PHOTODETECTION STATISTICS OF SELF PHASE MODULATED FIELDS

A THESIS
SUBMITTED TO THE DEPARTMENT OF ELECTRICAL ELECTRONICS
ENGINEERING
AND THE INSTITUTE OF ENGINEERING AND SCIENCES
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
Mustafa Çelik
May, 1991

TK
6553
.C45
1991

PHOTODETECTION STATISTICS OF SELF PHASE MODULATED FIELDS

A THESIS
SUBMITTED TO THE DEPARTMENT OF ELECTRICAL & ELECTRONICS ENGINEERING
AND THE INSTITUTE OF ENGINEERING AND SCIENCES
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
Mustafa Çelik
May 1991

TK
6553
.C45
1991
B_ 8914

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assist. Prof. Gürhan Şaplakoğlu (Principal Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Ayhan Altıntaş

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Erdal Arıkan

Approved for the Institute of Engineering and Sciences:

Prof. Dr. Mehmet Baray
Director of Institute of Engineering and Sciences

ABSTRACT

PHOTODETECTION STATISTICS OF SELF PHASE MODULATED FIELDS

Mustafa Çelik
M.S. in Electrical & Electronics Engineering
Supervisor: Assist. Prof. Gürhan Şaplakoğlu
May 1991

In this thesis, photodetection statistics of self phase modulated fields are examined. First and second order homodyne detection statistics are calculated and it is observed that self phase modulated multimode fields can exhibit more squeezing than single mode fields. A method is derived whereby heterodyne detection statistics of multimode self phase modulated fields can be calculated for any given modal expansion set.

Keywords : Self phase modulated field, photodetection statistics, homodyne and heterodyne detection, squeezed light.

ÖZET

KENDİLİĞİNDEN EVRE MODÜLASYONUNA UĞRAMIŞ ALANLARIN FOTO ALGILAMA İSTATİSTİKLERİ

Mustafa Çelik
Elektrik ve Elektronik Mühendisliği Bölümü Yüksek Lisans
Tez Yöneticisi: Yrd. Doç. Dr. Gürhan Şaplakoğlu
Mayıs 1991

Bu tezde kendiliğinden evre modülasyonuna uğramış alanların foto algılama istatistikleri incelenmiştir. Birinci ve ikinci dereceden homodin algılama istatistikleri hesaplanmış ve kendiliğinden evre modülasyonuna uğramış çok modlu alanların tek modlu alanlardan daha fazla sıkışma özelliğine sahip oldukları gözlenmiştir. Verilen her mod açılımına göre, kendiliğinden evre modülasyonuna uğramış çok modlu alanların heterodin istatistiklerinin hesaplanabileceği bir yöntem geliştirilmiştir.

Anahtar kelimeler : Kendiliğinden evre modülasyonuna uğramış alan, foto algılama istatistikleri, homodin ve heterodin algılama, sıkıştırılmış ışık.

ACKNOWLEDGMENT

I would like to thank my supervisor Assistant Professor Gürhan Şaplakoğlu for his guidance and encouragement during the development of this thesis.

I am also indebted to Associate Professor Ayhan Altıntaş for his help and advices. My thanks are also due to NATO-AGARD for their partial support.

Contents

List of Figures

Chapter 1

Introduction

It is well known that an electromagnetic field is self phase modulated if it passes through a nonlinear medium that exhibits Kerr effect. The optical Kerr effect can be characterized by an intensity dependent refractive index $n(I)$, which can be expressed as [1]

$$n(I) = n + n_2 I, \tag{1.1}$$

where $n_2 = 12\pi\chi^{(3)}/n$ is the nonlinear index coefficient, $\chi^{(3)}$ is the third order nonlinear susceptibility, and I is the intensity of the light. Since the variations in the refractive index affect only the phase of the light, the intensity remains constant during the propagation. The complex envelope of a self phase modulated field can be expressed as

$$E_{out} = E_{in} \exp(j\kappa \mid E_{in} \mid^2) \tag{1.2}$$

where E_{in} is the complex envelope of the field incident to the Kerr medium, $\kappa = 2\pi L n_2/\lambda$ is the nonlinear coupling constant and L is the length of the medium. As it is seen from (1.2), the phase of the field is modulated by its own intensity as it travels through the nonlinear medium, hence the name self phase modulation. Self phase modulation and its effects, such as spectral broadening, frequency chirping, pulse compressing and pulse broadening, etc., have been the subject of many experimental and theoretical research efforts in recent years [2] [3].

In this thesis, we will mainly deal with the photodetection statistics of self phase modulated fields. Since the fundamental noise in photodetection systems is quantum noise, the quantum mechanical formulation of light wave propagation and photodetection is needed. The quantum optical aspects of self phase modulation have also been investigated widely. It has been shown that the homodyne detection statistics of single mode self phase modulated fields are squeezed [4]. Another important fact of self phase modulation is that, it makes the quantum nondemolition measurements possible by using a nonlinear Mach-Zehnder interferometer with a Kerr medium in one arm [5]. In [6], the probability density function governing the heterodyne detection statistics of self phase modulated fields have been derived. But all these work deal with single mode fields.

In this work, we first derive the known photodetection results of single mode fields by a method different than the one used in [6]. We then develop some tools useful in manipulating multidimensional operators. Finally we calculate the quantum detection statistics of multimode self phase modulated fields. Our main contribution to this subject was in the case of multimode fields. To the best of our knowledge none of the multimode results have appeared in the literature. Although the single mode results were published previously our derivation of those results are original.

The organization of this thesis is as follows. In chapter 2 we briefly introduce the quantum field formulation and then present the single mode results. The mean and the variance of the first quadrature homodyne measurement of a self phase modulated field are calculated. The number state expansion for the quantum state of a self phase modulated field [from which we derive the probability density function of the heterodyne detection statistics] is developed. In chapter 3, we generalize the results to the self phase modulated multimode fields. The first and second order moments of homodyne detection statistics are calculated and the squeezing properties of the single and multimode fields are discussed. In chapter 3 we also determine the probability density function of the heterodyne statistics of a multimode self phase modulated field from its antinormally ordered characteristic function. Finally we conclude in chapter 4 with a summary and a brief discussion of our main results.

Chapter 2

Self Phase Modulation Of Single Mode Fields

In this chapter, we review the quantum formulation of electromagnetic fields and their measurements, then calculate the quantum detection statistics of self phase modulated single mode fields.

2.1 Quantum mechanical formulation of optical fields

In quantum mechanics, the electromagnetic fields are represented as operators defined over a Hilbert space. The Hilbert space operators that the quantum theory deal with obey the same algebra as infinite dimensional square matrices. Suppressing the polarization and spatial dependencies, a single mode quantized electric field can be written as

$$\hat{E}(z,t) = j[\frac{h\nu_0}{2\epsilon V}]^{1/2}(\hat{a}e^{j(kz-w_0 t)} - \hat{a}^{\dagger}e^{-j(kz-w_0 t)}), \tag{2.1}$$

where caret signs indicate Hilbert space operators and " $\dagger$ " indicates transposition plus complex conjugation. Furthermore ϵ is the dielectric constant of the medium, V is the quantization volume, ν_0 is the radiation frequency, h is the Planck's constant. The annihilation and creation operators $\hat{a}$ and $\hat{a}^{\dagger}$ obey the canonic commutator relation

$$[\hat{a}, \hat{a}^{\dagger}] \equiv \hat{a}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a} = 1. \tag{2.2}$$

We also introduce the self adjoint operators $\hat{a}_1$, $\hat{a}_2$ and $\hat{n}$, such that,

$$\hat{a}_1 \equiv Re\{\hat{a}\} = \frac{\hat{a}+\hat{a}^{\dagger}}{2}, \tag{2.3}$$

$$\hat{a}_2 \equiv Im\{\hat{a}\} = \frac{\hat{a}-\hat{a}^{\dagger}}{2j}, \tag{2.4}$$

$$\hat{n} \equiv \hat{a}^{\dagger}\hat{a}, \tag{2.5}$$

where (2.3) and (2.4) represent the quadrature components, and (2.5) represents the number of photons in the quantization volume of the field. In Dirac notation, the set of vectors that constitute the Hilbert space are referred to as "kets" and are denoted by the symbol "$| \cdot >$". Kets are invariably the eigenvectors of some operator and are labeled by their eigenvalues, for example, the coherent state ket $| \alpha >$ satisfies the eigenvalue equation

$$\hat{a} \mid \alpha >= \alpha \mid \alpha > . \tag{2.6}$$

The conjugate of a ket is a "bra" and is denoted by the symbol "$< \cdot |$". By convention the norm of all kets are unity, i.e.,

$$(| \cdot >)^{\dagger} \mid \cdot > \equiv < \cdot \mid \cdot >= 1. \tag{2.7}$$

Note that the norm is represented by a pair of brackets, this is the reason for adopting the names "bra" and "ket".

2.2 Quantum measurements

Let $\hat{M}$ be an operator defined over a Hilbert Space representing a physical quantity that is going to be measured. If $\hat{M}$ has a continuous set of eigenkets $\{| m >\}$, then a resolution of the identity in terms of these eigenkets can be written [7],

$$\hat{I} = \int_{-\infty}^{\infty} c \mid m >< m \mid dm, \tag{2.8}$$

where c is a positive constant. The set of operators $\{c \mid m >< m \mid\}$ constitute a probability operator measure (POM). The actual measurement of the physical quantity represented by the operator $\widehat{M}$ will yield one of the eigenvalues of $\widehat{M}$ governed by the probability density function

$$p_M(m) = Tr[\hat{\rho} c \mid m >< m \mid], \tag{2.9}$$

where $\hat{\rho}$ is the density operator of the electric field[1] just before the measurement and $Tr[\cdot]$ represents the Hilbert Space trace.

For example, in heterodyne detection the receiver measures the operator

$$\widehat{M}_{HET} = \hat{a} \tag{2.10}$$

which implies the POM $\{\frac{1}{\pi} \mid \alpha >< \alpha \mid\}$. Note that, since the real and imaginary parts of (2.10) do not commute, it is a non observable representation of the measurement performed by the heterodyne receiver. To make it observable we replace (2.10) by

$$\widehat{M}_{HET}^{(OBS)} = \hat{a} + \hat{a}_{im}^{\dagger}. \tag{2.11}$$

[1]Mathematically, the equations governing the electromagnetic field are identical to the equations of a simple LC circuit. Hence the state of an electromagnetic field is analogous to specifiying the voltage of the capacitor and current of the inductor. In quantum mechanics the state of a quantum system is represented by the density operator $\hat{\rho}$.

where $\hat{a}_{im}$ is a vacuum state field representing the noise contribution from the image band. The POM implied by (2.11) is still $\{\frac{1}{\pi} \mid \alpha >< \alpha \mid\}$ [8]. Hence the probability density function governing the heterodye detection statistics of a coherent state field becomes

$$P_Y(\alpha) = \frac{1}{\pi} e^{-|\alpha - \alpha_0|^2} \tag{2.12}$$

where we have assumed that $\hat{\rho} = \mid \alpha_0 >< \alpha_0 \mid$.

A homodyne detector measures the operator

$$\hat{M}_{HOM} = Re\{e^{-j\Phi}\hat{a}\} \tag{2.13}$$

where Φ is the phase of the mean local oscillator field. $\Phi = 0$ corresponds to the measurement of the first quadrature, $\hat{a}_1$, which implies the POM $\{\mid \alpha_1 >< \alpha_1 \mid\}$ Similarly $\Phi = \frac{\pi}{2}$ corresponds to the measurement of the second quadrature, $\hat{a}_2$, which implies the POM $\{\mid \alpha_2 >< \alpha_2 \mid\}$ where $\mid \alpha_1 >$ and $\mid \alpha_2 >$ are the eigenkets of $\hat{a}_1$ and $\hat{a}_2$ respectively.

If $\widehat{M}$ is an operator with a discrete set of eigenkets $\{\mid m >\}$, the resolution of the identity in terms of these eigenkets can be written as

$$\hat{I} = \sum c \mid m >< m \mid . \tag{2.14}$$

In this case the measurement of $\widehat{M}$ yields a random variable m, whose probability mass function is

$$Pr[M = m] = Tr[\hat{\rho} c \mid m >< m \mid]. \tag{2.15}$$

For example a direct detection receiver measures the photon number operator

$$\widehat{M}_{DD} = \widehat{N} \tag{2.16}$$

which implies the POM $\{\mid n >< n \mid\}$. Therefore the probability of detecting m photons when the field is in coherent state is given by

$$Pr[M = m] = \frac{e^{-|\alpha_0|^2} \mid \alpha_0 \mid^{2m}}{m!}. \tag{2.17}$$

Using the POM formulation, expressions for the first and second order moments of a measurement can easily be derived. The expected value of the measurement of an operator $\widehat{M}$ is given by

$$< \widehat{M} > \equiv Tr[\hat{\rho}\widehat{M}]. \tag{2.18}$$

If $\hat{\rho} = \mid \alpha >< \alpha \mid$, the expected value becomes

$$< \widehat{M} > = < \alpha \mid \widehat{M} \mid \alpha > . \tag{2.19}$$

The variance of the measurement of an operator $\widehat{M}$ can be written as

$$< \Delta\widehat{M}^2 > \equiv < \widehat{M}^2 > - < \widehat{M} >^2 . \tag{2.20}$$

For example, in the direct detection case, i.e., measurement of the photon number operator, we have

$$< \widehat{N} > = < \Delta\widehat{N}^2 > = | \alpha |^2 . \tag{2.21}$$

Similarly the first and second order moments of the quadrature operator measuremements (homodyne detection) can be calculated as

$$< \hat{a}_1 > = Re\{\alpha\} \tag{2.22}$$

$$< \hat{a}_2 > = Im\{\alpha\} \tag{2.23}$$

$$< \Delta\hat{a}_1^2 > = < \Delta\hat{a}_2^2 > = \frac{1}{4} \tag{2.24}$$

and finally for the heterodyne detection we have

$$< \hat{a} > = \alpha \tag{2.25}$$

$$< \Delta\hat{a}^2 > = \frac{1}{2}. \tag{2.26}$$

The number $\frac{1}{4}$ in (2.24) is referred as the coherent state noise level and it is a fundamental limit in quantum photodetection. For some non-classical states of light, known as squeezed states [10] [11], the variance in one quadrature is below the coherent state level.

Similar to the classical probability theory, the characteristic functions are also used commonly in quantum measurement theory. The antinormally ordered characteristic function of a density operator $\hat{\rho}$ is defined by,

$$X_A^{\hat{\rho}}(\zeta^*, \zeta) \equiv tr[\hat{\rho} e^{-\zeta^* \hat{a}} e^{\zeta \hat{a}^\dagger}] \tag{2.27}$$

where $\hat{\rho}$ is the density operator of an electromagnetic field represented by the annihilation operator $\hat{a}$ and $\zeta = \zeta_1 + j\zeta_2$ is a complex number. It can be shown that, (2.27) is the Fourier transform of $\rho^{(n)}(\alpha^*, \alpha)$, the normally ordered form of $\hat{\rho}$,

$$X_A^{\hat{\rho}}(\zeta^*, \zeta) = \int_{-\infty}^{\infty} \rho^{(n)}(\alpha^*, \alpha) e^{\zeta\alpha^* - \zeta^*\alpha} \frac{d^2\alpha}{\pi} \tag{2.28}$$

where the integral is taken over the complex α plane i.e.,

$$d^2\alpha \equiv d\alpha_1 d\alpha_2. \tag{2.29}$$

The normally ordered form of the density operator $\hat{\rho}$, is defined as

$$\rho^{(n)}(\alpha^*, \alpha) \equiv < \alpha \mid \hat{\rho} \mid \alpha >, \tag{2.30}$$

and its scaled version is the probability density function describing the heterodyne detection statistics of a single mode field

$$P_Y(\alpha) = \frac{1}{\pi}\rho^{(n)}(\alpha^*, \alpha). \tag{2.31}$$

The inverse transform of (2.28) can be readily written as

$$\rho^{(n)}(\alpha^*, \alpha) = \int_{-\infty}^{\infty} X_A^{\hat{\rho}}(\zeta^*, \zeta) e^{-\zeta\alpha^* + \zeta^*\alpha} \frac{d^2\zeta}{\pi}. \tag{2.32}$$

2.3 Self phase modulation of single mode fields

In [6], it is shown that the field operator at the output of the Kerr medium , which is characterized by (1.1), can be expressed as

$$\hat{a}_{out} = e^{j\kappa\hat{a}^\dagger\hat{a}}\hat{a}, \tag{2.33}$$

where $\hat{a}$ is the annihilation operator associated with the single mode input field. The input field is assumed to be in the coherent state $\mid \alpha_0 >< \alpha_0 \mid$. The nonlinear coupling constant κ is shown to be [6]

$$\kappa = \frac{\hbar w_0^2 n_2 L}{c\epsilon_0 n^2 V}, \tag{2.34}$$

where L is the length of the Kerr medium. One can easily show that the annihilation and the creation operators associated with the output field still obeys the canonic commutator relation

$$[\hat{a}_{out}, \hat{a}_{out}^\dagger] = 1 \tag{2.35}$$

as expected since the output field should have a correct quantum mechanical field representation. Furthermore the photon number operators of the input and output fields are identical, that is,

$$\hat{a}_{out}^\dagger \hat{a}_{out} = \hat{a}^\dagger \hat{a}, \tag{2.36}$$

which is again expected since there is no power gain or loss in the Kerr medium. We now introduce the operator $\hat{a}_{out}^{(1)}$ which represents the in-phase quadrature of the output mode,

$$\hat{a}_{out}^{(1)} \equiv Re[\hat{a}_{out}] = \frac{\hat{a}_{out} + \hat{a}_{out}^\dagger}{2}. \tag{2.37}$$

Using the operator theorem [9]

$$< \alpha \mid e^{x\hat{a}^\dagger\hat{a}} \mid \alpha >= exp[(e^x - 1) \mid \alpha \mid^2], \tag{2.38}$$

it can be shown that,

$$< \hat{a}^{(1)}_{out} >= e^{|\alpha_0|^2(\cos\kappa-1)} Re\{\alpha_0 e^{j|\alpha_0|^2 \sin\kappa}\}. \tag{2.39}$$

Similarly using (2.38), (2.39) and the operator theorem [9]

$$\hat{a} f(\hat{a}^\dagger \hat{a}) = f(\hat{a}^\dagger \hat{a} + 1)\hat{a}, \tag{2.40}$$

the variance of the measurement of the operator $\hat{a}^{(1)}_{out}$ can be found as

$$\begin{aligned} < \Delta\hat{a}^{(1)2}_{out} > &= \frac{1}{4} + \frac{1}{2} \mid \alpha_0 \mid^2 + \frac{1}{4} e^{|\alpha_0|^2(\cos 2\kappa - 1)} Re\{\alpha_0^2 e^{j\kappa} e^{j|\alpha_0|^2 \sin 2\kappa}\} \\ & - e^{2|\alpha_0|^2(\cos\kappa-1)} [Re\{\alpha_0 e^{j|\alpha_0|^2 \sin\kappa}\}]^2. \end{aligned} \tag{2.41}$$

The variance is calculated numerically and plotted in Figure 2.1 for various values of α_0, as a function of κ. As it is seen from Figure 2.1, for some values of κ, variances are below the coherent state level, that is, the self phase modulated single mode field exhibits squeezing.

Next we will derive the probability density function of the measurement resulting from the heterodyne detection of self phase modulated single mode fields. We define a new continous set of eigenstates $\{\mid \alpha, \kappa >\}$, for the output mode operator $\hat{a}_{out}$,

$$e^{j\kappa \hat{a}^\dagger \hat{a}} \hat{a} \mid \alpha, \kappa >= \alpha \mid \alpha, \kappa > . \tag{2.42}$$

The solution of this eigenvalue problem will give us the probability density function of the experimental outcome α via (2.9). Since the number states $\{\mid n >\}$ constitute a complete orthonormal set, the eigenstate $\mid \alpha, \kappa >$ can be expanded in terms of number states as follows:

$$\hat{I} = \sum_{n=0}^{\infty} \mid n >< n \mid, \tag{2.43}$$

$$\hat{I} \mid \alpha, \kappa >=\mid \alpha, \kappa > = \sum_{n=0}^{\infty} \mid n >< n \mid \alpha, \kappa >= \sum_{n=0}^{\infty} c_n \mid n >, \tag{2.44}$$

where $c_n \equiv< n \mid \alpha, \kappa >$ is the expansion coefficient. If we substitute (2.44) in (2.42), we obtain

$$\sum_{n=0}^{\infty} c_n e^{j\kappa \hat{a}^\dagger \hat{a}} \hat{a} \mid n >= \alpha \sum_{n=0}^{\infty} c_n \mid n > . \tag{2.45}$$

Using the relations [9], $\hat{a} \mid n >= \sqrt{n} \mid n-1 >$ and $e^{x\hat{a}^\dagger \hat{a}} \mid n >= e^{xn} \mid n >$, we have

$$\sum_{n=1}^{\infty} c_n e^{j\kappa(n-1)} \sqrt{n} \mid n-1 >= \alpha \sum_{n=0}^{\infty} c_n \mid n > . \tag{2.46}$$

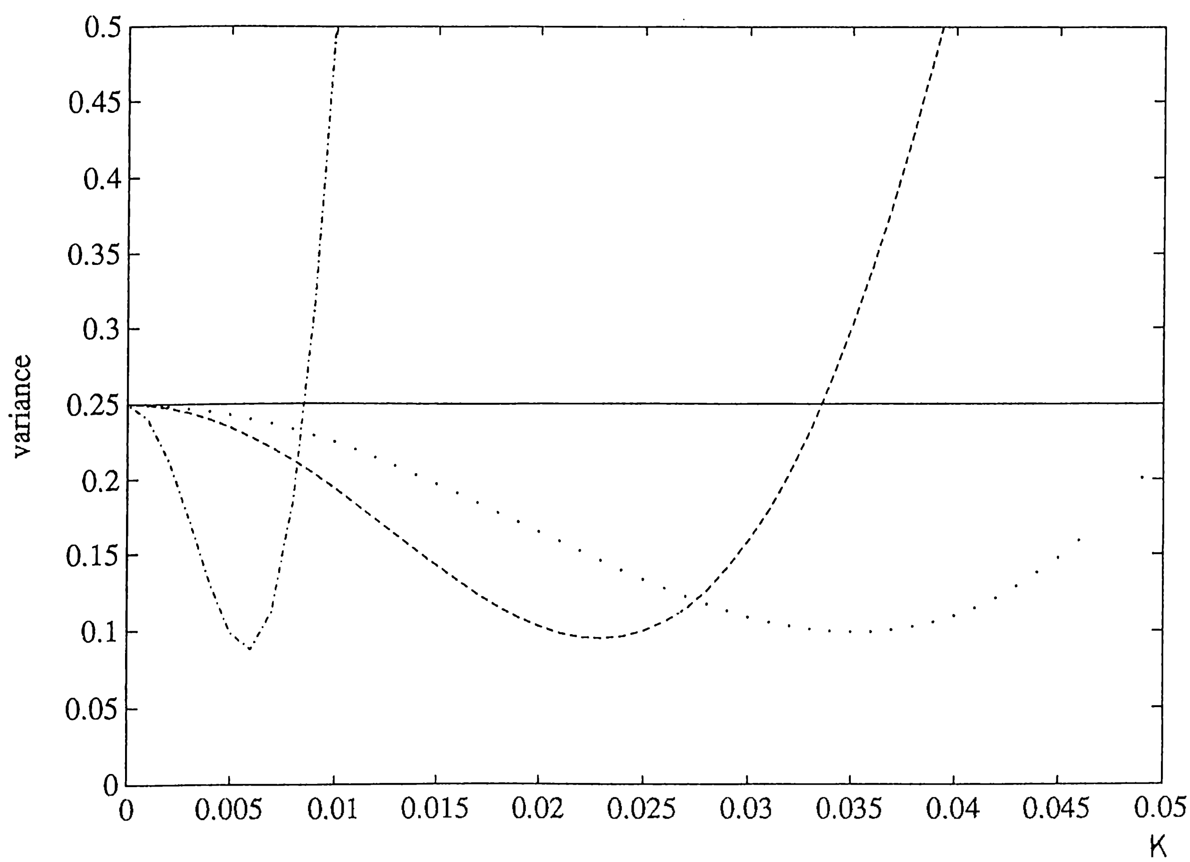

Figure 2.1: Variance of the first quadrature measurement of a single mode self phase modulated field versus κ for $\alpha_0 = 4$ (dotted line), $\alpha_0 = 5$ (dashed line) and $\alpha_0 = 10$ (dotted-dashed line). Solid line indicates the coherent state noise level.

Shifting the indices in the left hand side of (2.46) yields,

$$\sum_{n=0}^{\infty} c_{n+1} e^{j\kappa n}\sqrt{n+1} \mid n >= \alpha \sum_{n=0}^{\infty} c_n \mid n > . \tag{2.47}$$

If we multiply both sides of eqn (2.47) from the left by $< m \mid$ we obtain

$$c_{n+1} = \frac{\alpha e^{-j\kappa n}}{\sqrt{n+1}} c_n, \tag{2.48}$$

which follows from the fact that $< m \mid n >= \delta_{mn}$. Consequently c_n can be written in terms of c_0 recursively as

$$c_n = c_0 \frac{e^{\frac{-j\kappa n(n-1)}{2}} \alpha^n}{\sqrt{n!}}. \tag{2.49}$$

The norm of $\mid \alpha, \kappa >$ is given by

$$\begin{aligned} < \alpha, \kappa \mid \alpha, \kappa > &= \mid c_0 \mid^2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{e^{\frac{-j\kappa[n(n-1)-m(m-1)]}{2}} \alpha^n \alpha^{*m}}{\sqrt{n!m!}} < m \mid n >, & (2.50) \\ &= \mid c_0 \mid^2 \sum_{n} \frac{\mid \alpha \mid^{2n}}{n!}, & (2.51) \\ &= \mid c_0 \mid^2 e^{|\alpha|^2}. & (2.52) \end{aligned}$$

The coefficient c_0 is choosen so that the norm of $\mid \alpha, \kappa >$ is normalized to unity. Since the phase of c_0 is arbitrary, it is choosen as $e^{\frac{-|\alpha|^2}{2}}$. Thus the eigenket associated with the output state can be written as

$$\mid \alpha, \kappa >= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{e^{-\frac{j\kappa n(n-1)}{2}} \alpha^n}{\sqrt{n!}} \mid n > . \tag{2.53}$$

Note that, in the absence of nonlinearity, (2.53) is equivalent to the well known expansion of the coherent state ket $\mid \alpha >$,

$$\mid \alpha, 0 >=\mid \alpha >= e^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \mid n > . \tag{2.54}$$

The completeness relation of the output state can be verified as follows. Let

$$\hat{B} = \int_{-\infty}^{\infty} c \mid \alpha, \kappa >< \alpha, \kappa \mid d^2\alpha, \tag{2.55}$$

where c is an arbitrary constant. The use of (2.53) and its adjoint in (2.55) gives

$$\hat{B} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{c \mid n >< m \mid e^{\frac{j\kappa}{2}[m(m-1)-n(n-1)]}}{\sqrt{n!m!}} \int_{-\infty}^{\infty} e^{-|\alpha|^2} \alpha^{*m} \alpha^n d^2\alpha. \tag{2.56}$$

The above integral can be easily evaluated in polar coordinates,

$$\int_{-\infty}^{\infty} e^{-|\alpha|^2}\alpha^{*m}\alpha^n d^2\alpha = \int_0^{\infty} e^{-r^2} r^{m+n+1} dr \int_0^{2\pi} e^{j(n-m)\theta} d\theta. \quad (2.57)$$

Since

$$\int_0^{2\pi} e^{j(n-m)\theta} d\theta = 2\pi\delta_{nm} \quad (2.58)$$

and

$$\int_0^{\infty} e^{-r^2} r^{2n+1} dr = \frac{n!}{2} \quad (2.59)$$

we have

$$\hat{B} = c\pi \sum_{n=0}^{\infty} | n >< n | = c\pi \hat{I}, \quad (2.60)$$

where we have used the completeness relation of the number states. So, if we let $c = \frac{1}{\pi}$ we obtain

$$\hat{I} = \int_{-\infty}^{\infty} \frac{1}{\pi} | \alpha, \kappa >< \alpha, \kappa | d^2\alpha. \quad (2.61)$$

We next obtain the probability density function associated with the heterodyne detection of the output field. The POM implied by the output operator is $\{\frac{1}{\pi} | \alpha, \kappa >< \alpha, \kappa |\}$. Consequently the heterodyne measurement will yield the following probability density function

$$P_Y(\alpha) = Tr[\frac{1}{\pi} | \alpha_0 >< \alpha_0 | \alpha, \kappa >< \alpha, \kappa |], \quad (2.62)$$

$$= \frac{1}{\pi} |< \alpha_0 | \alpha, \kappa >|^2 . \quad (2.63)$$

Replacing the states $| \alpha_0 >$ and $| \alpha, \kappa >$, with their number state expansions in (2.63), we obtain

$$P_Y(\alpha) = \frac{1}{\pi} e^{-|\alpha|^2 - |\alpha_0|^2} | \sum_{n=0}^{\infty} \frac{e^{-\frac{j\kappa n(n-1)}{2}} \alpha^n \alpha_0^{*n}}{n!} |^2 . \quad (2.64)$$

The probability density functions are calculated numerically for $\alpha_0 = 5$ and shown in Figure 2.2 for several values of κ. The contour plots of the same distributions are given in Figure 2.3. Initially, since the field is in coherent state, the density is Gaussian centered at $\alpha = \alpha_0$. As κ increases, the density function is deformed to a crescent shape and rotates on the $| \alpha |= 5$ circle. It can be also observed that, for small values of κ the curve becomes narrower in the direction of the first quadrature, but afterwards its width enlarges rapidly. We had observed the same trend before in Figure 2.1.

So far we have examined the detection statistics of the self phase modulated fields using quantum mechanical tools. At this point one can wonder weather the same results could have been obtained by purely classical means. Now we will look at the

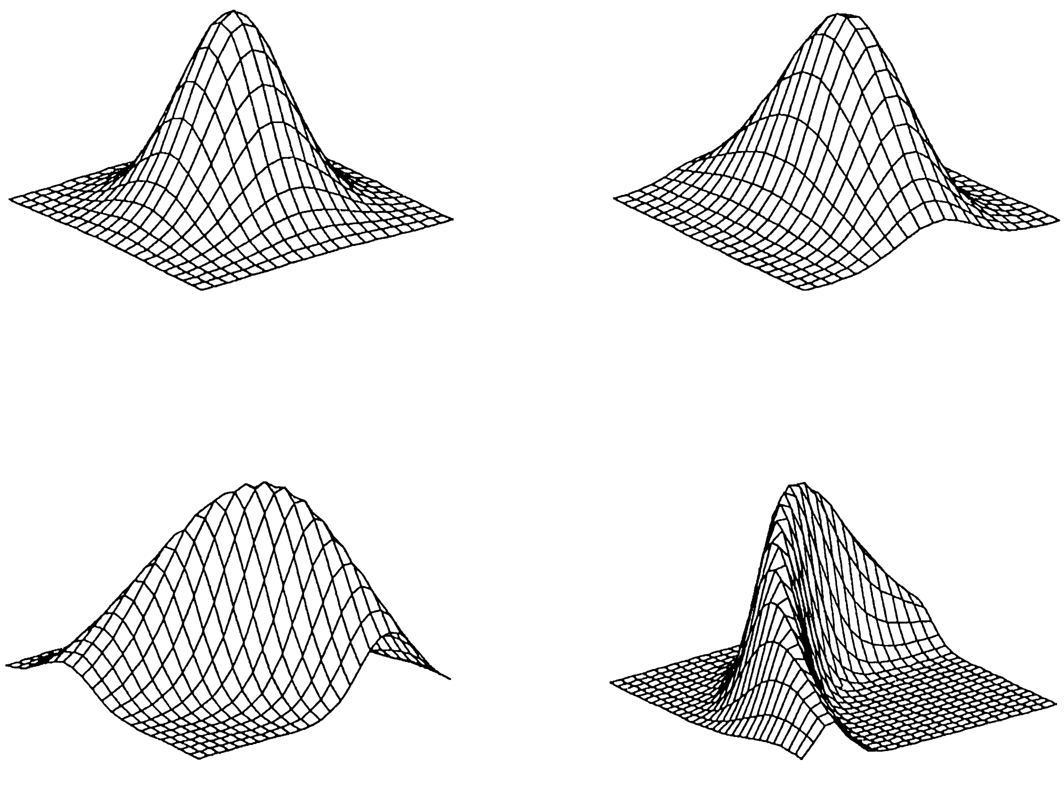

Figure 2.2: Probability density function of the single mode self phase modulated field for $\kappa = 0$ (a), $\kappa = 0.025$ (b), $\kappa = 0.05$ (c) and $\kappa = 0.1$ (d).

same problem from classical point of view. Assuming that the field incident to the Kerr medium is a classical coherent field, the real and imaginary parts of its envelope are jointly independent Gaussian random variables, that is,

$$P_{X_1,X_2}(x_1,x_2) = \frac{1}{\pi} e^{-(x_1 - m\cos\theta_m)^2 - (x_2 - m\sin\theta_m)^2} \tag{2.65}$$

where $E_{in} = X_1 + jX_2$. Using classical probability theory methods [14], the joint probability density function of the output field can be found as,

$$P_{Y_1,Y_2}(y_1,y_2) = \frac{1}{\pi} e^{-(y_1^2+y_2^2+m^2)+2m\sqrt{y_1^2+y_2^2}\cos(\arctan(\frac{y_2}{y_1})-\theta_m-\kappa(y_1^2+y_2^2))}. \tag{2.66}$$

In (2.66) the real random variables Y_1 and Y_2 are the first and second quadratures of the output field,

$$E_{out} = Y_1 + jY_2 = (X_1 + jX_2)e^{j\kappa(x_1^2+x_2^2)}. \tag{2.67}$$

In order to make a comparison with quantum theory results, the contour plots of (2.66) are plotted in Figure 2.4 for $m = 5$ and $\theta_m = 0$. Although the general behavior of both results are similar, for large values of κ, classical approach does not work, as expected. The fundamental difference between two approaches is that, in quantum theory the intensity is a discrete random variable, however it is assumed to be the square of a continuous variable in classical analysis.

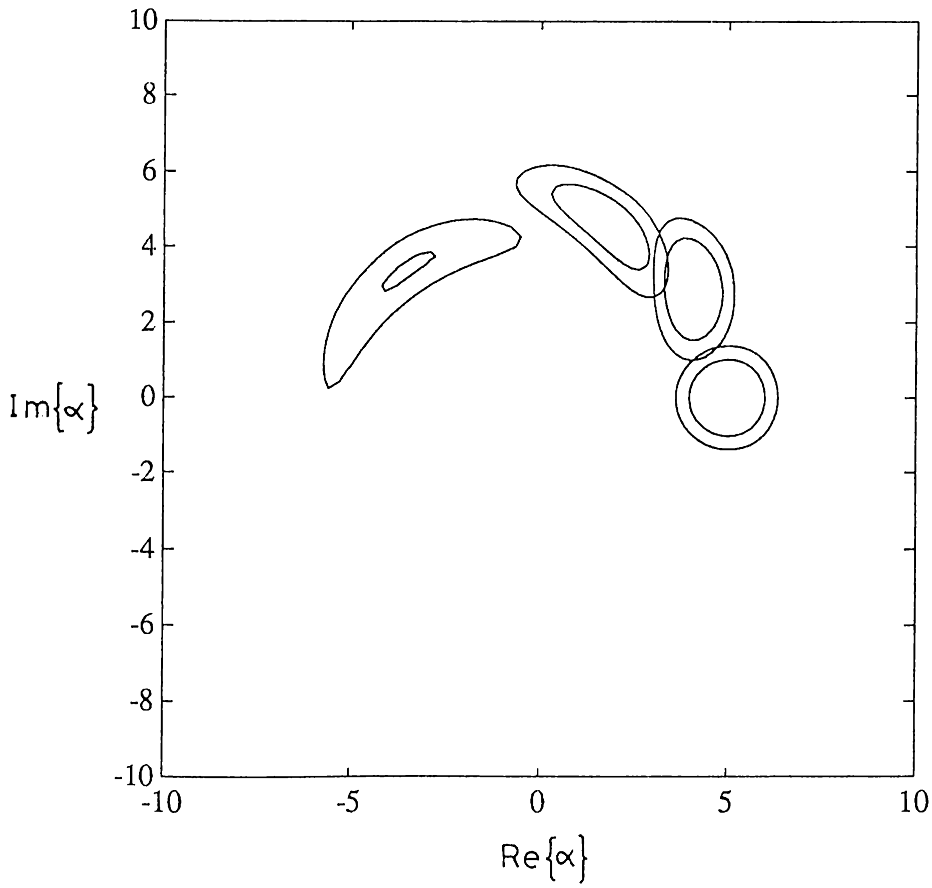

Figure 2.3: Contour plots of the probability density function of the single mode self phase modulated field for $\kappa = 0$ (a), $\kappa = 0.025$ (b), $\kappa = 0.05$ (c) and $\kappa = 0.1$ (d). Contours represent the 15% and 35% of the maximum value of $\kappa = 0$ (single mode coherent state) pdf.

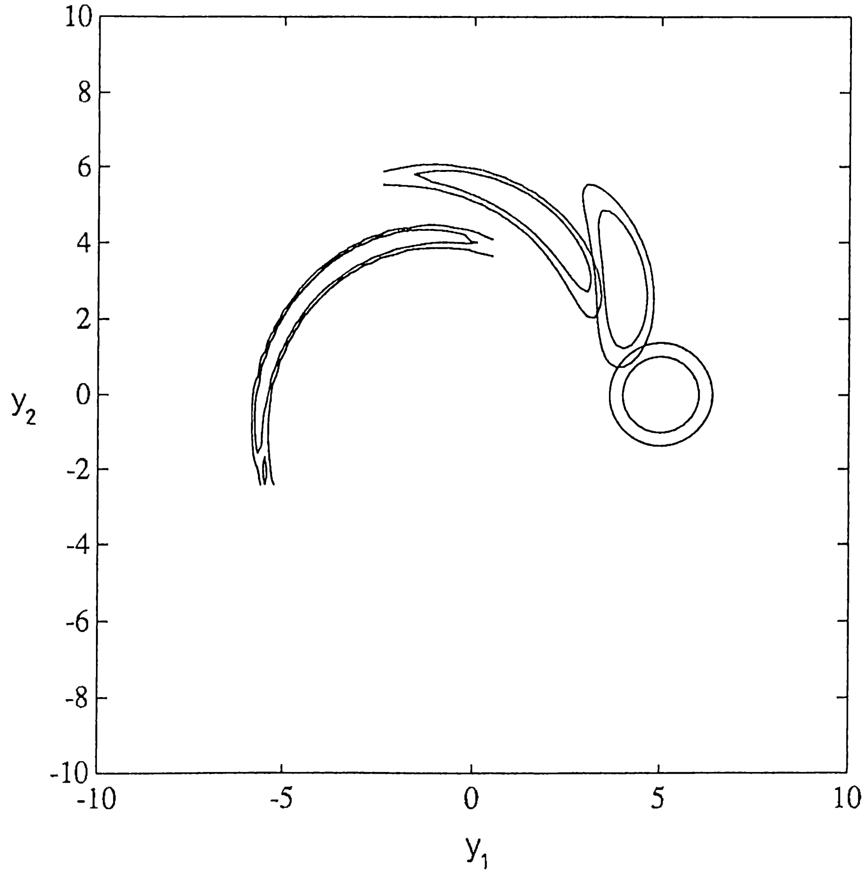

Figure 2.4: Contour plots of the probability density function (intuitive classical analysis) for $\kappa = 0$ (a), $\kappa = 0.025$ (b), $\kappa = 0.05$ (c) and $\kappa = 0.1$ (d). Contours represent the 15% and 35% of the maximum value of $\kappa = 0$ (single mode coherent state) pdf.

Chapter 3

Self Phase Modulation Of Multimode Fields

In this chapter we introduce the multimode quantum field formulation and then derive the quantum heterodyne detection statistics of a self phase modulated field.

3.1 Multimode quantum field

In quantum optics the field operator of a quantized multimode field can be written as [12]

$$\hat{E}(t) = \sum_n \hat{a}_n \Phi_n(t), \tag{3.1}$$

where the polarization and spatial dependencies have been suppressed. In (3.1), $\{\Phi_n(t)\}$ is an arbitrary, possibly complex valued, complete orthonormal set defined over the signaling interval $[0, T]$ and $\{\hat{a}_n\}$ is the associated set of modal annihilation operators obeying the canonical commutator relation

$$[\hat{a}_n, \hat{a}_m^\dagger] = \delta_{nm}, \tag{3.2}$$

where δ_{nm} is the Kronecker delta function. The completeness and orthonormality of the set $\{\Phi_n(t)\}$ and (3.2) yield the following commutator relation for the field operator given in (3.1),

$$[\hat{E}(t), \hat{E}^\dagger(t')] = \delta(t - t'). \tag{3.3}$$

Usually a finite number of temporal modes are sufficient in actual applications, hence in our derivations the modal expansion (3.1) is terminated at $n = N$. For this purpose we use the vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{a}}^\dagger$, to represent the N dimensional column vector of annihilation operators $\{\hat{a}_n : 1 \leq n \leq N\}$ and N dimensional row vector of creation operators $\{\hat{a}_n^\dagger : 1 \leq n \leq N\}$ respectively. Similarly we define, the vector $\mathbf{u}$ to denote the N dimensional column vector whose nth element is $\Phi_n^*(t)$ and the vector $\mathbf{u}^\dagger$ to denote the N dimensional row vector whose nth element is $\Phi_n(t)$, i.e.,

$$\mathbf{u} = \begin{bmatrix} \Phi_1^*(t) \\ \vdots \\ \Phi_N^*(t) \end{bmatrix}, \tag{3.4}$$

$$\mathbf{u}^\dagger = [\Phi_1(t) \cdots \Phi_N(t)] \tag{3.5}$$

3.2 Self phase modulation of multimode fields

We now assume that a quasimonochromatic field which is represented by the operator $\hat{E}_{in}(t)$ is incident to the Kerr medium. Then the operator representing the field emerging from the Kerr medium can be written as

$$\hat{E}_{out}(t) = e^{j\kappa \hat{E}_{in}^\dagger(t)\hat{E}_{in}(t)} \hat{E}_{in}(t), \tag{3.6}$$

where the coupling constant κ is the same with the one given for single mode case (2.34). In the vector notation that we have introduced above, the input and output field operators become

$$\begin{aligned} \hat{E}_{in}(t) &= \mathbf{u}^\dagger \hat{\mathbf{a}} && (3.7) \\ \hat{E}_{out}(t) &= e^{j\kappa \hat{\mathbf{a}}^\dagger \mathbf{A} \hat{\mathbf{a}}} \mathbf{u}^\dagger \hat{\mathbf{a}} && (3.8) \end{aligned}$$

where the $N \times N$ compex matrix $\mathbf{A}$ is defined as

$$\mathbf{A} \equiv \mathbf{u}\mathbf{u}^\dagger. \tag{3.9}$$

The matrix A has some useful properties that will be utilized in the sequel;

$$\begin{aligned} \mathbf{A}^n &= (\mathbf{u}^\dagger \mathbf{u})^{n-1} \mathbf{A} \qquad n \geq 1, && (3.10) \\ e^{x\mathbf{A}} &= \mathbf{I} + \frac{\mathbf{A}}{\mathbf{u}^\dagger \mathbf{u}} (e^{x\mathbf{u}^\dagger \mathbf{u}} - 1), && (3.11) \\ \mathbf{u}^\dagger e^{x\mathbf{A}} &= \mathbf{u}^\dagger e^{x\mathbf{u}^\dagger \mathbf{u}}, && (3.12) \end{aligned}$$

where (3.10) can be easily shown by induction, (3.11) follows directly from (3.10) and (3.12) can be obtained by multiplying (3.11) by $\mathbf{u}^\dagger$ from the left side.

From (3.7) and (3.8) we can immediately see that the photon number operators of the input and the output fields are equal to each other,

$$\begin{aligned} \hat{E}_{out}^\dagger(t)\hat{E}_{out}(t) &= \hat{\mathbf{a}}^\dagger \mathbf{u} e^{-j\kappa \hat{\mathbf{a}}^\dagger \mathbf{A} \hat{\mathbf{a}}} e^{j\kappa \hat{\mathbf{a}}^\dagger \mathbf{A} \hat{\mathbf{a}}} \mathbf{u}^\dagger \hat{\mathbf{a}} \\ &= \hat{\mathbf{a}}^\dagger \mathbf{A} \hat{\mathbf{a}} \\ &= \hat{E}_{in}^\dagger(t)\hat{E}_{in}(t). \end{aligned} \tag{3.13}$$

Using (3.12), (3.13) and the following operator theorem[1]

$$\hat{\mathbf{a}} f(\hat{\mathbf{a}}^\dagger \mathbf{A} \hat{\mathbf{a}}) = f(\hat{\mathbf{a}}^\dagger \mathbf{A} \hat{\mathbf{a}} \mathbf{I} + \mathbf{A})\hat{\mathbf{a}}, \tag{3.14}$$

we can show that

$$[\hat{E}_{out}(t), \hat{E}^\dagger_{out}(t)] = [\hat{E}_{in}(t), \hat{E}^\dagger_{in}(t)] = \mathbf{u}^\dagger \mathbf{u}. \tag{3.15}$$

Note that, since we are working with a finite modal expansion set the commutator relation for the multimode field operator is not a delta function, rather it is a function that depends on the choice of the modal expansion set.

3.3 First and second order homodyne statistics of self phase modulated multimode fields

In this section we will derive the first and second order homodyne statistics of self phase modulated multimode fields. We define the in phase quadrature operator of the output field as the real part of the output field operator,

$$\hat{E}^{(1)}_{out} \equiv Re\{\hat{E}_{out}\}. \tag{3.16}$$

We need the normally ordered representation of Gaussian type operators to calculate the mean and the variance of the quadrature operator measurement. The normally ordered representation of a multidimensional operator $\hat{M}(\hat{\mathbf{a}}, \hat{\mathbf{a}}^\dagger)$ is defined as [13]

$$< \boldsymbol{\alpha} \mid \hat{M}(\hat{\mathbf{a}}, \hat{\mathbf{a}}^\dagger) \mid \boldsymbol{\alpha} >= M(\boldsymbol{\alpha}, \boldsymbol{\alpha}^\dagger), \tag{3.17}$$

where $\mid \boldsymbol{\alpha} >$ is the multimode coherent state ket

$$\mid \boldsymbol{\alpha} >= \otimes_n \mid \alpha_n > . \tag{3.18}$$

In vector notation $\boldsymbol{\alpha}$ is a column vector whose nth element is α_n and $\boldsymbol{\alpha}^\dagger$ is a row vector whose nth element is α^*_n. The complex number α_n is the eigenvalue of the operator $\hat{a}_n$, that is,

$$\hat{a}_n \mid \boldsymbol{\alpha} >= \alpha_n \mid \boldsymbol{\alpha} > . \tag{3.19}$$

For Gaussian type operators, we have [13],

$$< \boldsymbol{\alpha} \mid \exp(\hat{\mathbf{a}}^\dagger \mathbf{A} \hat{\mathbf{a}}) \mid \boldsymbol{\alpha} >= \exp(\boldsymbol{\alpha}^\dagger (e^{\mathbf{A}} - \mathbf{I})\boldsymbol{\alpha}) \tag{3.20}$$

[1] (3.14) is proved in appendix A.

where $\mathbf{A}$ is an arbitrary complex matrix. Hence the mean of the in phase quadrature operator measurement can be found,

$$\begin{aligned} < \hat{E}^{(1)}_{out} > &\equiv < \alpha \mid \hat{E}^{(1)}_{out} \mid \alpha > \\ &= \frac{1}{2} < \alpha \mid e^{j\kappa \hat{\mathbf{a}}^{\dagger}\mathbf{A}\hat{\mathbf{a}}} \mathbf{u}^{\dagger}\hat{\mathbf{a}} \mid \alpha > + \frac{1}{2} < \alpha \mid \hat{\mathbf{a}}^{\dagger}\mathbf{u} e^{-j\kappa \hat{\mathbf{a}}^{\dagger}\mathbf{A}\hat{\mathbf{a}}} \mid \alpha > \\ &= \frac{1}{2}\mathbf{u}^{\dagger}\boldsymbol{\alpha} e^{\boldsymbol{\alpha}^{\dagger}(e^{j\kappa\mathbf{A}}-\mathbf{I})\boldsymbol{\alpha}} + \frac{1}{2}\boldsymbol{\alpha}^{\dagger}\mathbf{u} e^{\boldsymbol{\alpha}^{\dagger}(e^{-j\kappa\mathbf{A}}-\mathbf{I})\boldsymbol{\alpha}}. \end{aligned} \tag{3.21}$$

Using (3.11) the above expression can be further simplified as

$$< \hat{E}^{(1)}_{out} >= e^{\boldsymbol{\alpha}^{\dagger}\mathbf{A}\boldsymbol{\alpha}(\frac{\cos\kappa\mathbf{u}^{\dagger}\mathbf{u}-1}{\mathbf{u}^{\dagger}\mathbf{u}})} Re\{\mathbf{u}^{\dagger}\boldsymbol{\alpha} e^{j\boldsymbol{\alpha}^{\dagger}\mathbf{A}\boldsymbol{\alpha}(\frac{\sin\kappa\mathbf{u}^{\dagger}\mathbf{u}}{\mathbf{u}^{\dagger}\mathbf{u}})}\}. \tag{3.22}$$

Similarly, we develop an expression for the variance of the quadrature measurement as follows:

$$\begin{aligned} < \Delta\hat{E}^{(1)2}_{out} > &\equiv < \hat{E}^{(1)2}_{out} > - < \hat{E}^{(1)}_{out} >^2 \\ &= \frac{1}{4} < \hat{E}^{2}_{out} > + \frac{1}{4} < \hat{E}^{\dagger 2}_{out} > + \frac{1}{2} < \hat{E}^{\dagger}_{out}\hat{E}_{out} > + \frac{1}{4}\mathbf{u}^{\dagger}\mathbf{u} - < \hat{E}^{(1)}_{out} >^2 \\ &= \frac{1}{2} e^{\boldsymbol{\alpha}^{\dagger}\mathbf{A}\boldsymbol{\alpha}\frac{\cos 2\kappa\mathbf{u}^{\dagger}\mathbf{u}-1}{\mathbf{u}^{\dagger}\mathbf{u}}} Re\{(\mathbf{u}^{\dagger}\boldsymbol{\alpha})^2 e^{j\kappa\mathbf{u}^{\dagger}\mathbf{u}} e^{j\boldsymbol{\alpha}^{\dagger}\mathbf{A}\boldsymbol{\alpha}\frac{\sin 2\kappa\mathbf{u}^{\dagger}\mathbf{u}}{\mathbf{u}^{\dagger}\mathbf{u}}}\} + \frac{1}{2}\boldsymbol{\alpha}^{\dagger}\mathbf{A}\boldsymbol{\alpha} \\ &\quad + \frac{1}{4}\mathbf{u}^{\dagger}\mathbf{u} - e^{2\boldsymbol{\alpha}^{\dagger}\mathbf{A}\boldsymbol{\alpha}(\frac{\cos\kappa\mathbf{u}^{\dagger}\mathbf{u}-1}{\mathbf{u}^{\dagger}\mathbf{u}})}[Re\{\mathbf{u}^{\dagger}\boldsymbol{\alpha} e^{j\boldsymbol{\alpha}^{\dagger}\mathbf{A}\boldsymbol{\alpha}(\frac{\sin\kappa\mathbf{u}^{\dagger}\mathbf{u}}{\mathbf{u}^{\dagger}\mathbf{u}})}\}]^2 \end{aligned} \tag{3.23}$$

where we have used (3.20), (3.11), (3.15), (3.12) and (3.14) respectively.

In order to compare the squeezing properties of multimode self phase modulated fields with that of a single mode case, (3.23) is calculated numerically assuming a two mode field. We choose $\Phi_1 = e^{-\frac{jwt}{2}}$ with $\alpha_1 = 5$ and $\Phi_2 = e^{\frac{jwt}{2}}$ with $\alpha_2 = 2$, where $w \ll w_0$. The variance of the first quadrature measurement of this field is shown in Figure 3.1 for several values of κ as a function of time. For this specific example, it can be observed that the maximum squeezing occures for $\kappa = .019$ and $wt = 1.2$. At this point noise is about 35% of the coherent state level. This squeezing is larger than that of a single mode field with the same intensity [i.e., $\alpha = \sqrt{25+4}$] (Figure 3.2), which gives a maximum squeezing of 60% below the coherent state level. Note that while the coherent level is .25 for single mode field, it is equal to .5 in the two mode case. Physically we conclude that in the Kerr medium the modes are correlated such that the uncertainty in the measurement of the total field is reduced with respect to the single mode field.

3.4 Heterodyne statistics of self phase modulated multimode fields

In this section we will obtain the probability density function governing the heterodyne detection statistics of the self phase modulated multimode field.

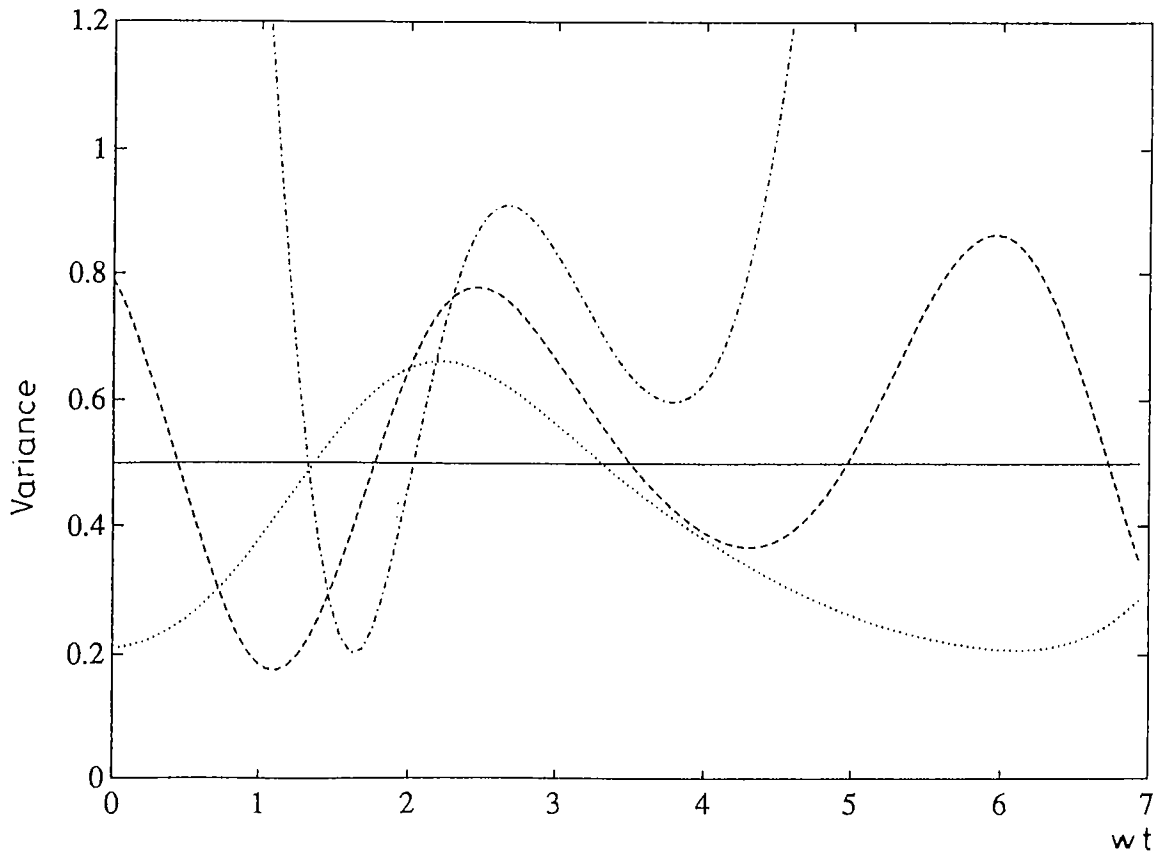

Figure 3.1: Variance of the first quadrature measurement of the two mode field versus wt for $\kappa = 0.01$ (dotted curve), $\kappa = 0.019$ (dashed curve), $\kappa = 0.03$ (dashed-dotted curve). Coherent state noise level is indicated by solid line ($\kappa = 0$).

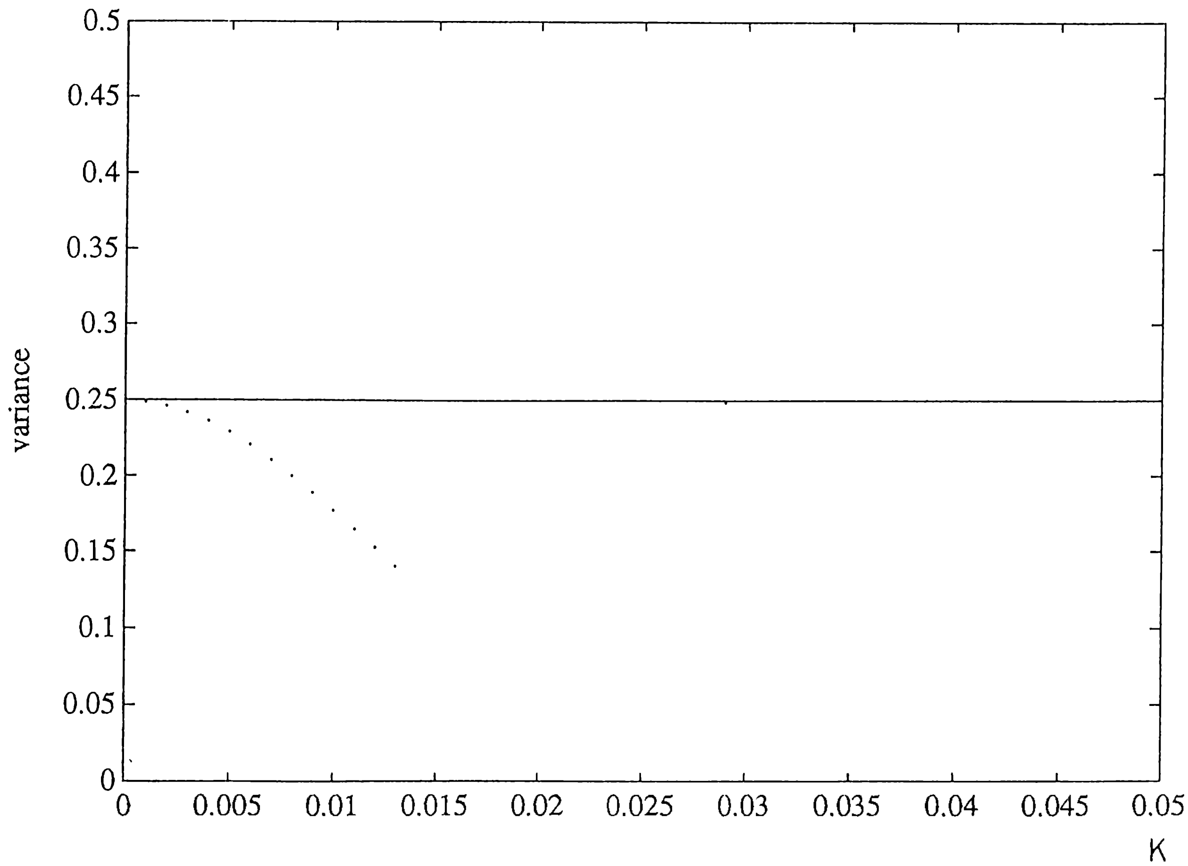

Figure 3.2: Variance of the first quadrature measurement of a single mode field versus κ for $\alpha_0 = \sqrt{29}$. Solid line indicates the coherent state noise level.

Using the multimode coherent state eigenvalue equation $(\mathbf{u}^\dagger \hat{\mathbf{a}})^k \mid \boldsymbol{\alpha}_o >= (\mathbf{u}^\dagger \boldsymbol{\alpha}_o)^k \mid \boldsymbol{\alpha}_o >$ and the operator theorem (3.20), we obtain

$$< \boldsymbol{\alpha}_o \mid \widehat{M} \mid \boldsymbol{\alpha}_o >= e^{\frac{j\kappa}{2}[n(n-1)-m(m-1)]}(\mathbf{u}^\dagger \boldsymbol{\alpha}_o)^n(\boldsymbol{\alpha}_o^\dagger \mathbf{u})^m e^{\boldsymbol{\alpha}_o^\dagger (e^{j\kappa(n-m)\mathbf{A}} - \mathbf{I})\boldsymbol{\alpha}_o}. \quad (3.34)$$

Substitution of (3.34) into (3.29) and using (3.11) gives the following expression for the antinormally ordered characteristic function,

$$\begin{aligned} X_A^{\hat{\rho}_{out}}(\zeta^*, \zeta) &= e^{-|\zeta|^2 \mathbf{u}^\dagger \mathbf{u}} \sum_m \sum_n \frac{(\zeta \boldsymbol{\alpha}_o^\dagger \mathbf{u})^m (-\zeta^* \mathbf{u}^\dagger \boldsymbol{\alpha}_o)^n}{m!n!} e^{\frac{j\kappa}{2}[n(n-1)-m(m-1)]}. \\ &\quad . e^{\boldsymbol{\alpha}_o^\dagger \mathbf{A} \boldsymbol{\alpha}_o (\frac{e^{j\kappa(n-m)\mathbf{u}^\dagger \mathbf{u}} - 1}{\mathbf{u}^\dagger \mathbf{u}})} \end{aligned} \quad (3.35)$$

We next take the inverse Fourier transform of the characteristic function to obtain the probability density function,

$$\begin{aligned} P(\alpha^*, \alpha) &= \frac{1}{\pi} e^{-\frac{|\alpha|^2}{\mathbf{u}^\dagger \mathbf{u}}} \sum_m \sum_n \frac{(\boldsymbol{\alpha}_o^\dagger \mathbf{u} \alpha)^m (\mathbf{u}^\dagger \boldsymbol{\alpha}_o \alpha^*)^n}{(\mathbf{u}^\dagger \mathbf{u})^{m+n+1}} e^{\frac{j\kappa}{2}[n(n-1)-m(m-1)]}. \\ &\quad . e^{\boldsymbol{\alpha}_o^\dagger \mathbf{A} \boldsymbol{\alpha}_o (\frac{e^{j\kappa(n-m)\mathbf{u}^\dagger \mathbf{u}} - 1}{\mathbf{u}^\dagger \mathbf{u}})} \sum_{k=0}^{min(m,n)} \frac{(-1)^{-k} (\frac{|\alpha|}{\sqrt{\mathbf{u}^\dagger \mathbf{u}}})^{-2k}}{k!(n-k)!(m-k)!} \end{aligned} \quad (3.36)$$

where we have used[2]

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-|\zeta|^2 \mathbf{u}^\dagger \mathbf{u} - \zeta \alpha^* + \zeta^* \alpha} \zeta^m (-\zeta^*)^n d^2\zeta &= \frac{\pi \alpha^{*n} \alpha^m e^{-\frac{|\alpha|^2}{\mathbf{u}^\dagger \mathbf{u}}} m!n!}{(\mathbf{u}^\dagger \mathbf{u})^{m+n+1}}. \\ &\quad \sum_{k=0}^{min(m,n)} \frac{(-1)^{-k} (\frac{|\alpha|}{\sqrt{\mathbf{u}^\dagger \mathbf{u}}})^{-2k}}{k!(n-k)!(m-k)!}. \end{aligned} \quad (3.37)$$

A very simple case is considered to interpret the above probability density function. We choose a two mode field such that $\Phi_1 = e^{jwt}$, $\alpha_1 = 4$, $\Phi_2 = e^{-jwt}$ and $\alpha_2 = 1$. By terminating n and m at sufficiently large numbers, (3.36) is calculated numerically. The contour plots of this time dependent probability density function are given in Figure 3.3, 3.4 and 3.5 for several values of wt and κ. In Figure 3.3, $\kappa = 0$, hence the probability density function is Gaussian centered at $m = \alpha_1 \Phi_1 + \alpha_2 \Phi_2$. Probability density curves rotate in the counterclockwise direction with a period of $wt = 2\pi$. It can be observed from both Figures 3.3 , 3.4 and 3.5 that, the contours of $wt = 0$ and π are farther away from the origin than those of $wt = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. The reason is that, when $wt = 0$ or π, the modes are in phase hence the intensity is maximized. However, when $wt = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, the modes are out of phase hence the intensity is minimum. The curves are deformed to crescent shapes as κ increases, similar to the single mode result and they move on the constant $\mid m \mid$ circle in the counterclockwise direction as it is seen from Figures 3.3 and 3.5.

[2]The proof of (3.37) is given in appendix C.

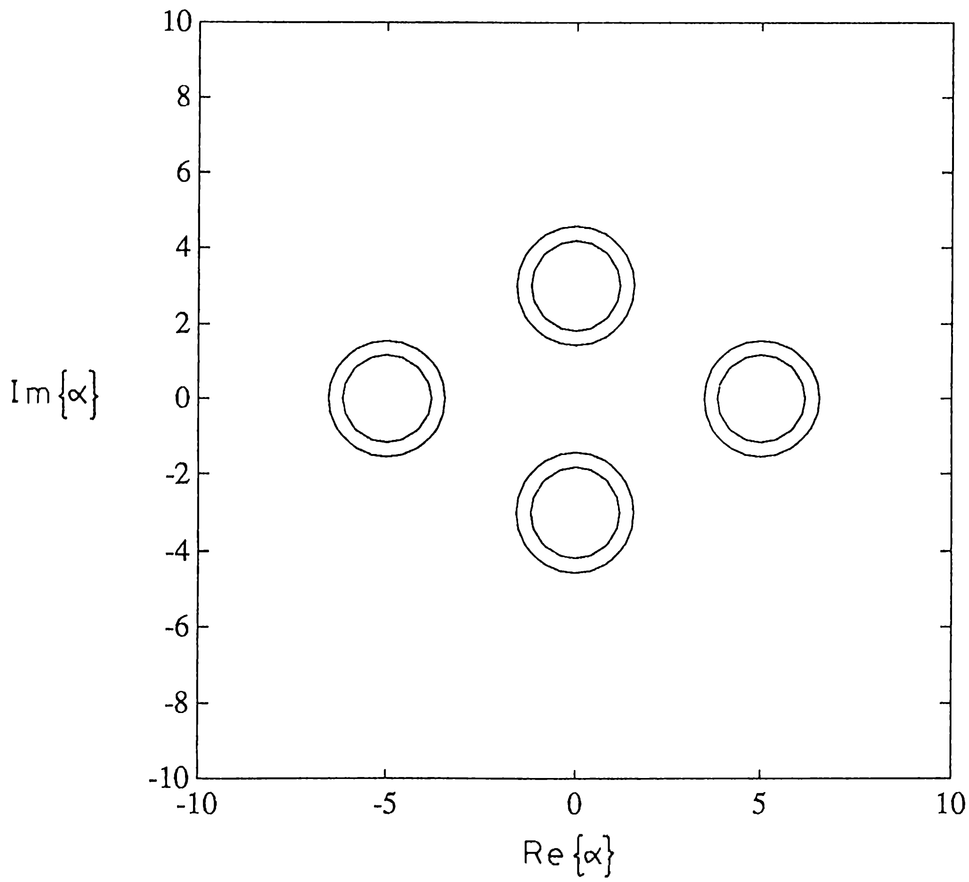

Figure 3.3: Contour plots of the probability density function for $\kappa = 0$ at $\omega t = 0$ (a), $\omega t = \frac{\pi}{2}$ (b), $\omega t = \pi$ (c), $\omega t = \frac{3\pi}{2}$ (d). Contours represent the 30% and 60% of the maximum value of $\kappa = 0$ (multimode coherent state) pdf.

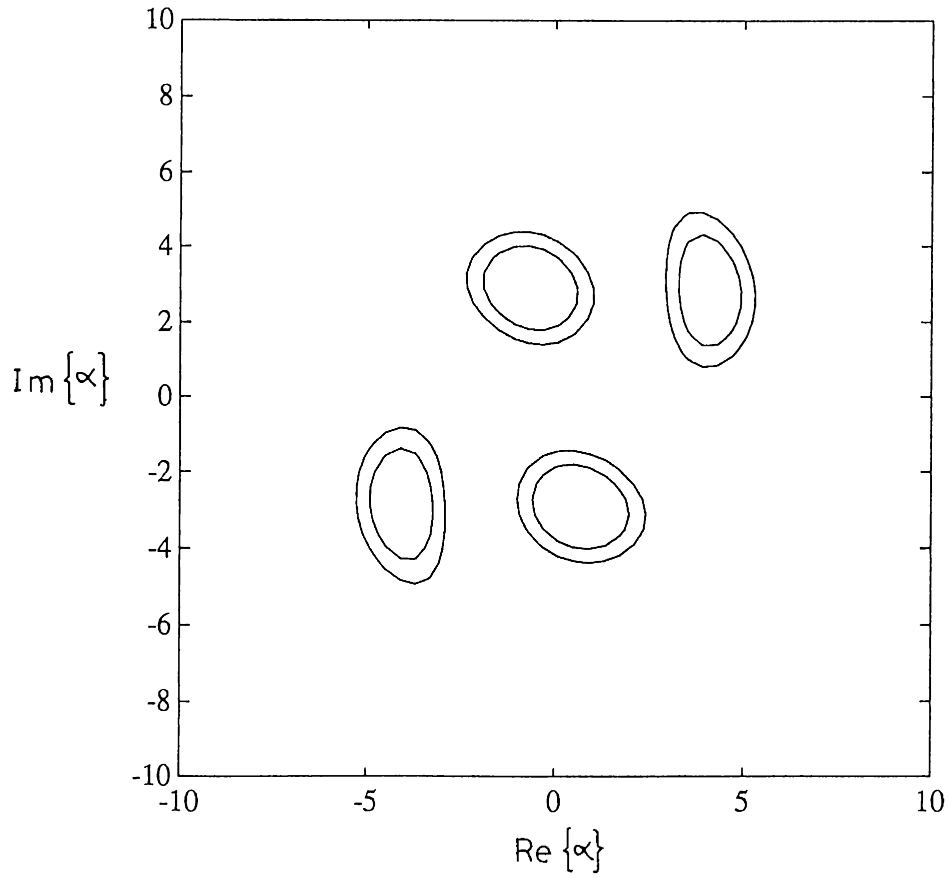

Figure 3.4: Contour plots of the probability density function for $\kappa = 0.025$ at $wt = 0$ (a), $wt = \frac{\pi}{2}$ (b), $wt = \pi$ (c), $wt = \frac{3\pi}{2}$ (d). Contours represent the 30% and 60% of the maximum value of $\kappa = 0$ (multimode coherent state) pdf.

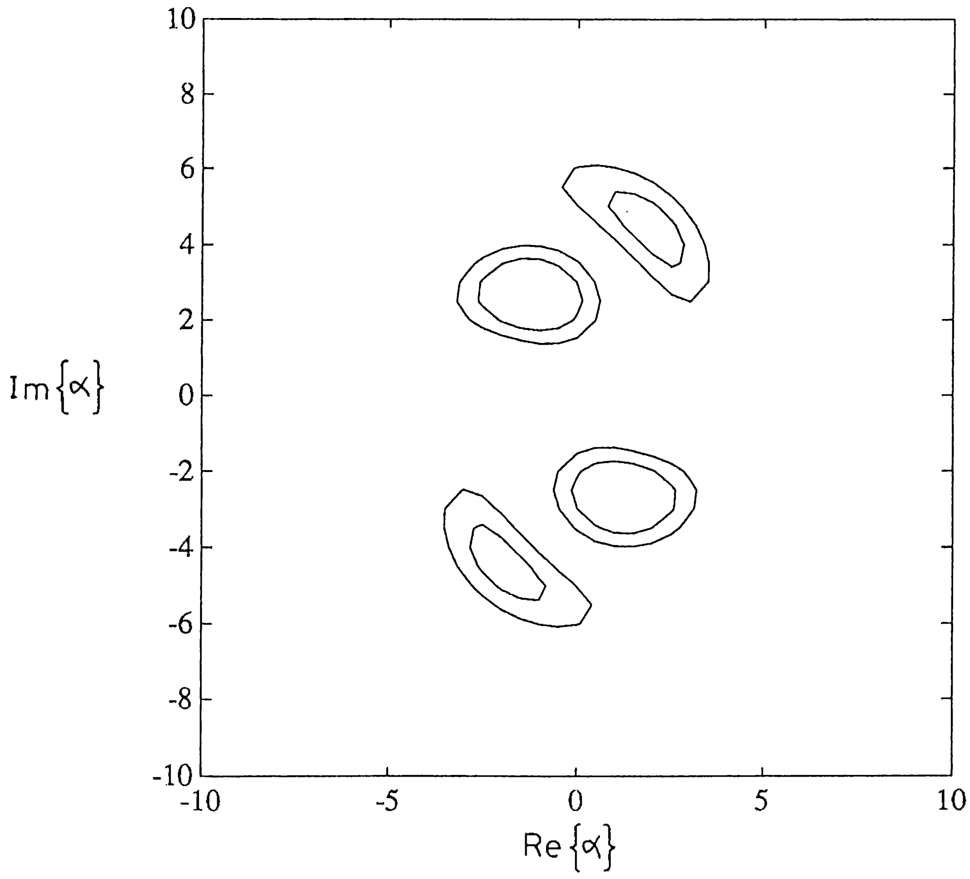

Figure 3.5: Contour plots of the probability density function for $\kappa = 0.05$ at $wt = 0$ (a), $wt = \frac{\pi}{2}$ (b), $wt = \pi$ (c), $wt = \frac{3\pi}{2}$ (d). Contours represent the 30% and 60% of the maximum value of $\kappa = 0$ (multimode coherent state) pdf.

Chapter 4

Conclusions

In this thesis, we have been mainly concerned with the photodetection statistics of the self phase modulated fields. Our main contribution to this subject was in the case of multimode fields. To the best of our knowledge none of the multimode results have appeared in the literature. Although the single mode results were published previously our derivation of those results are original.

Our main result is the fact that self phase modulated multimode fields can exhibit more squeezing than the single mode fields. This is a consequence of the fact that in the multimode case, the net contribution to the photodetection statistics can be attributed to two sources. One is the regular self phase modulation phenomenon that we encountered in the single mode case and the other one is the introduction of a nonzero correlation between two modes. It appears that this correlation can increase the squeezing. Although this fact was verified for a special case we believe it is true in general. This statement can be checked using different basis and the formulas derived in chapter 3.

Future research efforts can concentrate on evaluating the detection statistics for a variety of basis functions especially the prolate spheroidal wave functions. Also novel quantum state generation via similar nonlinear media can be investigated using the analytical tools developed in this thesis.

Appendix A

In this appendix we shall prove the property

$$\hat{\mathrm{a}} f(\hat{\mathrm{a}}^\dagger \mathbf{A} \hat{\mathrm{a}}) = f(\hat{\mathrm{a}}^\dagger \mathbf{A} \hat{\mathrm{a}} \mathbf{I} + \mathbf{A}) \hat{\mathrm{a}} \tag{A.1}$$

where A is an arbitrary $N \times N$ complex matrix. The single mode case of (A.1)

$$\hat{a} f(\hat{a}^\dagger \hat{a}) = f(\hat{a}^\dagger \hat{a} + 1) \hat{a} \tag{A.2}$$

is a well known property in quantum optics [9]. The proof of (A.1) is as follows. Using the canonical commutator relations $[\hat{a}_m, \hat{a}_n^\dagger] = \delta_{mn}$ and $[\hat{a}_m, \hat{a}_n] = 0$, we can obtain

$$\begin{aligned} \hat{a}_i(\hat{\mathrm{a}}^\dagger \mathbf{A} \hat{\mathrm{a}}) &= \hat{a}_i \sum_{j=1}^{N} \sum_{k=1}^{N} A_{jk} \hat{a}_j^\dagger \hat{a}_k \\ &= \hat{a}_i (\sum_{j \neq i} \sum_{k} A_{jk} \hat{a}_j^\dagger \hat{a}_k + \sum_{k} A_{ik} \hat{a}_i^\dagger \hat{a}_k) \\ &= (\sum_{j} \sum_{k} A_{jk} \hat{a}_j^\dagger \hat{a}_k) \hat{a}_i + \sum_{k} A_{ik} \hat{a}_k \\ &= (\hat{\mathrm{a}}^\dagger \mathbf{A} \hat{\mathrm{a}}) \hat{a}_i + A_i \hat{\mathrm{a}}. \end{aligned} \tag{A.3}$$

where A_i is the ith row of the matrix $\mathbf{A}$. Since (A.3) is true for every i, in vector notation we have,

$$\begin{aligned} \hat{\mathrm{a}}(\hat{\mathrm{a}}^\dagger \mathbf{A} \hat{\mathrm{a}}) &= (\hat{\mathrm{a}}^\dagger \mathbf{A} \hat{\mathrm{a}}) \hat{\mathrm{a}} + \mathbf{A} \hat{\mathrm{a}} \\ &= (\hat{\mathrm{a}}^\dagger \mathbf{A} \hat{\mathrm{a}} \mathbf{I} + \mathbf{A}) \hat{\mathrm{a}} \end{aligned} \tag{A.4}$$

The repetition of (A.4) k times yield

$$\hat{\mathrm{a}}(\hat{\mathrm{a}}^\dagger \mathbf{A} \hat{\mathrm{a}})^k = (\hat{\mathrm{a}}^\dagger \mathbf{A} \hat{\mathrm{a}} \mathbf{I} + \mathbf{A})^k \hat{\mathrm{a}} \tag{A.5}$$

Since a function of an operator may be expanded in a power series [9], we can write the function $f(\cdot)$ as,

$$f(\hat{\mathrm{a}}^\dagger \mathbf{A} \hat{\mathrm{a}}) = \sum c_k (\hat{\mathrm{a}}^\dagger \mathbf{A} \hat{\mathrm{a}})^k \tag{A.6}$$

where c_k are the expansion coefficients. Thus the proof is completed as follows:

$$\begin{aligned} \hat{\mathrm{a}} f(\hat{\mathrm{a}}^\dagger \mathbf{A} \hat{\mathrm{a}}) &= \sum_k c_k \hat{\mathrm{a}} (\hat{\mathrm{a}}^\dagger \mathbf{A} \hat{\mathrm{a}})^k \\ &= \sum_k c_k (\hat{\mathrm{a}}^\dagger \mathbf{A} \hat{\mathrm{a}} \mathbf{I} + \mathbf{A})^k \hat{\mathrm{a}} \\ &= f(\hat{\mathrm{a}}^\dagger \mathbf{A} \hat{\mathrm{a}} \mathbf{I} + \mathbf{A}) \hat{\mathrm{a}}. \end{aligned} \tag{A.7}$$

Appendix B

In this appendix we will prove

$$(e^{j\kappa \hat{a}^\dagger \mathbf{A}\hat{a}}\mathrm{u}^\dagger \hat{a})^n = e^{j\kappa n \hat{a}^\dagger \mathbf{A}\hat{a}} e^{j\kappa \mathrm{u}^\dagger \mathrm{u}\frac{n(n-1)}{2}}(\mathrm{u}^\dagger \hat{a})^n. \tag{B.1}$$

The proof will be done by induction;
i-) For $n = 2$, we have

$$e^{j\kappa \hat{a}^\dagger \mathbf{A}\hat{a}}\mathrm{u}^\dagger \hat{a} e^{j\kappa \hat{a}^\dagger \mathbf{A}\hat{a}}\mathrm{u}^\dagger \hat{a} = e^{j\kappa \mathbf{A}}\mathrm{u}^\dagger e^{j\kappa \hat{a}^\dagger \mathbf{A}\hat{a}} e^{j\kappa \mathbf{A}} \mathrm{au}^\dagger \hat{a}, \tag{B.2}$$

where we have utilized

$$\hat{a} f(\hat{a}^\dagger \mathbf{A}\hat{a}) = f(\hat{a}^\dagger \mathbf{A}\hat{a}\mathbf{I} + \mathbf{A})\hat{a}. \tag{B.3}$$

Using

$$\mathrm{u}^\dagger e^{j\kappa \mathbf{A}} = \mathrm{u}^\dagger e^{j\kappa \mathrm{u}^\dagger \mathrm{u}}, \tag{B.4}$$

we obtain

$$(e^{j\kappa \hat{a}^\dagger \mathbf{A}\hat{a}}\mathrm{u}^\dagger \hat{a})^2 = e^{j2\kappa \hat{a}^\dagger \mathbf{A}\hat{a}} e^{j\kappa \mathrm{u}^\dagger \mathrm{u}}(\mathrm{u}^\dagger \hat{a})^2 \tag{B.5}$$

ii-) We now assume that, (B.1) is true for $n = k$, and we will prove it for $n = k + 1$. If we substitute (B.1) with $n = k$ into

$$(e^{j\kappa \hat{a}^\dagger \mathbf{A}\hat{a}}\mathrm{u}^\dagger \hat{a})^{k+1} = e^{j\kappa \hat{a}^\dagger \mathbf{A}\hat{a}}\mathrm{u}^\dagger \hat{a}(e^{j\kappa \hat{a}^\dagger \mathbf{A}\hat{a}}\mathrm{u}^\dagger \hat{a})^k \tag{B.6}$$

we obtain

$$(e^{j\kappa \hat{a}^\dagger \mathbf{A}\hat{a}}\mathrm{u}^\dagger \hat{a})^{k+1} = e^{j\kappa \hat{a}^\dagger \mathbf{A}\hat{a}}\mathrm{u}^\dagger \hat{a} e^{j\kappa k \hat{a}^\dagger \mathbf{A}\hat{a}} e^{j\kappa \frac{k(k-1)}{2}\mathrm{u}^\dagger \mathrm{u}}(\mathrm{u}^\dagger \hat{a})^k. \tag{B.7}$$

Substitution of (B.3) and (B.4) into (B.1) yields

$$(e^{j\kappa \hat{a}^\dagger \mathbf{A}\hat{a}}\mathrm{u}^\dagger \hat{a})^{k+1} = e^{j\kappa(k+1)\hat{a}^\dagger \mathbf{A}\hat{a}} e^{j\kappa \mathrm{u}^\dagger \mathrm{u}\frac{k(k+1)}{2}}(\mathrm{u}^\dagger \hat{a})^{k+1}. \tag{B.8}$$

Appendix C

In this appendix, we will evaluate the integral

$$I = \int_{-\infty}^{\infty} e^{-|\xi|^2 c - \xi\alpha^* + \xi^*\alpha} \xi^m (-\xi^*)^n d^2\xi. \tag{C.1}$$

If we let $\alpha = | \alpha | e^{j\phi}$ and $\xi = re^{j\theta}$, then $d^2\xi = rdrd\theta$. Therefore, in polar coordinates, the integral becomes

$$I = (-1)^n \int_0^{\infty} dre^{-r^2 c} r^{m+n+1} \int_{-\pi}^{\pi} d\theta e^{j(m-n)\theta + 2jr|\alpha| sin(\phi-\theta)}. \tag{C.2}$$

By substituting $\phi - \theta = \pi - \theta'$, we obtain

$$I = (-1)^n e^{j(m-n)(\phi-\pi)} \int_0^{\infty} dre^{-r^2 c} r^{m+n+1} \int_{-\phi}^{2\pi-\phi} d\theta' e^{j[2r|\alpha| sin\theta' - (n-m)\theta']}. \tag{C.3}$$

Since the argument of the inner integral is periodic with 2π, the limits of the integral can be shifted by $\phi - \pi$. So we have

$$I = (-1)^n 2\pi e^{j(m-n)(\phi-\pi)} \int_0^{\infty} dre^{-r^2 c} r^{m+n+1} J_{n-m}(2r \mid \alpha \mid), \tag{C.4}$$

where $J_n(x)$ is the n'th order Bessel function of the first kind. If we make the substitution $r' = r\sqrt{c}$, we obtain

$$\int_0^{\infty} dre^{-r^2 c} r^{m+n+1} J_{n-m}(2r \mid \alpha \mid) = \int_0^{\infty} dr' \frac{e^{-r'^2} r'^{m+n+1}}{c^{\frac{m+n+2}{2}}} J_{n-m}\left(\frac{2r' \mid \alpha \mid}{\sqrt{c}}\right). \tag{C.5}$$

For $n \geq m$ we have [15],

$$\int_0^{\infty} dr' e^{-r'^2} r'^{m+n+1} J_{n-m}(2r'x) = \frac{n! e^{-x^2}}{2(n-m)!} x^{n-m+1} \Phi(-m, n-m+1, x^2) \tag{C.6}$$

where $\Phi(\alpha, \beta; z)$ is degenerate hypergeometric function. The series expansion of $\Phi(\alpha, \beta; z)$ is given as [15],

$$\Phi(\alpha, \beta; z) = 1 + \frac{\alpha}{\beta}\frac{z}{1!} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}\frac{z^2}{2!} + \ldots \tag{C.7}$$

By using (C.7) in (C.6), we obtain

$$\int_0^\infty dr e^{-r^2} r^{m+n+1} J_{n-m}(2rx) = \frac{n!}{2} e^{-x^2} x^{n-m} \sum_{i=0}^{m} \frac{m!(-1)^i x^{2i}}{(m-i)!i!(n-m+i)!}. \qquad \text{(C.8)}$$

Therefore, for $n \geq m$, (C.4) can be written as

$$I = (-1)^{m+n} \pi e^{j(m-n)(\phi-\pi)} e^{-\frac{|\alpha|^2}{c}} \mid \alpha \mid^{m+n} \frac{m!n!}{c^{m+n+1}} \sum_{k=0}^{m} \frac{(-1)^{-k} (\frac{|\alpha|}{\sqrt{c}})^{-2k}}{k!(n-k)!(m-k)!} \qquad \text{(C.9)}$$

where we have changed in (C.8) $m-i$ by k. Similarly, for $n \leq m$, we find

$$I = (-1)^{m+n} \pi e^{j(m-n)(\phi-\pi)} e^{-\frac{|\alpha|^2}{c}} \frac{\mid \alpha \mid^{m+n} m!n!}{c^{m+n+1}} \sum_{k=0}^{n} \frac{(-1)^{-k} (\frac{|\alpha|}{\sqrt{c}})^{-2k}}{k!(n-k)!(m-k)!}. \qquad \text{(C.10)}$$

Therefore, the final result takes the form

$$I = \pi \alpha^{*n} \alpha^m e^{-\frac{|\alpha|^2}{c}} \frac{m!n!}{c^{m+n+1}} \sum_{k=0}^{min(m,n)} \frac{(-1)^{-k} (\frac{|\alpha|}{\sqrt{c}})^{-2k}}{k!(n-k)!(m-k)!}. \qquad \text{(C.11)}$$

References

[1] R. H. Stolen and Chinlon Lin, "Self-phase-modulation in silica optical fibers," *Phys. Rev. A* **17**, 1448 (1978).

[2] R. H. Stolen, "Nonlinearity in Fiber Transmission," *Proceedings of the IEEE* **68**, 1232 (1980).

[3] W. J. Tomlinson, R. H. Stolen and C. V. Shank, " Compression of optical pulses chirped by self-phase modulation in fibers," *J. Opt. Soc. Am. B* **1**, 139 (1984).

[4] R. Tanas and S. Kielich, "Self-squeezing of light propagating through nonlinear optically isotropic media," *Optics Communications* **45**, 351 (1983).

[5] N. Imomoto, H. A. Haus and Y. Yamamoto, "Quantum nondemolition measurement of the photon number via the optical Kerr effect," *Phy. Rev. A* **32**, 2287 (1985).

[6] M. Kitagawa and Y. Yamamoto, "Number-phase minimum -uncertainty state with reduced number uncertainty in a Kerr nonlinear interferometer," *Phys. Rev. A* **34**, 3974 (1986).

[7] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York, 1976).

[8] J. H. Shapiro,"Quantum Noise and Excess Noise in Optical Homodyne And Heterodyne Receivers," *J. Quantum Electron.* **QE-21**, 237 (1985).

[9] W. H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973).

[10] D. F. Walls, "Squeezed states of light," *Nature* **306**, 141 (1983).

[11] H. P. Yuen, "Two-photon coherent states of the radiation field," *Phy. Rev. A* **13**, 2226 (1976).

[12] H. P. Yuen and J. H. Shapiro, "Optical Communication with Two-photon Coherent states - Part I:Quantum-State Propagation and Quantum-Noise Reduction," *IEEE Trans. on Inform. Theory* **IT-24**, 657 (1978).

[13] H. P. Yuen and J. H. Shapiro, "Optical Communication with Two-photon Coherent states - Part III:Quantum Measurements Realizable with Photoemissive Detectors," *IEEE Trans. on Inform. Theory* **IT-26**, 78 (1980).

[14] A. Papoulis, *Probability, Random Variables, And Stochastic Processes* (McGraw-Hill, New York, 1965).

[15] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, London, 1965).