{r} = \begin{bmatrix} x_1 & \dots & x_N \end{bmatrix}^{\mathrm{T}}, \quad \mathbf{r}' = \begin{bmatrix} x'_1 & & x'_N \end{bmatrix}^{\mathrm{T}}$$

If the kernel $h(\mathbf{r}, \mathbf{r}')$ is separable, i.e.

$$h(\mathbf{r}, \mathbf{r}') = h_1(x_1, x'_1)\, h_2(x_2, x'_2) \dots h_N(x_N, x'_N), \tag{3.26}$$

then the response in the x_i direction is the result of the one-dimensional transform

$$g_i(x_i, x'_1, x'_2 \dots x'_N) = \int_{-\infty}^{\infty} h_i(x_i, x'_i)\, f(x'_1, \dots x'_N)\, dx'_i \quad \text{for } i = 1 \text{ to } N. \tag{3.27}$$

Moreover if the function is also separable i.e.

$$f(\mathbf{r}) = f_1(x_1, x'_1)\, f_2(x_2, x'_2) \dots f_N(x_N, x'_N), \tag{3.28}$$

the overall response of the system is

$$g(\mathbf{r}) = g_1(x_1)\, g_2(x_2) \dots g_N(x_N) \tag{3.29}$$

where

$$g_i(x_i) = \int_{-\infty}^{\infty} h_i(x_i, x'_i)\, f_i(x'_i)\, dx'_i \qquad \text{for } i = 1 \text{ to } N. \tag{3.30}$$

Proof: If 3.28 and 3.26 is substituted in 3.26, then we have

$$g(\mathbf{r}) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h_1(x_1, x'_1) \dots h_N(x_N, x'_N) f_1(x'_1) \dots f_N(x'_N) dx'_1 \dots dx'_N$$

Rearranging terms will give us the desired result.

This simple theorem has a nice interpretation in optics which makes the analysis of the multi-dimensional systems easier. For example, in order to

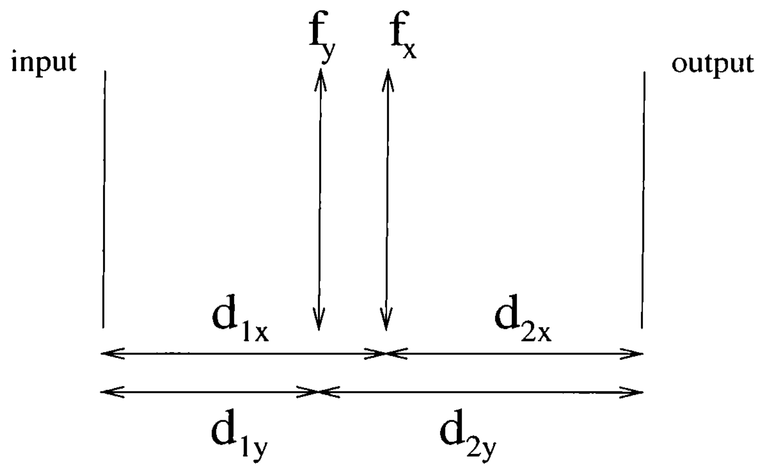


Figure 3.3: Type-1 system that realizes two-dimensional quadratic-phase systems

design an optical set-up that realizes imaging in x direction and Fourier transform in y direction, one can design two one-dimensional systems that realize the given transformations. When these two systems are put together, the overall effect of the system is imaging in x direction and Fourier transformation in y direction. Similarly if we can find a system that realizes fractional Fourier transform with order a_x in x direction and another system which realizes fractional Fourier transform with order a_y in y direction, then these two optical set-ups will together implement two-dimensional fractional Fourier transform. So the problem of designing a two-dimensional fractional Fourier transformer reduces to the problem of designing two one-dimensional fractional Fourier transformers.

TYPE-1:

According to Theorem 1, x and y directions can be considered independent of each other. Hence if two optical set-ups realizing one-dimensional quadratic-phase systems are put together, one can implement the desired two-dimensional fractional Fourier transform. The suggested optical system can be found in Fig. 3.3.

Parameters of type-1 system:

$$d_{1x} = \frac{\beta_x - \alpha_x}{\lambda(\beta_x^2 - \gamma_x\alpha_x)}, \quad d_{2x} = \frac{\beta_x - \gamma_x}{\lambda(\beta_x^2 - \gamma_x\alpha_x)}, \quad f_x = \frac{\beta_x}{\lambda(\beta_x^2 - \gamma_x\alpha_x)}, \tag{3.31}$$

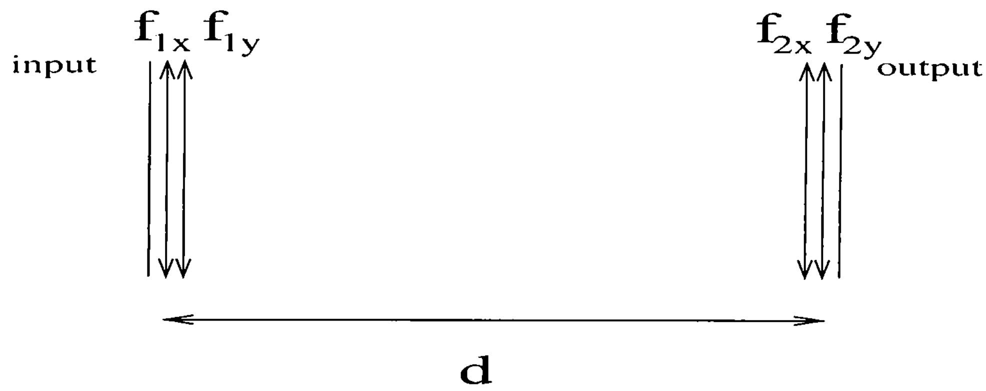


Figure 3.4: Type-2 system that realizes two-dimensional quadratic-phase systems

$$d_{1y} = \frac{\beta_y - \alpha_y}{\lambda(\beta_y^2 - \gamma_y\alpha_y)}, \quad d_{2x} = \frac{\beta_y - \gamma_y}{\lambda(\beta_y^2 - \gamma_y\alpha_y)}, \quad f_y = \frac{\beta_y}{\lambda(\beta_y^2 - \gamma_y\alpha_y)}. \quad (3.32)$$

The parameters of the optical system are given in equations 3.31 and 3.32. Even though the analysis is carried out by using the independence of x and y directions, the total length of the optical system is fixed. Thus $d_{1x}+d_{2x} = d_x = d_{1y} + d_{2y} = d_y$ should always be satisfied. The other constraint to be satisfied is the positivity of the lengths of the free space sections. d_{1x}, d_{1y}, d_{2x} and d_{2y} should always be positive. These two constraints restrict the set of quadratic-phase systems that can be implemented. The solution to this problem is to try to simulate anamorphic sections of free space which provides us a propagation of d_x in x direction and d_y in y direction where d_x and d_y may take negative values. The simulation of anamorphic free space will be given after type-2 system is analyzed. Besides different propagation distances, our free space should also simulate propagation with negative distances.

TYPE-2:

Two type-2 systems can also perform the desired two-dimensional quadratic-phase system.

Parameters of type-2 system:

$$d_x = \frac{1}{\lambda\beta_x}, \quad f_{1x} = \frac{1}{\lambda(\beta_x - \gamma_x)}, \quad f_{2x} = \frac{1}{\lambda(\beta_x - \alpha_x)}, \quad (3.33)$$

$$d_y = \frac{1}{\lambda\beta_y}, \quad f_{1y} = \frac{1}{\lambda(\beta_y - \gamma_y)}, \quad f_{2y} = \frac{1}{\lambda(\beta_y - \alpha_y)}. \quad (3.34)$$

The optical set-up in Fig. 3.4 with parameters given in the above equations implement two-dimensional quadratic-phase systems. In this optical set-up the constraint becomes $d_x = d_y = d$ which is even more restrictive. d_x and d_y can again be negative. In order to overcome these difficulties, we will try to design an optical set-up which simulates anamorphic sections of free space.

3.2.3 Simulation of anamorphic sections of free space

While designing optical set-ups that implement one-dimensional quadratic-phase systems, we treat the lengths of free space sections as free parameters. But some quadratic-phase systems specified by parameters α, γ, β, may require the use of free space sections with negative length. This problem is again encountered in the optical set-ups realizing two-dimensional quadratic-phase systems. Besides, the two-dimensional optical systems require different propagation distances in x and y directions. In order to implement all possible one-dimensional and two-dimensional quadratic-phase systems, we will design an optical system simulating the desired free space suitable for our purposes. The optical system in Fig. 3.5 which is composed of a Fourier block, an anamorphic lens and an inverse Fourier block simulates two-dimensional free space with propagation distances d_x in x direction and d_y in y direction. We will call the optical system in Fig. 3.5 as 'anamorphic free space'. When the analysis of the system in Fig. 3.5 is made the relation between the input light distribution $f(x,y)$ and output light distribution $g(x,y)$ is given as

$$g(x,y) = C \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[i\pi(x-x')^2/\lambda d_x + (y-y')^2/\lambda d_y] f(x',y')\, dx'\, dy' \quad (3.35)$$

where

$$d_x = \frac{s^4}{\lambda^2 f_x}, \quad d_y = \frac{s^4}{\lambda^2 f_y}. \quad (3.36)$$

where s is the scale of the Fourier and inverse Fourier blocks. f_x and f_y can take any real value including negative ones. Thus it is possible to obtain any combination of d_x and d_y by using the optical set-up in Fig. 3.5. The anamorphic lens which is used to control d_x and d_y, may be composed of two orthogonally situated cylindrical thin lenses with different focal lengths. The Fourier block and

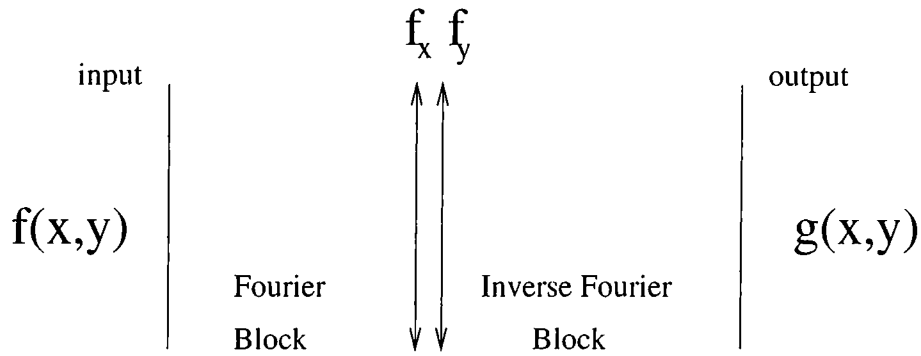


Figure 3.5: Optical system that simulates anamorphic free space propagation

inverse Fourier block are 2-f systems with a spherical lens between two sections of free space. Thus, a section of free space uses 2 cylindrical and 2 spherical lenses. The system in Fig. 3.5, simulates two-dimensional anamorphic free space. The same configuration is again a valid realization for one-dimensional case. When only one lens is used with one-dimensional Fourier and inverse Fourier blocks, it is possible to simulate propagation with negative distances. When the free space sections in the type-1 and type-2 systems are replaced by the optical set-up in Fig. 3.5, optical implementation of all separable quadratic-phase systems can be realized.

3.2.4 Optical implementation of two-dimensional fractional Fourier transform

In the previous section, we proposed two optical systems that realize any two-dimensional quadratic-phase system. It was discussed earlier that two-dimensional fractional Fourier transform is indeed a special quadratic-phase system with parameters

$$\alpha_x = \cot\phi_x/s_2^2, \qquad \gamma_x = \cot\phi_x/s_1^2, \qquad \beta_x = \csc\phi_x/s_1 s_2,$$

and

$$\alpha_y = \cot\phi_y/s_2^2, \qquad \gamma_y = \cot\phi_y/s_1^2, \qquad \beta_y = \csc\phi_y/s_1 s_2.$$

When these parameters are substituted in 2.34, the definition of two-dimensional fractional Fourier transform is obtained. Since fractional Fourier

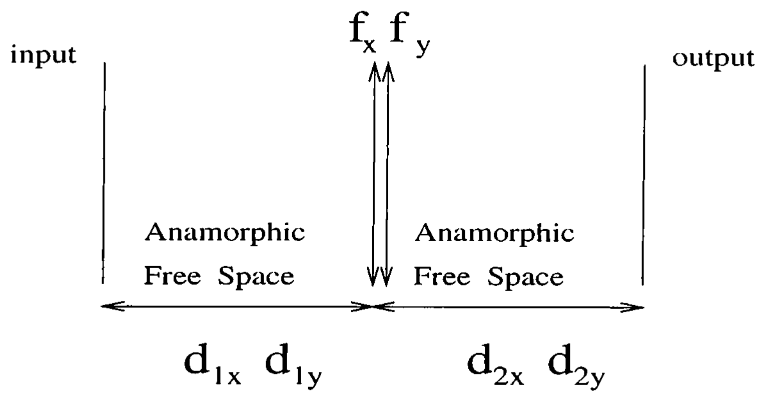


Figure 3.6: Type-1 optical system realizing two-dimensional fractional Fourier transform

transform belongs to the family of quadratic-phase systems, the optical set-ups suggested for the quadratic-phase systems are again valid realizations for fractional Fourier transform.

TYPE-1

The optical system in Fig. 3.6, realizes two-dimensional fractional Fourier transform with desired orders a_x, a_y, desired scale parameters s_1, s_2. There is no additional phase factors at the output. The system has 2 cylindrical lenses and 2 sections of anamorphic free space. Since a section of anamorphic free space consists of 2 cylindrical and 2 spherical lenses, the total number of lenses is 6 cylindrical and 4 spherical lenses. The system parameters are easily found from 3.31 and 3.32 as

$$d_{1x} = \frac{(s_1 s_2 - s_1^2 \cos\phi_x)}{\lambda \sin\phi_x}, \qquad d_{2x} = \frac{(s_1 s_2 - s_1^2 \cos\phi_x)}{\lambda \sin\phi_x}, \tag{3.37}$$

$$d_{1y} = \frac{(s_1 s_2 - s_1^2 \cos\phi_y)}{\lambda \sin\phi_y}, \qquad d_{2y} = \frac{(s_1 s_2 - s_1^2 \cos\phi_y)}{\lambda \sin\phi_y}, \tag{3.38}$$

$$f_x = \frac{s_1 s_2}{\lambda \sin\phi_x}, \qquad f_y = \frac{s_1 s_2}{\lambda \sin\phi_y}. \tag{3.39}$$

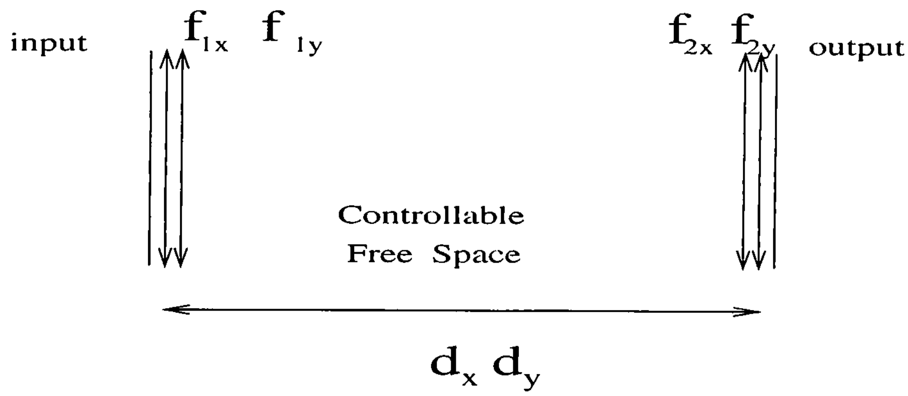


Figure 3.7: Type-2 optical system realizing two-dimensional fractional Fourier transform

TYPE-2

The analysis of the type-2 system is similar to type-1 system. If the free space sections in the type-2 system are replaced by sections of anamorphic free space, the two-dimensional fractional Fourier transform with the desired orders and scale parameters can be implemented. In this set-up, we have to use 6 cylindrical and 2 spherical lenses. The system parameters are

$$f_{1x} = \frac{s_1^2 s_2 \sin\phi_x}{\lambda(s_1 - s_2\cos\phi_x)}, \qquad f_{2x} = \frac{s_1 s_2^2 \sin\phi_x}{\lambda(s_2 - s_1\cos\phi_x)}, \tag{3.40}$$

$$f_{1y} = \frac{s_1^2 s_2 \sin\phi_y}{\lambda(s_1 - s_2\cos\phi_y)}, \qquad f_{2y} = \frac{s_1 s_2^2 \sin\phi_y}{\lambda(s_2 - s_1\cos\phi_y)}, \tag{3.41}$$

$$d_x = \frac{s_1 s_2}{\lambda\csc\phi_x}, \qquad d_y = \frac{s_1 s_2}{\lambda\csc\phi_y}. \tag{3.42}$$

Both type-1 and type-2 systems can implement all combinations of orders when the free space sections are replaced by sections of anamorphic free space. We have no additional phase factors at the output. Also the scale parameters can be specified by the designer. Thus, by using type-1 and type-2 systems, all combinations of orders a_x and a_y can be implemented with full control on scale parameters s_1, s_2 and phase factors p_x, p_y.

3.3 Other optical implementations of two-dimensional fractional Fourier transform

In the previous section, we presented a method of implementing the fractional Fourier transform optically. All combinations of a_x and a_y can be implemented with the proposed set-ups. However, both systems use 6 cylindrical lenses. In this section, we will consider simpler optical system having fewer lenses and try to see the limitations of these systems. We will not try to exhaust all possibilities, but offer several systems which we believe may be useful. Since the problem is solved in x and y directions independently, one lens is not adequate to control both directions. So the simplest set-up that we will consider has two cylindrical lenses.

3.3.1 Two-lens systems

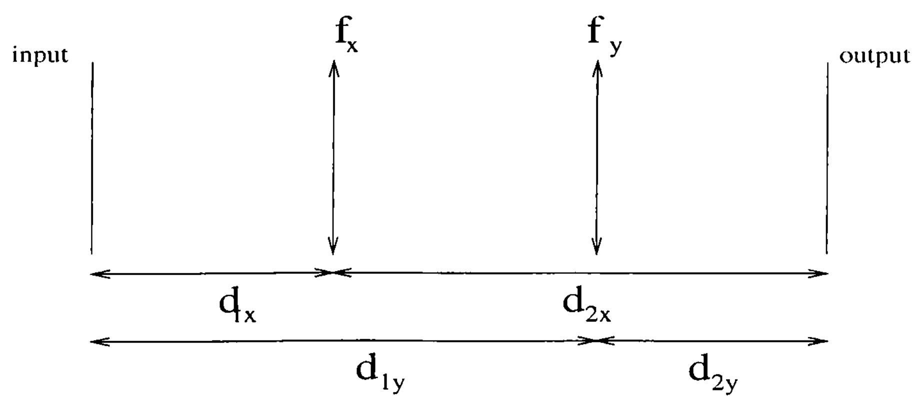


Figure 3.8: Optical set-up with 2 cylindrical lenses and 3 sections of free space

1. *Specified by the designer:* ϕ_x,ϕ_y, s_1, s_2, p_x, p_y.
   Design parameters: f_x, $f_y d_{1x}$, d_{1y}, d_{2x}, d_{2y}.
   Uncontrollable outcomes: None.
   The optical set-up in 3.3.1 has 6 design parameters and we also want to specify 6 parameters. It is possible to solve the design parameters in terms of the desired parameters determined by the designer. However, in order to have realizable set-up, the following constraints should be

satisfied:

- Total length of the system should be the same in both directions; $d_{1x} + d_{2x} = d_{1y} + d_{2y}$.
- The lengths of all free space sections should be positive; $d_{1x} \geq 0, d_{1y} \geq 0, d_{2x} \geq 0$ and $d_{2y} \geq 0$.
  These constraints are too restrictive and the range of orders a_x and a_y that can be implemented is very small. Thus we have to reduce the number of parameters that we want to control. This is considered next.

2. *Specified by the designer:* ϕ_x, ϕ_y, s_1, s_2.
   Design parameters: $f_x, f_y, d_{1x}, d_{1y}, d_{2x}, d_{2y}$.
   Uncontrollable outcomes: p_x, p_y.
   In this design, both the orders and the scale parameters can be specified. For given ϕ_x and ϕ_y, s_1 and s_2, the design parameters are

$$d_{1x} = d_{1y} = d_1 = \frac{s_1^2(\sin\phi_y - \sin\phi_x)}{\lambda(\cos\phi_y - \cos\phi_x)}, \tag{3.43}$$

$$d_{2x} = d_{2y} = d_2 = \frac{s_1 s_2 \sin(\phi_x - \phi_y)}{\lambda(\cos\phi_y - \cos\phi_x)}, \tag{3.44}$$

$$f_x = \frac{s_1^2 s_2 \sin(\phi_x - \phi_y)}{\lambda(s_1 - s_2\cos\phi_x)(\cos\phi_y - \cos\phi_x)}, \tag{3.45}$$

$$f_y = \frac{s_1^2 s_2 \sin(\phi_x - \phi_y)}{\lambda(s_1 - s_2\cos\phi_y)(\cos\phi_y - \cos\phi_x)}, \tag{3.46}$$

and the phase factors occuring at the output plane turn out to be

$$p_x = \frac{[s_2(\cos\phi_y - \cos\phi_x) + s_1(1 - \cos(\phi_y - \phi_x))]}{s_1 s_2^2 \sin(\phi_x - \phi_y)}, \tag{3.47}$$

$$p_y = \frac{[s_2(\cos\phi_y - \cos\phi_x) + s_1(\cos(\phi_y - \phi_x) - 1)]}{s_1 s_2^2 \sin(\phi_x - \phi_y)}. \tag{3.48}$$

In this optical set-up, d_1 and d_2 should always be positive. But for some values of ϕ_x, ϕ_y, s_1 and s_2, d_1 and d_2 may turn out to be negative. In

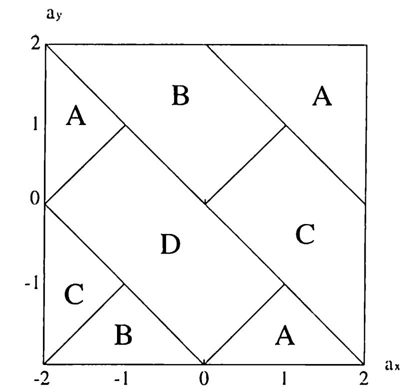


Figure 3.9: A:No flip, B:Flip of x axis, C:Flip of y axis, D: Flip of both axes

such cases we would have to deal with virtual objects and/or images. This would require the use of additional lenses. To avoid this, we must require that d_1 and d_2 be positive. This will then restrict the range of a_x and a_y, that can be realized. This range can be maximized by allowing the x or y axes to be flipped. For instance, if the given values of $d_{1x}, d_{2x}, d_{1y}, d_{2y}$ makes s_1 negative for $\phi_x = 60$ and $\phi_y = 30$, we flip one of the axes. This transform is equivalent to the fractional Fourier transform with $\phi_x = 60$ and $\phi_y = 210$ followed by a flip of the y axis or $\phi_x = 240$ and $\phi_y = 30$ followed by a flip of the x axis. (This is because a transform of order 2 corresponds to a flip of the coordinate axis.)In order to implement some orders, both axes should be flipped. Fig. 3.9 shows the necessary flip (s) required to realize different combinations of orders. This system allows us to specify the orders and scale parameters. However, the phase factors are arbitrary and out of our control. We should examine four-lens systems to control orders, scale parameters and phase factors at the same time.

3.3.2 Four-lens systems

We continue our discussion with the set-up in figure 3.10. The transformation matrix $\mathbf{T}_1$ of the system is found through multiplying the transformation matrices.

$$\mathbf{T}_1 = \mathbf{T}_{yl}(f_{y2})\,\mathbf{T}_{xl}(f_{x2})\,\mathbf{T}_f(d_2)\,\mathbf{T}_{yl}(f_{y1})\,\mathbf{T}_{xl}(f_{x1})\,\mathbf{T}_f(d_1), \qquad (3.49)$$

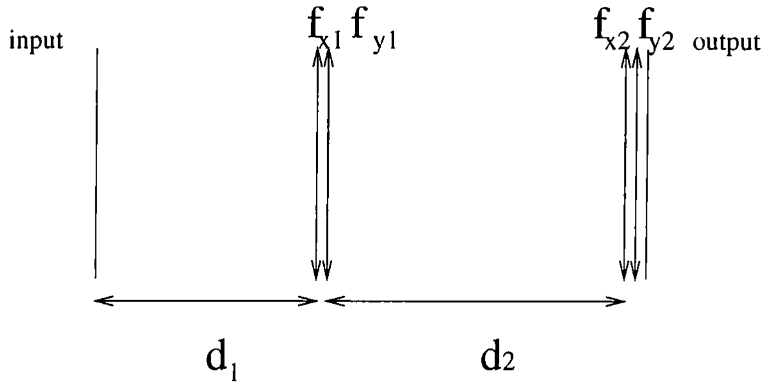


Figure 3.10: Optical set-up with 4 cylindrical lenses and 2 sections of free space

1. *Specified by the designer:* $\phi_x.\phi_y, s_1, s_2, p_x = p_y = 0$.
   Design parameters: $d_1, d_2, f_{x1}, f_{y1}, f_{x2}, f_{y2}$.
   Uncontrollable outcomes: None.
   In this configuration, we use the optical set-up in 3.10. In our previous design with 2 lenses, we managed to design an optical set-up that implements two-dimensional fractional Fourier transform with desired orders and scale parameters. However, additional phase factors at the output plane turned out to be arbitrary. If two cylindrical lenses are added to the output plane two-lens system, it is possible to remove the additional phase factor at the output. In this optical set-up d_1, d_2, f_{x1} and f_{x2} have the same expressions with the former two-lens system. Thus Fig. 3.9 is again valid and shows the necessary flips.

$$d_{1x} = d_{1y} = d_1 = \frac{s_1^2(\sin\phi_y - \sin\phi_x)}{\lambda(\cos\phi_y - \cos\phi_x)}, \tag{3.50}$$

$$d_{2x} = d_{2y} = d_2\frac{s_1 s_2 \sin(\phi_x - \phi_y)}{\lambda(\cos\phi_y - \cos\phi_x)}, \tag{3.51}$$

$$f_{x1} = \frac{s_1^2 s_2 \sin(\phi_x - \phi_y)}{\lambda(s_1 - s_2\cos\phi_x)(\cos\phi_y - \cos\phi_x)}, \tag{3.52}$$

$$f_{y1} = \frac{s_1^2 s_2 \sin(\phi_x - \phi_y)}{\lambda(s_1 - s_2\cos\phi_y)(\cos\phi_y - \cos\phi_x)}, \tag{3.53}$$

$$f_{x2} = \frac{s_1 s_2^2 \sin(\phi_x - \phi_y)}{\lambda[s_2(\cos\phi_y - \cos\phi_x) + s_1(1 - \cos(\phi_y - \phi_x))]}, \tag{3.54}$$

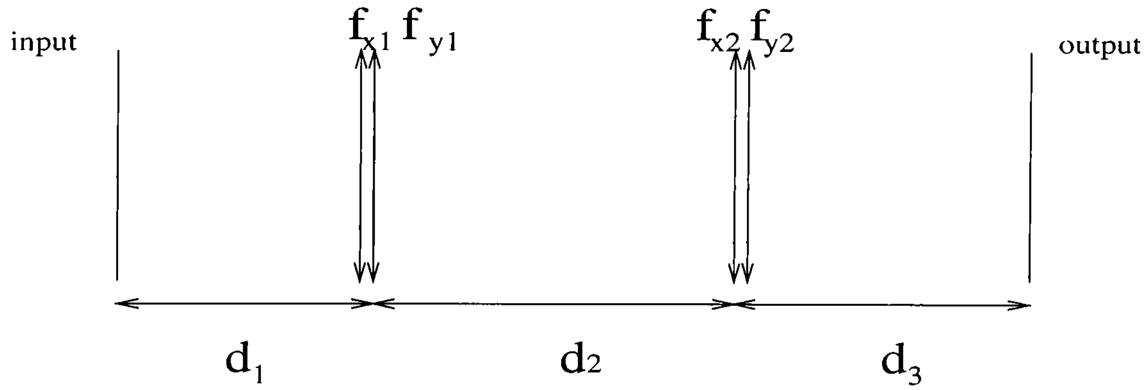


Figure 3.11: Optical set-up with 4 cylindrical lenses and 3 sections of free space

$$f_{y2} = \frac{s_1 s_2^2 \sin(\phi_x - \phi_y)}{\lambda[s_2(\cos\phi_y - \cos\phi_x) + s_1(\cos(\phi_y - \phi_x) - 1)]}. \tag{3.55}$$

This optical set-up implements two-dimensional fractional Fourier transform with the desired orders, scale parameters and phase factors.

2. *Specified by the designer:* $\phi_x, \phi_y, s_1, s_2, d_1, d_2, d_3$.
   Design parameters: $f_{x1}, f_{y1}, f_{x2}, f_{y2}$.
   Uncontrollable outcomes: p_x, p_y.
   For practical purposes, one may want to use a fixed system in which the lengths of all free space sections are fixed. For example, in [19], two-dimensional fractional Fourier transform is implemented by using cylindrical lenses with dynamically adjustable focal lengths in a fixed system. Both the location of lenses and the total length of the system is fixed. The only design parameters are the focal lengths of lenses which can be changed dynamically.
   Here we add one more section of free space to the system in 3.10 and obtain the set-up in Fig. 3.11. This fixed system has no control on phase factors while the orders and scale parameters can be specified by the designer. The parameters are

$$f_{x1} = \frac{s_1 s_2 d_2 \sin\phi_x/\lambda - (s_2/s_1) d_1 d_2 \cos\phi_x}{(s_2/s_1)(d_1 + d_2)\cos\phi_x - s_1 s_2 \sin\phi_x/\lambda + d_3}, \tag{3.56}$$

$$f_{y1} = \frac{s_1 s_2 d_2 \sin\phi_y/\lambda - (s_2/s_1) d_1 d_2 \cos\phi_y}{(s_2/s_1)(d_1 + d_2)\cos\phi_y - s_1 s_2 \sin\phi_y/\lambda + d_3}, \tag{3.57}$$

$$f_{x2} = \frac{d_2 d_3}{(s_2/s_1) d_1 \cos\phi_x - s_1 s_2 \sin\phi_x/\lambda + d_2 + d_3}, \tag{3.58}$$

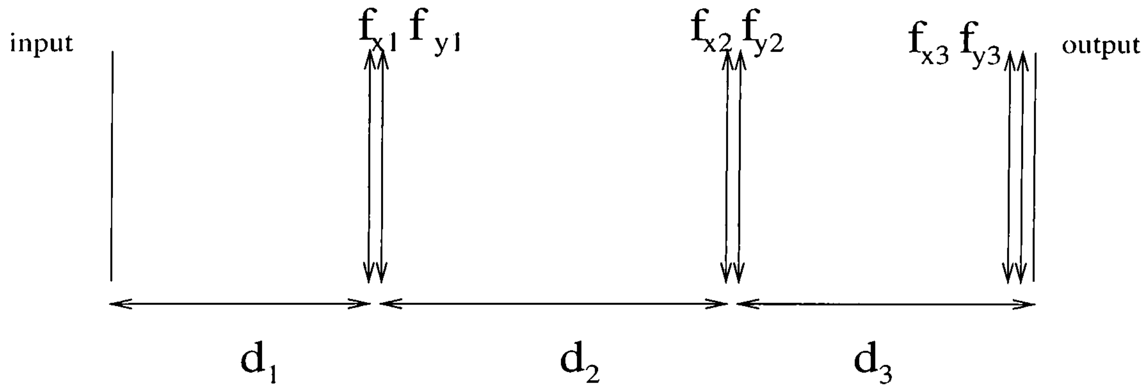


Figure 3.12: Optical system with 6 lenses and 3 sections of free space

and
$$f_{y2} = \frac{d_2 d_3}{(s_2/s_1)d_1 \cos\phi_y - s_1 s_2 \sin\phi_y/\lambda + d_2 + d_3}, \tag{3.59}$$
and the additional phase factors turn out to be
$$p_x = -\cos\phi_x + \frac{s_2}{s_1 \sin\phi_x}\left[1 - \frac{d_1}{f_{x1}} - \frac{d_1}{f_{x2}} - \frac{d_2}{f_{x2}} + \frac{d_1 d_2}{f_{x1} f_{x2}}\right], \tag{3.60}$$

$$p_y = -\cos\phi_y + \frac{s_2}{s_1 \sin\phi_y}\left[1 - \frac{d_1}{f_{y1}} - \frac{d_1}{f_{y2}} - \frac{d_2}{f_{y2}} + \frac{d_1 d_2}{f_{y1} f_{y2}}\right]. \tag{3.61}$$

This optical set-up can realize all combinations of a_x and a_y, however with additional uncontrollable phase factors observed at the output plane.

3.3.3 Six-Lens systems

Specified by the designer: $\phi_x, \phi_y, s_1, s_2, d_1, d_2, d_3, p_x = p_y = 0$.
Design parameters: $f_{x1}, f_{y1}, f_{x2}, f_{y2}$.
Uncontrollable outcomes: None.
The modified type-1 and type-2 systems use 6 cylindrical lenses. However, the lengths of the free space sections are not fixed. For practical purposes like we mentioned before, one may want to use a fixed system. In order to have control on all the parameters, a 6-lens system is required. The design that we made using the four-lens fixed system, has two uncontrollable outcomes, p_x and p_y. If

tw cylindrical lenses are added to the output plane, full control on parameters is achieved.

The system parameters f_{x1}, f_{y1}, f_{2x} and f_{y2} are the same with the 4-lens fixed system. The focal lengths of the additional lenses are

$$f_{x3} = \frac{1}{\lambda p_x}, \tag{3.62}$$

$$f_{y3} = \frac{1}{\lambda p_y}. \tag{3.63}$$

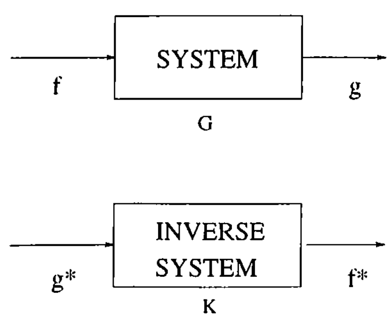
Thus, the fixed optical system in Fig. 3.12 can be used to implement desired fractional Fourier transform.

In the previous part, we proposed several optical set-ups. The following theorem may be useful in creating new set-ups by using our previous systems.

Theorem 3.2 *The reverse of any fractional Fourier transformer composed of thin lenses and sections of free space is also a fractional Fourier transformer.*

Proof:

The output g of the system fractional Fourier transform of f,i.e.



$$g = G(f) = F^{a}(f). \tag{3.64}$$

and from reciprocity, if g^* is the input to the reverse system, the output is f^*, i.e.

$$f^* = K(g^*). \tag{3.65}$$

If Eqn. 3.64 is substituted in Eqn. 3.65 we get,

$$K(g^*) = (F^{-a}g)^* \tag{3.66}$$

The same relation can be written for f as

$$K(f) = (F^{-a} f*)^* \tag{3.67}$$

It is known that $F^a(f^*) = (F^{-a} f)^*$. So

$$K(f) = (F^a f) = g \tag{3.68}$$

For example let us consider the four-lens optical set-up that realizes fractional Fourier transform. This system and its reverse can be seen in Fig. 3.13. According to Theorem 3.2, the reverse of the fractional Fourier transformer is again a fractional Fourier transformer.

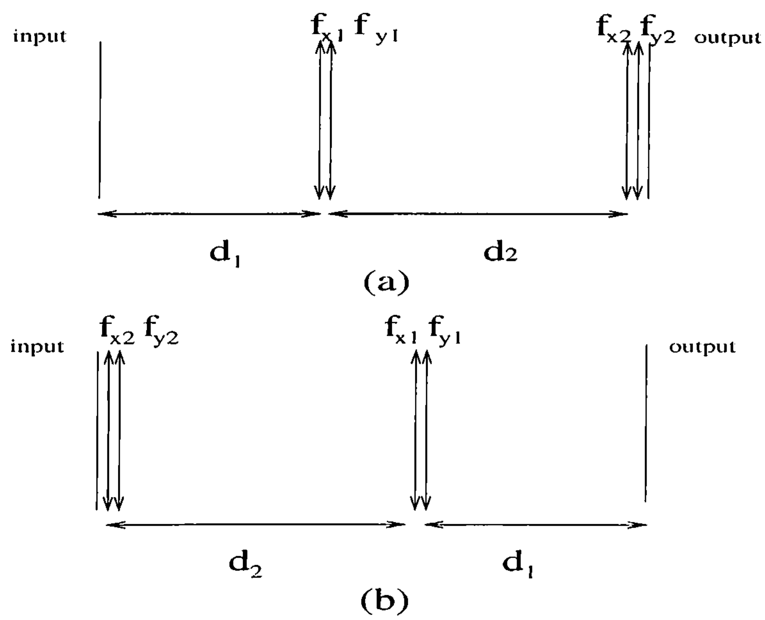


Figure 3.13: (a). The fractional Fourier transformer; (b) Its reverse which is also a fractional Fourier transformer

Chapter 4

A new, non-separable definition for two-dimensional fractional Fourier transform

4.1 Motivation

Many properties for the Fourier transform generalize to two dimensions, but new properties exist in two dimensions like the following affine property. The affine theorem states that [59]:
If $f(x, y)$ has two-dimensional Fourier transform $F(x, y)$, then $f(ax + by, cx + dy)$ has two-dimensional Fourier Transform

$$G(x, y) = \frac{1}{\Delta} F\left(\frac{ex - cy}{\Delta}, \frac{-bx + ay}{\Delta}\right) \tag{4.1}$$

where $\Delta = ad - bc$. Since Fourier transform is a special case of fractional Fourier transform we look for a similar property for two-dimensional fractional Fourier transform. However, the two-dimensional fractional Fourier transform does not have the affine property as property 13 suggests. If $F(x, y)$ is the two-dimensional fractional Fourier transform of $f(x, y)$ with the orders a_x and a_y, then $G(x, y)$ which is the two-dimensional fractional Fourier transform of $f(ax+by, cx+dy)$ cannot be represented in terms of a scaled version of $F(x, y)$ with a similar relation to 4.1. To see this, let us define the new coordinates as:

$$x' = ax + by \qquad\qquad y' = cx + dy. \tag{4.2}$$

It is easy to find x and y in terms of x' and y':

$$x = \frac{1}{\Delta}(dx' - by') \qquad\qquad y = \frac{1}{\Delta}(-cx' + ay') \tag{4.3}$$

where $\Delta = ad - bc$. If x and y are substituted in Eqn. 2.1 and compared with the definition of two-dimensional fractional Fourier transform, it can be easily seen that two-dimensional fractional Fourier transform of $f(ax + by, cx + dy)$ cannot be represented in terms of the scaled version of two-dimensional fractional Fourier transform of $f(x, y)$. This is because, the fractional Fourier transform of $f(ax + by, cx + dy)$ has cross terms, while our separable definition has none. It is even possible to say that any separable two-dimensional kernel fails to satisfy the affine property. The insufficiency of separable definition in satisfying the affine property is one of our motivations to look for a new, non-separable definition.

Our separable definition has two order parameters a_x and a_y. The directions

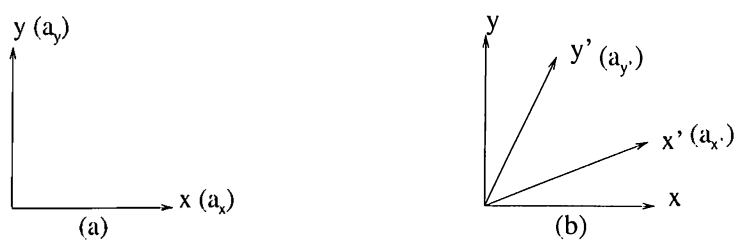


Figure 4.1: The transform orders and directions for (a) separable transform, (b) non-separable transform

along which the function is to be fractional Fourier transformed are fixed to the traditional x and y axes. Fig. 4.1.a shows the directions and the corresponding orders for the two-dimensional separable fractional Fourier transform. Since the two-dimensional fractional Fourier transform is a straightforward generalization from one-dimensional case, one cannot change the directions along which the orders are specified. However, we would like to specify both the directions x', y' and the orders a'_x and a'_y of the two-dimensional transform as can be seen in 4.1.b. This is another motivation for us to look for a new definition for two-dimensional fractional Fourier transform.

4.2 Definition

Here we present our new, non-separable definition for two-dimensional fractional Fourier transform. We define the non-separable fractional Fourier transform in such a manner that it corresponds to fractional Fourier transformation along arbitrary x' and y' directions with orders $a_{x'}$ and $a_{y'}$. It is equivalent to rotation of x and y axes followed by the separable definition. First, x axis is rotated by an angle θ_1 and y axis is rotated by an angle θ_2. Thus, x axis is mapped to x' which makes an angle θ_1 with the x axis and y axis is mapped to y' which makes an angle θ_2 with the y axis. This is equivalent to mapping $f(x,y)$ to $f(\cos\theta_1 x + \sin\theta_1 y, -\sin\theta_2 x + \cos\theta_2 y)$. Then the two-dimensional separable fractional Fourier transform with orders $a_{x'}$ and $a_{y'}$ is applied to $f(x',y') = f(\cos\theta_1 x + \sin\theta_1 y, -\sin\theta_2 x + \cos\theta_2 y)$. The resulting transformation is the new, non-separable two-dimensional fractional Fourier transform. The new definition has 4 parameters: a'_x, a'_y, θ_1 and θ_2. θ_1 is the angle between

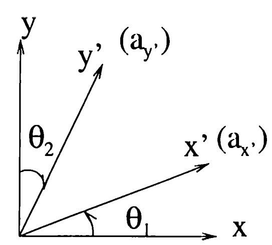


Figure 4.2: The parameters of the new definition

the standard x axis and x', a'_x is the order specified along x' direction, θ_2 indicates the angle between the standard y axis and y' and a'_y is the order along this direction as can be seen in Fig. 4.2. The non-separable fractional Fourier transform as defined above is given with the following equation

$$\mathcal{F}^{a_{x'},a_{y'}}_{\theta_1,\theta_2}\{f(\mathbf{r})\} = \int_{-\infty}^{\infty} B^{a_{x'},a_{y'}}_{\theta_1,\theta_2}(\mathbf{r},\mathbf{r}'')\, f(\mathbf{r}'')\, d\mathbf{r}'' \tag{4.4}$$

where

$$B_{\theta_1,\theta_2}^{a_{x'},a_{y'}}(\mathbf{r},\mathbf{r}'') = A_{\phi\mathbf{r}} \exp[i\pi(\mathbf{r}^{\mathrm{T}}\mathbf{A}\mathbf{r} + 2\mathbf{r}^{\mathrm{T}}\mathbf{B}\mathbf{r}'' + \mathbf{r}''^{\mathrm{T}}\mathbf{C}\mathbf{r}'')] \tag{4.5}$$

with

$$A_{\phi\mathbf{r}} = A_{\phi_{x'}}A_{\phi_{y'}}, \quad \mathbf{r} = \begin{bmatrix} x & y \end{bmatrix}^{\mathrm{T}}, \quad \mathbf{r}'' = \begin{bmatrix} x'' & y'' \end{bmatrix}^{\mathrm{T}},$$

$$\mathbf{A} = \begin{bmatrix} \cot\phi_{x'} & 0 \\ 0 & \cot\phi_{y'} \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} -\frac{\cos\theta_2 \csc\phi_{x'}}{\cos(\theta_1-\theta_2)} & \frac{\sin\theta_1 \csc\phi_{x'}}{\cos(\theta_1-\theta_2)} \\ -\frac{\sin\theta_2 \csc\phi_{y'}}{\cos(\theta_1-\theta_2)} & -\frac{\cos\theta_1 \csc\phi_{y'}}{\cos(\theta_1-\theta_2)} \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} \frac{\cos^2\theta_2}{\cos^2(\theta_1-\theta_2)}\cot\phi_{x'} + \frac{\sin^2\theta_2}{\cos^2(\theta_1-\theta_2)}\cot\phi_{y'} & -\frac{\sin\theta_1\cos\theta_2}{\cos^2(\theta_1-\theta_2)}\cot\phi_{x'} + \frac{\sin\theta_2\cos\theta_1}{\cos^2(\theta_1-\theta_2)}\cot\phi_{y'} \\ -\frac{\sin\theta_1\cos\theta_2}{\cos^2(\theta_1-\theta_2)}\cot\phi_{x'} + \frac{\sin\theta_2\cos\theta_1}{\cos^2(\theta_1-\theta_2)}\cot\phi_{y'} & \frac{\cos^2\theta_1}{\cos^2(\theta_1-\theta_2)}\cot\phi_{y'} + \frac{\sin^2\theta_1}{\cos^2(\theta_1-\theta_2)}\cot\phi_{x'} \end{bmatrix}.$$

Here it is important to note that x' and y' determine the directions along which we specify the orders while, x'' and y'' are dummy variables of the integration having no relation with them.

From now on we will call our new definition as non-separable fractional Fourier transform and use $\mathcal{F}_{\theta_1,\theta_2}^{a'_x,a'_y}$ to represent it, while the separable definition is represented by $\mathcal{F}^{a_x,a_y}$.

The new definition reduces to the separable definition for $\theta_1 = \theta_2 = 0$ which corresponds to fractional Fourier transformation along x and y axes.

This definition with 4 parameters is specified by its non-separable kernel. We constructed the definition in such a way that it corresponds to fractional Fourier transformation along arbitrary x' and y' directions. The next thing to do is to show that this definition satisfies the affine property. The following theorem states that, when an affine transform is applied to the function, its non-separable fractional Fourier transform can be represented in terms of the scaled version of the non-separable fractional Fourier transform of the original function.

Theorem 4.1 *Fractional Fourier transform of $f(ax+by, cx+dy)$ with orders a_x, a_y according to the new definition can be represented in terms of the* *fractional Fourier transform of $f(x,y)$ according to the new definition.*

$$F^{a_x,a_y}_{\theta_1,\theta_2}[f(ax+by,cx+dy)](x,y) = kF^{a'_x,a'_y}_{\theta'_1,\theta'_2}[f(x,y)](a_1x+b_1y,c_1x+d_1y) \quad (4.6)$$

with

$$k = \exp[C_{\phi_x}x^2 + C_{\phi_y}y^2 + C_{\phi'_x,\phi'_y}xy],$$

$$\phi'_x = \phi_x \cot^{-1}\left[\frac{D_{\theta_1\theta_2}}{(a\cos\theta_1 + b\sin\theta_2)^2 - (c\cos\theta_1 + d\sin\theta_2)^2)}\right],$$

$$\phi'_y = \phi_y \cot^{-1}\left[\frac{D_{\theta_1\theta_2}}{(d\cos\theta_2 + c\sin\theta_1)^2 - (b\cos\theta_2 + c\sin\theta_1)^2)}\right],$$

$$\theta_1 = \cos^{-1}\sqrt{\frac{(a\cos\theta_1 + b\sin\theta_2)^2[(d\cos\theta_2 + c\sin\theta_1)^2 - (b\cos\theta_2 + a\sin\theta_1)^2]}{D_{\theta_1\theta_2}}},$$

$$\theta_2 = \cos^{-1}\sqrt{\frac{(\mathrm{d}\cos\theta_2 + \mathrm{c}\sin\theta_1)^2[(\mathrm{a}\cos\theta_1 + \mathrm{b}\sin\theta_2)^2 - (\mathrm{c}\cos\theta_1 + \mathrm{d}\sin\theta_2)^2]}{\mathrm{D}_{\theta_1\theta_2}}},$$

$$a_1 = \frac{\csc\phi_x[(d\cos\theta_2 + c\sin\theta_1)\cos\theta_1 + (b\cos\theta_2 + a\sin\theta_2)\sin\theta_2]}{\csc\phi_{x'}\cos(\theta_1 - \theta_2)},$$

$$b_1 = \frac{\csc\phi_y[(c\cos\theta_1 + d\sin\theta_2)\cos\theta_1 + (a\cos\theta_1 + b\sin\theta_2)\sin\theta_2]}{\csc\phi_{x'}\cos(\theta_1 - \theta_2)},$$

$$c_1 = \frac{\csc\phi_x[(d\cos\theta_2 + c\sin\theta_1)\sin\theta_1 - (b\cos\theta_2 + a\sin\theta_1)\cos\theta_2]}{\csc\phi_{y'}\cos(\theta_1 - \theta_2)},$$

$$d_1 = \frac{\csc\phi_y[(a\cos\theta_1 + b\sin\theta_2)\cos\theta_2 - (c\cos\theta_1 + d\sin\theta_2)\sin\theta_1]}{\csc\phi_{y'}\cos(\theta_1 - \theta_2)},$$

where we employ the intermediate variables as

$$C_{\phi_x} = \frac{\cot\phi_x[((a\cos\theta_1 + b\sin\theta_2)^2 - (c\cos\theta_1 + d\sin\theta_2)^2) - D_{\theta_1\theta_2}]}{[((a\cos\theta_1 + b\sin\theta_2)^2 - (c\cos\theta_1 + d\sin\theta_2)^2 - \cot^2\phi_x D_{\theta_1\theta_2}]},$$

$$C_{\phi_y} = \frac{\cot\phi_y[((d\cos\theta_2 + c\sin\theta_1)^2 - (b\cos\theta_2 + c\sin\theta_1)^2) - D_{\theta_1\theta_2}]}{[((d\cos\theta_2 + c\sin\theta_1)^2 - (b\cos\theta_2 + a\sin\theta_1)^2) - \cot^2\phi_y D_{\theta_1\theta_2}]},$$

$$C_{\phi'_x,\phi'_y} = a_1 b_1 \cot\phi'_x + c_1 d_1 \cot\phi'_y,$$

and

$$D_{\theta_1\theta_2} = [(a\cos\theta_1 + b\sin\theta_2)^2(d\cos\theta_2 + c\sin\theta_1)^2 - (b\cos\theta_2 + a\sin\theta_1)^2(c\cos\theta_1 + d\sin\theta_2)].$$

It is important to note that a'_x and a'_y used in this theorem and $a_{x'}$ and $a_{y'}$ used in the definition of the non-separable fractional Fourier transform are different.

Proof:

The proof of this property follows directly from the definition of non-separable fractional Fourier transform.

The new definition satisfies the properties that we expected it to satisfy. But one might suggest a more general definition with a greater number of parameters. We now show that such a definition is not necessary. We know that the new definition corresponds to two-dimensional fractional Fourier transform of $f(\cos\theta_1 x + \sin\theta_1 y, -\sin\theta_2 x + \cos\theta_2 y)$. Let us propose another definition by applying the separable definition to $f(ax + by, cx + dy)$ where a, b, c and d are arbitrary. It is evident that this definition which has 6 parameters is more general than our non-separable definition. However, the next theorem states that such a definition is redundant since the fractional Fourier transform of $f(ax + by, cx + dy)$ for any a, b, c, d can be represented as scaled version of our new definition.

Theorem 4.2 *Fractional Fourier transform of $f(ax+by, cx+dy)$ with orders a_x, a_y according to the separable definition can be represented as a scaled version of the non-separable fractional Fourier transform of $f(x, y)$.*

$$\mathcal{F}^{a_x,a_y}_{\theta_1=\theta_2=0}[f(ax + by, cx + dy)](x, y) = k\mathcal{F}^{a'_x,a'_y}_{\theta'_1,\theta'_2}[f(x, y)](a_1 x + b_1 y, c_1 x + d_1 y)$$

with

$$\Delta = (ad - bc), \qquad k = \frac{1}{\Delta}\exp[C_{\phi_x}x^2 + C_{\phi_y}y^2 + C_{\phi'_x,\phi'_y}xy],$$

$$\phi'_x = \phi_x \cot^{-1}\left[\frac{(a^2d^2 - b^2c^2)}{\Delta^2(a^2 - c^2)}\right], \qquad \phi'_y = \phi_y \cot^{-1}\left[\frac{(a^2d^2 - b^2c^2)}{\Delta^2(d^2 - b^2)}\right],$$

$$\theta_1 = \cos^{-1}\sqrt{\frac{a^2(d^2 - b^2)}{(a^2d^2 - b^2c^2)}}, \qquad \theta_2 = \cos^{-1}\sqrt{\frac{d^2(a^2 - c^2)}{(a^2d^2 - b^2c^2)}},$$

$$a_1 = \frac{\csc\phi_x(d\cos\theta_1 + b\sin\theta_2)}{\Delta\csc\phi_{x'}\cos(\theta_1 - \theta_2)}, \qquad b_1 = \frac{\csc\phi_y(c\cos\theta_1 + a\sin\theta_2)}{\Delta\csc\phi_{x'}\cos(\theta_1 - \theta_2)},$$

$$c_1 = \frac{\csc\phi_x(d\sin\theta_1 - b\cos\theta_2)}{\Delta\csc\phi_{y'}\cos(\theta_1 - \theta_2)}, \qquad d_1 = \frac{\csc\phi_y(a\cos\theta_2 - c\sin\theta_1)}{\Delta\csc\phi_{y'}\cos(\theta_1 - \theta_2)},$$

where the intermediate variables are

$$C_{\phi_x} = \frac{\cot\phi_x[\Delta^4(a^2 - c^2) - (a^2d^2 - b^2c^2)^2]}{[\Delta^4(a^2 - c^2) - \cot^2\phi_x(a^2d^2 - b^2c^2)^2]},$$

$$C_{\phi_y} = \frac{\cot\phi_y[\Delta^4(d^2 - b^2) - (a^2d^2 - b^2c^2)^2]}{[\Delta^4(d^2 - b^2) - \cot^2\phi_y(a^2d^2 - b^2c^2)^2]},$$

$$C_{\phi'_x,\phi'_y} = a_1b_1\cot\phi'_x + c_1d_1\cot\phi'_y.$$

Proof:

This theorem can easily be proved by using the definitions of separable and non-separable two-dimensional fractional Fourier transform through straightforward yet lengthy algebraic manipulations.

Theorem 4.2 states that the separable fractional Fourier transform of any affine-transformed function $f(ax+by, cx+dy)$ can be represented as a scaled version of the non-separable fractional Fourier transform of the original function. This result indicates that a definition with more parameters will be redundant. An analogy with the common Fourier transform might be useful. We know that when the function is scaled, its Fourier transform can be represented as a scaled version of the Fourier transform of the original function. Thus, it is redundant to define a transform called the scaled Fourier transform. Just like this example, a definition for two-dimensional fractional Fourier transform with more than 4 parameters, will be redundant.

4.3 Properties of the non-separable fractional Fourier transform

Theorem 4.3 *The kernel of the inverse transform is*

$$\{B_{\theta_1,\theta_2}^{a_{x'},a_{y'}}\}^{-1}(\mathbf{r},\mathbf{r}'') = A_{-\phi\mathbf{r}} \exp[-i\pi(\mathbf{r}^{\mathrm{T}}\mathbf{A}\mathbf{r} + 2\mathbf{r}^{\mathrm{T}}\mathbf{B}\mathbf{r}'' + \mathbf{r}''^{\mathrm{T}}\mathbf{C}\mathbf{r}'')] \tag{4.7}$$

where

$$A_{-\phi\mathbf{r}} = A_{-\phi_{x'}}A_{-\phi_{y'}} \qquad \mathbf{r} = \begin{bmatrix} x & y \end{bmatrix}^{\mathrm{T}}, \quad \mathbf{r}'' = \begin{bmatrix} x'' & y'' \end{bmatrix}^{\mathrm{T}},$$

$$\mathbf{A} = \begin{bmatrix} \frac{\cos^2\theta_2}{\cos^2(\theta_1-\theta_2)}\cot\phi_{x'} + \frac{\sin^2\theta_2}{\cos^2(\theta_1-\theta_2)}\cot\phi_{y'} & -\frac{\sin\theta_1\cos\theta_2}{\cos^2(\theta_1-\theta_2)}\cot\phi_{x'} + \frac{\sin\theta_2\cos\theta_1}{\cos^2(\theta_1-\theta_2)}\cot\phi_{y'} \\ -\frac{\sin\theta_1\cos\theta_2}{\cos^2(\theta_1-\theta_2)}\cot\phi_{x'} + \frac{\sin\theta_2\cos\theta_1}{\cos^2(\theta_1-\theta_2)}\cot\phi_{y'} & \frac{\cos^2\theta_1}{\cos^2(\theta_1-\theta_2)}\cot\phi_{y'} + \frac{\sin^2\theta_1}{\cos^2(\theta_1-\theta_2)}\cot\phi_{x'} \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} -\frac{\cos\theta_2\csc\phi_{x'}}{\cos(\theta_1-\theta_2)} & \frac{\sin\theta_1\csc\phi_{x'}}{\cos(\theta_1-\theta-2)} \\ -\frac{\sin\theta_2\csc\phi_{y'}}{\cos(\theta_1-\theta_2)} & -\frac{\cos\theta_1\csc\phi_{y'}}{\cos(\theta_1-\theta-2)} \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} \cot\phi_{x'} & 0 \\ 0 & \cot\phi_{y'} \end{bmatrix}.$$

Notice that the kernel of the inverse transform $\{B_{\theta_1,\theta_2}^{a_{x'},a_{y'}}\}^{-1}$ *is not equivalent to* $\{B_{-\theta_1,-\theta_2}^{-a_{x'},-a_{y'}}\}$.

Proof:

We know that the fractional Fourier transform according to the new definition can be decomposed into an affine transform followed by the separable definition. Thus it is legitimate to write

$$\mathcal{F}^{a_{x'},a_{y'}}_{\theta_1,\theta_2}[f(x,y)] = \mathcal{F}^{a_{x'},a_{y'}}[f(\cos\theta_1 x + \sin\theta_1 y, -\sin\theta_2 x + \cos\theta_2 y)]. \quad (4.8)$$

By using the kernel of the inverse separable fractional Fourier transform given in 2.11

$$\begin{aligned} &f(\cos\theta_1 x + \sin\theta_1 y, -\sin\theta_2 x + \cos\theta_2 y) = \\ &= \textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[-i\pi(x^2\cot\phi_{x'} - 2xx''\csc\phi_{x'} + x''^2\cot\phi_{x'})] \\ &\times \exp[-i\pi(y^2\cot\phi_{y'} - 2yy''\csc\phi_{y'} + y''^2\cot\phi_{y'})]\, dx''\, dy'' \end{aligned} \quad (4.9)$$

$f(x,y)$ can be obtained in terms $g(x,y)$ by using a coordinate transformation. Thus, the kernel of the inverse transform is found as given in 4.7

Theorem 4.4 *The non-separable definition is unitary, i.e.,*

$$B^*_{a_{x'},a_{y'},\theta_1,\theta_2}(x,y;x'',y'') = B^{-1}_{a_{x'},a_{y'},\theta_1,\theta_2}(x'',y'';x,y). \quad (4.10)$$

Proof:

By using the kernel of the non-separable transform in 4.4 and its inverse in 4.7, the proof follows.

Theorem 4.5 *Let $W_f(x,y;\mu_x,\mu_y)$ be the Wigner distribution of $f(x,y)$. If $g(x,y)$ is the non-separable fractional Fourier transform of $f(x,y)$ with parameters $a_{x'}, a_{y'}, \theta_1$ and θ_2, then Wigner distribution of $g(x,y)$ is related to that of $f(x,y)$ through the following equation*

$$W_g(\mathbf{r},\mu) = W_f(\mathbf{Ar}+\mathbf{Bs}, \mathbf{Cr}+\mathbf{Ds}), \quad (4.11)$$

$$\mathbf{r} = \begin{bmatrix} x & y \end{bmatrix}^{\mathrm{T}}, \quad \mathbf{s} = \begin{bmatrix} \mu_x & \mu_y \end{bmatrix}^{\mathrm{T}}, \quad (4.12)$$

and

$$\mathbf{A} = \begin{bmatrix} \cos\phi_{x'}\cos\theta_1 & \cos\phi_{y'}\sin\theta_1 \\ -\cos\phi_{x'}\sin\theta_2 & \cos\phi_{y'}\cos\theta_2 \end{bmatrix}, \quad (4.13)$$

$$\mathbf{B} = \begin{bmatrix} -\sin\phi_{x'}\cos\theta_1 & -\sin\phi_{y'}\sin\theta_1 \\ \sin\phi_{x'}\sin\theta_2 & -\sin\phi_{y'}\cos\theta_2 \end{bmatrix}, \tag{4.14}$$

$$\mathbf{C} = \begin{bmatrix} \frac{\sin\phi_{x'}\cos\theta_2}{\cos(\theta_1-\theta_2)} & \frac{\sin\phi_{y'}\sin\theta_1}{\cos(\theta_1-\theta_2)} \\ -\frac{\sin\phi_{x'}\sin\theta_1}{\cos(\theta_1-\theta_2)} & \frac{\sin\phi_{y'}\cos\theta_1}{\cos(\theta_1-\theta_2)} \end{bmatrix}, \tag{4.15}$$

$$\mathbf{D} = \begin{bmatrix} \frac{\cos\phi_{x'}\cos\theta_2}{\cos(\theta_1-\theta_2)} & \frac{\cos\phi_{y'}\sin\theta_2}{\cos(\theta_1-\theta_2)} \\ -\frac{\cos\phi_{x'}\sin\theta_1}{\cos(\theta_1-\theta_2)} & \frac{\cos\phi_{y'}\cos\theta_1}{\cos(\theta_1-\theta_2)} \end{bmatrix}. \tag{4.16}$$

Proof:
This theorem is derived by using the general expressions for the transformation matrices derived by Bastiaans [37]. The proof is straightforward yet requires many matrix manipulations.

It is important to note that, for $\theta_1 = \theta_2 = 0$, the above theorem reduces to the Property 14 of Chapter 2 which characterizes the effect of separable definition on Wigner distribution.

4.4 Discrete-time implementation of the new definition

Due to the oscillatory nature of the fractional Fourier transform, its discrete-time implementation is very hard by simple integration techniques. However, in [43], a fast algorithm for the fractional Fourier transform is presented. While direct computation would require $O(N^2)$ multiplications, this fast algorithm computes the transform in $O(N \log N)$ time.

In order to use the non-separable definition for practical purposes, a fast discrete-time implementation is needed. By definition, it is composed of an affine transformation followed by the separable definition. In image processing several algorithms exist for affine transformation [60]. In order to implement the non-separable fractional Fourier transform of $f(x, y)$, with parameters $\theta_1, \theta_2, a_{x'}$ and $a_{y'}$, first $f(\cos\theta_1 x + \sin\theta_1 y, -\sin\theta_2 x + \cos\theta_2 y)$ is computed. In the computation of affine transform, bilinear interpolation method which was previously implemented is used [61]. Then the fast algorithm in [43], that computes the two-dimensional separable fractional Fourier transform, is applied to the affine-transformed function. The resulting transformation is the non-separable fractional Fourier transform. So we obtained a way of implementing our new definition in $O(N \log N)$ time.

Chapter 5

Optical implementation of the non-separable definition

In the previous chapter, a non-separable version of fractional Fourier transform is suggested. We are no longer restricted to define the orders along the traditional x and y directions. It is possible to specify the orders along arbitrary directions. Indeed the non-separable transformation is composed of an affine transformation followed by the separable transform. The non-separable definition has cross terms in its kernel. But the kernels of free space propagation and thin lenses have no cross terms as we can see in 3.7, 3.10 and 3.12. Let us first modify the kernel of free space propagation and assume that it is possible to have different propagation distances in x and y and also assume that we have a cross term. Then our new kernel becomes

$$h(x,y,x',y') = \exp\left(i\pi\left[\frac{(x-x')^2}{\lambda d_x} + \frac{(y-y')^2}{\lambda d_y} + \frac{(x-x')(y-y')}{\lambda d_{xy}}\right]\right). \tag{5.1}$$

We now assume that such sections of free space exist and call them as anamorphic and cross-termed sections of free space. In section 5.2 we will show that simulation of sections of free space with a kernel of the form is 5.1 is possible by using an optical set-up composed of thin lenses and sections of free space. Similarly the kernel for an anamorphic lens with focal length f_x along the x

direction, f_y along the y direction and f_{xy} along the xy direction is

$$h_{xyl}(x, y, x', y') = K_{xyl}\,\delta(x - x', y - y')\,\exp\left(-i\pi\left[\frac{x^2}{\lambda f_x} + \frac{y^2}{\lambda f_y} + \frac{xy}{\lambda f_{xy}}\right]\right). \quad (5.2)$$

It was further discussed that it is possible to construct an anamorphic lens with two cylindrical lenses located perpendicular to each other. When two cylindrical lenses are located with an arbitrary angle, we obtain the desired lens with the kernel in 5.2.

5.1 Optical set-ups that realize non-separable fractional Fourier transform

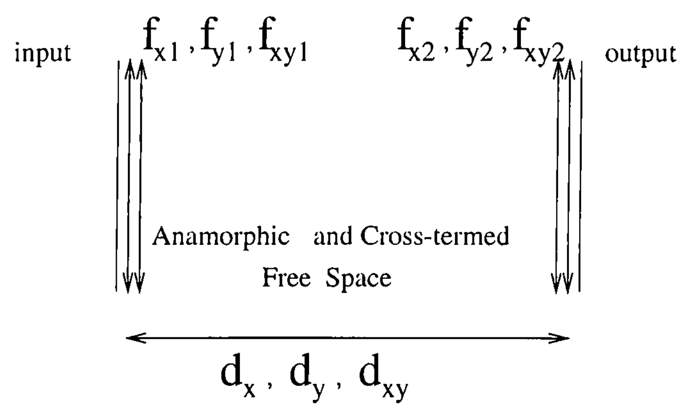


Figure 5.1: Optical set-up that realizes the non-separable fractional Fourier transform

The system in 5.1 is composed of 4 cylindrical lenses and 1 section of anamorphic and cross-termed free space. It is similar to the system in 3.7 which realizes two-dimensional fractional Fourier transform except for the cross terms of the lenses and free space sections. The analysis of the system is easy by just using the kernels defined in Eqn. 5.1 and 5.2. It is possible to implement the non-separable fractional Fourier transform with the desired orders, directions, input scale parameters and phase factors with the optical set-up in 5.1.

1. *Specified by the designer* :$\phi_x, \phi_y, \theta_1, \theta_2, s_{1x}, s_{1y}, p_x = p_y = 0$.
   Design parameters :$d_x, d_y, d_{xy}, f_{x1}, f_{x2}, f_{xy1}, f_{x2}, f_{y2}, f_{xy2}$.

In this case, one can implement all different combinations of $\phi_x, \phi_y, \theta_1, \theta_2$. The output scale parameters must be chosen in a way to satisfy the following equation

$$\frac{s_{2x}}{s_{2y}} = \frac{-s_{1x}\sin\theta_1\csc\phi_x}{s_{1y}\sin\theta_2\csc\phi_y}, \tag{5.3}$$

and the distances and focal lengths of lenses are

$$d_x = \frac{s_{1x}s_{2x}\cos(\theta_1-\theta_2)}{2\lambda\cos\theta_2\csc\phi_x}, \tag{5.4}$$

$$d_y = \frac{s_{1y}s_{2y}\cos(\theta_1-\theta_2)}{2\lambda\cos\theta_1\csc\phi_y}, \tag{5.5}$$

$$d_{xy} = \frac{s_{1x}s_{2y}\cos(\theta_1-\theta_2)}{2\lambda\sin\theta_2\csc\phi_y} = \frac{s_{1y}s_{2x}\cos(\theta_1-\theta_2)}{2\lambda\sin\theta_1\csc\phi_x}, \tag{5.6}$$

$$\frac{1}{\lambda f_{x1}} = \frac{2\cos\theta_2\csc\phi_x}{s_{1x}s_{2x}\cos(\theta_1-\theta_2)} - \frac{\cos^2\theta_2\cos\phi_x + \sin^2\theta_2\cot\phi_y}{s_{1x}^2\cos^2(\theta_1-\theta_2)}, \tag{5.7}$$

$$\frac{1}{\lambda f_{y1}} = \frac{2\cos\theta_1\csc\phi_y}{s_{1y}s_{2y}\cos(\theta_1-\theta_2)} - \frac{\cos^2\theta_1\cos\phi_y + \sin^2\theta_1\cot\phi_x}{s_{1y}^2\cos^2(\theta_1-\theta_2)}, \tag{5.8}$$

$$\frac{1}{\lambda f_{x2}} = \frac{2\cos\theta_2\csc\phi_x}{s_{1x}s_{2x}\cos(\theta_1-\theta_2)} - \frac{\cot\phi_x}{s_{2x}^2}, \tag{5.9}$$

$$\frac{1}{\lambda f_{y2}} = \frac{2\cos\theta_1\csc\phi_y}{s_{1y}s_{2y}\cos(\theta_1-\theta_2)} - \frac{\cot\phi_y}{s_{2y}^2}. \tag{5.10}$$

2. *Specified by the designer* :$\phi_x, \phi_y, \theta_1, \theta_2, s_{2x}, s_{2y}, p_x = p_y = 0$.
   Design parameters :$d_x, d_y, d_{xy}, f_{x1}, f_{x2}, f_{xy1}, f_{x2}, f_{y2}, f_{xy2}, s_{1x}, s_{1y}$.
   By using the same optical system it is possible to implement the non-separable fractional Fourier transform with the desired orders, directions, output scale parameters and phase factors. In this case the equation defining the input scale parameters is

$$\frac{s_{1x}}{s_{1y}} = \frac{-s_{2x}\sin\theta_2\csc\phi_y}{s_{2y}\sin\theta_1\csc\phi_x}. \tag{5.11}$$

   The other system parameters are the same with the former case.

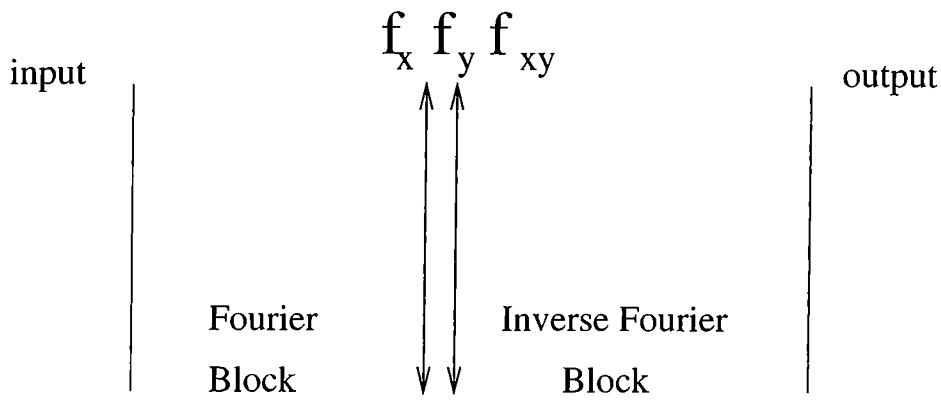


Figure 5.2: Optical set-up that simulates anamorphic free space with cross terms

5.2 Simulation of anamorphic and cross-termed sections of free space

The problem of designing the anamorphic and cross-termed sections of free space still exists. In chapter 3, we presented a way of simulating anamorphic free space with the desired d_x and d_y. The anamorphic lens in Fig. 3.5 is composed of two cylindrical lenses located perpendicular to each other. If they are located with an arbitrary angle, it becomes possible to simulate anamorphic and cross-termed free space with parameters d_x, d_y and d_{xy} by using the same optical system. The kernel of the optical system in Fig. 5.2 is

$$h(x,y,x',y') = \exp\left(i\pi\left[\frac{(x-x')^2}{\lambda d_x} + \frac{(y-y')^2}{\lambda d_y} + \frac{(x-x')(y-y')}{\lambda d_{xy}}\right]\right) \tag{5.12}$$

where

$$d_x = \frac{s^4(f_{xy}^2 - f_x f_y)}{\lambda^2 f_x f_{xy}^2}, \qquad d_y = \frac{s^4(f_{xy}^2 - f_x f_y)}{\lambda^2 f_y f_{xy}^2}, \tag{5.13}$$

$$d_{xy} = \frac{s^4(f_x f_y - f_{xy}^2)}{2\lambda^2 f_x f_y f_{xy}}. \tag{5.14}$$

where s is the scale of the Fourier and inverse Fourier blocks. Hence by controlling the focal lengths of the anamorphic lens, it is possible to control the parameters of the anamorphic free space with cross terms.

It is also possible to try to find different optical realizations for this definition. But since the kernel is non-separable, simpler set-ups turned out to be insufficient. 6-lens set-ups may be analyzed, but since the set-up in Fig. 5.2 with 6 cylindrical lenses is useful for our purposes, there is no need to discuss these systems here.

Chapter 6

An application of the new definition: Filtering in fractional Fourier domains

In Chapter 4, a non-separable definition is presented for two-dimensional fractional Fourier transform. This definition has four parameters where a_x and a_y define the orders and θ_1 and θ_2 specify the directions along which the function is to be fractional Fourier transformed. This non-separable definition enables us to define both the orders and the directions of the fractional Fourier transform. Hence, we are no longer restricted to the traditional x and y axes. Since we have come up with a more general and comprehensive definition, we expect to have improvements in problems where the separable fractional Fourier transform is being used.

The concept of filtering in fractional Fourier domains has been introduced to the problem of estimating images in the presence of space-varying noise [47]. The expressions for the optimal filter function in the fractional domains are derived in a manner analogous to the classical Wiener filtering problem. Here we modify the general formulation of optimal filtering in the fractional Fourier domains derived in [47].

Consider the following signal observation model:

$$\mathbf{o} = \mathcal{H}(\mathbf{f}) + \mathbf{n}, \tag{6.1}$$

where $\mathcal{H}(\cdot)$ is a linear system that degrades the input signal $\mathbf{f}$, and $\mathbf{n}$ is an additive noise term. Our problem is to filter the observed signal $\mathbf{o}$ to minimize the effect of noise. The error criteria to be minimized is the mean square error. It is assumed that the correlation functions of the input and noise processes are known : $R_f(x,y;x',y') = E[f(x,y)f(x',y')]$, $R_n(x,y;x',y') = E[n(x,y)n(x',y')]$.

The filters that we consider satisfy the following equation:

$$\mathcal{F}^{a_x,a_y}_{\theta_1,\theta_2}\{\mathcal{G}(\cdot)\} = \mathbf{g}\,.\,\mathcal{F}^{a_x,a_y}_{\theta_1,\theta_2}(\cdot)$$

corresponding to multiplication with a function $g(\cdot,\cdot)$ in the fractional Fourier domain. The estimate satisfies

$$\mathcal{F}^{a_x,a_y}_{\theta_1,\theta_2}\{\hat{\mathbf{f}}\} = \mathbf{g}\,.\,\mathcal{F}^{a_x,a_y}_{\theta_1,\theta_2}(\mathbf{o}),$$

and the mean square error is

$$\sigma_e^2 = E\left[|\mathbf{f} - \hat{\mathbf{f}}|^2\right].$$

Since the non-separable fractional Fourier transform is unitary, this MSE is equal to the error in the transform domain. It can be shown by modifying the solution in [47] that the optimal filter function that minimizes the MSE (see Appendix) is

$$g_{opt}(x,y) = \frac{R_{f_a,o_a}(x,y;x,y)}{R_{o_a,o_a}(x,y;x,y)} \tag{6.2}$$

In this equation f_a and o_a are the non-separable fractional Fourier transforms of $f(x,y)$ and $o(x,y)$ with parameters a_x, a_y, θ_1 and θ_2 respectively. $R_{f_a,o_a}(x,y;x',y')$ and $R_{o_a,o_a}(x,y;x',y')$ are the correlation functions in the transform domain (a_x, a_y) defined as

$$R_{f_a,o_a}(x,y;x',y') = E[f_a(x,y)o_a(x',y')] \tag{6.3}$$

and

$$R_{o_a,o_a}(x,y;x',y') = E[o_a(x,y)o_a(x',y')] \tag{6.4}$$

These correlation functions can easily be calculated from the correlation functions in the spatial domain. The optimal choice of a_x, a_y, θ_1 and θ_2 are those which result in the minimum MSE.

This derivation is a direct generalization from the formulation of optimal filtering in fractional Fourier domains derived in [47]. The modified derivation can be found in the Appendix, which makes use of the non-separable two-dimensional fractional Fourier transform.

Let us consider an example of noise separation problem. For purposes of illustration, we will choose the noise to be a deterministic function with well defined time-frequency characteristics:

$$n(x, y) = e^{1.6i\pi(x'-7.3)^2} + e^{1.4i\pi(y'+7.3)^2} \tag{6.5}$$

So that the distorted image is

$$f(x, y) + A[e^{1.6i\pi(x'-7.3)^2} + e^{1.4i\pi(y'+7.3)^2}] \tag{6.6}$$

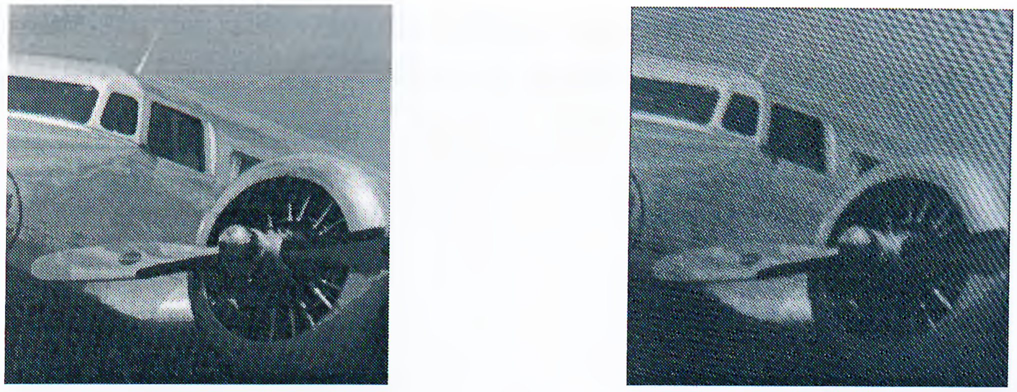
where the constant A takes different values to adjust signal-to-noise ratio (SNR) to the desired value. The original and distorted images can be seen in Fig. 6.1.a, 6.1.b and 6.3.b.

The two chirps which constitute the noise are not oriented along the x and y directions, but along arbitrary x' and y' directions. In our example, x' makes an angle of 15° with the x axis and y' makes an angle of 30° with the y axis. We will consider two cases with SNR=1 and SNR=0.1. For an $n \times m$ image SNR is defined as

$$\text{SNR} = \frac{\int\int |f(x, y)|^2 \, dx \, dy}{\int\int |n(x, y)|^2 dx \, dy}. \tag{6.7}$$

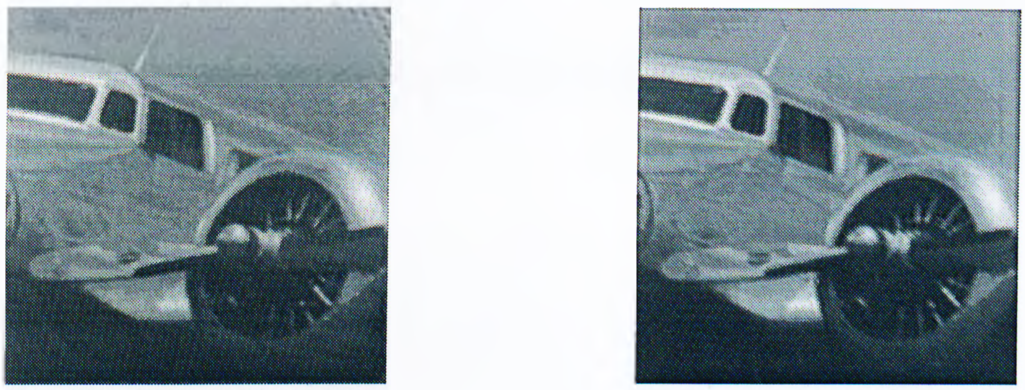
We will compare the use of our non-separable definition with the separable definition previously used by Kutay for the same problem [47].

The method used by Kutay [47] tries to minimize the MSE by optimizing over all possible combinations of a_x and a_y. The optimum orders are $a_x = 0.35$ and $a_y = -0.4$. Remember that the separable definition is a special case of the non-separable definition with $\theta_1 = \theta_2 = 0$. The restored images for SNR=1 and SNR=0.1 can be seen in Figure 6.2.a and 6.4.a respectively.



(a) (b)

Figure 6.1: (a) Original image; (b) Noisy image with SNR=1.

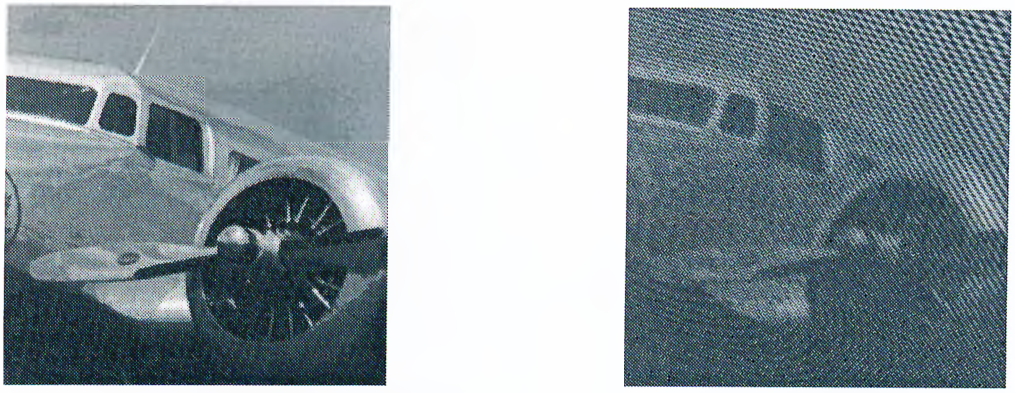


(a) (b)

Figure 6.2: (a) Image filtered by the separable definition; (b) Image filtered by the non-separable definition, for SNR=1.

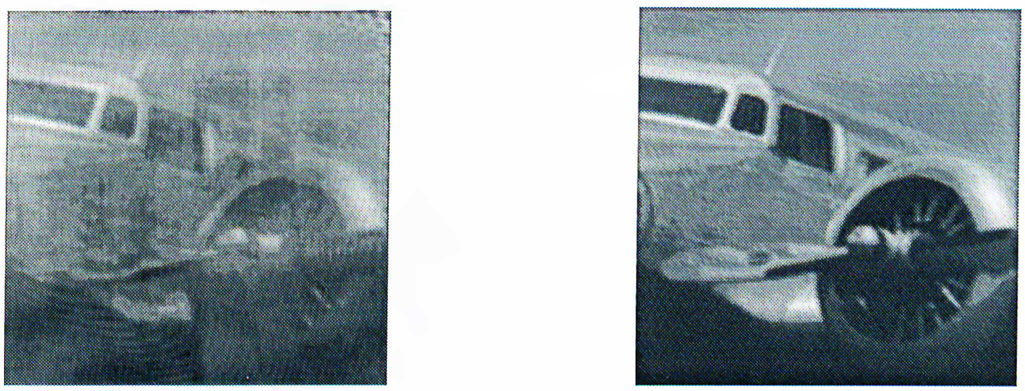
When we use the filtering method proposed in this thesis, we optimize over θ_1, θ_2 besides a_x and a_y. The optimum parameters are found as $a_x = 0.35$, $\theta_1 = 15°$ and $a_y = -0.4$, $\theta_2 = 30°$. Figure 6.2.b and 6.4.b show the restored images for SNR=1 and SNR=0.1 respectively. Due to computational constraints, we restricted our search to a local minimum only.

The improvement when SNR=0.1 is immediately visible when Fig. 6.4.a and 6.4.b are compared. In this case the non-separable definition gives an MSE of 0.020 where the separable definition results in an MSE of 0.101. Thus, MSE is reduced by a factor of 5. When SNR=1, the visible improvement is less evident, but nevertheless MSE has been decreased from 0.029 to 0.0084 and MSE is reduced by a factor of 3. For both cases, we achieved a remarkable reduction in the MSE when the non-separable definition is used. The MSE values given here are all normalized by the energy of the original image.



(a) (b)

Figure 6.3: (a) Original image; (b) Noisy image with SNR=0.1.



(a) (b)

Figure 6.4: (a) Image filtered by the separable definition; (b) Image filtered by the non-separable definition, for SNR=0.1.

Fig. 6.5 and 6.6 show minimum normalized MSE's for different θ_1 and θ_2 pairs for SNR=1. The corresponding MSE values in Fig. 6.5 and Fig. 6.6 are obtained. Fig. 6.5 shows normalized MSE as a function of θ_1, when θ_2 is fixed to 30°. The minimum value of the normalized MSE is attained for $\theta_1 = 15°$. In Fig. 6.6 normalized MSE is plotted as a function of θ_2, when θ_1 is fixed to 15°. The normalized MSE is minimum for $\theta_2 = 30°$. The local minimum for the MSE is obtained at $\theta_1 = 15°$ and $\theta_2 = 30°$.

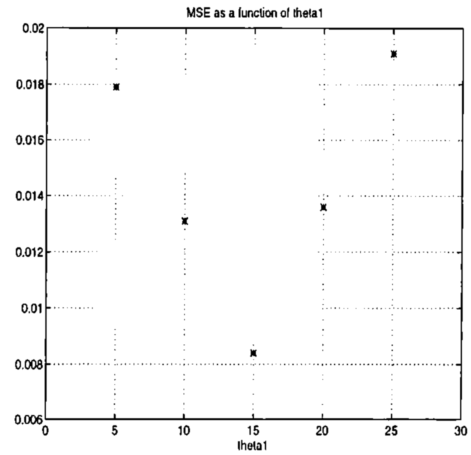


Figure 6.5: Normalized MSE as a function of θ_1 for SNR=1.

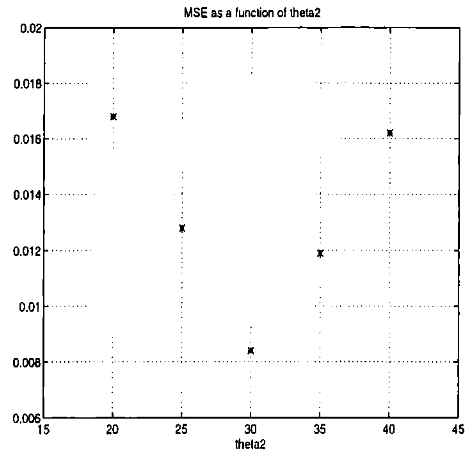


Figure 6.6: Normalized MSE as a function of θ_2 for SNR=1.

We expect fractional Fourier domain filtering in two dimensions to find greatest application in optical systems. This is because the types of noise for which fractional Fourier domain filtering achieves greatest benefits are often encountered in optical systems. For example, line defects on the lenses produce a chirp-like noise. Since the angle between the scratches are arbitrary, using non-separable fractional Fourier transform will result in greater improvements compared with the separable fractional Fourier transform and common Fourier transform. This filtering scheme may also find applications in optical systems to remove twin images in holography.

Appendix

The observation model is of the form

$$\mathbf{o} = \mathcal{H}(\mathbf{f}) + \mathbf{n}, \tag{6.8}$$

where $\mathcal{H}(\cdot)$ is a linear system that degrades the input signal $\mathbf{f}$, and $\mathbf{n}$ is an additive noise term. It is assumed that the correlation functions of the input and noise processes are known. Our problem is to filter the observed signal $\mathbf{o}$ to minimize the effect of noise. Our estimate satisfies the equation:

$$\mathcal{F}^{a_x,a_y}_{\theta_1,\theta_2}\{\hat{f}(x,y)\} = g(x,y).\mathcal{F}^{a_x,a_y}_{\theta_1,\theta_2}[o(x,y)].$$

The estimate is given by

$$\hat{f}(x,y) =$$

$$\int\int\{B^{a_x,-a_y}_{\theta_1,\theta_2}\}^{-1}(x,y;x'',y'')g(x'',y'')\int\int B^{a_x,a_y}_{\theta_1,\theta_2}(x'',y'';x',y')\,o(x',y')dx'\,dy'\,dx''\,dy''$$

and the error is

$$\sigma_e^2 = E\left[||\mathbf{f}-\hat{\mathbf{f}}||^2\right]$$

where $E[\cdot]$ denotes the expectation operator and $||\cdot||$ denotes the norm:

$$||\mathbf{f}|| = \int\int |f(x,y)|^2 dx\,dy.$$

Since the two-dimensional fractional Fourier transformation is unitary, this MSE is equal to the error in the transform domain:

$$\sigma_e^2 = E\left[||\mathbf{f}^{a_x,a_y}_{\theta_1,\theta_2} - \hat{\mathbf{f}}^{a_x,a_y}_{\theta_1,\theta_2}||^2\right].$$

The problem is to find the multiplicative filter $\mathbf{g}$ in the ath domain that minimizes the MSE in the above equation. We define the cost function J to be equal to the MSE:

$$J = \sigma_e^2 = E\left[\int\int (f(x,y)-\hat{f}(x,y))(f(x,y)-\hat{f}(x,y))^* dx\,dy\right].$$

J varies with the choice of the multiplicative filter $g(x,y)$ since $\hat{f}(x,y)$ varies. Thus, the functional J is to be minimized with respect to $g(\cdot)$. We substitute $g(\cdot) = g_o(\cdot) + \alpha\delta g_o(\cdot)$ in the expression of the estimate $\hat{f}(\cdot)$. In this equation,

α is a complex scalar parameter, $g(\cdot)$ is the optimum filter, and $\delta g_o(\cdot)$ is an arbitrary perturbation term. Since α is a complex constant, we can express it as $\alpha = \alpha_{re} + i\alpha_{im}$. Now $\hat{f}(\cdot)$ and J vary with α for each fixed $\delta g_o(\cdot)$. The optimum value of J will be obtained from the following conditions [62]:

$$\frac{\partial J(\alpha)}{\partial \alpha_{re}}|_{\alpha=0} = 0 \qquad \frac{\partial J(\alpha)}{\partial \alpha_{im}}|_{\alpha=0} = 0$$

By using the conditions above, the optimal filter function that minimizes the error can be shown to satisfy the following equation:

$$E\left[\left(f^{a_x,a_y}_{\theta_1,\theta_2}(x,y) - \hat{f}^{a_x,a_y}_{\theta_1,\theta_2}(x,y)\right)\{o^{a_x,a_y}_{\theta_1,\theta_2}\}^*(x,y)\right] = 0$$

which is nothing but the well-known orthogonality condition. The above equation states that the best linear mean-square estimate $\hat{f}^{a_x,a_y}_{\theta_1,\theta_2}(x,y)$ is an orthogonal projection of the signal $f^{a_x,a_y}_{\theta_1,\theta_2}(x,y)$ onto the space of observations.

The optimum filter function $g_{opt}(\cdot,\cdot)$ can be solved from the above equation by using the definition of $\hat{f}^{a_x,a_y}_{\theta_1,\theta_2}(x,y)$. The optimum filter function is found to be

$$g_{opt}(x,y) = \frac{R_{f_a,o_a}(x,y;x,y)}{R_{o_a,o_a}(x,y;x,y)}.$$

For simplicity, we use f_a and o_a as the non-separable fractional Fourier transforms of $f(x,y)$ and $o(x,y)$ with parameters a_x, a_y, θ_1 and θ_2. $R_{f_a,o_a}(x,y;x',y')$ and $R_{o_a,o_a}(x,y;x',y')$ are the correlation functions in the transform domain (a_x, a_y) defined as

$$R_{f_a,o_a}(x,y;x',y') = E[f_a(x,y)o_a(x',y')]$$

and

$$R_{o_a,o_a}(x,y;x',y') = E[o_a(x,y)o_a(x',y')].$$

These correlation functions can easily be calculated from the correlation functions in the spatial domain:

$$R_{f_a,o_a}(x,y;x,y) =$$
$$= \int\int\int\int B^{a_x,a_y}_{\theta_1,\theta_2}(x,y;x',y')\{B^{a_x,a_y}_{\theta_1,\theta_2}\}^{-1}(x,y;x'',y'')R_{f,o}(x',y';x'',y'')dx'\,dy'\,dx''\,dy''$$

and

$$R_{o_a,o_a}(x,y;x,y) =$$
$$= \int\int\int\int B^{a_x,a_y}_{\theta_1,\theta_2}(x,y;x',y')\{B^{a_x,a_y}_{\theta_1,\theta_2}\}^{-1}(x,y;x'',y'')R_{o,o}(x',y';x'',y'')dx'\,dy'\,dx''\,dy''$$

In order to find the domain in which the MSE is smallest, we substitute the optimum filter function into the MSE expression:

$$\begin{aligned}\sigma_{e,o} &= E\left[\int\int [f_a(x,y)-\hat{f}_{a,o}(x,y)][f_a(x,y)-\hat{f}_{a,o}(x,y)]^*\,dx\,dy\right]\\ &= \int [R_{f_a,f_a}(x,y;x,y) - 2Re(g_o^*(x,y)R_{f_a,o_a}(x,y;x,y))\\ &\quad +|g_o(x,y)|^2 R_{o_a,o_a}(x,y;x,y)]\,dx\,dy.\end{aligned}$$

The optimum values of a_x and a_y are those that minimizes the MSE. These values can be found by simply calculating the MSE for different values of a_x and a_y and choosing the values that minimizes the MSE or by using multivariate optimization techniques possibly involving simulated annealing or genetic algorithms.

Chapter 7

Conclusion

The fractional Fourier transform is generalized to two dimensions by applying the one-dimensional definition in x and y directions separately. Since the transform defined in this manner is separable, its properties are similar to that of one-dimensional transform. Several properties of the separable two-dimensional fractional Fourier transform are derived or collected. (Some of these properties were already known or are trivial generalizations of their one-dimensional counterparts.)

Separable two-dimensional fractional Fourier transform can be optically implemented by using optical systems composed of thin lenses and sections of free space. We presented several optical systems by taking two different approaches. The first approach is based on the optical implementation of quadratic-phase systems and the results are specified to fractional Fourier transform by using the fact that fractional Fourier transform is a special quadratic-phase system. The second approach is specific to fractional Fourier transform. Beginning from the simplest set-up with two cylindrical lenses, many optical systems are examined.

The separable definition fails to satisfy the affine property which the common Fourier transform satisfies. When an arbitrary affine transformation is applied to the function, its two-dimensional fractional Fourier transform can not be represented as a scaled version of the fractional Fourier transform of the original function. We also wanted to specify the direction of transformation, which will enable us to take the two-dimensional fractional Fourier transform of a function along two arbitrary directions with the specified orders. But the separable two-dimensional fractional Fourier transform is always defined along traditional x and y axes and has no control on the directions along which the orders are defined. These two reasons motivated us to search for a new and more comprehensive definition for two-dimensional transform. We presented a new, non-separable definition which corresponds to fractional Fourier transformation along arbitrary directions and showed that this definition satisfies the affine property. The discrete-time and optical implementations of non-separable definition are given and its properties are derived.

The last part of the study is devoted to an application which justifies the usefulness of our new definition. Chirp noise is added to the image and the filtering scheme is implemented by using both definitions. The non-separable definition enabled a significant reduction of the MSE compared to the separable one.

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