

TRANSFORMATION PROPERTIES OF PAINLEVE  
VI EQUATION

A THESIS  
SUBMITTED TO THE DEPARTMENT OF MATHEMATICS  
AND THE INSTITUTE OF ENGINEERING AND SCIENCES  
OF BILKENT UNIVERSITY  
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS  
FOR THE DEGREE OF  
MASTER OF SCIENCE

By  
Ayman Sakha  
June, 1995

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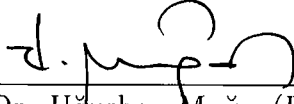
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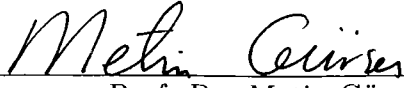
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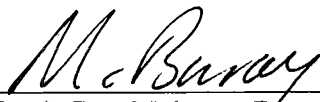
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# ABSTRACT

## TRANSFORMATION PROPERTIES OF PAINLEVÉ VI EQUATION

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M.S. in Mathematics

Supervisor: Asst. Prof. Dr. Uğurhan Muğan

June, 1995

In this thesis, we studied the Schlesinger transformations of Painlevé VI equation. We showed that Painlevé VI equation admits Schlesinger transformations which relate a given solution of Painlevé VI to solution of Painlevé VI but with different values of the parameters. Using these transformations we obtained the corresponding Bäcklund transformations for Painlevé VI. Also, we showed that the Schlesinger transformations and the corresponding Bäcklund transformations break down if and only if Painlevé VI has certain one-parameter family of solutions.

*Keywords : Painlevé Equations, Monodromy Data, Schlesinger Transformations, Riemann-Hilbert Problem.*

## ÖZET

### PAINLEVÉ VI DENKLEMİNİN DÖNÜŞÜM ÖZELLİKLERİ

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Matematik Bölümü Yüksek Lisans  
Tez Yöneticisi: Asst. Prof. Dr. Uğurhan Muğan  
Haziran, 1995

Bu tezde Painlevé VI denklemine ait Schlesinger dönüşümleri incelenmiştir. Bu çalışmanın sonununda, Painlevé VI denkleminin farklı parametre değerleri için olan çözümlerinin arasındaki ilişkiyi veren Schlesinger dönüşümleri elde edilmiştir. Buna ilaveten, Painlevé VI denkleminin tek parametrelili çözümlerinin Schlesinger dönüşümlerinin tanımsız olması halinde elde edilebileceği gösterilmiştir.

*Anahtar Kelimeler: Painlevé Denklemleri, Monodromy Data, Schlesinger Dönüşümleri, Riemann-Hilbert Problemi.*

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# Chapter 1

## Introduction

At the beginning of the century Painlevé and his school [1] classified the second order ODE of the form  $y'' = F(y', y, t)$ , where  $F$  is rational in  $y'$ , algebraic in  $y$  and locally analytic in  $t$ , which have the Painlevé property; i.e. their solutions are free from movable critical points. They found that, within a Möbius transformation, there exist fifty such equations. Distinguished among these fifty equations are the so called six Painlevé equations PI-PVI. The importance of these six equations arise from the fact that they are irreducible and they can not be integrated in terms of known transcendental functions, so they define new transcendents. Any other of the fifty equations can either be integrated in terms of known functions or can be reduced to one of these six equations. Although the six Painlevé equations were first discovered from strictly mathematical considerations, they have recently appeared in several physical applications [2],[3],[4].

Explicit transformations and relevant exact solutions admitted by the Painlevé equations first appeared in the Soviet literature. The main results can be summarized as follows [5],[6] :

- (i) For certain choices of the parameters, PII–PVI admit one-parameter family of solutions expressible in terms of the classical transcendental functions: Airy, Bessel, Weber-Hermite, Whittaker, and hypergeometric respectively.
- (ii) PII–PV admit transformations which map solutions of a given Painlevé equation to solutions of the same equation but with different values of the parameters.
- (iii) Using these transformations one can construct, for certain choices of the

parameters, various elementary solutions of PII–PV. These solutions are either rational or functions which are related, through repeated differentiations and multiplications, to the classical transcendental functions mentioned above.

Later, Fokas and Ablowitz [7] have developed an algorithmic method to study the transformation properties of second order ODE's of the Painlevé type. The algorithm is as follows:

Given one of the six Painlevé equations

$$y'' = P_1(y')^2 + P_2y' + P_3 \quad (1.1)$$

where  $P_1, P_2, P_3$ , are functions of  $y$ ,  $t$ , and a set of parameters  $\Theta$ . The first step is to find the discrete Lie-point symmetries of this equation, i.e., transformations of the form

$$\bar{y}(t; \bar{\Theta}) = F(y(t; \Theta), t) \quad (1.2)$$

where the function  $F$  is such that if  $y(t; \Theta)$  solves (1.1) with parameters  $\Theta$ , then  $\bar{y}(t; \bar{\Theta})$  solves (1.1) with parameters  $\bar{\Theta}$ . It is well known that the only transformation of the type (1.2) which preserve the Painlevé property is the Möbius transformation, hence one immediately replaces (1.2) by

$$\bar{y}(t; \bar{\Theta}) = \frac{a_1y + a_2}{a_3y + a_4} \quad (1.3)$$

where  $a_j$ ,  $j = 1, 2, 3, 4$ , are functions of  $t$  only. Using (1.3) the Lie-point discrete symmetries of (1.1) are easily obtained.

Next step is to find the generalized discrete symmetries of (1.1), i.e., transformations of the form

$$\bar{y}(t; \bar{\Theta}) = F(y'(t; \Theta), y(t; \Theta), t), \quad (1.4)$$

or more generally

$$v(t; \bar{\Theta}) = F(y'(t; \Theta), y(t; \Theta), t), \quad (1.5)$$

where  $F$  is such that  $v$  satisfies some second-order equation of the Painlevé type. The only transformation of the type (1.5), linear in  $y'$ , which preserve the Painlevé property is the one involving the Riccati equation, i.e.,

$$v(t; \bar{\Theta}) = \frac{y' + ay^2 + by + c}{ey^2 + fy + g}, \quad (1.6)$$

where  $a, b, c, e, f, g$  depend on  $t$  only. The aim is to find  $a, b, c, e, f, g$  such that (1.6) define a one-to-one invertible map between solutions  $y$  of (1.1) and

solutions  $v$  of some second order equation of the Painlevé type. In this process the equation for  $v$  is completely determined. To be more specific, define

$$I = ay^2 + by + c, \quad J = ey^2 + fy + g, \quad (1.7)$$

differentiating (1.6), and using (1.1) to replace  $y''$  and (1.6) to replace  $y'$ , one obtains

$$\begin{aligned} Jv' = [P_1J^2 - 2eyJ - fJ]v^2 + [-2P_1IJ + P_2J + 2ayJ + \\ bJ + 2eyI + fI - (e'y^2 + f'y + g')]v + \\ [P_1I^2 - P_2I + P_3 - 2ayI - bI + a'y^2 + b'y + c']. \end{aligned} \quad (1.8)$$

There are two cases to be distinguished :

(A) Find  $a, b, c, e, f, g$  such that (1.8) reduce to linear equation for  $y$ ,

$$A(v', v, t)y + B(v', v, t) = 0. \quad (1.9)$$

Having determined  $a, b, c, e, f$  upon substitution of  $y = -B/A$  in (1.6) one determines the equation for  $v$ , which will be one of the fifty equations of Painlevé.

(B) Find  $a, b, c, e, f, g$  such that (1.8) reduces to a quadratic equation for  $y$ ,

$$A(v', v, t)y^2 + B(v', v, t)y + C(v', v, t) = 0. \quad (1.10)$$

Then (1.6) yields an equation for  $v$  which is quadratic in the second derivative.

Using this method they have recovered most of the results given in the Soviet literature and obtained some new ones . For PVI they obtained the following results:

Let  $y(t; \alpha, \beta, \gamma, \delta)$  be a solution of PVI:

$$\begin{aligned} y'' = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) (y')^2 - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) y' \\ + \frac{y(y-1)(y-t)}{t^2(t-1)^2} \left( \alpha + \beta \frac{t}{y^2} + \gamma \frac{t-1}{(y-1)^2} + \delta \frac{t(t-1)}{(y-t)^2} \right), \end{aligned} \quad (1.11)$$

Then  $\bar{y}(t; \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta})$  are also solutions of PVI, where

$$\begin{aligned} \bar{y}(t; \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}) &= ty(\frac{1}{t}; \alpha, \beta, \gamma, \delta); \\ \bar{\alpha} &= \alpha, \quad \bar{\beta} = \beta, \quad \bar{\gamma} = -\delta + \frac{1}{2}, \quad \bar{\delta} = -\gamma + \frac{1}{2}, \end{aligned} \quad (1.12)$$

$$\begin{aligned} \bar{y}(t; \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}) &= 1 - y(1-t; \alpha, \beta, \gamma, \delta); \\ \bar{\alpha} &= \alpha, \quad \bar{\beta} = -\gamma, \quad \bar{\gamma} = -\beta, \quad \bar{\delta} = \delta, \end{aligned} \quad (1.13)$$

$$\begin{aligned}\bar{y}(t; \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}) &= 1 - (1-t)y\left(\frac{1}{1-t}; \alpha, \beta, \gamma, \delta\right); \\ \bar{\alpha} &= \alpha, \quad \bar{\beta} = \delta - \frac{1}{2}, \quad \bar{\gamma} = -\beta, \quad \bar{\delta} = -\gamma + \frac{1}{2},\end{aligned}\tag{1.14}$$

$$\begin{aligned}\bar{y} &= y + 2((t+1)y - 2t)\left(-\frac{2t(t-1)\Phi'}{\kappa\Phi} + \frac{(t-1)\Psi}{\kappa\Phi} - (t+1)\right)^{-1}; \\ \bar{\alpha} &= \frac{1}{2}[(-2\beta)^{1/2} - 1]^2, \quad \bar{\beta} = -\frac{1}{2}[(2\alpha)^{1/2} + 1]^2, \\ \bar{\gamma} &= \gamma + \frac{\kappa\mu}{4}, \quad \bar{\delta} = \delta + \frac{\kappa\mu}{4},\end{aligned}\tag{1.15}$$

where

$$\begin{aligned}\Phi &= t\frac{y'}{y} + \frac{(\lambda - \kappa - 1)}{2(t-1)}y + \frac{(\lambda + \kappa + 1)t}{2(t-1)y} - \frac{\lambda(t+1)}{2(t-1)} - \left(\frac{1}{2} + \frac{\mu}{4}\right), \\ \Psi &= \Phi^2 + \frac{\mu}{2}\Phi + \nu, \\ \kappa &= (-2\beta)^{1/2} - (2\alpha)^{1/2} - 1, \quad \lambda = (-2\beta)^{1/2} + (2\alpha)^{1/2}, \\ \mu &= \frac{4}{\kappa}\left(\frac{1}{2} - \gamma - \delta\right), \quad \nu = 2\delta - 1 + \left(\frac{\mu}{4} + \frac{\kappa}{2}\right)^2.\end{aligned}\tag{1.16}$$

These were the first transformations for PVI. It should be noticed that these transformations can not be used to generate infinite hierarchy of exact solutions. This follows from the fact that a finite number of applications of these transformations yields the identity. For example, one obtains the identity after three consecutive application of (1.14) and two consecutive applications of (1.15).

Another type of transformations for PVI was found by Kitaev [8]. These transformations, which can be considered as an analog of the well-known quadratic transformations for the hypergeometric functions, relate a given solution  $y(t)$  of PVI to a solution  $y(s)$  of PVI, where  $s$  is connected with  $t$  by a quadratic relation. The application of these transformations is limited, since they are only valid for specific values of the parameters,  $\alpha = \frac{1}{8}$  or  $\frac{9}{8}$ ,  $\beta = -\frac{1}{8}$ , and  $\delta \neq \frac{1}{2}$ .

The Schlesinger transformations of the Painlevé equations have been discovered during the implementation of the so-called inverse monodromic method, an extension of the inverse spectral method to ODE's [9],[10],[11],[12],[13],[14]. In order to apply the inverse monodromy method, it is necessary to study the analytical structure of the solution of the associated monodromy problem,  $Y_z = AY$ , where  $z$  plays the role of the spectral parameter. It turns out that there exists a sectionary meromorphic function  $Y(z)$ , with certain jumps across certain contours in the complex  $z$ -plane; these jumps are specified by the so-called monodromy data, MD. It turns out that it is possible to find an appropriate transformations for the parameters of the Painlevé equation such that the MD are invariant. These transformations can be found in closed form, by solving a certain simple Riemann-Hilbert problems [15].

The Schlesinger transformations of PII–PV have been studied by Muřan and Fokas [16]. Using these transformations they re-drive some of the well known Bäcklund transformations of Painlevé equations. Using the same procedure we will investigate the Schlesinger transformations of PVI [17].

This thesis is organized as follows:

In chapter 2 the monodromy problem associated with PVI is given and the analytic structure of  $Y(z)$  is obtained.

Chapter 3 consists of the Schlesinger transformations and the associated Bäcklund transformations of PVI.

In chapter 4 the one-parameter family of solutions of PVI is obtained from the associated transformations.

## Chapter 2

### The monodromy problem of PVI

In this chapter we present the linear equation associated with PVI and study the analytical structure of the solution of this equation.

#### 2.1 The sixth Painlevé equation

It is known that PVI,

$$\frac{d^2y}{dt^2} = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) \left( \frac{dy}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) \frac{dy}{dt} + \frac{y(y-1)(y-t)}{t^2(t-1)^2} \left( \alpha + \beta \frac{t}{y^2} + \gamma \frac{t-1}{(y-1)^2} + \delta \frac{t(t-1)}{(y-t)^2} \right), \quad (2.1)$$

can be written as the compatibility condition of the following linear system of equations [10],

$$\frac{\partial Y}{\partial z} = A(z)Y(z, t), \quad \frac{\partial Y}{\partial t} = B(z)Y(z, t), \quad (2.2)$$

where

$$A(z) = \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t} = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix},$$

$$A_0 = \begin{pmatrix} u_0 + \theta_0 & -w_0 u_0 \\ w_0^{-1}(u_0 + \theta_0) & -u_0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} u_1 + \theta_1 & -w_1 u_1 \\ w_1^{-1}(u_1 + \theta_1) & -u_1 \end{pmatrix}, \quad (2.3)$$

$$A_t = \begin{pmatrix} u_t + \theta_t & -w_t u_t \\ w_t^{-1}(u_t + \theta_t) & -u_t \end{pmatrix}, \quad B(z) = -\frac{A_t}{z-t}.$$

Setting,

$$\begin{aligned}
A_\infty &= -(A_0 + A_1 + A_t) = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}, \\
\kappa_1 + \kappa_2 &= -(\theta_0 + \theta_1 + \theta_t), & \kappa_1 - \kappa_2 &= \theta_\infty, \\
a_{12}(z) &= -\frac{w_0 u_0}{z} - \frac{w_1 u_1}{z-1} - \frac{w_t u_t}{z-t} = \frac{k(z-y)}{z(z-1)(z-t)}, \\
u &= a_{11}(y) = \frac{u_0 + \theta_0}{y} + \frac{u_1 + \theta_1}{y-1} + \frac{u_t + \theta_t}{y-t}, \\
\bar{u} &= -a_{22}(y) = u - \frac{\theta_0}{y} - \frac{\theta_1}{y-1} - \frac{\theta_t}{y-t}.
\end{aligned} \tag{2.4}$$

Then

$$\begin{aligned}
u_0 + u_1 + u_t &= \kappa_2, & w_0 u_0 + w_1 u_1 + w_t u_t &= 0, \\
\frac{u_0 + \theta_0}{w_0} + \frac{u_1 + \theta_1}{w_1} + \frac{u_t + \theta_t}{w_t} &= 0, \\
(t+1)w_0 u_0 + t w_1 u_1 + w_t u_t &= k, & t w_0 u_0 &= k(t)y,
\end{aligned} \tag{2.5}$$

which are solved as,

$$\begin{aligned}
w_0 &= \frac{ky}{tu_0}, & w_1 &= -\frac{k(y-1)}{u_1(t-1)}, & w_t &= \frac{k(y-t)}{t(t-1)u_t}, \\
u_0 &= \frac{y}{t\theta_\infty} \{y(y-1)(y-t)\bar{u}^2 \\
&\quad + [\theta_1(y-t) + t\theta_t(y-1) - 2\kappa_2(y-1)(y-t)]\bar{u} \\
&\quad + \kappa_2^2(y-t-1) - \kappa_2(\theta_1 + t\theta_t)\}, \\
u_1 &= -\frac{y-1}{(t-1)\theta_\infty} \{y(y-1)(y-t)\bar{u}^2 \\
&\quad + [(\theta_1 + \theta_\infty)(y-t) + t\theta_t(y-1) - 2\kappa_2(y-1)(y-t)]\bar{u} \\
&\quad + \kappa_2^2(y-t) - \kappa_2(\theta_1 + t\theta_t) - \kappa_1\kappa_2\}, \\
u_t &= \frac{y-t}{t(t-1)\theta_\infty} \{y(y-1)(y-t)\bar{u}^2 \\
&\quad + [\theta_1(y-t) + t(\theta_t + \theta_\infty)(y-1) - 2\kappa_2(y-1)(y-t)]\bar{u} \\
&\quad + \kappa_2^2(y-1) - \kappa_2(\theta_1 + t\theta_t) - t\kappa_1\kappa_2\}.
\end{aligned} \tag{2.6}$$

The equation  $Y_{zt} = Y_{tz}$  implies

$$\begin{aligned}
\frac{dy}{dt} &= \frac{y(y-1)(y-t)}{t(t-1)} \left( 2u - \frac{\theta_0}{y} - \frac{\theta_1}{y-1} - \frac{\theta_t-1}{y-t} \right), \\
\frac{du}{dt} &= \frac{1}{t(t-1)} \{ [-3y^2 + 2(t+1)y - t]u^2 \\
&\quad + [(2y-t-1)\theta_0 + (2y-t)\theta_1 + (2y-1)(\theta_t-1)]u \\
&\quad - \kappa_1(\kappa_2 + 1) \}, \\
\frac{1}{k} \frac{dk}{dt} &= (\theta_\infty - 1) \frac{y-t}{t(t-1)}.
\end{aligned} \tag{2.7}$$

Thus  $y$  satisfies the six Painlevé equation (2.1), with the parameters

$$\alpha = \frac{1}{2}(\theta_\infty - 1)^2, \quad \beta = -\frac{1}{2}\theta_0^2, \quad \gamma = \frac{1}{2}\theta_1^2, \quad \delta = \frac{1}{2}(1 - \theta_t^2). \quad (2.8)$$

## 2.2 Direct Problem

The essence of the direct problem is to establish the analytic structure of  $Y$  with respect to  $z$ , in the entire complex  $z$ -plane. Since (2.2) is a linear ODE in  $z$ , the analytic structure is completely determined by its singular points. The equation (2.2) has regular singular points at  $z = 0, 1, t, \infty$ .

It is well known that if the coefficient matrix of the linear ODE has an isolated singularity at  $z = 0$ , then the solution in the neighborhood of  $z = 0$  can be obtained via a convergent power series. In this particular case the solution  $Y_0(z) = (Y_0^{(1)}(z), Y_0^{(2)}(z))$ , for  $\theta_0 \neq n, n \in \mathbb{Z}$  has the form

$$Y_0 = \hat{Y}_0(z)z^{D_0} = G_0(I + Y_{01}z + Y_{02}z^2 + \dots)z^{D_0}, \quad (2.9)$$

where

$$G_0 = \begin{pmatrix} 2k_0 & l_0 w_0 u_0 \\ \frac{2k_0}{w_0} & l_0(u_0 + \theta_0) \end{pmatrix}, \quad \det G_0 = 1, \quad D_0 = \begin{pmatrix} \theta_0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (2.10)$$

$$k_0 = \tilde{k}_0 e^{\sigma_0(t)}, \quad l_0 = \tilde{l}_0 e^{-\sigma_0(t)}, \quad \tilde{k}_0, \tilde{l}_0 = \text{constant},$$

$$\sigma_0 = \int^t \frac{1}{s} \left[ u_t + \theta_t - \frac{w_t u_t}{w_0} \right] ds;$$

and  $Y_{01}$  satisfies the following equation :

$$Y_{01} + [Y_{01}, D_0] = -G_0^{-1} \left( A_1 G_0 - \frac{dG_0}{dt} \right). \quad (2.11)$$

If  $\theta_0 = n, n \in \mathbb{Z}$  then the solution  $Y_0(z)$  may or may not have the  $\log z$  term.

The monodromy matrix about  $z = 0$  is given as

$$Y_0(z e^{2i\pi}) = Y_0(z) e^{2i\pi D_0}. \quad (2.12)$$

The solution  $Y_1(z) = (Y_1^{(1)}(z), Y_1^{(2)}(z))$ , of equation (2.2) in the neighborhood of the regular singular point  $z = 1$  for  $\theta_1 \neq n, n \in \mathbb{Z}$  has the form

$$Y_1 = \hat{Y}_1(z)(z-1)^{D_1} = G_1(I + Y_{11}(z-1) + Y_{12}(z-1)^2 + \dots)(z-1)^{D_1}, \quad (2.13)$$



where

$$G_1 = \begin{pmatrix} 2k_1 & l_1 w_1 u_1 \\ \frac{2k_1}{w_1} & l_1(u_1 + \theta_1) \end{pmatrix}, \quad \det G_1 = 1, \quad D_1 = \begin{pmatrix} \theta_1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$k_1 = \tilde{k}_1 e^{\sigma_1(t)}, \quad l_1 = \tilde{l}_1 e^{-\sigma_1(t)}, \quad \tilde{k}_1, \tilde{l}_1 = \text{constant},$$

$$\sigma_1 = \int^t \frac{1}{s-1} \left[ u_t + \theta_t - \frac{w_t u_t}{w_1} \right] ds; \quad (2.14)$$

and  $Y_{11}$  satisfies the following equation :

$$Y_{11} + [Y_{11}, D_1] = G_1^{-1} \left( A_0 G_1 - \frac{dG_1}{dt} \right). \quad (2.15)$$

If  $\theta_1 = n, n \in \mathbb{Z}$  then the solution  $Y_1(z)$  may or may not have the  $\log(z-1)$  term.

The monodromy matrix about  $z=1$  is given as

$$Y_1(z e^{2i\pi}) = Y_1(z) e^{2i\pi D_1}. \quad (2.16)$$

The solution  $Y_t(z) = (Y_t^{(1)}(z), Y_t^{(2)}(z))$ , of equation (2.2) in the neighborhood of the regular singular point  $z=t$  for  $\theta_t \neq n, n \in \mathbb{Z}$  has the form

$$Y_t = \hat{Y}_t(z)(z-t)^{D_t} = G_t (I + Y_{t1}(z-t) + Y_{t2}(z-t)^2 + \dots)(z-t)^{D_t}, \quad (2.17)$$

where

$$G_t = \begin{pmatrix} 2k_t & l_t w_t u_t \\ \frac{2k_t}{w_t} & l_t(u_t + \theta_t) \end{pmatrix}, \quad \det G_t = 1, \quad D_t = \begin{pmatrix} \theta_t & 0 \\ 0 & 0 \end{pmatrix}, \quad (2.18)$$

$$k_t = \tilde{k}_t e^{\sigma_t(t)}, \quad l_t = \tilde{l}_t e^{-\sigma_t(t)}, \quad \tilde{k}_t, \tilde{l}_t = \text{constant},$$

$$\sigma_t = \int^t \left[ \frac{1}{s} (u_0 + \theta_0 - \frac{w_0 u_0}{w_t}) + \frac{1}{s-1} \left( (u_1 + \theta_1 - \frac{w_1 u_1}{w_t}) \right) \right] ds;$$

and  $Y_{t1}$  satisfies the following equation :

$$Y_{t1} + [Y_{t1}, D_t] = G_t^{-1} \left( \frac{d}{dt} G_t \right). \quad (2.19)$$

If  $\theta_t = n, n \in \mathbb{Z}$  then the solution  $Y_t(z)$  may or may not have the  $\log(z-t)$  term.

The monodromy matrix about  $z = t$  is given as

$$Y_t(ze^{2i\pi}) = Y_t(z)e^{2i\pi D_t}. \quad (2.20)$$

The solution  $Y_\infty(z) = (Y_\infty^{(1)}(z), Y_\infty^{(2)}(z))$ , of equation (2.2) in the neighborhood of the regular singular point  $z = \infty$  for  $\theta_\infty \neq n, n \in \mathbb{Z}$  has the form

$$Y_\infty = \hat{Y}_\infty(z) \left(\frac{1}{z}\right)^{D_\infty} = \left(I + Y_{\infty 1} \frac{1}{z} + Y_{\infty 2} \left(\frac{1}{z}\right)^2 + \dots\right) \left(\frac{1}{z}\right)^{D_\infty}, \quad (2.21)$$

where

$$D_\infty = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}, \quad (2.22)$$

$$\kappa_1 = u_0 + u_1 + u_t, \quad \kappa_1 - \kappa_2 = \theta_\infty, \quad \kappa_1 + \kappa_2 = -(\theta_0 + \theta_1 + \theta_t);$$

and  $Y_{\infty 1}$  satisfies the following equation :

$$Y_{\infty 1} + [Y_{\infty 1}, D_\infty] = -(A_1 + tA_t). \quad (2.23)$$

If  $\theta_\infty = n, n \in \mathbb{Z}$  then the solution  $Y_\infty(z)$  may or may not have the  $\log \frac{1}{z}$  term.

The monodromy matrix about  $z = \infty$  is given as

$$Y_\infty(ze^{2i\pi}) = Y_\infty(z)e^{-2i\pi D_\infty}. \quad (2.24)$$

## 2.2.1 Monodromy Data

$Y_0, Y_1, Y_t$  and  $Y_\infty$  are solutions of the same linear equation (2.2), therefore there are matrices,  $E_j, j = 0, 1, t$ , independent of  $z$  such that

$$Y_\infty(z) = Y_i(z)E_i, \quad E_i = \begin{pmatrix} \mu_i & \nu_i \\ \zeta_i & \eta_i \end{pmatrix}, \quad \det E_i = 1, \quad i = 0, 1, t. \quad (2.25)$$

Let  $Y_\infty(z_0)$  be the solution of equation (2.2) at  $z = z_0$  where  $z_0 \neq 0, 1, t$  is a point in the complex  $z$ -plane. Starting from the point  $z = z_0$ , if we describe a closed path around the branch point  $z = 0$ , then equations (2.12) and (2.25) imply

$$Y_\infty(z_0) = Y_\infty(z_0 e^{2i\pi}) E_0^{-1} e^{2i\pi D_0} E_0. \quad (2.26)$$

If we continue and describe a closed path around the branch point  $z = 1$ , then using the analyticity of  $Y_1(z)$  at  $z = 0$  and equations (2.16), (2.25), (2.27) we find

$$Y_\infty(z_0) = Y_\infty(z_0 e^{2i\pi}) E_1^{-1} e^{2i\pi D_1} E_1 E_0^{-1} e^{2i\pi D_0} E_0. \quad (2.27)$$

Similarly, after enclosing the branch point  $z = t$ , equations (2.24),(2.25) and (2.27) give

$$Y_\infty(z_0) = Y_\infty(z_0 e^{2i\pi}) E_t^{-1} e^{2i\pi D_t} E_t E_1^{-1} e^{2i\pi D_1} E_1 E_0^{-1} e^{2i\pi D_0} E_0. \quad (2.28)$$

Therefore, comparing the equations (2.24) and (2.28) we find that the monodromy data  $MD = \{\mu_0, \nu_0, \zeta_0, \eta_0, \mu_1, \nu_1, \zeta_1, \eta_1, \mu_t, \nu_t, \zeta_t, \eta_t\}$  should satisfy the following consistency condition:

$$(E_0^{-1} e^{2i\pi D_0} E_0)(E_1^{-1} e^{2i\pi D_1} E_1)(E_t^{-1} e^{2i\pi D_t} E_t) = e^{-2i\pi D_\infty}, \quad (2.29)$$

The trace of (2.29) reads

$$\begin{aligned} & \cos \pi(\theta_0 - \theta_1)(\zeta_0 \mu_0 \eta_1 \nu_1 + \eta_0 \nu_0 \mu_1 \zeta_1 - \eta_0 \mu_0 \nu_1 \zeta_1 - \zeta_0 \nu_0 \mu_1 \eta_1) + \\ & \cos \pi(\theta_0 + \theta_1)(\nu_0 \zeta_0 \nu_1 \zeta_1 + \mu_0 \eta_0 \mu_1 \eta_1 - \mu_0 \zeta_0 \nu_1 \eta_1 - \nu_0 \eta_0 \mu_1 \zeta_1) = \\ & \mu_t \eta_t \cos \pi(\theta_t + \theta_\infty) - \nu_t \zeta_t \cos \pi(\theta_t - \theta_\infty). \end{aligned} \quad (2.30)$$

## Chapter 3

### Transformations Of PVI

In this chapter we will study the Schlesinger transformations of the linear system (2.2). Using these transformations we will obtain Bäcklund transformations for PVI.

#### 3.1 Schlesinger Transformations

Let  $R(z)$  be the transformation matrix which transforms the solution of the linear problem (2.2) as ;

$$Y' = R(z)Y(z), \quad (3.1)$$

but leaves the monodromy data associated with  $Y(z)$  the same. Let  $u'_i, w'_i, \theta'_i = \theta_i + \lambda_i$  be the transformed quantities of  $u_i, w_i, \theta_i, i = 0, 1, t, \infty$ . The consistency condition of the monodromy data (2.29) or (2.30) is invariant under the transformation if  $\lambda_1 + \lambda_0 = k, \lambda_1 - \lambda_0 = l, \lambda_\infty + \lambda_t = m, \lambda_\infty - \lambda_t = n$ , where  $k, l, m, n$ , are either all odd or all even integers. It is enough to consider

the following three cases;

$$\begin{aligned}
 a : & \begin{cases} \theta'_0 = \theta_0 + \lambda_0 \\ \theta'_1 = \theta_1 \\ \theta'_t = \theta_t \\ \theta'_\infty = \theta_\infty + \lambda_\infty, \end{cases} \\
 b : & \begin{cases} \theta'_0 = \theta_0 \\ \theta'_1 = \theta_1 + \lambda_1 \\ \theta'_t = \theta_t \\ \theta'_\infty = \theta_\infty + \lambda_\infty, \end{cases} \quad (3.2) \\
 c : & \begin{cases} \theta'_0 = \theta_0 \\ \theta'_1 = \theta_1 \\ \theta'_t = \theta_t + \lambda_t \\ \theta'_\infty = \theta_\infty + \lambda_\infty, \end{cases}
 \end{aligned}$$

for  $\lambda_j = \pm 1$ ,  $j = 0, 1, t, \infty$ .

Let the complex  $z$ -plane be divided into two sectors  $S^\pm$  by an infinite contour  $C$  passing through the points  $z = 0, 1, t$  and let,

$$R(z) = R^\pm(z), \quad \text{when } z \in S^\pm. \quad (3.3)$$

Then the transformation (3.1) can be written as

$$[Y^\pm]' = R^\pm(z)Y^\pm(z) \quad \text{when } z \in S^\pm, \quad (3.4)$$

and the monodromy matrices (2.12), (2.16), (2.20) and (2.24) about  $z = 0, 1, t, \infty$  imply that the transformation matrix  $R(z)$  satisfies the following RH-problems;

$$\begin{aligned}
 a : & \begin{cases} R^+(z) = R^-(z) & \text{on } C_0^- \\ R^+(z) = R^-(ze^{2i\pi}) & \text{on } C_0^+, \end{cases} \\
 b : & \begin{cases} R^+(z) = R^-(z) & \text{on } C_1^- \\ R^+(z) = R^-(ze^{2i\pi}) & \text{on } C_1^+, \end{cases} \quad (3.5) \\
 c : & \begin{cases} R^+(z) = R^-(z) & \text{on } C_t^- \\ R^+(z) = R^-(ze^{2i\pi}) & \text{on } C_t^+, \end{cases}
 \end{aligned}$$

where  $C_i^\pm$ ,  $i = 0, 1, t$  are parts of the contour  $C$  with the initial points  $z = 0, 1, t$  respectively. The boundary conditions for the RH-problems are as follows;

$$\begin{aligned}
a: & \begin{cases} R^+ \sim \hat{Y}'_0(z) z^{\Lambda_0} \hat{Y}_0^{-1}(z) & \text{as } z \rightarrow 0, & z \in S^+ \\ R^+ \sim \hat{Y}'_1(z) \hat{Y}_1^{-1}(z) & \text{as } z \rightarrow 1, & z \in S^+ \\ R^+ \sim \hat{Y}'_t(z) \hat{Y}_t^{-1}(z) & \text{as } z \rightarrow t, & z \in S^+ \\ R^+ \sim \hat{Y}'_\infty(z) \left(\frac{1}{z}\right)^{\Sigma_0} \hat{Y}_\infty^{-1}(z) & \text{as } |z| \rightarrow \infty, & z \in S^+, \end{cases} \\
b: & \begin{cases} R^+ \sim \hat{Y}'_0(z) \hat{Y}_0^{-1}(z) & \text{as } z \rightarrow 0, & z \in S^+ \\ R^+ \sim \hat{Y}'_1(z) (z-1)^{\Lambda_1} \hat{Y}_1^{-1}(z) & \text{as } z \rightarrow 1, & z \in S^+ \\ R^+ \sim \hat{Y}'_t(z) \hat{Y}_t^{-1}(z) & \text{as } z \rightarrow t, & z \in S^+ \\ R^+ \sim \hat{Y}'_\infty(z) \left(\frac{1}{z}\right)^{\Sigma_1} \hat{Y}_\infty^{-1}(z) & \text{as } |z| \rightarrow \infty, & z \in S^+, \end{cases} \quad (3.6) \\
c: & \begin{cases} R^+ \sim \hat{Y}'_0(z) \hat{Y}_0^{-1}(z) & \text{as } z \rightarrow 0, & z \in S^+ \\ R^+ \sim \hat{Y}'_1(z) \hat{Y}_1^{-1}(z) & \text{as } z \rightarrow 1, & z \in S^+ \\ R^+ \sim \hat{Y}'_t(z) (z-t)^{\Lambda_t} \hat{Y}_t^{-1}(z) & \text{as } z \rightarrow t, & z \in S^+ \\ R^+ \sim \hat{Y}'_\infty(z) \left(\frac{1}{z}\right)^{\Sigma_t} \hat{Y}_\infty^{-1}(z) & \text{as } |z| \rightarrow \infty, & z \in S^+, \end{cases}
\end{aligned}$$

where

$$\begin{aligned}
\Lambda_i &= \begin{pmatrix} \lambda_i & 0 \\ 0 & 0 \end{pmatrix}, \\
\Sigma_i &= \begin{pmatrix} \frac{1}{2}(\lambda_\infty - \lambda_i) & 0 \\ 0 & -\frac{1}{2}(\lambda_\infty + \lambda_i) \end{pmatrix}, \quad i = 0, 1, t.
\end{aligned} \quad (3.7)$$

For each case a, b and c there exist a function  $R(z)$  which is analytic everywhere and the boundary conditions (3.6) specify  $R(z)$ .

Solving the RH-problem for each case we find the following transformation matrices  $R_j(z)$ ,  $j = 1, 2, \dots, 12$  :

$$\begin{cases} \theta'_0 = \theta_0 + 1 \\ \theta'_1 = \theta_1 \\ \theta'_t = \theta_t \\ \theta'_\infty = \theta_\infty + 1, \end{cases} \quad R_{(1)}(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z + \begin{pmatrix} 1 & -w_0 \\ -r_1 & w_0 r_1 \end{pmatrix}, \quad (3.8)$$

$$\begin{cases} \theta'_0 = \theta_0 - 1 \\ \theta'_1 = \theta_1 \\ \theta'_t = \theta_t \\ \theta'_\infty = \theta_\infty - 1, \end{cases} \quad R_{(2)}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{u_0 + \theta_0}{u_0 w_0} r_2 & -r_2 \\ -\frac{u_0 + \theta_0}{u_0 w_0} & 1 \end{pmatrix} \frac{1}{z}, \quad (3.9)$$

$$\begin{cases} \theta'_0 = \theta_0 - 1 \\ \theta'_1 = \theta_1 \\ \theta'_t = \theta_t \\ \theta'_\infty = \theta_\infty + 1, \end{cases} R_{(3)}(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -\frac{u_0 w_0}{u_0 + \theta_0} \\ -r_1 & \frac{u_0 w_0}{u_0 + \theta_0} r_1 \end{pmatrix} \frac{1}{z}, \quad (3.10)$$

$$\begin{cases} \theta'_0 = \theta_0 + 1 \\ \theta'_1 = \theta_1 \\ \theta'_t = \theta_t \\ \theta'_\infty = \theta_\infty - 1, \end{cases} R_{(4)}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z + \begin{pmatrix} \frac{r_2}{w_0} & -r_2 \\ -\frac{1}{w_0} & 1 \end{pmatrix}, \quad (3.11)$$

$$\begin{cases} \theta'_0 = \theta_0 \\ \theta'_1 = \theta_1 + 1 \\ \theta'_t = \theta_t \\ \theta'_\infty = \theta_\infty + 1, \end{cases} R_{(5)}(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} (z - 1) + \begin{pmatrix} 1 & -w_1 \\ -r_1 & w_1 r_1 \end{pmatrix}, \quad (3.12)$$

$$\begin{cases} \theta'_0 = \theta_0 \\ \theta'_1 = \theta_1 - 1 \\ \theta'_t = \theta_t \\ \theta'_\infty = \theta_\infty - 1, \end{cases} R_{(6)}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{u_1 + \theta_1}{u_1 w_1} r_2 & -r_2 \\ -\frac{u_1 + \theta_1}{u_1 w_1} & 1 \end{pmatrix} \frac{1}{z - 1}, \quad (3.13)$$

$$\begin{cases} \theta'_0 = \theta_0 \\ \theta'_1 = \theta_1 - 1 \\ \theta'_t = \theta_t \\ \theta'_\infty = \theta_\infty + 1, \end{cases} R_{(7)}(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -\frac{u_1 w_1}{u_1 + \theta_1} \\ -r_1 & \frac{u_1 w_1}{u_1 + \theta_1} r_1 \end{pmatrix} \frac{1}{z - 1}, \quad (3.14)$$

$$\begin{cases} \theta'_0 = \theta_0 \\ \theta'_1 = \theta_1 + 1 \\ \theta'_t = \theta_t \\ \theta'_\infty = \theta_\infty - 1, \end{cases} R_{(8)}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (z - 1) + \begin{pmatrix} \frac{r_2}{w_1} & -r_2 \\ -\frac{1}{w_1} & 1 \end{pmatrix}, \quad (3.15)$$

$$\begin{cases} \theta'_0 = \theta_0 \\ \theta'_1 = \theta_1 \\ \theta'_t = \theta_t + 1 \\ \theta'_\infty = \theta_\infty + 1, \end{cases} R_{(9)}(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} (z - t) + \begin{pmatrix} 1 & -w_t \\ -r_1 & w_t r_1 \end{pmatrix}, \quad (3.16)$$

$$\begin{cases} \theta'_0 = \theta_0 \\ \theta'_1 = \theta_1 \\ \theta'_t = \theta_t - 1 \\ \theta'_\infty = \theta_\infty - 1, \end{cases} \quad R_{(10)}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{u_t + \theta_t}{u_t w_t} r_2 & -r_2 \\ -\frac{u_t + \theta_t}{u_t w_t} & 1 \end{pmatrix} \frac{1}{z - t}, \quad (3.17)$$

$$\begin{cases} \theta'_0 = \theta_0 \\ \theta'_1 = \theta_1 \\ \theta'_t = \theta_t - 1 \\ \theta'_\infty = \theta_\infty + 1, \end{cases} \quad R_{(11)}(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -\frac{u_t w_t}{u_t + \theta_t} \\ -r_1 & \frac{u_t w_t}{u_t + \theta_t} r_1 \end{pmatrix} \frac{1}{z - t}, \quad (3.18)$$

$$\begin{cases} \theta'_0 = \theta_0 \\ \theta'_1 = \theta_1 \\ \theta'_t = \theta_t + 1 \\ \theta'_\infty = \theta_\infty - 1, \end{cases} \quad R_{(12)}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (z - t) + \begin{pmatrix} \frac{r_2}{w_t} & -r_2 \\ -\frac{1}{w_t} & 1 \end{pmatrix}, \quad (3.19)$$

where

$$r_1 = -\frac{1}{1 + \theta_\infty} \left( \frac{u_1 + \theta_1}{w_1} + t \frac{u_t + \theta_t}{w_t} \right), \quad (3.20)$$

$$r_2 = \frac{1}{1 - \theta_\infty} (w_1 u_1 + t w_t u_t),$$

and  $u_i, w_i, i = 0, 1, t$  are given in (2.6).

The linear equation (2.2.a) is transformed under any transformation matrix  $R(z)$  as follows:

$$\frac{\partial Y'}{\partial z} = A'(z)Y', \quad A'(z) = [R(z)A(z) + \frac{\partial}{\partial z} R(z)]R^{-1}(z). \quad (3.21)$$

Therefore, the entries  $u'_i, w'_i, i = 0, 1, t$  of the coefficient matrix  $A'(z)$  can be determined in terms of the entries  $u_i, w_i, i = 0, 1, t$  of  $A(z)$ . Let  $R_{(j)}(z)$  and  $R_{(k)}(z)$  be any transformation matrices which shift the parameters  $\theta_0, \theta_1, \theta_t, \theta_\infty$  to  $\theta_0 + \lambda_0, \theta_1 + \lambda_1, \theta_t + \lambda_t, \theta_\infty + \lambda_\infty$  and  $\theta_0 + \lambda'_0, \theta_1 + \lambda'_1, \theta_t + \lambda'_t, \theta_\infty + \lambda'_\infty$  respectively. The solution  $Y(z, t; u_i, w_i)$  of equation (2.2) is transformed under the transformation matrix  $R_{(j)}(z)$  as;

$$Y'(z, t; u'_i, w'_i) = R_{(j)}(z, t; u_i, w_i)Y(z, t; u_i, w_i). \quad (3.22)$$

Applying the transformation matrix  $R_{(k)}(z)$  to  $Y'(z)$  one obtains;

$$\begin{aligned} Y''(z, t; u''_i, w''_i) &= R_{(k)}(z, t; u'_i, w'_i)Y'(z, t; u'_i, w'_i) \\ &= R_{(k)}(z, t; u'_i, w'_i)R_{(j)}(z, t; u_i, w_i)Y(z, t; u_i, w_i). \end{aligned} \quad (3.23)$$



Since  $u'_i, w'_i$  can be determined in terms of  $u_i, w_i, i = 0, 1, t$ , one can obtain a transformation matrix  $R(z, t; u_i, w_i) = R_{(k)}(z, t; u'_i, w'_i)R_{(k)}(z, t; u_i, w_i)$  which shifts the parameters  $\theta_0, \theta_1, \theta_t, \theta_\infty$  to  $\theta_0 + \lambda_0 + \lambda'_0, \theta_1 + \lambda_1 + \lambda'_1, \theta_t + \lambda_t + \lambda'_t, \theta_\infty + \lambda_\infty + \lambda'_\infty$ . Therefore, using the transformation matrices  $R_{(j)}, j = 1, 2, \dots, 12$ , one can obtain the transformation matrix  $R(z)$  which shifts the parameters  $\theta_0, \theta_1, \theta_t, \theta_\infty$  by any integers. For examples, the transformation matrices  $R_{(3,6)}(z) = R_{(3)}(z)R_{(6)}(z), R_{(4,8)}(z) = R_{(4)}(z)R_{(8)}(z)$  and  $R_{(1,7)}(z) = R_{(1)}(z)R_{(7)}(z)$  are given as follows:

$$\begin{cases} \theta'_0 = \theta_0 - 1 \\ \theta'_1 = \theta_1 + 1 \\ \theta'_t = \theta_t \\ \theta'_\infty = \theta_\infty, \end{cases} R_{(3,6)}(z) = I + \frac{1}{r_3} \begin{pmatrix} -w_1(u_0 + \theta_0) & w_1w_0u_0 \\ -(u_0 + \theta_0) & w_0u_0 \end{pmatrix} \frac{1}{z}, \quad (3.24)$$

$$r_3 = w_1(u_0 + \theta_0) - u_0w_0$$

$$\begin{cases} \theta'_0 = \theta_0 + 1 \\ \theta'_1 = \theta_1 - 1 \\ \theta'_t = \theta_t \\ \theta'_\infty = \theta_\infty, \end{cases} R_{(4,8)}(z) = I + \frac{1}{r_4} \begin{pmatrix} -w_0(u_1 + \theta_1) & w_0w_1u_1 \\ -(u_1 + \theta_1) & w_1u_1 \end{pmatrix} \frac{1}{z-1},$$

$$r_4 = w_0(u_1 + \theta_1) - u_1w_1 \quad (3.25)$$

$$\begin{cases} \theta'_0 = \theta_0 + 1 \\ \theta'_1 = \theta_1 + 1 \\ \theta'_t = \theta_t \\ \theta'_\infty = \theta_\infty, \end{cases} R_{(1,7)}(z) = Iz + \frac{1}{w_1 - w_0} \begin{pmatrix} -w_1 & w_1w_0 \\ -1 & w_0 \end{pmatrix}. \quad (3.26)$$

Note that, if

$$Y'(z, t; \theta'_0, \theta'_1, \theta'_t, \theta'_\infty) = R_{(j)}(z, t; \theta_0, \theta_1, \theta_t, \theta_\infty)Y(z, t; \theta_0, \theta_1, \theta_t, \theta_\infty), \quad (3.27)$$

and

$$Y''(z, t; \theta''_0, \theta''_1, \theta''_t, \theta''_\infty) = R_{(j)}(z, t; \theta'_0, \theta'_1, \theta'_t, \theta'_\infty)Y'(z, t; \theta'_0, \theta'_1, \theta'_t, \theta'_\infty), \quad (3.28)$$

then

$$R_{(j+1)}(z, t; \theta'_0, \theta'_1, \theta'_t, \theta'_\infty)R_{(j)}(z, t; \theta_0, \theta_1, \theta_t, \theta_\infty) = I \quad (3.29)$$

for  $j = 1, 3, 5, 7, 9, 11$ .

### 3.2 Bäcklund Transformations For PVI

As we have shown in the previous section, equation (3.21) gives the relation between  $u_i$ ,  $w_i$  and the transformed quantities  $u'_i$ ,  $w'_i$ ,  $i = 0, 1, t$ . Using these relations and the equation (2.5.d)

$$k' = (t+1)u'_0w'_0 + tu'_1w'_1 + u'_tw'_t, \quad (3.30)$$

one obtains  $u'_0w'_0$  and  $k'$  in terms of  $u_i$ 's,  $w_i$ 's. Thus, the transformation between the solution  $y(t)$  for the parameters  $\alpha, \beta, \gamma, \delta$  and the solution  $y'(t)$  for the parameters  $\alpha', \beta', \gamma', \delta'$  of PVI can be obtained using the equation (2.5.e):

$$y' = \frac{tu'_0w'_0}{k'}. \quad (3.31)$$

The transformations between the solutions of PVI obtained via the Schlesinger transformation matrices  $R_{(j)}(z)$ ,  $j = 1, 2, \dots, 12$  may be listed as follows:

$$R_{(1)}(z) : u'_0w'_0 = w_0 \left[ (w_1 - w_0) \left( \frac{u_1 + \theta_1}{w_1} - \frac{u_1}{w_0} \right) + \frac{1}{t} (w_t - w_0) \left( \frac{u_t + \theta_t}{w_t} - \frac{u_t}{w_0} \right) \right], \quad (3.32)$$

$$k' = -\theta_\infty w_0,$$

$$\alpha' = \frac{1}{2}[\sqrt{2\alpha} + 1]^2, \quad \beta' = -\frac{1}{2}[\sqrt{-2\beta} + 1]^2, \quad \gamma' = \gamma, \quad \delta' = \delta.$$

$$R_{(2)}(z) : u'_0w'_0 = (\theta_0 - 1)r_2 + \left[ u_1w_1 \left( \frac{u_1 + \theta_1}{u_1w_1} - \frac{u_0 + \theta_0}{u_0w_0} \right) \left( \frac{u_0 + \theta_0}{u_0w_0} - \frac{1}{w_1} \right) + \frac{u_tw_t}{t} \left( \frac{u_t + \theta_t}{u_tw_t} - \frac{u_0 + \theta_0}{u_0w_0} \right) \left( \frac{u_0 + \theta_0}{u_0w_0} - \frac{1}{w_t} \right) \right] r_2^2,$$

$$k' = (t-1)u_1w_1 + [\theta_1 + t(\theta_0 - \theta_1 - 1) +$$

$$2(t-1)u_1w_1 \left( \frac{u_0 + \theta_0}{u_0w_0} - \frac{1}{w_1} \right)] r_2 - \theta_\infty \frac{u_0 + \theta_0}{u_0w_0} r_2^2,$$

$$\alpha' = \frac{1}{2}[\sqrt{2\alpha} - 1]^2, \quad \beta' = -\frac{1}{2}[\sqrt{-2\beta} - 1]^2, \quad \gamma' = \gamma, \quad \delta' = \delta. \quad (3.33)$$

$$R_{(3)}(z) : u'_0w'_0 = \frac{u_0w_0}{w_1} \left( \frac{u_0w_0}{u_0 + \theta_0} - w_1 \right) \left( \frac{u_1w_1}{u_0w_0} - \frac{u_1 + \theta_1}{u_0 + \theta_0} \right) + \frac{u_0w_0}{tw_t} \left( \frac{u_0w_0}{u_0 + \theta_0} - w_t \right) \left( \frac{u_tw_t}{u_0w_0} - \frac{u_t + \theta_t}{u_0 + \theta_0} \right), \quad (3.34)$$

$$k' = -\theta_\infty \frac{u_0w_0}{u_0 + \theta_0},$$

$$\alpha' = \frac{1}{2}[\sqrt{2\alpha} - 1]^2, \quad \beta' = -\frac{1}{2}[\sqrt{-2\beta} - 1]^2, \quad \gamma' = \gamma, \quad \delta' = \delta.$$

$$\begin{aligned}
R_{(4)}(z) : u'_0 w'_0 &= -(\theta_0 + 1)r_2 + \left[ \left( \frac{u_1 + \theta_1}{w_1} - \frac{u_1}{w_0} \right) \left( \frac{w_1}{w_0} - 1 \right) \right. \\
&\quad \left. + \frac{1}{t} \left( \frac{u_t + \theta_t}{w_t} - \frac{u_t}{w_0} \right) \left( \frac{w_t}{w_0} - 1 \right) \right] r_2^2, \\
k' &= -tu_0 w_0 - [t(\theta_0 + \theta_1 + 1) + \theta_0 + \theta_t + 1 + \\
&\quad 2\frac{u_t}{w_0}(w_0 - w_t) + 2t\frac{u_1}{w_0}(w_0 - w_1)]r_2 - \frac{\theta_\infty}{w_0}r_2^2, \\
\alpha' &= \frac{1}{2}[\sqrt{2\alpha} - 1]^2, \quad \beta' = -\frac{1}{2}[\sqrt{-2\beta} + 1]^2, \quad \gamma' = \gamma, \quad \delta' = \delta.
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
R_{(5)}(z) : u'_0 w'_0 &= w_1(w_0 - w_1) \left( \frac{u_0 + \theta_0}{w_0} - \frac{u_0}{w_1} \right), \\
k' &= -\theta_\infty w_1, \\
\alpha' &= \frac{1}{2}[\sqrt{2\alpha} + 1]^2, \quad \beta' = \beta, \quad \gamma' = \frac{1}{2}[\sqrt{2\gamma} + 1]^2, \quad \delta' = \delta.
\end{aligned} \tag{3.36}$$

$$\begin{aligned}
R_{(6)}(z) : u'_0 w'_0 &= -u_0 w_0 - \left[ \theta_0 - 2u_0 w_0 \left( \frac{u_1 + \theta_1}{u_1 w_1} - \frac{1}{w_0} \right) \right] r_2 + \\
&\quad u_0 w_0 \left( \frac{u_0 + \theta_0}{u_0 w_0} - \frac{u_1 + \theta_1}{u_1 w_1} \right) \left( \frac{u_1 + \theta_1}{u_1 w_1} - \frac{1}{w_0} \right) r_2^2, \\
k' &= -tu_0 w_0 - [\theta_1 - 1 + t(1 - \theta_0 + \theta_1) - \\
&\quad 2tu_0 w_0 \left( \frac{u_1 + \theta_1}{u_1 w_1} - \frac{1}{w_0} \right)] r_2 - \theta_\infty \frac{u_1 + \theta_1}{u_1 w_1} r_2^2, \\
\alpha' &= \frac{1}{2}[\sqrt{2\alpha} - 1]^2, \quad \beta' = \beta, \quad \gamma' = \frac{1}{2}[\sqrt{2\gamma} - 1]^2, \quad \delta' = \delta.
\end{aligned} \tag{3.37}$$

$$\begin{aligned}
R_{(7)}(z) : u'_0 w'_0 &= \frac{u_1 w_1}{w_0} \left( \frac{u_0 + \theta_0}{u_1 + \theta_1} - \frac{u_0 w_0}{u_1 w_1} \right) \left( w_0 - \frac{u_1 w_1}{u_1 + \theta_1} \right), \\
k' &= -\theta_\infty \frac{u_1 w_1}{u_1 + \theta_1}, \\
\alpha' &= \frac{1}{2}[\sqrt{2\alpha} + 1]^2, \quad \beta' = \beta, \quad \gamma' = \frac{1}{2}[\sqrt{2\gamma} - 1]^2, \quad \delta' = \delta.
\end{aligned} \tag{3.38}$$

$$\begin{aligned}
R_{(8)}(z) : u'_0 w'_0 &= -u_0 w_0 - \left[ \theta_0 - 2u_0 \left( \frac{w_0}{w_1} - 1 \right) \right] r_2 + \\
&\quad \left( \frac{u_0 + \theta_0}{w_0} - \frac{u_0}{w_1} \right) \left( \frac{w_0}{w_1} - 1 \right) r_2^2, \\
k' &= -tu_0 w_0 + [\theta_1 + 1 - t(1 + \theta_0 + \theta_1) + \\
&\quad 2tu_0 \left( \frac{w_0}{w_1} - 1 \right)] r_2 - \frac{\theta_\infty}{w_1} r_2^2, \\
\alpha' &= \frac{1}{2}[\sqrt{2\alpha} - 1]^2, \quad \beta' = \beta, \quad \gamma' = \frac{1}{2}[\sqrt{2\gamma} + 1]^2, \quad \delta' = \delta.
\end{aligned} \tag{3.39}$$

$$\begin{aligned}
R_{(9)}(z) : u'_0 w'_0 &= w_t(w_0 - w_t) \left( \frac{u_0 + \theta_0}{w_0} - \frac{u_0}{w_t} \right), \\
k' &= -\theta_\infty w_t, \\
\alpha' &= \frac{1}{2}[\sqrt{2\alpha} + 1]^2, \quad \beta' = \beta, \quad \gamma' = \gamma, \quad \delta' = \frac{1}{2} - \frac{1}{2}[\sqrt{1 - 2\delta} + 1]^2.
\end{aligned} \tag{3.40}$$

$$\begin{aligned}
R_{(10)}(z) : \quad u'_0 w'_0 &= -tu_0 w_0 - \left[ \theta_0 - 2u_0 w_0 \left( \frac{u_t + \theta_t}{u_t w_t} - \frac{1}{w_0} \right) \right] r_2 + \\
&\quad \frac{u_0 w_0}{t} \left( \frac{u_0 + \theta_0}{u_0 w_0} - \frac{u_t + \theta_t}{u_t w_t} \right) \left( \frac{u_t + \theta_t}{u_t w_t} - \frac{1}{w_0} \right) r_2^2, \\
k' &= (t-1)u_1 w_1 + \left[ \theta_1 - t(\theta_t - 1) - 2u_1 w_1 \left( \frac{u_t + \theta_t}{u_t w_t} - \frac{1}{w_1} \right) \right] r_2 \\
&\quad - \theta_\infty \frac{u_t + \theta_t}{u_t w_t} r_2^2, \\
\alpha' &= \frac{1}{2}[\sqrt{2\alpha} - 1]^2, \quad \beta' = \beta, \quad \gamma' = \gamma, \quad \delta' = \frac{1}{2} - \frac{1}{2}[\sqrt{1 - 2\delta} - 1]^2.
\end{aligned} \tag{3.41}$$

$$\begin{aligned}
R_{(11)}(z) : \quad u'_0 w'_0 &= \frac{1}{t} \left[ \frac{u_t w_t}{u_t + \theta_t} (2u_0 + \theta_0) - \frac{1}{w_0} \left( \frac{u_t w_t}{u_t + \theta_t} \right)^2 (u_0 + \theta_0) - u_0 w_0 \right], \\
k' &= -\theta_\infty \frac{u_t w_t}{u_t + \theta_t}, \\
\alpha' &= \frac{1}{2}[\sqrt{2\alpha} + 1]^2, \quad \beta' = \beta, \quad \gamma' = \gamma, \quad \delta' = \frac{1}{2} - \frac{1}{2}[\sqrt{1 - 2\delta} - 1]^2.
\end{aligned} \tag{3.42}$$

$$\begin{aligned}
R_{(12)}(z) : \quad u'_0 w'_0 &= -tu_0 w_0 - \left[ \theta_0 - 2u_0 \left( \frac{w_0}{w_t} - 1 \right) \right] r_2 + \\
&\quad \frac{1}{t} \left( \frac{u_0 + \theta_0}{w_0} - \frac{u_0}{w_t} \right) \left( \frac{w_0}{w_t} - 1 \right) r_2^2, \\
k' &= -tu_0 w_0 - \left[ \theta_0 + \theta_t + 1 - t(1 + \theta_t) - 2u_0 \left( \frac{w_0}{w_t} - 1 \right) \right] r_2 \\
&\quad - \frac{\theta_\infty}{w_t} r_2^2, \\
\alpha' &= \frac{1}{2}[\sqrt{2\alpha} - 1]^2, \quad \beta' = \beta, \quad \gamma' = \gamma, \quad \delta' = \frac{1}{2} - \frac{1}{2}[\sqrt{1 - 2\delta} + 1]^2.
\end{aligned} \tag{3.43}$$

where  $u_i, w_i, i = 0, 1, t$  and  $r_1, r_2$  are given in the equations (2.6) and (3.20) respectively.

The transformations (3.32)–(3.43) give implicit relations between  $y'(t)$  and  $y(t)$ . Therefore, in order to obtain  $y'(t)$  one should proceed as follows: Firstly, one uses the equation (2.8) to obtain the parameters  $\theta_i, i = 0, 1, t, \infty$ , and then equation (2.4.b) to obtain  $\kappa_1, \kappa_2$ . Second step, using equation (2.7.a) one gets  $u$  which is substituted in (2.4.e) to obtain  $\bar{u}$ . Next step, substituting  $\theta_i$ 's,  $\kappa_1, \kappa_2$  and  $\bar{u}$  in (2.6)  $u_i, w_i, i = 0, 1, t$  can be obtained in terms of  $y(t)$ . Having obtained  $u_i$ 's,  $w_i$ 's one easily calculates  $u'_0 w'_0$ , and  $k'$ . Lastly, using equation (3.31) we obtain  $y'(t)$  in terms of  $y(t)$ .

## Chapter 4

### One-Parameter Families Of Solutions Of PVI

Lukashevich and Yablonskii [6] have proved that PVI admit one-parameter family of solutions characterized by the Riccati equation

$$t(t-1)\frac{dy}{dt} = \sqrt{2\alpha}y^2 + (\lambda t - \mu)y + \sqrt{-2\beta}t, \quad (4.1)$$

if

$$\sqrt{2\alpha} - \sqrt{-2\beta} - 1 \neq 0 \quad (4.2)$$

and

$$\begin{aligned} \alpha^2 + \beta^2 - 6\alpha\beta + 2(\alpha - \beta)(\delta - \gamma) + (\delta + \gamma)^2 + 2(\alpha - \beta - \gamma) \\ + 2\sqrt{2\alpha}(-\alpha + 3\beta + \gamma - \delta) + 2\sqrt{-2\beta}(3\alpha - \beta - \gamma + \delta) = 0 \end{aligned} \quad (4.3)$$

where

$$\lambda = \frac{\sqrt{2\alpha} - \alpha - \beta - \gamma - \delta}{\sqrt{2\alpha} - \sqrt{-2\beta} - 1}, \quad \mu = \frac{\sqrt{-2\beta} - \alpha - \beta + \gamma + \delta}{\sqrt{2\alpha} - \sqrt{-2\beta} - 1}, \quad (4.4)$$

If we define  $v$  as

$$y = -\frac{t(t-1)(dv/dt)}{\sqrt{2\alpha}v}, \quad s = \frac{1}{1-t}, \quad \alpha \neq 0, \quad (4.5)$$

then  $v(s)$  satisfy the hypergeometric equation

$$s(s-1)\frac{d^2v}{ds^2} + [(1 + \alpha_1 + \beta_1)s - \gamma_1]\frac{dv}{ds} + \alpha_1\beta_1v = 0 \quad (4.6)$$

where

$$\alpha_1 = \sqrt{2\alpha}, \quad \beta_1 = \sqrt{-2\beta}, \quad \gamma_1 = \lambda. \quad (4.7)$$

The same result has been redrived by Fokas and Ablowitz [7]. They noticed that the transformation (1.15) breaks down if and only if  $\phi = 0, \nu = 0$  (see equation (1.16)), which nothing but (4.1).

Following the observation of Fokas and Ablowitz it is possible to redrive the one-parameter family of solution (4.1) and to find some new ones using the Schlesinger transformations and the corresponding Bäcklund transformations of PVI. First of all the linear problem and hence the Schlesinger transformations are well defined if and only if  $u_j, w_j \neq 0, j = 0, 1, t$  and  $\theta_\infty \neq 0$ . Using equation (2.5), one can find that this restriction is violated if one of the following is true:

$$u_0 = u_1 = u_t = 0 \text{ and } \kappa_2 = 0,$$

$$u_0 = u_1 = u_t + \theta_t = 0 \text{ and } \kappa_2 + \theta_t = 0,$$

$$u_0 = u_1 + \theta_1 = u_t = 0 \text{ and } \kappa_2 + \theta_1 = 0,$$

$$u_0 = u_1 + \theta_1 = u_t + \theta_t = 0 \text{ and } \kappa_2 + \theta_1 + \theta_t = 0,$$

$$u_0 + \theta_0 = u_1 = u_t = 0 \text{ and } \kappa_2 + \theta_0 = 0,$$

$$u_0 + \theta_0 = u_1 = u_t + \theta_t = 0 \text{ and } \kappa_2 + \theta_0 + \theta_t = 0,$$

$$u_0 + \theta_0 = u_1 + \theta_1 = u_t = 0 \text{ and } \kappa_2 + \theta_0 + \theta_1 = 0,$$

$$\text{or } u_0 + \theta_0 = u_1 + \theta_1 = u_t + \theta_t = 0 \text{ and } \kappa_2 + \theta_0 + \theta_1 + \theta_t = 0.$$

Using equations (2.4),(2.7) we find the following one-parameter families of solutions respectively :

$$\begin{aligned} t(t-1)\frac{dy}{dt} &= (1-\theta_\infty)y^2 - [\theta_0 + \theta_t + 1 + t(\theta_0 + \theta_1)]y + \theta_0 t, \\ \theta_0 + \theta_1 + \theta_t + \theta_\infty &= 0. \end{aligned} \quad (4.8)$$

$$\begin{aligned} t(t-1)\frac{dy}{dt} &= (1-\theta_\infty)y^2 - [1 + \theta_0 - \theta_t + t(\theta_0 + \theta_1)]y + \theta_0 t, \\ \theta_0 + \theta_1 - \theta_t + \theta_\infty &= 0. \end{aligned} \quad (4.9)$$

$$\begin{aligned} t(t-1)\frac{dy}{dt} &= (1-\theta_\infty)y^2 - [1 + \theta_0 + \theta_t + t(\theta_0 - \theta_1)]y + \theta_0 t, \\ \theta_0 - \theta_1 + \theta_t + \theta_\infty &= 0. \end{aligned} \quad (4.10)$$

$$\begin{aligned} t(t-1)\frac{dy}{dt} &= (1-\theta_\infty)y^2 - [1 + \theta_0 - \theta_t + t(\theta_0 - \theta_1)]y + \theta_0 t, \\ \theta_0 - \theta_1 - \theta_t + \theta_\infty &= 0. \end{aligned} \quad (4.11)$$

$$\begin{aligned} t(t-1)\frac{dy}{dt} &= (1-\theta_\infty)y^2 - [1 + \theta_t - \theta_0 - t(\theta_0 - \theta_1)]y - \theta_0 t, \\ -\theta_0 + \theta_1 + \theta_t + \theta_\infty &= 0. \end{aligned} \quad (4.12)$$

$$\begin{aligned} t(t-1)\frac{dy}{dt} &= (1-\theta_\infty)y^2 - [1 - \theta_t - \theta_0 - t(\theta_0 - \theta_1)]y - \theta_0 t, \\ \theta_0 - \theta_1 + \theta_t - \theta_\infty &= 0. \end{aligned} \quad (4.13)$$

$$\begin{aligned} t(t-1)\frac{dy}{dt} &= (1-\theta_\infty)y^2 - [1 + \theta_t - \theta_0 - t(\theta_0 + \theta_1)]y - \theta_0 t, \\ \theta_0 + \theta_1 - \theta_t - \theta_\infty &= 0. \end{aligned} \quad (4.14)$$

$$\begin{aligned} t(t-1)\frac{dy}{dt} &= (1-\theta_\infty)y^2 - [1 - \theta_t - \theta_0 - t(\theta_0 + \theta_1)]y - \theta_0 t, \\ \theta_0 + \theta_1 + \theta_t - \theta_\infty &= 0. \end{aligned} \quad (4.15)$$

It also possible to obtain other one parameter family of solutions from some of the Bäcklund transformations of PVI. For example, the transformation (3.35) break downs if and only if  $k' = u'_0 w'_0 = 0$ . This implies  $k = 0$  and hence one of the equations (4.8)–(4.15), or

$$(\theta_0 + 1) = \left[ \left( \frac{u_1 + \theta_1}{w_1} - \frac{u_1}{w_0} \right) \left( \frac{w_1}{w_0} - 1 \right) + \frac{1}{t} \left( \frac{u_t + \theta_t}{w_t} - \frac{u_t}{w_0} \right) \left( \frac{w_t}{w_0} - 1 \right) \right] r_2, \quad (4.16)$$

$$t u_0 w_0 + [t(\theta_0 + \theta_1 + 1) + \theta_0 + \theta_t + 1 + 2 \frac{u_t}{w_0} (w_0 - w_t) + 2t \frac{u_1}{w_0} (w_0 - w_1)] r_2 + \frac{\theta_\infty}{w_0} r_2^2 = 0. \quad (4.17)$$

Using the equation (2.6), the equations (4.16) and (4.17) become

$$(y - 1)(y - t)\bar{u}^2 + [\theta_1(y - t) + \theta_t(y - 1)]\bar{u} - \kappa_2^2 - \kappa_2(\theta_1 + \theta_t) + (\theta_0 + 1)(\theta_\infty - 1) = 0 \quad (4.18)$$

and

$$y(y - 1)(y - t)\bar{u}^2 + [\theta_1(y - t) + t\theta_t(y - 1) - 2(\kappa_2 + \theta_\infty - 1)(y - 1)(y - t)]\bar{u} + (\kappa_2 + \theta_\infty - 1)^2 y - t[\kappa_2^2 + \kappa_2\theta_t - (\theta_\infty - 1)(\theta_1 + \theta_0 + 1)] - [\kappa_2^2 + \kappa_2\theta_1 - (\theta_\infty - 1)(\theta_t + \theta_0 + 1)] = 0 \quad (4.19)$$

Solving these two equations we find

$$t(t - 1) \frac{dy}{dt} = (\theta_\infty - 1)y^2 - \frac{1}{2\kappa} [t(\kappa^2 + 2\kappa\theta_0 + \theta_1^2 - \theta_t^2) + \kappa^2 + 2\kappa(\theta_0 + 1) - \theta_1^2 + \theta_t^2]y + t\theta_0, \quad (4.20)$$

and

$$(\kappa - \theta_1 - \theta_t)(\kappa + \theta_1 + \theta_t)(\kappa - \theta_1 + \theta_t)(\kappa + \theta_1 - \theta_t) = 0, \quad (4.21)$$

if

$$\kappa = \theta_\infty - \theta_0 - 2 \neq 0. \quad (4.22)$$

If

$$\kappa = \theta_\infty - \theta_0 - 2 = 0, \quad (4.23)$$

then equations (4.16) and (4.17) give

$$\theta_1^2 = \theta_t^2, \quad (4.24)$$

and

$$t(t - 1) \frac{dy}{dt} = (\theta_\infty - 1)y^2 - [t(\theta_0 + a) + \theta_0 - a + 1]y + t\theta_0 \quad (4.25)$$

where  $a$  is such that  $a^2 = \theta_1^2$ . This result coincides with the result of Lukashevich and Yablonskii [6] with the choice  $\sqrt{\beta} = \theta_0$ . The choice  $\sqrt{\beta} = -\theta_0$  can be obtained by using the transformation (3.33) instead of (3.35). The transformations (3.37) and (3.39) ((3.41) and (3.43)) give similar results to the transformations (3.33) and (3.35) but with the roles of  $\theta_0$  and  $\theta_1(\theta_t)$  be exchanged.

## 4.1 Rational solutions of PVI

Using the one-parameter family of solutions and the transformations (3.32)–(3.43) one can obtain infinite hierarchies of rational solutions of PVI. But to use these transformations it should be noticed that one should start with the solution  $y(t)$  of PVI for the parameters  $\alpha, \beta, \gamma, \delta$  ( $\theta_\infty, \theta_0, \theta_1, \theta_t$ ) such that  $\theta_j, j = 0, 1, t, \infty$  do not satisfy the certain conditions under which PVI can be reduced to Riccati equation. Since, under these restrictions on  $\theta_j, j = 0, 1, t, \infty$  the transformations break down. One can avoid these restrictions by using discrete symmetries (2.12)–(2.15). For example, if we choose  $\theta_0 = 0, \theta_1 = 1, \theta_t = -2, \theta_\infty = 1$ , then equation (4.4) implies  $y = \frac{c}{t}$ ,  $c = \text{constant}$ . Starting with the solution

$$\begin{aligned} y_0(t) &= \frac{c}{t}; \\ \alpha_0 &= 0, \quad \beta_0 = 0, \quad \gamma_0 = \frac{1}{2}, \quad \delta_0 = -\frac{3}{2}, \end{aligned} \tag{4.26}$$

then the transformation (1.14) yields,

$$\begin{aligned} y_1(t) &= 1 - c(t-1)^2; \\ \alpha_1 &= 0, \quad \beta_1 = -2, \quad \gamma_1 = 0, \quad \delta_1 = 0. \end{aligned} \tag{4.27}$$

Using (4.23) in the transformation (1.15) we obtain [7]

$$\begin{aligned} y_2(t) &= \frac{t(ct^2 - 2ct + c - 1)}{2ct^3 - 3ct^2 + c - 1}; \\ \alpha_2 &= \frac{9}{2}, \quad \beta_2 = -\frac{1}{2}, \quad \gamma_2 = \frac{1}{2}, \quad \delta_2 = \frac{1}{2}, \end{aligned} \tag{4.28}$$

Then the application of transformation (3.40) twice gives

$$\begin{aligned} y_3(t) &= \frac{t(ct^3 - 3ct^2 + 3ct - 3t - c + 1)}{2(ct^4 - 2ct^3 + 2ct - 2t - c + 1)}; \\ \alpha_3 &= 8, \quad \beta_3 = -\frac{1}{2}, \quad \gamma_3 = \frac{1}{2}, \quad \delta_3 = 0, \end{aligned} \tag{4.29}$$

and

$$\begin{aligned} y_4(t) &= \frac{t(ct^4 - 4ct^3 + 6ct^2 - 6t^2 - 4ct + 4t - 1)}{2ct^5 - 5ct^4 + 10ct^2 - 10t^2 + -10ct + 10t + 3c - 3}; \\ \alpha_4 &= \frac{25}{2}, \quad \beta_4 = -\frac{1}{2}, \quad \gamma_4 = \frac{1}{2}, \quad \delta_4 = -\frac{3}{2}. \end{aligned} \tag{4.30}$$

respectively. It can be verified that  $y_i(t)$ ,  $i = 1, 2, 3, 4$  satisfy PVI. Following the same procedure one can generate infinitely many rational solutions of PVI by using the transformations (3.32)–(3.43).



## Chapter 5

### Conclusion

Transformation properties of Painlevé equations was the subject of extensive investigations. However, the first transformations for PVI were obtained in 1982 [7]. Although these were important results, they were not enough to generate infinite hierarchies of exact solutions. This follows from the fact that a finite product of these transformations yields the identity. Another transformation for PVI was obtained in 1991 [8]. To use this transformation there are very strong restrictions, and this restrict the usage of this transformation.

It is well known[15] that one can use the Schlesinger transformations associated with a given Painlevé equation to obtain Bäcklund transformations for this equation . Using this fact we studied the Schlesinger transformation of PVI. We show that the linear problem associated with PVI admit transformations which shift the parameters of PVI,  $\theta_j$ ;  $j = 0, 1, t, \infty$ , by integers and leave the monodromy data the same. Among these transformations there are twelve basic transformations which can be obtained in closed form by solving some simple RH-problems. All other transformations can be obtained using these basic ones. Since these transformations shifts the parameters by integers, any finite product of them does not give the identity. Moreover, these transformations break down if and only if the solution  $y$  of PVI satisfies certain one parameter families of solutions. Therefore, using these transformations and the discrete symmetries [7] one can generate infinite hierarchies of exact solutions. In addition, one should notice that if the solution of PVI is known for some intervals,  $a_j \leq \theta_j \leq a_j + 1$ ;  $j = 0, 1, t, \infty$ , then one can use the Schlesinger transformations to obtain the solution for any other values of  $\theta_j$ 's.

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