UNIFIED APPROACH TO TRANSFORMATIONS OF PAINLEVÉ EQUATIONS

A THESIS

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS

AND THE INSTITUTE OF ENGINEERING AND SCIENCES

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

By

Ali Reza Modaressi Chahardehi

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Asst. Prof. Uğurhan Muğan(Principal Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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Abstract

In this thesis, we find the explicit form of some transformations associated with the second, third, fourth and fifth Painlevé equations. These transformations are obtained by using the Schlesinger transformations associated with the linear system of equations of Painlevé equations.The application of such transformations enables us to generate the new solutions of the given Painlevé equation with different values of parameters, from the known solutions.

Özet

Bu tezde ikinci, üçüncü, dördüncü ve beşinci Painlevé denklemlerinin çözümlerine ait dönüşümler elde edilmektedir. Bu dönüşümler uyumluk şartı Painlevé denklemlerini veren lineer denklem sistemlerine (monodromy problemi) ait Schlesinger dönüşümlerinden elde edilmektedir. Elde edilen bu dönüşümler Painlevé denklemlerinin bilinen çözümlerinden yeni çözümler bulmaya imkan verir.

Acknowledgement

I feel fortunate to have had Professor Uğurhan Muğan as my advisor for this thesis. He provided guidance, comments, suggestions and encouragement during the process. He is an excellent mathematician as well as a wonderful human being.

The kindness and patience showed by Mrs.Behnaz Sadr in the process of typing my thesis was greatly appreciated.

Finally, I wish to express my thanks and love to my wife and children, without whose understanding and support, this project would have been most difficult. I owe you more than simple words can say.

Contents

1 Introduction

The most significant difference between the linear and nonlinear ordinary differential equations is the singularity structure of the solutions. For given linear ordinary differential equation if the solution has singularity, that must be the singularity of the differential equation. In other words, the singularity of the solution is fixed. For example, the first order linear ordinary differential equation,

$$y' = -\frac{y}{t^2}, \tag{1.1}$$

has singularity at $t = 0$, and its general solution is given as follows;

$$y(t) = ce^{1/t}, \tag{1.2}$$

where c is an arbitrary integration constant. From (1.2), $t = 0$ is an essential singular point of the solution, which is independent of the integration constant.

For the nonlinear ordinary differential equation, the behavior of the solution is unpredictable. If the solution has singularity, the location of singularity may depend on the arbitrary integration constant, i.e. depend on the initial or boundary conditions. For example,

$$\frac{dy}{dt} + y^2 = 0, \tag{1.3}$$

has the solution,

$$y(t) = \frac{1}{t-c}, \tag{1.4}$$

where, c is the constant of integration. Clearly, the location of singularity (in this case, it is a pole) depends on the integration constant c. Hence, as the initial condition changes, the singular point moves in the complex t-plane. Similar examples can be found for the case of the critical points (branch points and essential singularities). Consider the following second order nonlinear ordinary differential equation [1]:

$$\frac{d^2y}{dt^2} = \left(\frac{dy}{dt}\right)^2 \frac{2y-1}{2y+1}, \tag{1.5}$$

which has the solution,

$$y(t) = \tan\left\{\log(c_1 t - c_2)\right\}, \tag{1.6}$$

where c_1 and c_2 are arbitrary constants. In this particular example, the point $t = \frac{c_2}{c_1}$ is both a branch point and an essential singularity. Its location is given in terms of the constants of integration, i.e. it is a critical movable point.

One can consider the class of differential equations whose movable singularities are only poles. In this class, the only first order nonlinear ordinary differential equation of the form,

$$\frac{dy}{dt} = f(t, y), \tag{1.7}$$

where, f is rational in y, and locally analytic in t, is the Riccati equation, [1],[2],

$$\frac{dy}{dt} = P_0(t) + P_1(t)y + P_2(t)y^2. \tag{1.8}$$

During the late 19th and early 20th century Painlevé [3] and his school [4] examined the second order ordinary differential equations,

$$\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt}), \tag{1.9}$$

where f is rational in $\frac{dy}{dt}$, algebraic in y and analytic in t, with the property of having no movable critical points. This property is called the Painlevé property. They showed that, within a Möbius transformation, there are fifty such equations [1]. The most interesting of the fifty equations are those which are irreducible (that is, cannot be mapped to a simpler equation or combination of simpler equations), and serve to define new transcendents. These irreducible six equations are called Painlevé equations(PI-PVI)

$$PI: \frac{d^2y}{dt^2} = 6y^2 + t, \tag{1.10}$$

$$PII: \frac{d^2y}{dt^2} = 2y^3 + ty + \alpha, \tag{1.11}$$

$$PIII: \frac{d^2y}{dt^2} = \frac{1}{y}(\frac{dy}{dt})^2 - \frac{1}{t}\frac{dy}{dt} + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}, \tag{1.12}$$

$$PIV: \frac{d^2y}{dt^2} = \frac{1}{2y}(\frac{dy}{dt})^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}, \tag{1.13}$$

$$PV: \frac{d^2y}{dt^2} = (\frac{1}{2y} + \frac{1}{y-1})(\frac{dy}{dt})^2 - \frac{1}{t}\frac{dy}{dt}$$
$$+\frac{(y-1)^2}{t^2}(\alpha y + \frac{\beta}{y}) + \frac{\gamma y}{t} + \frac{\delta y(y+1)}{y-1}, \tag{1.14}$$

$$PVI: \frac{d^2y}{dt^2} = \frac{1}{2}\left\{\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right\}(\frac{dy}{dt})^2$$
$$-\left\{\frac{1}{t} + \frac{1}{t-1} + \frac{1}{t-y}\right\}\frac{dy}{dt} \tag{1.15}$$
$$+\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left\{\alpha + \frac{\beta t}{y^2} + \frac{\gamma(t-1)}{(y-1)^2} + \frac{\delta t(t-1)}{(y-t)^2}\right\}.$$

The remaining forty-four equations can either be integrated in terms of known elementary transcendental functions or can be reduced to one of these six equations.

Besides having the Painlevé property, these equations have rich structure. The properties of these equations can be summarized as follows:

a) For particular choices of the parameters, all the Painlevé equations except the first Painlevé equation, admit rational solutions, as well as one-parameter family of solutions expressible in terms of elementary transcendental functions. For $\alpha = -\frac{1}{2}$, a one-parameter family of solutions of the second Painlevé equation can be expressed in terms of the Airy function [4],[5]. The one-parameter family of solutions of the third Painlevé equation can be expressed in terms of Bessel function [6]. For any integer value of the parameter α, such that $\beta + 2(\alpha + 1)^2 = 0$ or $\beta + 2(\alpha - 1)^2 = 0$, the fourth

Painlevé equation has a solution expressible rationally in terms of the Hermite polynomials, for non-integer value of α, which itself is expressible in terms of Weber-Hermite functions [7]. One parameter family of solutions of the fifth and sixth Painlevé equations are expressed in terms of Whittaker [8], and hypergeometric functions [8], [9] respectively.

b) All the Painlevé equations except the first one, admit transformations which map the solution of a given Painlevé equation to solutions of the same equation with different values of parameters. For example, if $y(t)$ is a solution of the fourth Painlevé equation with the parameters α, β then

$$\bar{y}(t) = \frac{1}{2y}\left[\frac{dy}{dt} - y^2 - 2ty - (-2\beta)^{1/2}\right], \tag{1.16}$$

is also a solution of the fourth Painlevé equation with the parameter values,

$$\bar{\alpha} = \frac{1}{4}\left[2 - 2\alpha + 3(-2\beta)^{1/2}\right], \quad \bar{\beta} = -\frac{1}{2}\left[1 + \alpha + \frac{1}{2}(-2\beta)^{1/2}\right]^2.$$

The transformations for PII-PV were obtained in the Soviet Literature [10],[11],[12],[13], and for PVI by Fokas and Yortsos [14], by using different methods.

c) It is possible to obtain the PI-PV from PVI by a certain type of limit process(contraction) [1]. Also the transformations associated with PI-PV can be obtained from the transformations associated with PVI, by using the same limit process [15]. As an example, in the sixth Painlevé equation, substitute,

$$\begin{aligned} y &= y', \quad t = 1 + \epsilon t', \quad \alpha = \alpha', \\ \beta &= \beta', \quad \gamma = \tfrac{\gamma'}{\epsilon} - \tfrac{\delta'}{\epsilon^2}, \quad \delta = \tfrac{\delta'}{\epsilon}. \end{aligned} \tag{1.17}$$

In the limit as $\epsilon \to 0$, the fifth Painlevé equation arises. By using a similar procedure, PIII and PIV can be obtained from PV and PIII yields PII. The first and fourth Painlevé equations may also be obtained from the second Painlevé equation.

d) The Painlevé equations can also be obtained on the integrability conditions of a certain kind of deformation problem; so called monodromy preserving deformation problem [16],[17]. Consider a first order system of ordinary differential equations [18]

$$\begin{aligned} \frac{dY(x)}{dx} &= A(x)Y(x), \\ A(x) &= \sum_{i=1}^{n}\sum_{j=0}^{m} \frac{A_{i,-j}}{(x-a_i)^{j+1}} + \sum_{k=1}^{l} A_{\infty,-k}x^{k-1}, \end{aligned} \tag{1.18}$$

having regular or irregular singularities of arbitrary rank. Let $Y(x)$ be fundamental solution of (1.18), in general $Y(x)$ is multi-valued with $a_1, a_2, ..., a_n, \infty$ as its branch points. As x describes a closed path Γ avoiding these singular points, the solutions $Y(x)$ is mapped to,

$$Y(x) \to Y(x)M_\Gamma, \tag{1.19}$$

where M_Γ is a constant, nonsingular matrix. The matrix M_Γ depends on the closed path Γ, and is called the monodromy matrix of $Y(x)$ corresponding to Γ. The monodormy preserving deformation problem is: to deform the coefficient $A(x)$ in (1.18) as a function of the deformation parameter t in such a way that the monodromy matrices remain the same.

Recently, the Painlevé equations have appeared in physical problems. E. Barouch et al. [19] showed that the correlation function of the rectangular two-dimensional Ising model in the scaling limit admit closed form solutions in terms of the solution of the third Painlevé equation. In the 1970's, M. J. Ablowitz et al. have discovered a connection between the nonlinear partial differential equations (PDE) solvable by inverse scattering transform (IST) and Painlevé equations [20]. By exploiting this connection, they reduced a special case of the second Painlevé equation to a linear integral equation [21]. The special case of an equation, which is related to the sixth Painlevé equation via one-to-one transformations, has been obtained from the equations satisfied by the scaling invariant solutions of the three-wave resonant system in one spatial and one temporal dimensions, by A. S. Fokas et al. [22].

The similarity solution (a solution which is invariant under certain scaling) of the modified Korteweg-de Vries (MKdV) equation

$$q_t - 6q^2q_x + q_{xxx} = 0, (MKdV) \tag{1.20}$$

is given as,

$$q(x,t) = (3t)^{-1/3} y(x(3t)^{-1/3}). \tag{1.21}$$

It follows from (1.20) that y satisfies the second Painlevé equation. One may apply the usual inverse method for the MKdV, and formally obtain an integral equation (Gel'fand-Levitan'Marchenko integral equation) by which $q(x,t)$ is determined from the scattering data. At that point, using the self similarity of $q(x,t)$ and scaling out the variable t yield the Fredholm integral equation by which the solution of the second Painlevé equation for $\alpha = 0$ can be obtained.

Similarly, special cases of the third Painlevé equation and of the fourth Painlevé equation can be obtained from the exact similarity reduction of the Sine-Gordan and of the nonlinear Schrödinger equations respectively. It is also interesting that exact reductions of the Korteweg-de Vries (KdV) equation leads to the first and second Painlevé equations [9].

Besides the connection between the Painlevé equations and nonlinear PDE's solvable by IST, they have other common properties. A nonlinear PDE solvable by IST appears as the integrability condition of an isospectral deformation problem: Coefficients of a linear spectral operator can be deformed as a function of an additional parameter, such that the eigenvalues of the spectral operators remain invariant. The best known isospectral operator is the Schrödinger operator $L = -\frac{d^2}{dx^2} + q(x)$. If the potential $q(x)$ as a function of the deformation parameter t, satisfies the KdV equation,

$$q_t - 6qq_x + q_{xxx} = 0, (KdV) \tag{1.22}$$

then, the eigenvalues of the Schrödinger operator remain invariant. Writing the Schrödinger eigenvalue problem in matrix form one obtains [23]:

$$\psi_x(z,x,t) = z \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \psi(z,x,t) + \begin{pmatrix} 0 & q(x,t) \\ r(x,t) & 0 \end{pmatrix} \psi(z,x,t), \tag{1.23}$$

where,

$$r(x,t) = -q(x,t). \tag{1.24}$$

Solving the initial value problem for $q(x,t)$,amounts to solving an inverse problem for $\psi(x,t;z)$, namely ,for given (appropriate) scattering data, reconstruct $\psi(x,t;z)$. The solution of the inverse problem is obtained via a Riemann-Hilbert problem for a function ψ sectionally meromorphic with respect to the variable z. To define the Riemann-Hilbert problem, the analyticity properties of ψ with respect to z must be examined by using (1.23). However this result can be used to solve the initial value problem of $q(x,t)$ only if $q(x,t)$ evolves in such a way in t that the scattering data is known for all t. That is, $q(x,t)$ satisfies the integrability condition for the isospectral deformation problem.

In this thesis we obtain the transformation associated with the second, third, fourth, and fifth Painlevé equations, from the Schlesinger transformations associated with the linear system of equations. These transformations enables us to obtain the new solution of the Painlevé equation from the known ones. Actually the transformations for the Painlevé equations have been obtained before by using different methods [9],[11]. The procedure that we have used to obtain these transformations gives us the unified approach to derive all known properties of the Painlevé equations. It is well known that it is possible to find the Painlevé transcendentals and the rational solutions for the particular choice of the parameters in Painlevé equations by using the associated linear system of equations.

2 Transformation from Painlevé II to Painlevé II

In this section , we will present the procedure to obtain the Schlesinger transformations associated with the linear system of the second Painlevé equation [24].
The second Painlevé equation

$$\frac{d^2y}{dt^2} = 2y^3 + ty + \alpha, \tag{2.1}$$

can be obtained as the compatibility condition of the following linear system of equations

$$Y_z(z) = A(z)Y(z), \qquad Y_t(z) = B(z)Y(z), \tag{2.2}$$

where [25]

$$A(z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z^2 + \begin{pmatrix} 0 & u \\ -\frac{2v}{u} & 0 \end{pmatrix} z +$$

$$\begin{pmatrix} v + \frac{t}{2} & -uy \\ -\frac{2}{u}(\theta + yv) & -(v + \frac{t}{2}) \end{pmatrix}, \tag{2.3}$$

$$B(z) = \tfrac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z + \tfrac{1}{2} \begin{pmatrix} 0 & u \\ -2\frac{v}{u} & 0 \end{pmatrix}.$$

The compatibility condition $Y_{zt} = Y_{tz}$ implies

$$\frac{dv}{dt} = -2yv - \theta, \qquad \frac{du}{dt} = -uy, \qquad \frac{dy}{dt} = v + y^2 + \frac{t}{2}. \tag{2.4}$$

Thus, y satisfies the second Painlevé equation (2.1), with the parameter,

$$\alpha = \frac{1}{2} - \theta. \tag{2.5}$$

The two linearly independent formal solutions $\tilde{Y}_\infty(z) = \left(\tilde{Y}_\infty^{(1)}(z),\ \tilde{Y}_\infty^{(2)}(z)\right)$, about $z = \infty$ of the system (2.2a) have the expansions,

$$\tilde{Y}_\infty^{(1)}(z) = \left(\frac{1}{z}\right)^\theta e^{\eta(z)} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -K \\ \frac{v}{u} \end{pmatrix} \frac{1}{z} + \ldots \right\} = \left(\frac{1}{z}\right)^\theta e^{\eta(z)} \hat{Y}_\infty^{(1)}(z), \tag{2.6}$$

$$\tilde{Y}^{(2)}_{\infty}(z) = \left(\frac{1}{z}\right)^{-\theta} e^{-q(z)} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -\frac{u}{2} \\ K \end{pmatrix} \frac{1}{z} + \dots \right\} = \left(\frac{1}{z}\right)^{-\theta} e^{-q(z)} \hat{Y}^{(2)}_{\infty}(z), \tag{2.7}$$

where

$$K = \frac{1}{2}v^2 + (y + \frac{t}{2})v + \theta y, \qquad q(z) = \frac{z^3}{3} + \frac{t}{2}z. \tag{2.8}$$

The formal solution $\tilde{Y}_{\infty}(z)$ is an asymptotic expansion of the actual solution $Y(z)$ as $|z| \to \infty$, in certain sectors of the complex plane. The sectors S_j, $j = 1,2,3,4,5,6$ are given by the central angle $\frac{\pi}{3}$ and vertex centered at the origin, and each sector S_j contains the initial boundary line C_j on which both formal solutions are neutral.The sectors S_j are given by;

$$S_1 : -\frac{\pi}{6} \le argz < \frac{\pi}{6}, \quad S_2 : \frac{\pi}{6} \le argz < \frac{\pi}{2}, \quad S_3 : \frac{\pi}{2} \le argz < \frac{5\pi}{6},$$

$$S_4 : \frac{5\pi}{6} \le argz < \frac{7\pi}{6}, \quad S_5 : \frac{7\pi}{6} \le argz < \frac{3\pi}{2}, \quad S_6 : \frac{3\pi}{2} \le argz < \frac{11\pi}{2}. \tag{2.9}$$

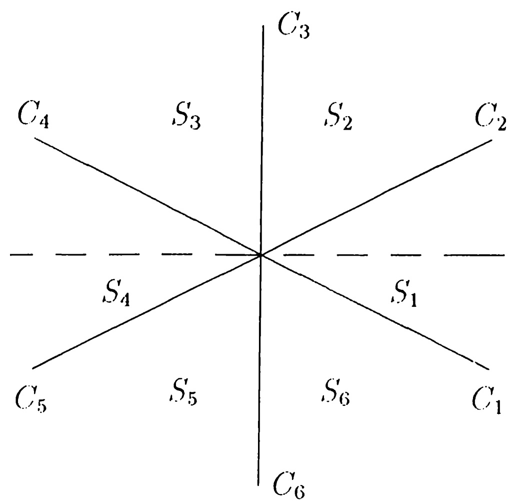

Figure 2.1

Corresponding to formal solutions $\tilde{Y}_{\infty}(z)$ and each sector S_j, $j = 1,2,3,4,5,6$, there exists a function $Y_j(z)$ holomorphic in the sector S_j such that, $Y_j(z) \sim \tilde{Y}_{\infty}(z)$ as $|z| \to \infty$ in S_j and $detY_j(z) = 1$. The solution matrix $Y_j(z)$ is related to its neighbors $Y_{j+1}(z)$ and $Y_{j-1}(z)$ via Stokes matrices G_j,

$$Y_{j+1}(z) = Y_j(z)G_j, \quad j = 1, ..., 5, \qquad Y_1(z) = Y_6(ze^{2i\pi})G_6e^{2i\pi\theta\sigma_3}, \quad (2.10)$$

where the Stokes matrices G_j and σ_3 are;

$$G_1 = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad G_3 = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix},$$

$$G_4 = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, \quad G_5 = \begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix}, \quad G_6 = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}, \quad (2.11)$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and a, b, c, d, e, f are complex constants with respect to z. The entries of the Stokes matrices, a, b, c, d, e, f, form the set of monodromy data MD,

$$MD = \{a, b, c, d, e, f\} \quad (2.12)$$

The monodromy data satisfy the following consistency condition;

$$\prod_{j=1}^{6} G_j e^{2i\pi\theta\sigma_3} = I. \quad (2.13)$$

The Schlesinger transformation associated with the linear system (2.2) allows us to shift the parameter θ by integer such that the MD are invariant.If $Y'(z)$ corresponds to θ' and $Y(z)$ corresponds to θ, the transformation matrix $R(z)$ can be defined by,

$$Y'(z) = R(z)Y(z). \quad (2.14)$$

Let $R(z) = R_j(z)$ when z in S_j; then the definition of the Stokes matrices (2.10) implies that the transformation matrix $R(z)$ satisfies the Rieamann-Hilbert problem along the contour C_j, $j = 1, ..., 6$, indicated in Figure 2.1,

$$\begin{aligned} R_{j+1}(z) &= R_j(z) \quad \text{on } C_{j+1}, \quad j = 1, ..., 5, \\ R_1(z) &= R_6(ze^{2i\pi}) \quad \text{on } C_1, \end{aligned} \quad (2.15)$$

with the boundary condition,

$$R_j(z) \sim \hat{Y}'_\infty(z)\left(\frac{1}{z}\right)^{n\sigma_3} \hat{Y}_\infty^{-1}(z), \quad \text{as} \quad z \to \infty, \quad z \text{ in } S_j. \tag{2.16}$$

The shifts $\theta \to \theta' = \theta \pm 1$ are enough to obtain all possible integer shifts in θ. The transformation matrices $R_{(1)}(z)$ and $R_{(2)}(z)$ for $\theta' = \theta + 1$ and $\theta' = \theta - 1$ respectively are given as follows:

$$\begin{aligned} \theta' = \theta + 1, \qquad & R_{(1)}(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z + \begin{pmatrix} 0 & -\frac{u}{v} \\ \frac{v}{u} & -\frac{\theta}{v} - y \end{pmatrix}, \\ \theta' = \theta - 1, \qquad & R_{(2)}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z + \begin{pmatrix} y & \frac{u}{2} \\ -\frac{2}{u} & 0 \end{pmatrix}. \end{aligned} \tag{2.17}$$

Successive applications of the transformation matrices $R_{(i)}(z), \quad i = 1,2$ map θ to $\theta' = \theta + n, \ n \in \mathbb{Z}$. If, $y', u', v', \theta' = \theta + 1$ are the transformed quantities of y, u, v, θ under the transformation given by $R_{(1)}(z)$, i.e.

$$Y'(z; t, y', u', v', \theta') = R_{(1)}(z; t, y, u, v, \theta) Y(z; t, y, u, v, \theta), \tag{2.18}$$

and if $y'', u'', v'', \theta'' = \theta' - 1$ are the transformed quantities of y', u', v', θ' under the transformation given by $R_{(2)}(z)$, i.e.

$$Y''(z; t, y'', u'', v'', \theta'') = R_{(2)}(z; t, y', u', v', \theta') Y(z; t, y', u', v', \theta'), \tag{2.19}$$

then,

$$R_{(2)}(z; t, y'(y, u, v, \theta), ...) R_{(1)}(z; t, y, u, v, \theta) = I. \tag{2.20}$$

Also,

$$\begin{aligned} R_{(1)}(z; t, y'(y, u, v, \theta), ...) R_{(1)}(z; t, y, u, v, \theta) &= R_{(3)}(z), \\ R_{(2)}(z; t, y'(y, u, v, \theta), ...) R_{(2)}(z; t, y, u, v, \theta) &= R_{(4)}(z), \end{aligned} \tag{2.21}$$

where $R_{(3)}(z)$ and $R_{(4)}(z)$ shift the exponents $\theta \to \theta' = \theta + 2$ and $\theta \to \theta' = \theta - 2$ respectively.

The linear equation (2.2.a) under the Schlesinger transformation given by eq. (2.14), is transformed as follows,

$$Y'_z(z) = A'(z) Y', \qquad A'(z) = \left[R(z) A(z) + R_z(z)\right] R^{-1}(z). \tag{2.22}$$

For the particular case of $R_{(1)}(z)$, the quantities y, u, v, θ are transformed by,

$$\theta' = \theta + 1, \quad y' = -y - \frac{\theta}{v},$$
$$u' = \frac{2u}{v}, \quad v' = -t - \frac{2(\theta + yv)^2}{v^2}. \tag{2.23}$$

From the equations (2.23), the following transformation for the solutions of P.II can be obtained;

$$y' = -y + \frac{2\alpha - 1}{2y_t - 2y^2 - t}, \quad \alpha' = \alpha - 1, \quad \alpha \neq \frac{1}{2}. \tag{2.24}$$

Similarly, the Schlesinger transformation given by the transformation matrix $R_{(2)}(z)$ transforms the quantities y, u, v, θ as follows:

$$\theta' = \theta - 1, \quad y' = -y + \frac{\theta'}{2y^2 + v + t},$$
$$u' = \frac{u}{2}v', \quad v' = -v - 2y^2 - t. \tag{2.25}$$

The transformation for the solutions of P.II can be obtained from (2.25) as follows;

$$y' = -y + \frac{2\alpha + 1}{2y^2 + 2y_t + t}, \quad \alpha' = \alpha + 1, \quad \alpha \neq -\frac{1}{2}. \tag{2.26}$$

The transformation (2.24) and (2.26) were also obtained in [27] and [9] respectively. The transformations (2.24) and (2.26) allows us to obtain the new solution of the second Painlevé equation from the known solutions. For example, $y = 0$ for $\alpha = 0$ solves P.II. By using the transformation (2.24),the new solution

$$y' = 1/t \quad for \quad \alpha = -1, \tag{2.27}$$

of P.II can be obtained.

3 Transformation from Painlevé III to Painlevé III

The third Painlevé equation

$$\frac{d^2y}{dt^2} = \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}, \tag{3.1}$$

can be obtained as the compatibility condition of the following linear system of equations,

$$Y_z(z) = A(z)Y(z), \qquad Y_t(z) = B(z)Y(z), \tag{3.2}$$

where,[25]

$$A(z) = \tfrac{t}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} -\theta_\infty/2 & u \\ v & \theta_\infty/2 \end{pmatrix}\tfrac{1}{z} +$$

$$\begin{pmatrix} s-\frac{t}{2} & -ws \\ \frac{1}{w}(s-t) & -(s-\frac{t}{2}) \end{pmatrix}\tfrac{1}{z^2}, \tag{3.3}$$

$$B(z) = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z + \tfrac{1}{t}\begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix} - \tfrac{1}{t}\begin{pmatrix} s-\frac{t}{2} & -ws \\ \frac{1}{w}(s-t) & -(s-\frac{t}{2}) \end{pmatrix}\tfrac{1}{z}.$$

The compatibility condition, $Y_{zt} = Y_{tz}$, implies

$$\begin{aligned}
\frac{du}{dt} &= \frac{\theta_\infty}{t}u - 2ws, \qquad \frac{dv}{dt} = -\frac{\theta_\infty}{t}v + \frac{2}{w}(t-s), \\
t\frac{ds}{dt} &= -4ys^2 + (4yt - 2\theta_\infty + 1)s + (\theta_0 + \theta_\infty)t, \\
t\frac{dw}{dt} &= w\left[\frac{t}{s}(\theta_0 + \theta_\infty) - 2ty + \theta_\infty\right], \\
t\frac{dy}{dt} &= 4sy^2 - 2ty^2 + (2\theta_\infty - 1)y + 2t,
\end{aligned} \tag{3.4}$$

where $y = -\frac{u}{sw}$ and

$$\frac{\theta_0}{2} = -\frac{s-t}{wt}(u - \frac{\theta_\infty}{2}w) + \frac{s}{t}(wv + \frac{\theta_\infty}{2}). \tag{3.5}$$

Thus y satisfies the third Painlevé equation (3.1) with the parameters,

$$\alpha = 4\theta_0, \qquad \beta = 4(1-\theta_\infty), \qquad \gamma = 4, \qquad \delta = -4. \tag{3.6}$$

The Schlesinger transformations associated with the linear system (3.2) allows us to shift the parameters θ_0 and θ_∞ as $\theta_0' = \theta_0 + m - n$, $\theta_\infty' = \theta_\infty + m + n$, provided that $n + m = 2k, m, n, k \in \mathbb{Z}$. It is enough to consider the following four cases,

$$\begin{aligned}
&(1) : \begin{cases} \theta_0{}' = \theta_0 - 1 \\ \theta_\infty{}' = \theta_\infty + 1 \end{cases}, \qquad (2) : \begin{cases} \theta_0{}' = \theta_0 + 1 \\ \theta_\infty{}' = \theta_\infty + 1 \end{cases}, \\
&(3) : \begin{cases} \theta_0{}' = \theta_0 - 1 \\ \theta_\infty{}' = \theta_\infty - 1 \end{cases}, \qquad (4) : \begin{cases} \theta_0{}' = \theta_0 + 1 \\ \theta_\infty{}' = \theta_\infty - 1 \end{cases},
\end{aligned} \tag{3.7}$$

and the explicit form of the transformation matrices $R_{(i)}(z), i = 1,2,3,4$ are;

$$R_{(1)}(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z^{1/2} + \begin{pmatrix} 1 & -\frac{ws}{s-t} \\ -\frac{v}{t} & \frac{v}{t}\frac{ws}{s-t} \end{pmatrix} z^{-1/2}, \tag{3.8}$$

$$R_{(2)}(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z^{1/2} + \begin{pmatrix} 1 & -w \\ -\frac{v}{t} & \frac{wv}{t} \end{pmatrix} z^{-1/2}, \tag{3.9}$$

$$R_{(3)}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z^{1/2} + \begin{pmatrix} -\frac{u}{t}\frac{s-t}{ws} & \frac{u}{t} \\ -\frac{s-t}{ws} & 1 \end{pmatrix} z^{-1/2}, \tag{3.10}$$

$$R_{(4)}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z^{1/2} + \begin{pmatrix} -\frac{u}{tw} & \frac{u}{t} \\ -\frac{1}{w} & 1 \end{pmatrix} z^{-1/2}. \tag{3.11}$$

All the possible shifts can be obtained by the successive application of $R_{(i)}(z), i = 1,2,3,4$. Since, if $y', u', v', w', s', \theta_0', \theta_\infty'$ are transformed quantities of $y, u, v, w, s, \theta_0, \theta_\infty$ under the transformation given by $R_{(k)}(z)$,i.e.

$$\begin{aligned}
&Y'(z,t;y',u',v',\bar{u}',\bar{v}',w',s',\theta_\infty{}',\theta_0{}') = \\
&\qquad\qquad R_{(k)}(z,t;y,...,\theta_0)Y(z,t;y,...,\theta_0),
\end{aligned} \tag{3.12}$$

and if $y'', u'', v'', w'', s'', \theta_0'', \theta_\infty''$ are transformed quantities of $y', u', v', w', s', \theta_0', \theta_\infty'$ under the transformation given by $R_{(l)}(z)$, i.e.

$$Y''(z,t;y'',u'',v'',\bar{u}'',\bar{v}'',w'',s'',\theta_\infty'',\theta_0'') = R_{(l)}(z,t;y',...,\theta_0')Y'(z,t;y',...,\theta_0'), \tag{3.13}$$

then

$$R_{(k)}(z,t;y'(y,u,...\theta_0),...)R_{(l)}(z,t;y,...,\theta_0) = I, \tag{3.14}$$

for $k,l = 2,3$ and $k,l = 1,4$.

It is possible to obtain the transformations between the solution of PIII, corresponding to different values of the parameters α, β, γ and δ from the Schlesinger transformation associated with the linear system (3.2). In Particular, the transformation matrix $R_{(1)}(z)$, transforms the quantities y, u, v, w, s, θ_0, θ_∞ as follows:

$$\begin{gathered}
\theta_0' = \theta_0 - 1, \qquad \theta_\infty' = \theta_\infty + 1, \\
ts' + vw's' = 0, \\
t(s'-t) + vw's' = tvw', \\
u' = \frac{tws}{s-t}, \qquad s \neq t \\
vwsu' - t(s-t)(w's'-u) + tws\theta_\infty = 0, \\
w(s-t)(tv' - v\theta_\infty - v) - t(s-t)^2 - v^2w^2s = 0, \\
ws(tv'-v) + (s-t)(ts' - uv - ts) = 0.
\end{gathered} \tag{3.15}$$

By eliminating s, w, u and v from the above equations and writing θ_0 and θ_∞ in terms of α, β, γ and δ, one can obtain the following transformation for the solutions of PIII;

$$y' = \frac{2m(m-4ty^2)}{m[(\alpha - 3\beta + 12)y^2 + 2my - 8ty^3] - 8t(\alpha - \beta + 4)y^4}, \tag{3.16}$$

with

$$\alpha' = \alpha - 4, \qquad \beta' = \beta - 4, \qquad \gamma' = 4, \qquad \delta' = -4, \tag{3.17}$$

where

$$m = ty_t + 2ty^2 - (1 - \frac{\beta}{2})y - 2t. \tag{3.18}$$

The transformation (3.16) allows us to obtain the new solution y' corresponding to the parameters α', β', γ' and δ' from the solution y corresponding to the parameters α, β, γ and δ of PIII. For example,$y = -1$ for $\alpha = -2, \beta = 2, \gamma = -\delta = 4$, solves PIII. By using the transformation (3.16) one can obtain the new solution;

$$y' = \frac{-2t}{1+2t}, \qquad for \qquad \alpha' = -6, \quad \beta' = -2, \quad \gamma' = -\delta' = 4. \tag{3.19}$$

Similarly, from the Schlesinger transformations given by the transformation matrix $R_{(2)}(z)$, (3.2) transforms the quantities $y, u, v, w, s, \theta_0, \theta_\infty$ as follows:

$$\begin{aligned}
&\theta_0' = \theta_0 + 1, \qquad \theta_\infty' = \theta_\infty + 1, \\
&ts' + vw's' = t^2, \\
&u' = tw, \\
&vwu' - tw's' - tu + tw\theta_\infty = 0, \\
&twv' - t(s-t) + vw(\theta_\infty + 1) - v^2w^2 = 0, \\
&t(wv' + s' - s) - v(u + w) = 0.
\end{aligned} \tag{3.20}$$

By eliminating s, w, u and v from the above equations and writing θ_0 and θ_∞ in terms of α, β, γ and δ, one can obtain the following transformations for the solution of PIII;

$$y' = \frac{2m}{(\alpha - \beta + 4)y^2 - 2my}, \tag{3.21}$$

with

$$\alpha' = \alpha + 4, \qquad \beta' = \beta - 4, \qquad \gamma' = -\delta' = 4, \tag{3.22}$$

where

$$m = ty_t + 2ty^2 - (1 - \frac{\beta}{2})y - 2t. \tag{3.23}$$

The transformation (3.21) allows us to obtain the new solution y' corresponding to the parameters α', β', γ' and δ' from the solution y corresponding to the parameters α, β, γ and δ of PIII. For example, by using the transformation (3.21) and the solution (3.19) of PIII, the new solution,

$$y' = 1 + \frac{1}{2t} \qquad for \qquad \alpha' = -2, \quad \beta' = -6, \quad \gamma' = -\delta' = 4. \tag{3.24}$$

can be obtained.

The Schlesinger transformations associated with the linear equation (3.2) and given by the transformation matrix $R_{(3)}(z)$ transforms the quantities $y, u, v, w, s, \theta_0, \theta_\infty$ as follows:

$$\begin{aligned}
&\theta_0' = \theta_0 - 1, \qquad \theta_\infty' = \theta_\infty - 1, \\
&u = tw', \\
&tw's' - u(s' - t) - t^2 w' = 0, \\
&wsv' + t(s - t) = 0, \\
&(s - t)(tu' + u) + ws(uv + ts - ts') = 0, \\
&ws(tu' + u + tws - u\theta_\infty) + u^2(s - t) = 0, \\
&w'(s - t)(t\theta_\infty + uv') + tws(vw' - s' + t) = 0,
\end{aligned} \tag{3.25}$$

From the equations (3.25), the following transformation for the solution of PIII can be obtained;

$$y' = \frac{2tyn}{2tn + \frac{1}{2}(1 - y^2) - \frac{ty_t}{2}(\frac{1}{y} + y - \frac{y_t}{2y}) + \frac{1-\beta}{4} y_t + \frac{t}{4} y_{tt}}, \tag{3.26}$$

with

$$\alpha' = \alpha - 4, \qquad \beta' = \beta + 4, \qquad \gamma' = -\delta' = 4, \tag{3.27}$$

where

$$n = \frac{1}{y} + \frac{1}{t}(1 - \frac{\beta}{4}) - \frac{1}{t} + \frac{m}{4ty} - y,$$
$$m = ty_t + 2ty^2 - (1 - \frac{\beta}{2})y - 2t. \tag{3.28}$$

The transformation (3.26) allows us to obtain the new solution y' corresponding to the Parameters α', β', γ' and δ' from the solution y corresponding to the parameters α, β, γ and δ of PIII.The transformation (3.26) generates the following new solution from the solution (3.19);

$$y' = \frac{-(2t+1)}{2(t+1)} \qquad for \qquad \alpha' = -10, \quad \beta' = 2, \quad \gamma = -\delta = 4. \tag{3.29}$$

Similarly, from the Schlesinger transformations associated with the linear equation (3.2), the transformation matrix $R_{(4)}(z)$ transforms the quantities $y, u, v, w, s, \theta_0, \theta_\infty$ as follows:

$$\theta_0' = \theta_0 + 1, \qquad \theta_\infty' = \theta_\infty - 1,$$
$$u(s' - t) - tw's' = 0,$$
$$wv' + t = 0,$$
$$t(ws' - ws - u') - u(1 + vw) = 0, \tag{3.30}$$
$$tw(u' + ws) + u(u - w\theta_\infty + w) = 0,$$
$$w'(uv' + t\theta_\infty + tvw) - tw(s - t) = 0.$$

The transformation for the solutions of PIII can be obtained from the (3.30) by eliminating s, w, u and v and writing θ_0 and θ_∞ in terms of α, β, γ and δ;

$$y' = \frac{t + \frac{1}{4}(m - \beta y)}{y[t + \frac{m}{4} - \frac{\alpha - \beta + 4}{8} y]}, \tag{3.31}$$

with

$$\alpha' = \alpha + 4, \qquad \beta' = \beta + 4, \qquad \gamma' = -\delta' = 4, \tag{3.32}$$

where

$$m = ty_t + 2ty^2 - (1 - \tfrac{\beta}{2})y - 2t. \tag{3.33}$$

That is, if y solves the third Painlevé equation corresponding to the parameters α, β, γ and δ, then y' also satisfies the third Painlevé equation with the parameters α', β', γ' and δ'.

It is possible to obtain the transformation which ganerates the new solution y' corresponding to the parameters α', β', γ' and δ' from the solution y corresponding to the parameters α, β, γ and δ of PIII. For example, by using the transformation (3.31) and the solution (3.19) of PIII, the new solution,

$$y' = -1 \qquad for \qquad \alpha' = -2, \qquad \beta' = 2, \qquad \gamma' = -\delta' = 4, \tag{3.34}$$

can be generated.

4 Transformation from Painlevé IV to Painlevé IV

The fourth Painlevé equation

$$\frac{d^2y}{dt^2} = \frac{1}{2y}\left(\frac{dy}{dt}\right)^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}, \tag{4.1}$$

can be obtained as the compatibility condition of the following linear system of equations,

$$Y_z(z) = A(z)Y(z), \qquad Y_t(z) = B(z)Y(z), \tag{4.2}$$

where[25]

$$A(z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z + \begin{pmatrix} t & u \\ \frac{2}{u}(v - \theta_0 - \theta_\infty) & -t \end{pmatrix} + \begin{pmatrix} \theta_0 - v & -\frac{uy}{2} \\ 2\frac{v}{uy}(v - 2\theta_0) & -(\theta_0 - v) \end{pmatrix} \frac{1}{z}, \tag{4.3}$$

$$B(z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z + \begin{pmatrix} 0 & u \\ \frac{2}{u}(v - \theta_0 - \theta_\infty) & 0 \end{pmatrix}.$$

The compatibility condition, $Y_{zt} = Y_{tz}$, implies

$$\begin{aligned} \frac{du}{dt} &= -u(y + 2t), \\ \frac{dv}{dt} &= -\frac{2}{y}v^2 + \left(\frac{4\theta_0}{y} - y\right)v + (\theta_0 + \theta_\infty)y, \\ \frac{dy}{dt} &= -4v + y^2 + 2ty + 4\theta_0. \end{aligned} \tag{4.4}$$

Thus y satisfies the fourth Painlevé equation (4.1) with the parameters

$$\alpha = 2\theta_\infty - 1, \qquad \beta = -8\theta_0{}^2. \tag{4.5}$$

The Schlesinger transformation matrices; $R(z)$; associated to the linear system (4.2) , are given as follows;

$$\begin{cases} \theta_0{}' = \theta_0 - \frac{1}{2} \\ \theta_\infty{}' = \theta_\infty + \frac{1}{2}, \end{cases}$$
$$R_{(1)}(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z^{1/2} + \begin{pmatrix} 1 & \frac{uy}{2(v-2\theta_0)} \\ -\frac{v-\theta_0-\theta_\infty}{u} & -\frac{y(v-\theta_0-\theta_\infty)}{2(v-2\theta_0)} \end{pmatrix} z^{-1/2}, \tag{4.6}$$

$$\begin{cases} \theta_0{}' = \theta_0 + \frac{1}{2} \\ \theta_\infty{}' = \theta_\infty - \frac{1}{2}, \end{cases}$$
$$R_{(2)}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z^{1/2} + \begin{pmatrix} \frac{v}{y} & \frac{u}{2} \\ \frac{2v}{uy} & 1 \end{pmatrix} z^{-1/2}, \tag{4.7}$$

$$\begin{cases} \theta_0{}' = \theta_0 + \frac{1}{2} \\ \theta_\infty{}' = \theta_\infty + \frac{1}{2}, \end{cases}$$
$$R_{(3)}(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z^{1/2} + \begin{pmatrix} 1 & \frac{uy}{2v} \\ -\frac{v-\theta_0-\theta_\infty}{u} & -\frac{y(v-\theta_0-\theta_\infty)}{2v} \end{pmatrix} z^{-1/2}, \tag{4.8}$$

$$\begin{cases} \theta_0{}' = \theta_0 - \frac{1}{2} \\ \theta_\infty{}' = \theta_\infty - \frac{1}{2}, \end{cases}$$
$$R_{(4)}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z^{1/2} + \begin{pmatrix} \frac{v-2\theta_0}{y} & \frac{u}{2} \\ \frac{2}{uy}(v-2\theta_0) & 1 \end{pmatrix} z^{-1/2}. \tag{4.9}$$

The transformation matrices $R_{(i)}(z)$, $i = 1, ..., 4$ are enough to cover all possible shifts in the exponents θ_0, θ_∞. Since, if $y', u', v', \theta_0{}', \theta_\infty{}'$ are transformed quantities of $y, u, v, \theta_0, \theta_\infty$ under the transformation given by $R_{(k)}(z)$, i.e.

$$Y'(z,t;y',u',v',\bar{u}',\bar{v}',\theta_\infty{}',\theta_0{}') = R_{(k)}(z,t;y,...,\theta_0)Y(z,t;y,...,\theta_0), \tag{4.10}$$

and if $y'', u'', v'', \theta_0{}'', \theta_\infty{}''$ are transformed quantities of $y', u', v', \theta_0{}', \theta_\infty{}'$ under the transformation given by $R_{(l)}(z)$, i.e.

$$Y''(z,t;y'',u'',v'',\bar{u}'',\bar{v}'',\theta_\infty{}'',\theta_0{}'') = R_{(l)}(z,t;y',...,\theta_0{}')Y'(z,t;y',...,\theta_0{}'), \tag{4.11}$$

then

$$R_{(k)}(z,t;y'(y,u,...\theta_0),...)R_{(l)}(z,t;y,...,\theta_0) = I, \tag{4.12}$$

for $k,l=2,1$ and $k,l=3,4$. Also,

$$\begin{aligned} R_{(1)}(z,t;y'(y,u,...\theta_0),...)R_{(3)}(z,t;y,...,\theta_0) &= R_{(5)}(z), \\ R_{(2)}(z,t;y'(y,u,...\theta_0),...)R_{(4)}(z,t;y,...,\theta_0) &= R_{(6)}(z), \end{aligned} \tag{4.13}$$

where $R_{(5)}(z)$ and $R_{(6)}(z)$ are;

$$\begin{cases} {\theta_0}' = \theta_0 \\ {\theta_\infty}' = \theta_\infty + 1, \end{cases}$$
$$R_{(5)}(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z + \begin{pmatrix} 0 & \frac{u}{v-\theta_0-\theta_\infty} \\ -\frac{v-\theta_0-\theta_\infty}{u} & -\frac{v(v-2\theta_0)}{y(v-\theta_0-\theta_\infty)} + t \end{pmatrix}, \tag{4.14}$$

$$\begin{cases} {\theta_0}' = \theta_0 + 1 \\ {\theta_\infty}' = \theta_\infty, \end{cases}$$
$$R_{(6)}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{2\theta_0+1}{N} \begin{pmatrix} -1 & -\frac{uy}{2v} \\ \frac{2v}{uy} & 1 \end{pmatrix} z^{-1}, \tag{4.15}$$

where

$$N = 2[t + \frac{v}{y} + \frac{y}{2v}(v - \theta_0 - \theta_\infty)]. \tag{4.16}$$

It is possible to obtain the transformation which generates the new solution y' corresponding to the parameters α' and β' from the known solution y corresponding to the parameters α and β of PIV, from the Schlesinger transformation associated with the linear system (4.2). In particular, the transformation matrix $R_{(1)}(z)$, transforms the quantities $y, u, v, \theta_0, \theta_\infty$ as follows:

$$\begin{aligned} &\theta_0' = \theta_0 - \tfrac{1}{2}, \qquad \theta_\infty' = \theta_\infty + \tfrac{1}{2}, \\ &u' = -\frac{uy}{v-2\theta_0}, \qquad v \neq 2\theta_0, \\ &2(v-2\theta_0)^2(v'+v-\theta_0-\theta_\infty) \\ &\qquad -2ty(v-2\theta_0)(v-\theta_0-\theta_\infty) - y^2(v-\theta_0-\theta_\infty)^2 = 0, \\ &yu'(v-\theta_0-\theta_\infty) + (v-2\theta_0)(u'y'+2u) - 2tuy = 0. \end{aligned} \tag{4.17}$$

By eliminating u and v from the above equations and writing θ_0 and θ_∞ in terms of α and β, one can obtain the following transformation for the solution of P.IV;

$$y' = -2t - y + \frac{r}{2} - \frac{(-2\beta)^{\frac{1}{2}} - 2\alpha - 2}{r}, \tag{4.18}$$

with

$$\alpha' = \alpha + 1, \qquad \beta' = \beta + 2(-2\beta)^{\frac{1}{2}} - 2, \tag{4.19}$$

where

$$r = y + 2t - \frac{y_t}{y} - \frac{(-2\beta)^{\frac{1}{2}}}{y}. \tag{4.20}$$

The transformation (4.18) allows us to obtain the new solution y' corresponding to the parameters α' and β' from the solution y corresponding to the parameters α and β of P.IV. For example,

$$y = \frac{1}{t} \qquad for \qquad \alpha = 2, \quad \beta = -2, \tag{4.21}$$

solves P.IV. By using the transformation (4.18), one can obtain the new solution $y' = 0$ for the parameters $\alpha' = 3, \beta' = 0$.

Similarly, from the Schlesinger transformations associated with the linear equation (4.2) , and given by the transformation matrix $R_{(2)}(z)$ transforms the quantities $y, u, v, \theta_0, \theta_\infty$ as follows:

$$\begin{aligned} \theta_0' &= \theta_0 + \tfrac{1}{2}, \qquad \theta_\infty' = \theta_\infty - \tfrac{1}{2}, \\ v' &= \tfrac{2vu'}{uy} + \theta_0 + \theta_\infty, \\ u' &= -u(\tfrac{y}{2} - \tfrac{v}{y} + t), \\ y' &= -\tfrac{u}{u'}(v' - 2\theta_0 - 1). \end{aligned} \tag{4.22}$$

The transformation for the solution of PIV can be obtained from the equation (4.22) by eliminating u and v and writing θ_0 and θ_∞ in terms of α and β;

$$y' = \frac{(-2\beta)^{\frac{1}{2}} - 2\alpha + 2}{2(-2t - y) + p} - \frac{p}{2}, \tag{4.23}$$

with

$$\alpha' = \alpha - 1, \qquad \beta' = \beta - 2(-2\beta)^{\frac{1}{2}} - 2, \tag{4.24}$$

where

$$p = y + 2t - \frac{y_t}{y} + \frac{(-2\beta)^{\frac{1}{2}}}{y}. \tag{4.25}$$

The transformation (4.23) enables us to use a known solution of (4.1) to construct the new solutions with new values of the parameters. For example, by using the transformation (4.23) and the solution (4.21) of PIV, the new solution,

$$y' = -(2t + \frac{1}{t}), \qquad for \qquad \alpha' = 1, \quad \beta' = -8, \tag{4.26}$$

can be generated.

Also, from the Schlesinger transformation defined by the transformation matrix $R_{(3)}(z)$, transforms the quantities $y, u, v, \theta_0, \theta_\infty$ as follows:

$$\begin{aligned}
&\theta_0' = \theta_0 + \tfrac{1}{2}, \qquad \theta_\infty' = \theta_\infty + \tfrac{1}{2}, \\
&u' = -\frac{uy}{v}, \qquad v \neq 0, \\
&2u(ty - v) - yu'(v - \theta_0 - \theta_\infty) - vu'y' = 0, \\
&2v^2(v' + v - 3\theta_0 - \theta_\infty - 1) - 2tyv(v - \theta_0 - \theta_\infty) - \\
&\qquad\qquad y^2(v - \theta_0 - \theta_\infty)^2 = 0.
\end{aligned} \tag{4.27}$$

From the equations (4.27), the follwing transformation for the solutions of PIV can be obtained,

$$y' = \frac{p - 2(2t + y)}{2} + \frac{(-2\beta)^{1/2} + 2\alpha + 2}{p}, \tag{4.28}$$

with

$$\alpha' = \alpha + 1, \qquad \beta' = \beta - 2(-2\beta)^{\frac{1}{2}} - 2, \tag{4.29}$$

where

$$p = y + 2t - \frac{y_t}{y} + \frac{(-2\beta)^{\frac{1}{2}}}{y}. \tag{4.30}$$

The transformation (4.28) in conjunction with the known solution y corresponding to the parameters α and β leads to new solutions y' of P.IV for the parameters α', β'. For example, by using the transformation (4.28) and the solution (4.21) of PIV one can obtain the new solution,

$$y' = \frac{4t}{2t^2+1}, \qquad for \qquad \alpha' = 3, \quad \beta' = -8. \tag{4.31}$$

Similarly, from the Schlesinger transformation associated with the linear equation (4.2) , the transformation matrix $R_{(4)}(z)$ transforms the quantities $y, u, v, \theta_0, \theta_\infty$ as follows:

$$\begin{aligned} &\theta_0' = \theta_0 - \tfrac{1}{2}, \qquad \theta_\infty' = \theta_\infty - \tfrac{1}{2}, \\ &v' = \frac{2u'(v-2\theta_0)}{uy} + \theta_0 + \theta_\infty - 1, \qquad u \neq 0, \\ &u' = u(\frac{v-2\theta_0}{y} - t - \frac{y}{2}), \qquad y \neq 0, \\ &y' = -\frac{uv'}{u'}, \qquad u' \neq 0. \end{aligned} \tag{4.32}$$

The equations (4.32) gives the following transformation for the solution of PIV,

$$y' = \frac{(-2\beta)^{\frac{1}{2}} + 2\alpha - 2}{2(2t+y) - r} - \frac{r}{2}, \tag{4.33}$$

with

$$\alpha' = \alpha - 1, \qquad \beta' = \beta + 2(-2\beta)^{\frac{1}{2}} - 2, \tag{4.34}$$

where

$$r = y + 2t - \frac{y_t}{y} - \frac{(-2\beta)^{\frac{1}{2}}}{y}. \tag{4.35}$$

The transformation (4.33) allows us to obtain the new solution y' corresponding to the parameters α' and β' from the solution y corresponding to the parameters α and β of P.IV. For example, the equation (4.21) solves PIV. By using the transformation (4.33) one can obtain the new solution,

$$y' = 0, \qquad for \qquad \alpha' = 1, \quad \beta' = 0. \tag{4.36}$$

The transformation (4.33) associated with PIV can be obtained by the successive applications of the following transformation given in [6]

$$\bar{y} = -\frac{y_t - (-2\beta)^{1/2} + 2ty + y^2}{2y},$$

$$\bar{\alpha} = -\tfrac{1}{4}\left[2 + 2\alpha - 3(-2\beta)^{1/2}\right], \tag{4.37}$$

$$\bar{\beta} = -\tfrac{1}{2}\left[\alpha - 1 + \tfrac{1}{2}(-2\beta)^{1/2}\right],$$

provided that,

$$\alpha - 1 - \frac{1}{2}(-2\beta)^{1/2} = 0. \tag{4.38}$$

5 Transformation from Painlevé V to Painlevé V

The fifth Painlevé equation

$$\frac{d^2y}{dt^2} = (\frac{1}{2y} + \frac{1}{y-1})(\frac{dy}{dt})^2 - \frac{1}{t}\frac{dy}{dt} + \frac{(y-1)^2}{t^2}(\alpha y + \frac{\beta}{y}) + \frac{\gamma y}{t} + \frac{\delta y(y+1)}{y-1}, \quad (5.1)$$

can be obtained as the compatibility condition of the following linear system of equations,

$$Y_z(z) = A(z)Y(z), \qquad Y_t(z) = B(z)Y(z), \quad (5.2)$$

where [25]

$$A(z) = \tfrac{1}{2}\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} +$$

$$\begin{pmatrix} v + \frac{\theta_0}{2} & -u(v+\theta_0) \\ \frac{v}{u} & -(v + \frac{\theta_0}{2}) \end{pmatrix}\frac{1}{z} + \begin{pmatrix} -w & uy(w - \frac{\theta_1}{2}) \\ -\frac{1}{uy}(w + \frac{\theta_1}{2}) & w \end{pmatrix}\frac{1}{z-1},$$

$$B(z) = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z + \quad (5.3)$$

$$\frac{1}{t}\begin{pmatrix} 0 & u\left[v + \theta_0 - y(w - \frac{\theta_1}{2})\right] \\ \frac{1}{u}\left[v - \frac{1}{y}(w + \frac{\theta_1}{2})\right] & 0 \end{pmatrix}.$$

$$w = v + \tfrac{1}{2}(\theta_0 + \theta_\infty).$$

The compatibility condition, $Y_{zt} = Y_{tz}$,implies

$$t\frac{dy}{dt} = ty - 2v(y-1)^2 - \frac{1}{2}(y-1)\left[(\theta_0 - \theta_1 + \theta_\infty)y - (3\theta_0 + \theta_1 + \theta_\infty)\right],$$

$$t\frac{dv}{dt} = yv\left(w - \frac{\theta_1}{2}\right) - \frac{1}{y}(v + \theta_0)\left(w + \frac{\theta_1}{2}\right),$$

$$t\frac{du}{dt} = u\left[-2t - \theta_0 + y\left(w - \frac{\theta_1}{2}\right) + \frac{1}{y}\left(w + \frac{\theta_1}{2}\right)\right]. \quad (5.4)$$

Thus y satisfies the fifth Painlevé equation (5.1) with the parameters

$$\alpha = \frac{1}{2}\left(\frac{\theta_0 - \theta_1 + \theta_\infty}{2}\right)^2, \quad \beta = -\frac{1}{2}\left(\frac{\theta_0 - \theta_1 - \theta_\infty}{2}\right)^2, \quad \gamma = 1-\theta_0-\theta_1, \quad \delta = -\frac{1}{2}. \tag{5.5}$$

The Schlesinger transformation matrices $R(Z)$ associated to the linear system (5.2) , are given as follows

$$\begin{cases} \theta_0' = \theta_0 + 1 \\ \theta_1' = \theta_1 \\ \theta_\infty' = \theta_\infty + 1, \end{cases}$$
$$R_{(1)}(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z^{1/2} + \tag{5.6}$$
$$\begin{pmatrix} 1 & -\frac{u}{v}(v+\theta_0) \\ -\frac{1}{tu}\left[v - \frac{1}{y}(w + \theta_1/2)\right] & \frac{1}{tv}(v+\theta_0)\left[v - \frac{1}{y}\left(w + \frac{\theta_1}{2}\right)\right] \end{pmatrix} z^{-1/2},$$

$$\begin{cases} \theta_0' = \theta_0 - 1 \\ \theta_1' = \theta_1 \\ \theta_\infty' = \theta_\infty - 1, \end{cases}$$
$$R_{(2)}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z^{1/2} + \tag{5.7}$$
$$\begin{pmatrix} \frac{1}{t}\left[v + \theta_0 - y\left(w - \frac{\theta_1}{2}\right)\right] & -\frac{u}{t}\left[v + \theta_0 - y\left(w - \frac{\theta_1}{2}\right)\right] \\ -\frac{1}{u} & 1 \end{pmatrix} z^{-1/2},$$

$$\begin{cases} \theta_0' = \theta_0 + 1 \\ \theta_1' = \theta_1 \\ \theta_\infty' = \theta_\infty - 1, \end{cases}$$
$$R_{(3)}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z^{1/2} + \tag{5.8}$$
$$\begin{pmatrix} \frac{1}{t}\left[v + \theta_0 - y\left(w - \frac{\theta_1}{2}\right)\right]\frac{v}{v+\theta_0} & -\frac{u}{t}\left[v + \theta_0 - y\left(w - \frac{\theta_1}{2}\right)\right] \\ -\frac{v}{u(v+\theta_0)} & 1 \end{pmatrix} z^{-1/2},$$

$$\begin{cases} \theta_0' = \theta_0 - 1 \\ \theta_1' = \theta_1 \\ \theta_\infty' = \theta_\infty + 1, \end{cases}$$

$$R_{(4)}(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z^{1/2} + \tag{5.9}$$

$$\begin{pmatrix} 1 & -u \\ -\frac{1}{tu}\left[v - \frac{1}{y}\left(w + \frac{\theta_1}{2}\right)\right] & \frac{1}{t}\left[v - \frac{1}{y}\left(w + \frac{\theta_1}{2}\right)\right] \end{pmatrix} z^{-1/2},$$

$$\begin{cases} \theta_0' = \theta_0 \\ \theta_1' = \theta_1 + 1, \\ \theta_\infty' = \theta_\infty + 1, \end{cases}$$

$$R_{(5)}(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} (z-1)^{1/2} + \tag{5.10}$$

$$\begin{pmatrix} 1 & -\frac{uy}{w_1} \\ -\frac{1}{tu}\left[v - \frac{1}{y}\left(w + \frac{\theta_1}{2}\right)\right] & \frac{y}{t}\left[v - \frac{1}{y}\left(w + \frac{\theta_1}{2}\right)\right]\frac{1}{w_1} \end{pmatrix} (z-1)^{-1/2},$$

$$\begin{cases} \theta_0' = \theta_0 \\ \theta_1' = \theta_1 - 1 \\ \theta_\infty' = \theta_\infty - 1, \end{cases}$$

$$R_{(6)}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (z-1)^{1/2} + \tag{5.11}$$

$$\begin{pmatrix} \frac{1}{ty}\left[v + \theta_0 - y\left(w - \frac{\theta_1}{2}\right)\right] & -\frac{u}{t}\left[v + \theta_0 - y\left(w - \frac{\theta_0}{2}\right)\right] \\ -\frac{1}{uy} & 1 \end{pmatrix} (z-1)^{-1/2},$$

$$\begin{cases} \theta_0' = \theta_0 \\ \theta_1' = \theta_1 + 1 \\ \theta_\infty' = \theta_\infty - 1, \end{cases}$$

$$R_{(7)}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (z-1)^{1/2} +$$

$$\begin{pmatrix} \frac{w_1}{ty}\left[v + \theta_0 - y\left(w - \frac{\theta_1}{2}\right)\right] & -\frac{u}{t}\left[v + \theta_0 - y\left(w - \frac{\theta_1}{2}\right)\right] \\ -\frac{1}{uy}w_1 & 1 \end{pmatrix} (z-1)^{-1/2}, \tag{5.12}$$

$$\begin{cases} \theta_0' = \theta_0 \\ \theta_1' = \theta_1 - 1 \\ \theta_\infty' = \theta_\infty + 1, \end{cases}$$

$$R_{(8)}(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} (z-1)^{1/2} + \tag{5.13}$$

$$\begin{pmatrix} 1 & -uy \\ -\frac{1}{tu}\left[v - \frac{1}{y}\left(w + \frac{\theta_1}{2}\right)\right] & \frac{y}{t}\left[v - \frac{1}{y}\left(w + \frac{\theta_1}{2}\right)\right] \end{pmatrix} (z-1)^{-1/2},$$

where,

$$w_1 = \frac{w + \theta_1/2}{w - \theta_1/2}. \tag{5.14}$$

The transformation matrices $R_{(i)}(z)$, $i = 1, ..., 8$ are enough to cover all possible shifts in the exponents $\theta_0, \theta_1, \theta_\infty$. Since, if $y', u', v', \theta_0', \theta_1', \theta_\infty'$ are transformed quantities of $y, u, v, \theta_0, \theta_1, \theta_\infty$ under the transformation given by $R_{(k)}(z)$, i.e.

$$Y'(z, t; y', u', v', \theta_0', \theta_1', \theta_\infty') = R_{(k)}(z, t; y, ..., \theta_\infty) Y(z, t; y, ..., \theta_\infty), \tag{5.15}$$

and if $y'', u'', v'', \theta_0'', \theta_1'', \theta_\infty''$ are transformed quantities of $y', u', v', \theta_0', \theta_1', \theta_\infty'$ under the transformation given by $R_{(l)}(z)$, i.e.

$$Y''(z, t; y'', u'', v'', \theta_0'', \theta_1'', \theta_\infty'') = R_{(l)}(z, t; y', ..., \theta_\infty') Y(z, t; y', ..., \theta_\infty'). \tag{5.16}$$

then

$$R_{(l)}(z, t; y'(y, u, ...\theta_\infty), ...) R_{(k)}(z, t; y, ..., \theta_\infty) = I, \tag{5.17}$$

for $l = k + 1$, $k = 1, 3, 5, 7$.

Also, $R_{(1)}(z) R_{(7)}(z) = R_{(9)}(z)$ shifts the exponents as $\theta_0' = \theta_0 + 1$, $\theta_1' = \theta_1 + 1$, $\theta_\infty' = \theta_\infty$, and $R_{(2)}(z) R_{(8)}(z) = R_{(10)}(z)$ shifts the exponents as, $\theta_0' = \theta_0 - 1$, $\theta_1' = \theta_1 - 1$, $\theta_\infty = \theta_\infty$. The explicit form of $R_{(9)}$ and $R_{(10)}$ are

$$\begin{cases} \theta_0' = \theta_0 + 1 \\ \theta_1' = \theta_1 + 1 \\ \theta_\infty' = \theta_\infty, \end{cases}$$

$$R_{(9)}(z) = z^{1/2}(z-1)^{-1/2} \left[I + \frac{1}{g_{21}f_{11} - g_{12}f_{21}} \begin{pmatrix} g_{21}f_{11} & -g_{11}f_{11} \\ g_{21}f_{21} & -g_{11}f_{21} \end{pmatrix} \frac{1}{z} \right], \tag{5.18}$$

$$\begin{cases} \theta_0' = \theta_0 - 1 \\ \theta_1' = \theta_1 - 1 \\ \theta_\infty' = \theta_\infty \end{cases}$$

$$R_{(10)}(z) = z^{-1/2}(z-1)^{1/2}\left[I + \frac{1}{g_{22}f_{12} - g_{12}f_{22}}\begin{pmatrix} -g_{12}f_{22} & g_{12}f_{12} \\ -g_{22}f_{22} & g_{22}f_{12} \end{pmatrix}\frac{1}{z-1}\right], \tag{5.19}$$

where,

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{\theta_0}(v+\theta_0)\,e^{-\sigma_0(t)} & ue^{\sigma_0(t)} \\ \frac{v}{u\theta_0}e^{-\sigma_0(t)} & e^{\sigma_0(t)} \end{pmatrix},$$

$$F = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} -\frac{2w-\theta_1}{2\theta_1}e^{-\sigma_1(t)} & uye^{\sigma_1(t)} \\ -\frac{1}{uyw_1}\left(\frac{2w-\theta_1}{2\theta_1}\right)e^{-\sigma_1(t)} & e^{\sigma_1(t)} \end{pmatrix}, \tag{5.20}$$

with

$$\sigma_0(t) = \int^t \left\{\frac{1}{t'}\left[v - \frac{1}{y}(w+\theta_1/2)\right] - \frac{1}{2}\right\} dt',$$

$$\sigma_1(t) = \int^t \left\{\frac{y}{t'}\left[v - \frac{1}{y}(w+\theta_1/2)\right] - \frac{1}{2}\right\} dt'. \tag{5.21}$$

It is possible to obtain the transformation for P.V, from the Schlesinger transformation associated with the linear system (5.2) .In Particular, the transformation matrix $R_{(3)}(z)$, transforms the quantities $y, u, v, \theta_0, \theta_1$ and θ_∞ as follows:

$$\theta_0' = \theta_0 + 1, \qquad \theta_1' = \theta_1, \qquad \theta_\infty' = \theta_\infty - 1,$$

$$u' = -\frac{uA}{t},$$

$$\frac{u(v' + \frac{\theta_0+\theta_1+\theta_\infty}{2})A}{tu'y'} + \frac{yv(v + \frac{\theta_0-\theta_1+\theta_\infty}{2})}{v+\theta_0} + v' - v = 0, \tag{5.22}$$

$$\left(\frac{v' + \frac{\theta_0+\theta_1+\theta_\infty}{2}}{u'y'}\right)\left[1 + \frac{vA}{t(v+\theta_0)}\right] +$$

$$\frac{v(v'+v+\theta_0+\theta_\infty)}{u(v+\theta_0)} - \frac{v + \frac{\theta_0+\theta_1+\theta_\infty}{2}}{uy} = 0,$$

where

$$A = v + \theta_0 - y(w - \frac{\theta_1}{2}). \tag{5.23}$$

The transforamtion for the solutions of PV can be obtained from (5.22) by eliminating u, v and writing θ_0, θ_1 and θ_∞ in terms of α, β, γ and δ;

$$y' = \frac{E(D + (2\alpha)^{1/2}) + \frac{1}{y}(D + C)\left[A - (1 - y)(C + D + (-2\beta)^{1/2})\right] + A(v - F)}{A(F + (2\alpha)^{1/2})} \tag{5.24}$$

with

$$\alpha' = \alpha, \qquad \beta' = -\frac{1}{2}\left[(-2\beta)^{1/2} + 1\right]^2, \qquad \gamma' = \gamma - 1, \qquad \delta' = \delta = -\frac{1}{2} \tag{5.25}$$

where

$$A = v + \theta_0 - y(w - \tfrac{\theta_1}{2})$$

$$B = ty - ty_t - (2\alpha)^{1/2}(y - 1)^2 - (\gamma - 1)(y - 1)$$

$$C = \frac{(2\alpha)^{1/2} + (-2\beta)^{1/2} + 1 - \gamma}{2}$$

$$D = \frac{B}{2(y - 1)^2},$$

$$E = \left[yA + yD(1 - y) - \frac{yAD}{v + C + (-2\beta)^{1/2}} - ty_t\right]$$

$$F = \frac{\frac{A(v+C)\left[v+C-(-2\beta)^{1/2}\right]}{y} - vA\left[v+C-(-2\beta)^{1/2}+(2\alpha)^{1/2}\right] - t(v+C)\left[\frac{yv(v+(2\alpha)^{1/2}}{v+C} - v\right]\left[1+\frac{vA}{t(v+C)}\right]}{t(v+C)+vA}. \tag{5.26}$$

The transformation 5.24 can be used to construct the new solution y' with different values of parameters α', β', γ' and δ'.For example,

$$y = t + 1 \qquad for \qquad \alpha = 1/2, \quad \beta = -1/2 \quad , \gamma = 1 \quad , \delta = -1/2, \tag{5.27}$$

solves PV.By using the transformation 5.24, one can obtain the new solution,

$$y' = t + 2 \qquad for \qquad \alpha' = 1/2, \quad \beta' = -2, \quad \gamma' = 0, \quad \delta' = -1/2. \tag{5.28}$$

Similarly, from the Schlesinger transformations associated with the linear equation (5.2) , the transformation matrix $R_{(6)}(z)$ transforms the quantities

$y, u, v, \theta_0, \theta_1$ and θ_∞ as follows:

$$\theta'_0 = \theta_0, \qquad \theta'_1 = \theta_1 - 1, \qquad \theta'_\infty = \theta_\infty - 1,$$

$$\frac{uA}{t}(v' + v + \theta_0 + t) + u'(v' + \theta_0) + uy(w - \frac{\theta_1}{2}) - \frac{u(v + \theta_0)A}{ty} = 0,$$

$$(v' - v)(\frac{A}{ty} - 1) + \frac{u'(v' + \theta_0)}{uy} + \frac{vA}{t} = 0, \tag{5.29}$$

$$\frac{uv'A}{tu'} + \frac{v + \theta_0}{y} + v' - v = 0,$$

$$\frac{(w' + \frac{\theta'_1}{2})A}{t} + \frac{u'y'(w' - \frac{\theta'_1}{2})}{u} = 0,$$

where

$$A = v + \theta_0 - y(w - \frac{\theta_1}{2}) \tag{5.30}$$

By eliminating u and v from the above equations and writing θ_0, θ_1 and θ_∞ interms of α, β, γ and δ , one can obtain the following transformation for the solution of P.V,

$$y' = \frac{\left[H - \frac{B}{2(y-1)^2} + \frac{1}{y}(\frac{B}{2(y-1)^{1/2}} + C_1 + (-2\beta)^{1/2})\right](H + C_1 - 1)}{H(H + (2\alpha)^{1/2})} \tag{5.31}$$

with

$$\alpha' = \alpha, \quad \beta' = -\frac{1}{2}\left[(-2\beta)^{1/2} + 1\right]^2, \quad \gamma' = \gamma + 1, \quad \delta' = \delta = -1/2. \tag{5.32}$$

where

$$A = v + \theta_0 - y(w - \tfrac{\theta_1}{2}),$$

$$B = ty - ty_t - (2\alpha)^{1/2}(y - 1)^2 - (\gamma - 1)(y - 1),$$

$$C_1 = \frac{(2\alpha)^{1/2} - (-2\beta)^{1/2} + 1 - \gamma}{2}, \tag{5.33}$$

$$G = \left[v + C_1 + (-2\beta)^{1/2} - y(v + (2\alpha)^{1/2})\right],$$

$$H = \frac{G\left[v + C_1 + (-2\beta)^{1/2} + vy^2 - y(2v + C_1 + (-2\beta)^{1/2} + t)\right]}{ty^2}.$$

Using (5.31), one can obtain the new solution y' corresponding to the parameters α', β', γ' and δ' from the solution y corresponding to the parameters α, β, γ and δ of PV. For example, by using the transformation (5.31) and the solution (5.27) of PV, the new solution,

$$y' = \frac{t^2 + 2t + 2}{t + 2} \qquad for \qquad \alpha' = 1/2, \quad \beta' = -2, \quad \gamma' = 2, \quad \delta' = -1/2, \quad (5.34)$$

can be obtained.

The Schlesinger transformation associated with the linear equation (5.2) and given by the transformation matrix $R_{(9)}(z)$, transforms the quantities $y, u, v, \theta_0, \theta_1$ and θ_∞ as follows:

$$\theta_0' = \theta_0 - 1, \qquad \theta_1' = \theta_1 - 1, \qquad \theta_\infty' = \theta_\infty,$$

$$uyv' - u'(v' + \theta_0 - 1) = 0,$$

$$u(v' + \frac{\theta_0 + \theta_1 + \theta_\infty}{2} - 1) = u'y'(v' + \frac{\theta_0 - \theta_1 + \theta_\infty}{2}), \quad (5.35)$$

$$\frac{2uy(v' + \frac{\theta_0 + \theta_1 + \theta_\infty}{2} - 1)}{u'y'} - 2(v' + \frac{\theta_0 - \theta_1 + \theta_\infty}{2} + t)$$
$$- \frac{2u(y-1)v'}{u'} + 2v(y-1) - \frac{2(y-1)(v + \frac{\theta_0 + \theta_1 + \theta_\infty}{2})}{y} = 0.$$

From the equations (5.35), the following transformation for the solution of PV can be obtained.

$$y' = \frac{(I + C_1 + (-2\beta)^{1/2} - 1)(I + C_1 - 1)}{yI(I + (2\alpha)^{1/2})} \quad (5.36)$$

with

$$\alpha' = \alpha, \quad \beta' = \beta, \quad \gamma' = \gamma + 2, \quad \delta' = \delta = -1/2. \quad (5.37)$$

where

$$I = \frac{y}{2(y-1)^2}\left\{2t + \frac{2(y-1)}{y}\left[v + (2\alpha)^{1/2} - \gamma\right] - 2(y-1)(v + (2\alpha)^{1/2})\right\}, \quad (5.38)$$

and C_1 is the same as in (5.33). The transformation (5.36) can be used to costruct a new solution y' corresponding to the parameters α', β', γ' and

δ' from the known solution y corresponding to the parameters α, β, γ and δ of P.V. For example, $y = -1$ for $\alpha = 0, \beta = 0, \gamma = 0$, and $\delta = -1/2$ solves PV. By using the transformation (5.36) the new solution of PV,

$$y' = \frac{t^2 + 4t + 4}{-t^2 + 4t - 4} \quad for \quad \alpha' = 0, \beta' = 0, \gamma' = 2, \delta' = -1/2. \qquad (5.39)$$

can be generated.

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