THE DISTRIBUTION OF THE RESIDUAL LIFETIME AND ITS APPLICATIONS

A THESIS

SUBMITTED TO THE DEPARTMENT OF INDUSTRIAL ENGINEERING
AND THE INSTITUTE OF ENGINEERING AND SCIENCES
OF BİLKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By

Mine Alp Çağlar

March, 1991

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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Assoc. Prof. Cemal Dinçer

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

THE DISTRIBUTION OF THE RESIDUAL LIFETIME AND ITS APPLICATIONS

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M.S. in Industrial Engineering
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March, 1991

Let T be a continuous positive random variable representing the lifetime of an entity. This entity could be a human being, an animal or a plant, or a component of a mechanical or electrical system. For nonliving objects the lifetime is defined as the total amount of time for which the entity carries out its function satisfactorily. The concept of *aging* involves the adverse effects of age such as increased probability of failure due to wear. In this thesis, we consider certain characteristics of the residual lifetime distribution at age t, such as the mean, median, and variance, as describing aging. The following families of statistical distributions are studied from this point of view:

1. Gamma with two parameters,

2. Weibull with two parameters,

3. Lognormal with two parameters,

4. Inverse Polynomial with one parameter.

Gamma and Weibull distributions are fitted to actual data.

Keywords: Reliability, residual life distribution, mean, variance and percentile of residual life, Gamma distribution, Weibull distribution.

ÖZET

ARTAKALAN ÖMÜR DAĞILIMI VE UYGULAMALARI

Mine Alp Çağlar
Endüstri Mühendisliği Bölümü Yüksek Lisans
Tez Yöneticisi: Prof. Mohammed M. Siddiqui
Mart, 1991

T bir birimin ömrünü gösteren sürekli bir rassal değişken olsun. Bu birim bir canlı olabileceği gibi, mekanik ya da elektronik bir sistemin bileşeni de olabilir. Cansız varlıklar için ömür, birimin istenen düzeyde işlevini sürdürebildiği toplam zaman miktarı olarak tanımlanabilir. *Yaşlanma* kavramı, aşınmaya bağlı olarak artan bozulma olasılığı gibi zamanın olumsuz etkilerini içerir. Bu çalışmada, t zamanında artakalan ömrün ortalama, ortanca ve varyans gibi bazı özellikleri, yaşlanmayı tanımlamak üzere ele alınmıştır. Aşağıdaki dağılımlar bu açıdan incelenmiştir:

1. İki parametreli Gamma,
2. İki parametreli Weibull,
3. İki parametreli Lognormal,
4. Bir parametreli Inverse Polynomial.

Gamma ve Weibull dağılımları, gerçek verilere uygulanmıştır.

Anahtar sözcükler: Güvenilirlik, artakalan ömür dağılımı, artakalan ömrün ortalama, varyans ve yüzdelikleri, Gamma dağılımı, Weibull dağılımı.

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Chapter 1

Introduction

1.1 The Concept of Reliability

Reliability may be defined as the capability of a piece of equipment not to break down while in operation, or *quality over the long run.* Reliability and quality concepts are usually used together, but they are essentially different from each other. From the producer's point of view, the quality of a product is assessed against certain specifications or attributes, and, if they are met, the product is classified as good and is then delivered to the customer. A good product is considered to be reliable, i.e., capable of doing its job without failure. On the other hand, from the customer's point of view, the product is good only if it is reliable (i.e. does its job satisfactorily) over a period of time. We therefore must bring a time-based concept of quality in addition to an inspection-based concept of quality. The inspector's concept is not time dependent; the product either passes or fails a certain test. On the other hand, reliability is concerned with failures in time domain.

The reliability of a piece of equipment in most cases has vital importance to the user and by this means to the manufacturer, in terms of competetion in the market and decreasing the warranty costs. Hence the price of unreliability is very high. Here, the term 'equipment' may be applied to a simple device such as a switch, a diode or a connection, or it may be a very complex machine, such as a computer, a radar, an aircraft or a missile.

Reliability is, then, concerned with *failures* of items. We therefore need to understand why an item fails. Failures can be classified into three catagories [4]. First, the failures which occur early in the life of a component are called *early failures* and in most cases result from poor manufacturing and quality control techniques during the production process. Early failures can be eliminated by the so- called 'debugging' or 'burn-in' process. The debugging process consists of operating a piece of equipment for a number of hours under conditions simulating actual use. When weak, substandard components fail in these early hours of the equipment's operation, they are replaced by good components, and when assembly faults show up, they are corrected. Only then is the equipment released for service. The burn-in process consists of operating a large number of components under simulated conditions for a number of hours and then using the components which survive for the assembly of the equipment.

Secondly, there are failures which are caused by *wearout* of parts; wearout failures are a symptom of component aging. The age at which wearout occurs differs widely among components. In most cases wearout failures can be prevented. For instance, in repeatedly operated equipment, one method is to replace at regular intervals the accessible parts which are known to be subject to wearout, and to make the replacement intervals shorter than the mean wearout life of the parts. Otherwise, when the parts are inaccessible, they are designed for a longer life than the intended life of the equipment.

Thirdly, there are so-called *chance* failures which neither good debugging techniques nor the best maintanence practices can eliminate. These failures are caused by sudden stress accumulations beyond the strength of the component. Chance failures occur at random intervals, irregularly and unexpectedly. It is not normally easy to eliminate chance failures. However, reliability techniques have been developed which can reduce the incidence of their occurence and therefore reduce their number to a minimum within a given time interval, or even completely eliminate equipment breakdowns resulting from component chance failures.

Reliability theory and practice differentiates between early, wearout and chance failures for two main reasons. First, each of these types of failures follows a specific statistical distribution and therefore requires a different mathematical treatment. Secondly, different methods must be used for their elimination.

Defined mathematically, reliability is the probability that no failure will occur in a given time interval of operation. In notations, let T be the positive random variable denoting the lifetime (or time between failures) of an item and F be its cumulative distribution function. Then the reliability of the equipment (item) corresponding to a duration $t \geq 0$ is denoted by $R(t)$ and defined as

$$R(t) = P[T > t] = 1 - F(t)$$

1.2 The Scope of the Thesis

There is a great amount of literature on the subject of reliability because of its depth and breath. As discussed in section 1.1, there are three types of failures. The mathematical analysis varies with the type of the failure, and for each type we can define a number of functions, quantities, parameters, etc. In this thesis we confine ourselves to a consideration of the wearout failure of a piece of equipment which consists of a single component and concentrate on the related variable: Residual Lifetime. It may be mentioned that the failure process itself may be quite complex and its mathematical description very difficult. Consequently, we only deal with a statistical summary of the failure process in terms of a distribution function F. All concepts pertaining to failure process are, then, in terms of F. Among these are: reliability, conditional reliability, hazard rate, and mean residual lifetime. We have already defined the reliability function $R(t) = 1 - F(t)$. We now introduce the other functions. The conditional reliability of a component of age t is

$$R(x|t) = \frac{R(t+x)}{R(t)} \qquad x > 0 \quad ,$$

when $R(t) > 0$, and the conditional probability of failure is

$$1 - R(x|t) = \frac{F(t+x) - F(t)}{R(t)}$$

Then the hazard rate $h(t)$ at time t is defined as follows [3]:

$$h(t) = \lim_{x \to 0} \frac{1}{x} \frac{F(t+x) - F(t)}{R(t)}$$

If the probability density function $f(t)$ exists, i.e., $f(t) = F'(t) = -R'(t)$, then

$$h(t) = \frac{f(t)}{R(t)}$$

The hazard rate is widely used and studied in determining whether the item is wearing out, in other words, aging or not. If the hazard rate for a lifetime distribution is monotonically increasing, we can say that the item whose lifetime is distributed with that distribution wears out in time. In [5], a total seven criteria for aging were proposed. First let us define the related functions.

The specific aging factor $A(t, s)$:

$$A(t,s) = \frac{R(t)R(s)}{R(t+s)} \quad .$$

The mean residual lifetime $e(t)$:

$$e(t) = \int_t^\infty \frac{R(x)}{R(t)} \, dx \quad .$$

The specific interval-average hazard rate $H(t, s)$:

$$H(t,s) = t^{-1} \int_s^{s+t} h(x) \, dx$$

Then the hazard rate average can be defined as $H(t, 0)$. The criteria are:

1. *Increasing specific aging factor*:

$$A(t_2, s) \geq A(t_1, s) \quad \forall\, s \geq 0; t_2 \geq t_1 \geq 0.$$

2. *Increasing hazard rate (IHR)*:

$$h(t_2) \geq h(t_1) \quad \forall\, t_2 \geq t_1 \geq 0.$$

3. *Increasing interval average hazard rate*:

$$H(t_2, s) \geq H(t_1, s) \quad \forall\, t_2 \geq t_1 \geq 0, s \geq 0.$$

4. *Decreasing mean residual lifetime*:

$$e(t_2) \leq e(t_1) \quad \forall\, t_2 \geq t_1 \geq 0.$$

5. *Increasing hazard rate average*:

$$H(t_2, 0) \geq H(t_1, 0) \quad \forall\, t_2 \geq t_1 \geq 0.$$

6. *Positive aging*:

$$A(t, s) \geq A(0, s) \quad \forall\, t, s \geq 0.$$

7. *Net decreasing mean residual lifetime*:

$$e(t) \leq e(0) \ \ \forall t \geq 0.$$

From probability theory, we know that there are several functions which completely specify the distribution of a random variable. Examples of these are probability density function, characteristic function, Mellin transform and cumulative distribution function. However, in reliability context, five mathematically equivalent, popular representations have evolved: probability density function, reliability, hazard rate, cumulative hazard function and mean residual lifetime function. Each of these functions completely describes the distribution of a lifetime and any one of the functions determines the other four. The relationship between them can be summarized as follows [13]:

$$\begin{aligned} f(t) &= -R'(t) \\ H(t) &= -\log R(t) \\ h(t) &= H'(t) \\ e(t) &= \int_t^\infty \frac{R(x)}{R(t)}\,dx \\ h(t).e(t) &= 1 + e'(t) \end{aligned} \tag{1.1}$$

The distributions then are classified in terms of those functions so that we can determine whether the item is wearing out or not. Since the aging criteria defined above are also in terms of them, they represent classes of distributions, too. The classes are defined exactly; first, new better than used (NBU) and new worse than used (NWU) distribution classes are defined in terms of $R(t)$. A distribution is NBU (NWU) if and only if it is positively (negatively) aging, as defined in criterion 6.

Increasing hazard rate (IHR) was discussed above. Similarly, decreasing hazard rate(DHR) can be defined in terms of $h(t)$. Again, similar to IHRA, decreasing hazard rate average (DHRA) can be defined in terms of H(t,0). Moreover, increasing mean residual lifetime (IMRL) and decreasing mean residual lifetime (DMRL), new better than used in expectation (NBUE) and new worse than used in expectation (NWUE) classes are defined in terms of $e(t)$; DMRL is criterion 4 and NBUE is equivalent to criterion 7 above, and IMRL and NWUE are their analogues for a negative aging item.

In addition to the aging criteria defined up to now, some other criteria can also be defined [9]; a distribution is called new better than used in hazard rate (NBUHR) if and only if

$$h(0) \leq h(t) \quad \forall\, t > 0 \quad ,$$

and is called new better than used in hazard rate average (NBUHRA) if and only if

$$h(0) \leq H(t,0) \quad \forall\, t > 0.$$

In terms of e(t), a distribution is called decreasing mean residual life in harmonic average (DMRLHA) if $[(1/t)\int_0^t (1/e(u))du]$ is decreasing in t; it is called harmonically new better than used in expectation (HNBUE) if $\int_t^\infty \bar{F}(x)\,dx \leq e(0)\exp(-t/e(0))$ for $t > 0$.

Some implications exist among different classes of distributions, and hence among different aging criteria. These can be found in the mentioned references; [5] and [9], but from the point of view of this thesis, it can be mentioned that Bryson and Siddiqui [5] have shown that IHR implies DMRL; and the reverse is not true. The class of IHR distributions therefore forms a proper subset of the class of DMRL distributions. What is more, an essential difference between the hazard function and $e(t)$ is that the former accounts only for the immediate future in assessing the event component failure, whereas the latter accounts for the complete future [14]. This is readily seen from the expressions of $h(t)$ and $e(t)$. It explains why a component can experience positive aging, in the sense that its corresponding MRL function is decreasing, and yet have zero hazard, because its failure cannot occur in the immediate future. Such a situation is exemplified by the uniform distribution in $[a, b]$ for which the hazard function and the MRL function are shown in figure 1.1. In a reliability context, the component is clearly wearing out. On the other hand, the hazard function is zero and gives no indication. Also the actual age t, cannot be deduced from the hazard function prior to time a. So the MRL function provides a more descriptive measure of an aging process than the hazard rate function. Furthermore, the MRL function is very useful in decision making for replacement policies and in solving burn-in problems [6].

In statistical practice, the median and other percentiles are used as well as the mean, for example, in situations where the underlying distribution is skewed. So it should be of interest to study the median residual life function or more generally the α-percentile residual lifetime function. In this case, classes of distributions analogous to the previously described ones can also be defined [10]; an example is 'new better than used with respect to the α -percentile' NBUP-α. In comparison to α-percentile residual lifetime, MRL has some theoretical and practical shortcomings. In an experiment it

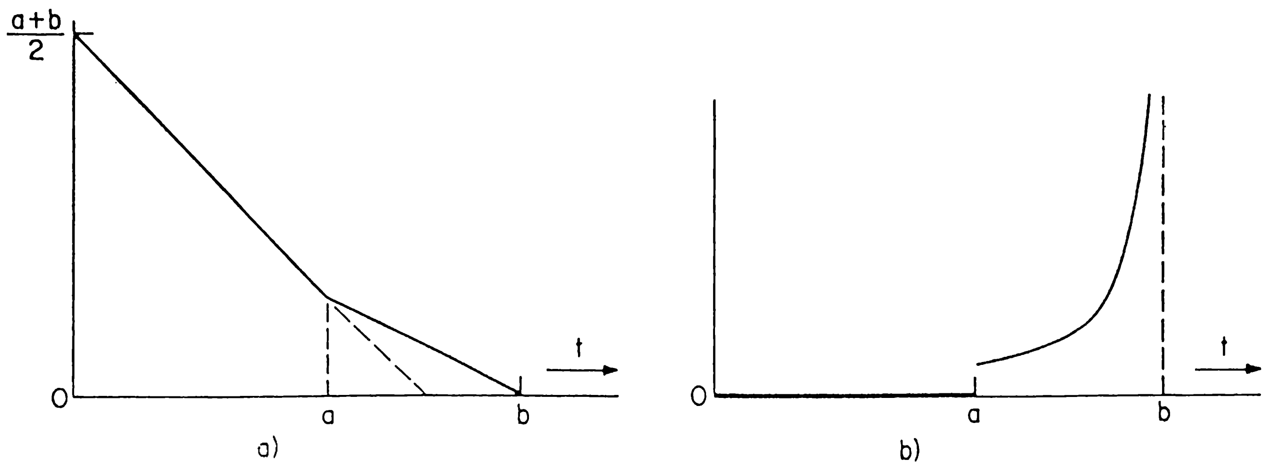

Figure 1.1: a) MRL function, and b) Hazard function of the uniform distribution with probability mass in $[a, b]$

is often impossible or impractical to wait until all items have failed. Estimation of empirical MRL is not straight forward in this case. However, if we consider for example, the median residual lifetime, calculation of this statistic with censored data posseses no difficulty as long as at least half of those remaining have recorded failure times. Moreover in some instances, MRL may not even exist [18].

In this thesis, residual lifetime distribution is discusssed in terms of its mean, variance and percentiles, and behavior of those quantities as t tends to infinity, for certain distributions. Lawless [11] provides a relationship between the asymptotic value of $e(t)$ and the probability density function $f(t)$:

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} -\frac{1}{\frac{d}{dt}\log[f(t)]}$$

Moreover, Calabria and Pulcini [6] provide a relationship between the asymptotic behaviors of $e(t)$ and $h(t)$:

$$\lim_{t\to\infty} e(t) = \frac{1}{\lim_{t\to\infty} h(t)}$$

They then conclude that $\lim_{t\to\infty} e'(t) = 0$, by referring to equation 1.1 which is due to Park [17]. In our study of asymptotic behavior of MRL, we have additionally found the order of convergence to these limits.

Because of its simplicity, Exponential distribution is widely used. This distribution has memoryless property which is equivalent to *no aging* in reliability context. However there are situations where the property of no memory (equivalently no aging or wearout) does not agree with the physical realities, as is the case when T represents a service time or a repair time, or when failure is due to wearout. Weibull and Gamma are two typical distributions which describe the lifetime of an aging piece of equipment when the corresponding shape parameter is greater than 1. Hence they are investigated in Chapters 2 and 3. Lognormal distribution whose MRL function is not monotonic and the Inverse Polynomial Distribution whose MRL function is linearly increasing are studied in Chapter 4. Finally, in the last chapter, several sets of data are analyzed to illustrate the use of the properties of residual life distributions, such as the mean and the variance, for selecting one or more theoretical distributions which adequately fit the data.

Chapter 2

Gamma Family of Distributions with Integral Shape Parameter

In this chapter, random variable T (the lifetime of an item) is considered to be distributed with Gamma Distribution which is defined by the following density function

$$f_T(x) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} \quad \forall x \geq 0, \alpha > 0, \lambda > 0$$

In this expression λ is the scale parameter and α is the shape parameter. In general, they both take positive real values, but for simplicity and gaining an insight into the problem in the first place, α is taken as an integer and is replaced by n in this case. In fact, there are applications of Gamma distribution with integer shape parameter. For example it is suitable in situations where shocks arrive by a Poisson process and failure of the equipment occurs exactly at the k^{th} shock; then the time between failures are distributed with Gamma distribution with shape parameter k.

2.1 Hazard Rate and Hazard Rate Average Functions

The hazard rate function for $\alpha > 0, \lambda > 0$ is

$$h(t) = \frac{f(t)}{R(t)} = \frac{\frac{\lambda^\alpha t^{\alpha-1} e^{-\lambda t}}{\Gamma(\alpha)}}{\int_t^\infty \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x} dx}{\Gamma(\alpha)}}$$

Then

$$[h(t)]^{-1} = \int_t^{\infty} (x/t)^{\alpha-1} e^{-\lambda(x-t)}\, dx$$

Putting u=x-t

$$[h(t)]^{-1} = \int_t^{\infty} (1 + u/t)^{\alpha-1} e^{-\lambda u}\, du \quad .$$

So h(t) is decreasing for $0 < \alpha \leq 1$ and $h(t)$ is increasing for $\alpha \geq 1$. That is, Gamma family of distributions are decreasing hazard rate (DHR) for $0 < \alpha \leq 1$ and increasing hazard rate (IHR) for $\alpha \geq 1$. We can conclude that the distribution is also DHRA for $0 < \alpha \leq 1$ and IHRA for $\alpha \geq 1$.

The hazard rate average function is as follows:

$$H(t,0) = (1/t) \int_0^t h(x)\, dx = -\frac{\log R(t)}{t}$$

2.2 Residual Lifetime Distribution

The density for residual lifetime is:

$$f_{T-t}(x) = \frac{f_T(x+t)}{R_T(t)}$$

(for the rest of the text, the subscript T will be supressed).

2.2.1 The Properties

For the Gamma Distribution with integer shape parameter n the density of the residual life time distribution is:

$$f_{T-t}(x) = \frac{\lambda^n (x+t)^{n-1} e^{-\lambda(x+t)}}{(n-1)! \sum_{k=0}^{n-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!}} \quad ,$$

because (for n integer),

$$R(t) = \int_t^{\infty} \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} = \sum_{k=0}^{n-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

The density function simplifies to

$$f_{T-t}(x) = \frac{\lambda^n (x+t)^{n-1} e^{-\lambda x}}{(n-1)! \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}}$$

The moment generating function

$$M(s) = \int_0^\infty \frac{e^{sx}\lambda^n (x+t)^{n-1} e^{-\lambda x}\, dx}{(n-1)! \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}}$$

which by integration by parts, is simplified as follows:

$$M(s) = \frac{(\frac{\lambda}{\lambda - s})^n \sum_{k=0}^{n-1} \frac{[(\lambda - s)t]^k}{k!}}{\sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}}$$

So the expectation of residual lifetime $e(t)$:

$$e(t) = M'(s)|_{s=0} = \frac{\sum_{k=0}^{n-1} \frac{(n-k)\lambda^{k-1} t^k}{k!}}{\sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}} \tag{2.1}$$

The second moment:

$$M''(s)|_{s=0} = \frac{\sum_{k=0}^{n-1} \frac{(n-k)(n-k+1)\lambda^{k-2} t^k}{k!}}{\sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}}$$

As a result the variance of residual lifetime $v(t)$:

$$v(t) = \frac{\sum_{k=0}^{n-1} \frac{(n-k)(n-k+1)\lambda^{k-2} t^k}{k!}}{\sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}} - \frac{(\sum_{k=0}^{n-1} \frac{(n-k)\lambda^{k-1} t^k}{k!})^2}{(\sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!})^2} . \tag{2.2}$$

The percentiles of the residual life distribution at age t can be found as follows. Since

$$P[T - t > x | T > t] = \frac{\int_x^\infty \frac{\lambda^n (z+t)^{n-1} e^{-\lambda(z+t)}\, dz}{(n-1)!}}{\sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}} ,$$

ξ_p, the p^{th} percentile, is the solution of the following equation in x:

$$\begin{aligned} 1 - p &= P[T - t > x | T > t] \\ &= \frac{e^{-\lambda x} \sum_{k=0}^{n-1} \frac{(x+t)^k \lambda^k}{k!}}{\sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}} \end{aligned} \tag{2.3}$$

This equation cannot be solved exactly for an arbitrary integer n. That is why the asymptotic behavior will be discussed in the next section.

2.2.2 Asymptotic Behavior of Mean, Variance and Percentiles of Residual Lifetime

Lifetime distribution with integer shape parameter

For n=1, the residual lifetime distribution is reduced to an exponential distribution, so $e(t) = 1/\lambda \ \ \forall t \geq 0$.

For n=2, from equation 2.1, $e(t) = \frac{(2/\lambda + t)}{1+\lambda t} = \frac{1}{\lambda(1+\lambda t)} + \frac{1}{\lambda}$. Then, at t=0,$e(t) = 2/\lambda$ and as $t \to \infty$ $e(t) = 1/\lambda + O(1/t)$.

For n=3, from equation 2.1,

$$e(t) = \frac{3/\lambda + 2t + \lambda t^2/2}{1 + \lambda t + (\lambda t)^2/2} = \frac{4 + 2\lambda t}{\lambda(2 + 2\lambda t + (\lambda t)^2)} + \frac{1}{\lambda}$$

Then, at t=0,$e(t) = 3/\lambda$ and as $t \to \infty$ $e(t) = 1/\lambda + O(1/t)$.

For $n \in Z^+$, from equation 2.1,

$$e(t) = \frac{n/\lambda + (n-1)t/1! + (n-2)\lambda t^2/2! + \cdots + \lambda^{n-2}t^{n-1}/(n-1)!}{1 + \lambda t/1! + \lambda^2 t^2/2! + \cdots + \lambda^{n-1}t^{n-1}/(n-1)!}$$

Then, at t=0,$e(t) = n/\lambda$ and as $t \to \infty$ $\quad e(t) = 1/\lambda + O(1/t)$.

Similarly, for the variance starting with n=1, 2 and 3:

For n=1, $v(t) = 1/\lambda \ \forall t \geq 0$.

For n=2, from equation 2.2, at t=0 $v(t) = 2/\lambda^2$ and as $t \to \infty$ $v(t) = 1/\lambda^2 + O(1/t)$.

For n=3, from equation 2.2, at t=0 $v(t) = 3/\lambda^2$ and as $t \to \infty$ $v(t) = 1/\lambda^2 + O(1/t)$.

For $n \in Z^+$ from equation 2.2, writing the smallest and highest orders of t with their coefficients in the summations:

$$v(t) = \frac{(\frac{n(n+1)}{\lambda^2} + \cdots + \frac{(\lambda t)^{n-1}}{(n-1)!}\frac{2\lambda^{n-3}t^{n-1}}{(n-1)!}) - (\frac{n^2}{\lambda} + \cdots + \frac{\lambda^{2(n-2)}t^{2(n-1)}}{[(n-1)!]^2})}{1 + \cdots + \frac{(\lambda t)^{2(n-1)}}{[(n-1)!]^2}},$$

we obtain

$$v(t) = \frac{\frac{n}{\lambda^2} + \cdots + \frac{\lambda^{2n-4}t^{2n-2}}{[(n-1)!]^2}}{1 + \cdots + \frac{\lambda^{2n-2}t^{2n-2}}{[(n-1)!]^2}}$$

Then, at $t = 0$ $v(t) = n/\lambda^2$ and as $t \to \infty$ $\quad v(t) = 1/\lambda^2 + O(1/t)$, because

$$v(t) - \frac{1}{\lambda^2} = \frac{(n-1) + \cdots + \frac{4\lambda^{2n-5}t^{2n-3}}{(n-1)!(n-2)!}}{1 + \cdots + \frac{\lambda^{2n-2}t^{2n-2}}{[(n-1)!]^2}} = O(1/t) \quad for \quad large \quad t.$$

As stated in the previous section, the percentiles must also be studied for large t. For $n = 1$, $\xi_p = -\log(1-p)/\lambda$. Let us denote the p^{th} percentile corresponding to the residual lifetime with shape parameter n as $\xi_p(n)$.

Then, for $n = 2$ from equation 2.3,

$$1 - p = \frac{e^{-\lambda\xi_p(2)}[1 + (\xi_p(2) + t)\lambda]}{1 + \lambda t}$$

By simplification, the following is found:

$$\lambda\xi_p(2) = \log(1 + \frac{\lambda\xi_p(2)}{1 + \lambda t}) + \lambda\xi_p(1)$$

But, for large t, the denominator of the second term in the logarithmic function also becomes large and the approximation $\log(1 + x) \cong x$ can be used. So for large t,

$$\xi_p(2)[1 - \frac{1}{1 + \lambda t}] \cong \xi_p(1)$$

or

$$\xi_p(2) = \xi_p(1) + O(1/t)$$

For $n = 3$, $\xi_p(3)$ can be found in the same way as

$$\xi_p(3) = \xi_p(1)[1 + 2/\lambda t + O(1/t^2)] \quad .$$

For arbitrary n, taking logarithm of both sides of equation 2.3,

$$\log(1-p) = -\lambda x + log[1 + \frac{\lambda(x+t)}{1!} + \cdots + \frac{(\lambda(x+t))^{n-1}}{(n-1)!}] - \log[1 + \frac{\lambda t}{1!} + \cdots + \frac{(\lambda t)^{n-1}}{(n-1)!}]$$

By simplification and by writing the smallest and the highest orders of t in the summations, we get

$$x = \frac{1}{\lambda}\log[1 + \frac{\lambda x + \frac{(\lambda x)^2}{2} + \lambda^2 xt + \ldots + \frac{\lambda^{n-1}t^{n-2}x}{(n-2)!}}{1 + \lambda t + \frac{(\lambda t)^2}{2} + \ldots + \frac{(\lambda t)^{n-1}}{(n-1)!}}] + \xi_p(1)$$

Then, using the approximation $\log(1 + x) \cong x$ as we did for $n = 2$ for large t, we have

$$x \cong \frac{x + \lambda x^2/2 + \ldots + (\lambda t)^{n-2}x/(n-2)!}{1 + \lambda t + \ldots + (\lambda t)^{n-1}/(n-1)!} + \xi_p(1)$$

But for large t, the last terms in the numerator and denominator respectively dominate, so that

$$x \cong \frac{(\lambda t)^{n-2}x/(n-2)!}{(\lambda t)^{n-1}/(n-1)!} + \xi_p(1)$$

This gives

$$x[1 - \frac{n-1}{\lambda t}] \cong \xi_p(1) \quad ,$$

or finally,

$$x = \xi_p(1)[1 + \frac{n-1}{\lambda t} + O(1/t^2)]$$

The last equation is written by the identity $\frac{1}{1-x} \cong \sum_{k=0}^{\infty} x^k$ for sufficiently small $x > 0$. Now x stands for $\xi_p(n)$, so for large t, $\xi_p(n) = \xi_p(1) + O(1/t)$.

These are interesting results, because this means that if an item whose life distribution is defined by a Gamma Distribution (with integer shape parameter) survives a long time, then it behaves asymptotically as if it has an exponential residual life time distribution, i.e., it behaves as if it does not age any more. Moreover, for $n \in \{1, 2, \ldots\}$ it can be concluded that both $e(t)$ and $v(t)$ are decreasing functions of t. And for large t,the order of decrease is $O(1/t)$.

Generalization to real shape parameter

The mean residual lifetime in general is

$$e(t) = \frac{\int_t^{\infty} x f(x)\, dx}{R(t)} - t \quad . \tag{2.4}$$

The relationship between the hazard rate and $e(t)$ is given in [16] for the Gamma distribution as one of its characteristic property.

By writing the density and the reliability functions explicitly in 2.4 and by simplification, we get

$$e(t) = \frac{\int_t^{\infty} x^{\alpha} e^{-\lambda x}\, dx}{\int_t^{\infty} x^{\alpha-1} e^{-\lambda x}} - t \quad .$$

After this point, we apply integration by parts to the integral in the numerator, and calling the integral in the denominator I_1, the result in terms of I_1 only, is:

$$e(t) = \frac{1}{\lambda} + \frac{(\alpha-1)I_1 + t^{\alpha} e^{-\lambda t} - \lambda t I_1}{\lambda I_1}$$

Now, let $I_k = \int_t^{\infty} x^{\alpha-k} e^{-\lambda x}\, dx$ and apply integration by parts several times:

$$e(t) = \frac{1}{\lambda} + \frac{\frac{(\alpha-1)t^{\alpha-2}}{\lambda^2} + \frac{(\alpha-1)^2(\alpha-2)e^{\lambda t}}{\lambda^2} I_3 - \frac{(\alpha-1)(\alpha-2)(\alpha-3)te^{\lambda t}}{\lambda^2} I_4}{t^{\alpha-1} + (\alpha-1)e^{\lambda t} I_2}$$

Finally, for large t,

$$e(t) = \frac{1}{\lambda} + O(\frac{\alpha-1}{\lambda^2 t}) = \frac{1}{\lambda} + O(\frac{1}{t})$$

as in the case of integer shape parameter.

The variance for the general case is studied in the same manner. It is explicitly:

$$\begin{aligned} v(t) &= E[T^2|T>t] - E^2[T|T>t] \\ &= \frac{\int_t^\infty x^2 f(x)\,dx}{R(t)} - \frac{(\int_t^\infty x f(x)\,dx)^2}{R^2(t)} \end{aligned}$$

Putting the expressions of $R(t)$ and $f(x)$,and applying integration by parts several times, we get

$$v(t) = \frac{1}{\lambda^2} + \frac{\frac{(\alpha-1)(\alpha-2)(\alpha-3)t^{\alpha+1}e^{\lambda t}}{\lambda} I_4 - \frac{2(\alpha-2)(\alpha-1)\alpha t^\alpha e^{\lambda t}}{\lambda^2} I_3}{e^{2\lambda t} I_1^2} \quad ,$$

so that

$$v(t) = \frac{1}{\lambda^2} + \frac{O(t^{2\alpha-3})}{O(t^{2\alpha-2})} = \frac{1}{\lambda^2} + O(\frac{1}{t})$$

Percentiles of the residual lifetime distribution can be obtained in the same way as the percentiles for integer shape parameter. Let $n \leq \alpha < n+1$. Then the real shape parameter version of 2.3 is

$$1-p = (\int_x^\infty \frac{\lambda^\alpha (z+t)^{\alpha-1} e^{-\lambda(z+t)}}{\Gamma(\alpha)}\, dz)/(\int_t^\infty \frac{\lambda^\alpha z^{\alpha-1} e^{-\alpha z}}{\Gamma(\alpha)}\, dz)$$

Putting $u = z + t$, and simplifying the righthand side, we obtain

$$1-p = \frac{\int_{x+t}^\infty u^{\alpha-1} e^{-\lambda u}\, du}{\int_t^\infty z^{\alpha-1} e^{-\lambda z}\, dz}$$

By integration by parts several times, it is found that

$$1-p = \frac{\frac{(x+t)^{\alpha-1}e^{-\lambda(x+t)}}{\lambda} + \frac{(\alpha-1)(x+t)^{\alpha-2}e^{-\lambda(x+t)}}{\lambda^2} + \cdots + \frac{(\alpha-1)\ldots(\alpha-n+1)I_1}{\lambda^{n-1}}}{\frac{t^{\alpha-1}e^{-\lambda t}}{\lambda} + \frac{(\alpha-1)t^{\alpha-2}e^{-\lambda t}}{\lambda^2} + \cdots + \frac{(\alpha-1)\ldots(\alpha-n+1)I_2}{\lambda^{n-1}}} \quad ,$$

where

$$\begin{aligned} I_1 &= \int_{x+t}^\infty u^{\alpha-n} e^{-\lambda u}\, du \\ I_2 &= \int_t^\infty u^{\alpha-n} e^{-\lambda u}\, du \end{aligned}$$

Let's call the numerator of last equation as a and the denominator as b . Then taking the logarithm of both sides,

$$\log(1-p) = -\lambda(x+t) + \log(e^{\lambda(x+t)}a) + \lambda t - \log(e^{\lambda t} b)$$

By rearranging the logarithm terms on the righthand side and writing the smallest and highest orders of t in the summations,

$$\log(1-p) = -\lambda x + \log[1 + \tfrac{(\alpha-1)xt^{\alpha-2}/\lambda + \cdots + (\alpha-1)\ldots(\alpha-n+1)e^{\lambda(x+t)}I_1/\lambda^{n-1}}{t^{\alpha-1}/\lambda + \cdots + (\alpha-1)\ldots(\alpha-n+1)e^{\lambda t}I_2/\lambda^{n-1}}]$$

Because, for large t, the algebraic term inside the logarithm function in the last equation becomes small; we can use the approximation $\log(1+x) \cong x$. As a result,

$$x \cong \frac{1}{\lambda}\tfrac{(\alpha-1)xt^{\alpha-2}/\lambda + \cdots + (\alpha-1)\ldots(\alpha-n+1)e^{\lambda(x+t)}I_1/\lambda^{n-1}}{t^{\alpha-1}/\lambda + \cdots + (\alpha-1)\ldots(\alpha-n+1)e^{\lambda t}I_2/\lambda^{n-1}} + \xi_p(1) \quad .$$

Moreover for large t, the highest orders of t are the dominating terms in both the numerator and the denominator, so that

$$x \cong \frac{1}{\lambda}\,\frac{(\alpha-1)xt^{\alpha-2}/\lambda}{t^{\alpha-1}/\lambda} + \xi_p(1)$$

or

$$x \cong \frac{(\alpha-1)x}{\lambda t} + \xi_p(1)$$

and finally

$$x = [1 + \frac{\alpha-1}{\lambda t} + O(1/t^2)]\,\xi_p(1) \quad \forall\,\alpha > 0$$

So, all results pertaining to integer shape parameter case are also valid for real shape parameter. Moreover, these results suggest that, in model selection we have to study the MRL plots for relatively small t, in order to discriminate between different shape parameters of Gamma distribution, because for large t they all converge to $1/\lambda$ and thus discrimination becomes difficult. What is more, the results can be used in debugging or burn-in processes. For example, some trade off can be found in terms of the testing period; the MRL of the items decreases when they are debugged, but on the contrary the residual lifetime becomes more stable in terms of both its mean and the variance.

Chapter 3

Weibull Family of Distributions

In this chapter, T is considered to be distributed with Weibull distribution which is defined by any of the following three functions:

$$F(t) = 1 - e^{-(\lambda t)^{\alpha}} \quad , \qquad R(t) = e^{-(\lambda t)^{\alpha}} \quad ,$$

and

$$f(t) = \alpha\lambda(\lambda t)^{\alpha-1} e^{-(\lambda t)^{\alpha}} \quad \alpha > 0, \lambda > 0, t \geq 0$$

3.1 Hazard Rate and Hazard Rate Average Functions

The hazard rate function,

$$h(t) = \frac{f(t)}{R(t)} = \alpha\lambda(\lambda t)^{\alpha-1} \quad \alpha > 0, \lambda > 0 \quad ,$$

and the hazard rate average function,

$$H(t, 0) = (1/t) \int_0^t h(x)\, dx = \lambda^{\alpha} t^{\alpha-1}$$

It can easily be seen from the above equations that Weibull Distribution is increasing hazard rate (average) for $\alpha \geq 1$ and decreasing hazard rate (average) for $0 < \alpha \leq 1$.

3.2 Residual Lifetime Distribution

3.2.1 The Properties

The residual lifetime density:

$$f_{T-t}(x) = \frac{f(x+t)}{R(t)} = \frac{\alpha\lambda^\alpha (x+t)^{\alpha-1} e^{-[\lambda(x+t)]^\alpha}}{e^{-(\lambda t)^\alpha}}$$

The moment generating function is

$$M(s) = \frac{\int_0^\infty e^{sx}[\lambda(x+t)]^{\alpha-1} e^{-[\lambda(x+t)]^\alpha} \lambda\alpha \, dx}{e^{-(\lambda t)^\alpha}}$$

by using the identity $e^x = \sum_{i=0}^\infty \frac{x^i}{i!}$ and by making a change of variable $u = [\lambda(x+t)]^\alpha$, we get

$$M(s) = \frac{e^{-st}}{e^{-(\lambda t)^\alpha}} \sum_{i=0}^{\infty} (s/\lambda)^i \frac{\int_{(\lambda t)^\alpha}^\infty u^{i/\alpha} e^{-u} \, du}{\Gamma(i+1)}$$

Let

$$\Gamma(a, x) = \int_x^\infty u^{a-1} e^{-u} \, du$$

Then

$$M(s) = \frac{e^{-st}}{e^{-(\lambda t)^\alpha}} \sum_{i=0}^{\infty} (s/\lambda)^i \frac{\Gamma(i/\alpha + 1, (\lambda t)^\alpha)}{\Gamma(i+1)}$$

and

$$e(t) = M'(s)|_{s=0} = \frac{\Gamma(1 + 1/\alpha, (\lambda t)^\alpha)}{\lambda e^{-(\lambda t)^\alpha}} - t$$

The second moment is

$$M''(s)|_{s=0} = t^2 - \frac{2t\Gamma(1 + 1/\alpha, (\lambda t)^\alpha)}{\lambda e^{-(\lambda t)^\alpha}} + \frac{\Gamma(1 + 2/\alpha, (\lambda t)^\alpha)}{\lambda^2 e^{-(\lambda t)^\alpha}} \quad ,$$

so the variance

$$v(t) = \frac{\Gamma(1 + 2/\alpha, (\lambda t)^\alpha)}{\lambda^2 e^{-(\lambda t)^\alpha}} - \frac{\Gamma^2(1 + 1/\alpha, (\lambda t)^\alpha)}{\lambda^2 e^{-2(\lambda t)^\alpha}} \tag{3.1}$$

The p^{th} percentile ξ_p is the solution of the following equation in x:

$$p = P[T - t \leq x | T > t] = \frac{\int_0^x \alpha\lambda[\lambda(y+t)]^{\alpha-1} e^{-[\lambda(y+t)]^\alpha} \, dy}{e^{-(\lambda t)^\alpha}}$$

Evaluating the integral in the last expression by putting $u = [\lambda(y+t)]^\alpha$, we get

$$p = \frac{e^{-(\lambda t)^\alpha} - e^{-[\lambda(x+t)]^\alpha}}{e^{(-\lambda t)^\alpha}} \quad ,$$

and finally

$$\xi_p = t[1 - \frac{\log(1-p)}{(\lambda t)^\alpha}]^{1/\alpha} - t \tag{3.2}$$

3.2.2 Asymptotic Behavior of Expectation, Variance and Percentiles of Residual Lifetime Distribution

At $t = 0$, $e(t) = \frac{\Gamma(1+1/\alpha)}{\lambda}$, which is the expected value of T. For $t > 0$,

$$e(t) = \frac{\Gamma(1+1/\alpha, (\lambda t)^\alpha)}{\lambda e^{-(\lambda t)^\alpha}} - t \quad , \tag{3.3}$$

and in open form, $\Gamma(1+1/\alpha, (\lambda t)^\alpha) = \int_{(\lambda t)^\alpha}^\infty u^{1/\alpha} e^{-u}\, du$. By integration by parts, this can be reduced to

$$\Gamma(1+1/\alpha, (\lambda t)^\alpha) = \lambda t e^{-(\lambda t)^\alpha} + \int_{\lambda t}^\infty e^{-x^\alpha}\, dx \tag{3.4}$$

Let $I_1 = \int_{\lambda t}^\infty e^{-x^\alpha}\, dx$. I_1 cannot be evaluated but an approximation for large t can be found. By integration by parts, it can be shown that

$$\frac{e^{-(\lambda t)^\alpha}}{\alpha(\lambda t)^{\alpha-1}}[1 - \frac{\alpha-1}{\alpha(\lambda t)^\alpha}] \le I_1 \le \frac{e^{-(\lambda t)^\alpha}}{\alpha(\lambda t)^{\alpha-1}}$$

Then, for large t,

$$I_1 = \frac{e^{-(\lambda t)^\alpha}}{\alpha(\lambda t)^{\alpha-1}}[1 + O(1/t^\alpha)] \tag{3.5}$$

By putting this first in equation 3.4, then in equation 3.3, and by simplification,for large t, we obtain:

$$e(t) = \frac{1}{\alpha\lambda(\lambda t)^{\alpha-1}}[1 + O(1/t^\alpha)]$$

This last expression is valid for large t only, but it is much more practical for computation than equation 3.3.

So

$$\lim_{t\to\infty} e(t) = \begin{cases} 0 & for \quad \alpha \geq 1 \\ \infty & for \quad 0 < \alpha < 1 \end{cases}$$

And for $\alpha = 1$, $e(t) = 1/\lambda$.

The variance (in equation 3.1) can be studied in a similar fashion, then in open form,

$$\Gamma(1+2/\alpha,(\lambda t)^\alpha) = \int_{(\lambda t)^\alpha}^{\infty} u^{2/\alpha} e^{-u}\, du$$

By change of variables as $u = \lambda(x+t)$ and by integration by parts, we have

$$\Gamma(1+2/\alpha,(\lambda t)^\alpha) = \lambda^2 t^2 e^{-(\lambda t)^\alpha} + 2\int_{\lambda t}^{\infty} x e^{-x^\alpha}\, dx \tag{3.6}$$

Let $I_2 = \int_{\lambda t}^{\infty} x e^{-x^\alpha}\, dx$. For large t, I_2 can be approximated as

$$I_2 = \frac{e^{-(\lambda t)^\alpha}}{\alpha(\lambda t)^{\alpha-2}}[1+O(1/t^\alpha)]$$

Cancellation occurs with $\Gamma^2(1+1/\alpha,(\lambda t)^\alpha)$ in which there is I_1,which is also approximated as in equation 3.5. So, one more step is taken in approximation for both I_1 and for I_2. The results are as follows:

$$\begin{aligned} I_1 &= \frac{e^{-(\lambda t)^\alpha}(\alpha(\lambda t)^\alpha - \alpha + 1)}{\alpha^2(\lambda t)^{2\alpha-1}}[1+O(1/t^{2\alpha})] \quad ; \\ I_2 &= \frac{e^{-(\lambda t)^\alpha}(\alpha(\lambda t)^\alpha - \alpha + 2)}{\alpha^2(\lambda t)^{2\alpha-2}}[1+O(1/t^{2\alpha})] \end{aligned}$$

Putting those expressions in equation 3.4 and equation 3.6 respectively, then putting the results in equation 3.1, we obtain

$$v(t) = \frac{\alpha(\lambda t)^\alpha[\alpha(\lambda t)^\alpha + 2(\alpha-1)] - (1-\alpha)^2}{\lambda^2\alpha^4(\lambda t)^{4\alpha-2}}[1+O(1/t^{2\alpha})]$$

and

$$\lim_{t\to\infty} v(t) = \begin{cases} 0 & for \quad \alpha > 1 \\ \infty & for \quad 0<\alpha<1 \end{cases}$$

For $\alpha = 1$, $v(t) = 1/\lambda^2$. The asymptotic behavior of the percentiles can be studied in relation to equation 3.2 as follows, using the fact that $-\log(1-p) < (\lambda t)^2$ for large t, by the binomial expansion formula,

$$\xi_p = t[1 - \frac{1}{\alpha}\frac{\log(1-p)}{1!\,(\lambda t)^\alpha} + \frac{1}{\alpha}(\frac{1}{\alpha}-1)\frac{\log^2(1-p)}{2!\,(\lambda t)^{2\alpha}} - \frac{1}{\alpha}(\frac{1}{\alpha}-1)(\frac{1}{\alpha}-2)\frac{\log^3(1-p)}{3!\,(\lambda t)^{3\alpha}} + \cdots] - t \quad ,$$

so that

$$\begin{aligned} \xi_p &= -\frac{t}{\alpha}\frac{\log(1-p)}{(\lambda t)^\alpha}[1 - (\frac{1}{\alpha}-1)\frac{\log(1-p)}{2!\,(\lambda t)^\alpha} + (\frac{1}{\alpha}-1)(\frac{1}{\alpha}-2)\frac{\log^2(1-p)}{3!\,(\lambda t)^{2\alpha}} - \cdots] \\ &= -\frac{t\,\log(1-p)}{\alpha\,(\lambda t)^\alpha}[1+O(1/t^\alpha)] \end{aligned}$$

So

$$\lim_{t\to\infty} \xi_p = \begin{cases} 0 & for \quad \alpha > 1 \\ \infty & for \quad 0<\alpha<1 \end{cases}$$

The above result is valid for $\alpha \neq 1$. For $\alpha = 1$, the distribution is reduced to the exponential distribution which has been discussed in the previous chapter.

Having found the expressions for large t, for $e(t)$, $v(t)$, and ξ_p we can conclude that the greater the shape parameter the faster the convergence to 0 (when $\alpha > 1$), and the same is valid for the variance and the percentiles of the residual lifetime. In other words, when the item's lifetime is explained by Weibull distribution, the aging process is much severe for larger shape parameters. Hence, this property can be considered in discrimination of different shape parameters of Weibull distribution or in discrimination between different types of distributions.

Chapter 4

Lognormal and Inverse Polynomial Distributions

4.1 Lognormal Distribution

In this section, the lifetime T of an item is considered to be distributed with Lognormal distribution, which is defined by the following density function:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} x^{-1} e^{-\frac{(\log x - \mu)^2}{2\sigma^2}} \qquad \forall\, x \geq 0, \mu > 0, \sigma > 0.$$

This family of distributions is suitable especially when the data are positively skewed.

4.1.1 Hazard Rate and Hazard Rate Average Functions

The hazard rate function is:

$$h(t) = \frac{f(t)}{R(t)} = \frac{e^{-\frac{(\log t - \mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}\; t\; \Phi(\frac{\mu - \log t}{\sigma})} \tag{4.1}$$

and the hazard rate average function:

$$H(t,0) = (1/t)\int_0^t h(x)\, dx = (1/t)\int_0^t \frac{e^{-\frac{(\log x - \mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}\; x\; \Phi(\frac{\mu - \log x}{\sigma})}\, dx \tag{4.2}$$

In both equations 4.1 and 4.2, $\Phi(x)$ denotes the cumulative distribution function for $Z \sim N(0,1)$. The hazard rate function is not monotonic for the lognormal distribution.

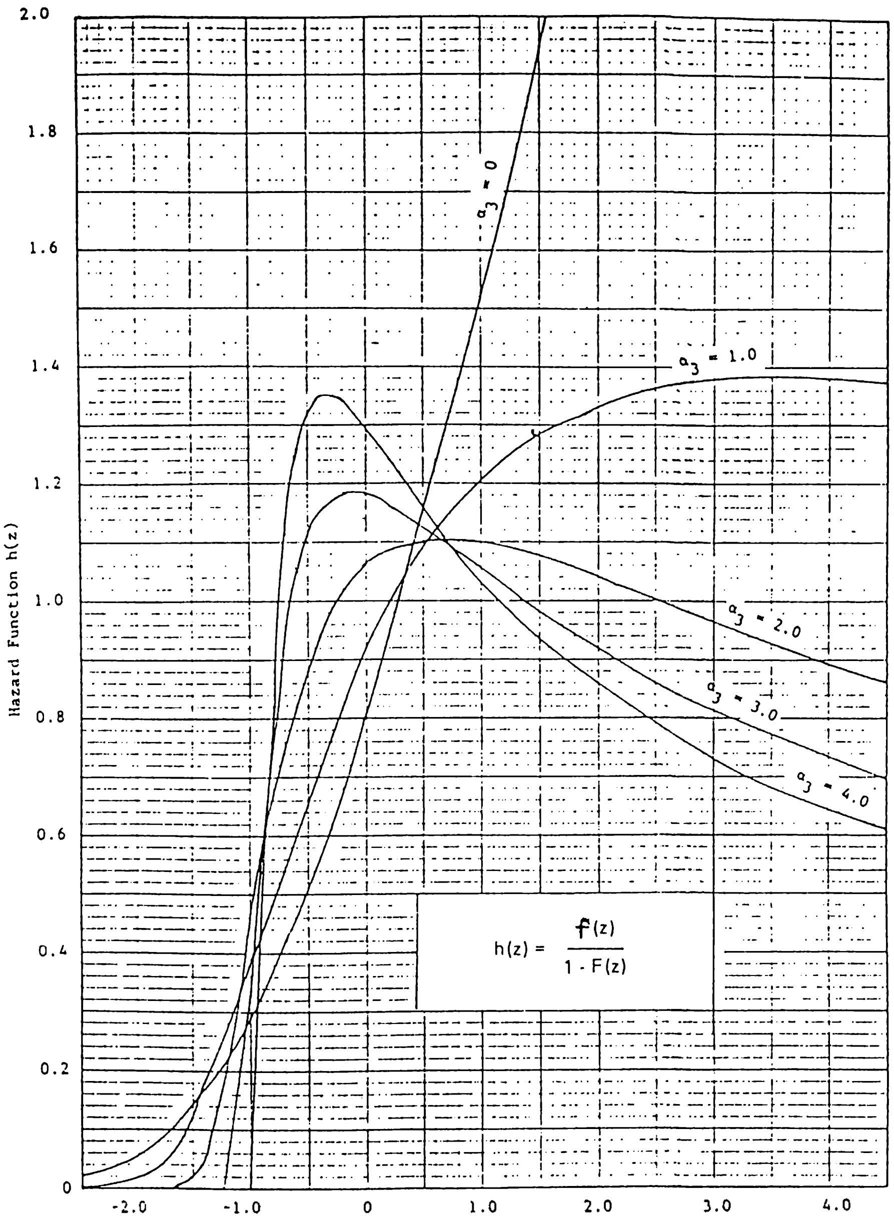

Figure 4.1: Lognormal hazard functions.

The plots of the hazard functions [7] identified with the value of α_3 where $\alpha_3 = E[(X - EX)^3]$ are shown in figure 4.1.

4.1.2 Residual Lifetime Distribution

The Properties

The density of the residual lifetime distribution is:

$$f_{T-t}(x) = \frac{f(x+t)}{R(t)}$$

$R(t)$ can be found in terms of Φ, having the definition of T as $T = e^Y$ where $Y \sim N(\mu, \sigma^2)$. as follows:

$$R(t) = P[T > t] = P[\log T > \log t] = P[Y > \log t] = \Phi(\frac{\mu - \log t}{\sigma})$$

So,

$$f_{T-t}(x) = \frac{(x+t)^{-1} e^{-\frac{(\log(x+t)-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}\,\Phi(\frac{\mu-\log t}{\sigma})}$$

Accordingly, the moment generating function $M(s)$ can be found by the change of variables $u = x + t$, then $z = \frac{\log u - \mu}{\sigma}$, and then $y = z - s\sigma$.

$$M(s) = \frac{e^{s(\mu - t + s\sigma^2/2)}\Phi(s\sigma - \frac{\log t - \mu}{\sigma})}{\Phi(\frac{\mu - \log t}{\sigma})}$$

$M(s)$ is not suitable for finding the moments, since it contains the function Φ. The moments are found directly by their definition. The first moment, i.e., the mean residual life:

$$\epsilon(t) = E[T - t | T > t] = \frac{1}{R(t)} \int_0^\infty \frac{x e^{-\frac{(\log(x+t)-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}(x+t)}\, dx \quad . \tag{4.3}$$

Equation 4.3 is simplified first by change of variables as $u = x + t$ then as $z = \frac{\log u - \mu}{\sigma}$. The result is:

$$e(t) = \frac{e^{\mu + \sigma^2/2}\Phi(\sigma - \frac{\log t - \mu}{\sigma})}{\Phi(\frac{\mu - \log t}{\sigma})} - t \tag{4.4}$$

Similarly. the second moment

$$E[(T-t)^2 | T > t] = \frac{1}{R(t)} \int_0^\infty \frac{x^2 e^{-\frac{(\log(x+t)-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}(x+t)}\, dx \tag{4.5}$$

By making the change of variables, first $u = x + t$ then $z = \frac{\log u - \mu}{\sigma}$ in equation 4.5, we have:

$$E[(T-t)^2|T>t] = \frac{e^{2(\mu+\sigma^2)}\Phi(2\sigma - \frac{\log t - \mu}{\sigma}) - 2te^{\mu+\sigma^2/2}\Phi(\sigma - \frac{\log t - \mu}{\sigma})}{\Phi(\frac{\mu - \log t}{\sigma})} + t^2$$

Consequently,

$$v(t) = \frac{e^{2(\mu+\sigma^2)}\Phi(2\sigma - \frac{\log t - \mu}{\sigma})}{\Phi(\frac{\mu - \log t}{\sigma})} - \frac{e^{2\mu+\sigma^2}\Phi^2(\sigma - \frac{\log t - \mu}{\sigma})}{\Phi^2(\frac{\mu - \log t}{\sigma})} \tag{4.6}$$

The p^{th} percentile ξ_p can be found in a similar fashion, and it is the solution of the following equation in x.

$$p = \frac{\Phi(\frac{\log(x+t)-\mu}{\sigma}) - \Phi(\frac{\log t - \mu}{\sigma})}{1 - \Phi(\frac{\log t - \mu}{\sigma})}$$

It can be found from the tables for $Z \sim N(0,1)$, as follows: Let $z = \frac{\log(x+t)-\mu}{\sigma}$. Then, by rearrangement:

$$\Phi(z) = p + (1-p)\Phi(\frac{\log t - \mu}{\sigma})$$

Thus, z can be found from the tables, and

$$\xi_p = x = e^{\sigma z + \mu} - t$$

Asymptotic Behavior of Expectation and Variance

At $t = 0$, by equation 4.4 $\lim_{t\to 0} e(t) = e^{\mu+\sigma^2/2}$, which is the expected value of T. For $t > 0$, by making the change of variable $u = x + t$ in equation 4.3, we get

$$e(t) = \frac{1}{\sigma\sqrt{2\pi}R(t)}\left[\int_t^\infty e^{-\frac{(\log u - \mu)^2}{2\sigma^2}}\,du - t\int_t^\infty \frac{e^{-\frac{(\log u-\mu)^2}{2\sigma^2}}}{u}\,du\right] \tag{4.7}$$

Let I_1 be the first and I_2 be the second integral, respectively in equation 4.7. Then I_1 can be simplified by making the change of variables $y = \frac{\log u - \mu}{\sqrt{2}\sigma}$ and $x = y - \sigma/\sqrt{2}$. The result is,

$$I_1 = \sqrt{2}\sigma e^{\mu+\sigma^2/2}\int_a^\infty e^{-x^2}\,dx \quad ,$$

where $a = \frac{\log t - \mu - \sigma^2}{\sqrt{2}\sigma}$. But, as in section 3.3.2, $I = \int_t^\infty e^{-x^2}\,dx$ can be approximated for large t as:

$$I = \frac{e^{-t^2}}{2t}[1 + O(1/t^2)] \tag{4.8}$$

Consequently, for large t, by simplification

$$I_1 = \frac{\sigma t e^{-\frac{(\log t-\mu)^2}{2\sigma^2}}}{\frac{\log t-\mu}{\sigma} - \sigma}[1 + O(1/\log^2 t)] \tag{4.9}$$

Similarly, by the change of variables $y = \frac{\log u-\mu}{\sqrt{2}\sigma}$, I_2 can be approximated for large t as follows:

$$I_2 = \frac{\sigma e^{-\frac{(\log t-\mu)^2}{2\sigma^2}}}{\frac{\log t-\mu}{\sigma}}[1 + O(1/\log^2 t)] \tag{4.10}$$

Then, $R(t)$ may also be approximated for large t. It was shown in the previous section that $R(t) = \Phi(\frac{\mu-\log t}{\sigma})$, i.e.:

$$R(t) = \int_{\frac{\log t-\mu}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

By change of variables $y = z/\sqrt{2}$ and applying approximation 4.8

$$R(t) = \frac{e^{-\frac{(\log t-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\frac{\log t-\mu}{\sigma}}[1 + O(1/\log^2 t)] \tag{4.11}$$

Putting equations 4.9, 4.10, 4.11 in equation 4.7, we obtain

$$e(t) = \frac{\sigma^3 t}{(\log t - \mu - \sigma^2)(\log t - \mu)} \frac{[1 + O(1/\log^2 t)]}{[1 + O(1/\log^2 t)]}$$

or

$$e(t) = \frac{\sigma^3 t}{(\log t - \mu - \sigma^2)(\log t - \mu)}[1 + O(1/\log^2 t)]$$

As a result , as $t \to \infty$, $e(t) \to \infty$ with $O(t/\log^2 t)$.

Now, the variance of the residual lifetime at $t = 0$, is found by taking the limit of both sides in equation 4.6 as t tends to 0. So, $\lim_{t\to 0} v(t) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$, which is the variance of T. For $t > 0$, by making the change of variables $u = x + t$ in equation 4.5, we obtain the second moment of the residual lifetime,

$$E[(T-t)^2|T > t] = \frac{1}{R(t)\sigma\sqrt{2\pi}} \int_t^{\infty} \frac{(u-t)^2}{u} e^{-\frac{(\log u-\mu)^2}{2\sigma^2}}\, du$$

Then by expanding $(u-t)^2$, and comparing the result with I_1 and I_2,we find the following expression

$$E[(T-t)^2|T > t] = \frac{1}{R(t)\sigma\sqrt{2\pi}} \int_t^{\infty} u e^{-\frac{(\log u-\mu)^2}{2\sigma^2}}\, du - 2tI_1 + t^2 I_2 \tag{4.12}$$

Moreover, comparing equations 4.11 and 4.10 , it can be seen that $I_2 = \sigma\sqrt{2\pi}R(t)$. Before finding $v(t)$, let us also approximate the first integral in equation 4.12 for large

t. Let it be I_3. By the change of variables $y = \frac{\log u - \mu}{\sqrt{2}\sigma}$, and then $x = y - \sqrt{2}\sigma$ and by use of equation 4.8, for large t, we get

$$I_3 = \frac{\sigma t^2 e^{-\frac{(\log t - \mu)^2}{2\sigma^2}}}{\frac{\log t - \mu}{\sigma} - 2\sigma}[1 + O(1/\log^2 t)]$$

Note that equation 4.7 can be rewritten as $e(t) = (1/I_2)(I_1 - tI_2)$. Now, replacing the first integral by I_3 and $\sigma\sqrt{2\pi}R(t)$ by I_2 in equation 4.12, we find the variance $v(t)$ as follows:

$$v(t) = E[(T-t)^2|T > t] - e^2(t) = \frac{I_2 I_3 - I_1^2}{I_2^2}$$

Then, for large t, by simplification:

$$v(t) = \frac{t^2\sigma^4(\log t - \mu)^2}{(\log t - \mu)(\log t - \mu - 2\sigma^2)(\log t - \mu - \sigma^2)^2}[1 + O(1/\log^2 t)]$$

As a result, as $t \to \infty$ $v(t) \to \infty$ with $O(t^2/\log^2 t)$.

4.2 Inverse Polynomial Distribution

In this section, the lifetime T of an item is considered to be distributed with a distribution which is entitled as *Inverse Polynomial Distribution* in this thesis. The density is:

$$f(x) = \frac{\beta}{(1+x)^{\beta+1}} \quad \forall x \geq 0, \beta > 0 .$$

The reliability function is $R(t) = 1/(1+x)^{\beta}$. This class of distributions are similar to the *Pareto Distribution* ,except that it does not have a truncation parameter.

4.2.1 Hazard Rate and Hazard Rate Average

The hazard rate function follow directly from the definition of $f(t)$ and $R(t)$:

$$h(t) = \frac{\beta}{1+t}$$

The hazard rate average function,

$$H(t, 0) = \frac{\beta \log(1+t)}{t}$$

So h(t) decreases as t increases, and the inverse polynomial distribution is DHR. Thus, this family of distributions can be a suitable model in situations where product or systems development results in improved performance as development proceeds.

4.2.2 Residual Lifetime Distribution

The residual lifetime density is:

$$f_{T-t}(x) = \frac{f(x+t)}{R(t)} = \frac{\beta(1+t)^{\beta}}{(1+x+t)^{\beta+1}}$$

The moments of residual lifetime distribution can be found by direct calculation as shown below.The r^{th} moment,

$$E[(T-t)^r|T>t] = \int_0^{\infty} \frac{x^r \beta(1+t)^{\beta}}{(1+x+t)^{\beta+1}}\, dx \quad ,$$

by integration by parts several times,

$$E[(T-t)^r|T>t] = \frac{(1+t)^r r!}{(\beta-1)(\beta-2)\ldots(\beta-r)} \quad for \quad \beta > r$$

So the r^{th} moment exists if and only if $r < \beta$. First, $e(t)$ is found for $\beta > 1$, as

$$e(t) = \frac{1+t}{\beta-1}$$

This implies that $e(t) = (1+t)e(0)$, i.e., mean residual lifetime function is increasing linearly with t . At $t = 0$, $e(t) = 1/(\beta-1)$, which is the expectation of T . And as t tends to ∞, obviously $e(t) \to \infty$ with $O(t)$.

Similarly, the variance exists for $\beta > 2$; it is:

$$v(t) = \frac{\beta(1+t)^2}{(\beta-1)^2(\beta-2)}$$

At $t = 0$, $v(t) = \frac{\beta}{(\beta-1)^2(\beta-2)}$ which is the variance of T. And as t tends to ∞, $v(t) \to \infty$ with $O(t^2)$. From the expression of $e(t)$, we can conclude that the item is negatively aging, as it is implied by the decreasing hazard rate. However, it should also be noted that the variance of the residual lifetime also increases rapidly in time. Inverse Polynomial distribution is thus, a suitable model when both the performance and its variance increases (provided that it exists). MRL plot can be used in distinguishing these features.

The analysis of the mean and the variance depends on β; however percentiles exist for all β. The p^{th} percentile ξ_p is the solution of the following equation in x:

$$p = \int_0^{x} \frac{\beta(1+t)^{\beta}}{(1+y+t)^{\beta+1}}\, dy$$

By the change of variables $u = 1 + y + t$, it is simplified as

$$p = 1 - (\frac{1+t}{1+x+t})^{\beta} \quad ,$$

that for fixed t,

$$\xi_p = \frac{(1+t)[1-(1-p)^{1/\beta}]}{(1-p)^{1/\beta}} \quad .$$

As t tends to infinity, $\xi_p \to \infty$ with $O(t)$.

Chapter 5

Data Analysis and Conclusion

In this chapter, four different sets of data are studied for the purpose of illustrating the use of mean residual lifetime function as a criterion for aging. The first two sets are lifetimes of 75 Watt bulbs produced in two different Turkish factories; the third one is the lifetime of Kevlar 49/Epoxy Strands [2] (tested at 70% stress level), and the last one is the service time between failures of the air-conditioning equipment in Boeing 720 jet aircraft [8]. For this last set, the essential assumption is that after repair the equipment becomes as good as new. The first three sets of data are examples of items with decreasing mean residual life. i.e., the items are aging in time. Conversely, the last data set is an example of exponentially distributed time between failures (lifetimes. in a sense). The reason for selecting decreasing mean residual lifetime distributions is the greater importance of aging items in production environment than those with no aging, in terms of quality. The data are given in tables 5.1 and 5.2.

At this point it must be stated that data set 1 and data set 2 are ordered statistics taken from random samples, tested under 320 Volts and 286 Volts, respectively, voltages which are never encountered during normal usages of those bulbs. These high voltages are only for shortening the test period and observing *all* of the bulbs' lifetime, since under 220 Volts the lifetime of a bulb may be in years. These two data sets are in fact, for the purpose of illustration rather than making inferences on the lifetimes of bulbs in ordinary usage.

Table 5.1: The Data Sets (description is given in the text)

Data Set 1 (min.)	Data Set 2 (hr.)	Data Set 3 (hr.)	Data Set 4 (hr.)
295	750	1051	3
420	778	1337	5
420	865	1389	5
445	904	1921	13
450	956	1942	14
470	983	2322	15
470	988	3629	22
500	1000	4006	22
500	1034	4012	23
502	1061	4063	30
525	1063	4921	36
540	1063	5445	39
550	1065	5620	44
550	1097	5817	46
555	1100	5905	50
555	1108	5956	72
560	1116	6068	79
570	1124	6121	88
580	1179	6473	97
580	1210	7501	102
600	1214	7886	139
605	1222	8108	188
610	1285	8546	197
630	1297	8666	210
630	1308	8831	
645	1308	9106	
660	1380	9711	
660	1399	9806	
675	1415	10205	
685	1466	10396	
690	1494	10861	
690	1533	11026	
690	1533	11214	
695	1580	11362	
700	1612	11604	
715	1698	11608	
715	1698	11745	
720	1765	11762	
720	1824	11895	
725	1946	12044	

Table 5.2: The Data Sets Continued

Data Set 1	Data Set 2	Data Set 3
750	1946	13520
755	1968	13670
768	2005	14110
780	2005	14496
780	2005	15395
785	2264	16179
800	2314	17092
810	2319	17568
810	2332	17568
830	2458	
840		
870		
870		
870		
885		
885		
890		
900		
909		
910		
915		
915		
915		
930		
950		
960		
960		
960		
960		
990		
990		
1005		
1005		
1020		
1050		
1050		
1080		
1185		
1200		
1200		

The natural estimator for the mean residual lifetime function is:

$$\hat{e}(t) = S^{-1}\sum_{j=1}^{S}(t_j - t) \quad ,$$

where S denotes the number of survivors at time t out of an initial population of size n, and $\{t_j : j = 1, \ldots, S\}$ is the set of data points which are greater than t.

Since the first three data sets show decreasing mean residual lifetime, Gamma and Weibull distributions were fitted to them. Moreover they reveal the presence of a truncation parameter, because the smallest value in the set is comparatively high. We fitted three parameter Weibull and Gamma distributions. In estimation procedure, we estimated truncation parameter by the smallest data point (let it be x_1), then transferred the data to a new one by subtracting x_1 from each data point. Then we fitted two parameter Weibull and Gamma distributions to the transferred data set. We have used moment estimators.

For random variable X distributed with Weibull distribution, let γ be the truncation parameter. Then the density function is:

$$f(x;\lambda,\alpha,\gamma) = \alpha\lambda^{\alpha}(x-\gamma)^{\alpha-1}e^{-[\lambda(x-\gamma)]^{\alpha}} \quad for \ \gamma < x < \infty, \alpha > 0, \lambda > 0 \, .$$

The interpretation of γ can be some type of guarantee period, in our concern. Then we estimate it as $\hat{\gamma} = x_1$. The first two moments of the distribution are:

$$E(X - x_1) = \frac{\Gamma_1}{\lambda} \quad V(X - x_1) = \frac{\Gamma_2 - {\Gamma_1}^2}{\lambda^2}$$

where $\Gamma_k = \Gamma(1 + k/\alpha)$.

On equating the first two sample moments to corresponding distribution moments as given above, the moment estimators become

$$\frac{\hat{\Gamma}_1}{\hat{\lambda}} = \bar{x} - x_1, \quad \frac{\hat{\Gamma}_2 - \hat{\Gamma}_1^{\,2}}{\hat{\lambda}^2} = s^2 \tag{5.1}$$

In [7], the α values versus the coefficient of variation (C.V.) are tabulated; it is reproduced in table 5.3. So the estimation procedure is simplified by first finding an approximation for $\hat{\alpha}$ from this table. This approximation is used to determine $\hat{\lambda}$ from either of the equations in 5.1. It is then, improved by trial and error until both of them gave the same result for $\hat{\lambda}$.

Table 5.3: Values of Mode, Median and Coefficient of Variation for the Weibull Distribution

α	C	D	α_3	α_4	Mo	Me	C.V.
.50	.22361	.44721	6.61876	87.72000	J-SHAPE	-.33973	2.23607
.55	.29893	.50890	5.43068	57.39817	J-SHAPE	-.35533	1.96502
.60	.37805	.56881	4.59341	40.48166	J-SHAPE	-.36357	1.75807
.65	.45895	.62706	3.97420	30.20718	J-SHAPE	-.36591	1.59475
.70	.54020	.68380	3.49837	23.54202	J-SHAPE	-.36379	1.46242
.75	.62082	.73917	3.12124	18.98700	J-SHAPE	-.35834	1.35236
.80	.70020	.79333	2.81465	15.74074	J-SHAPE	-.35048	1.26051
.85	.77796	.84638	2.56009	13.34657	J-SHAPE	-.34092	1.18150
.90	.85389	.89845	2.34496	11.53005	J-SHAPE	-.33020	1.11303
.95	.92791	.94963	2.16040	10.11872	J-SHAPE	-.31874	1.05305
1.00	1.00000	1.00000	2.00000	9.00000	J-SHAPE	-.30685	1.00000
1.05	1.07020	1.04965	1.85904	8.09795	-.99074	-.29478	.95270
1.10	1.13859	1.09864	1.73397	7.35985	-.96992	-.28269	.91022
1.15	1.20525	1.14703	1.62204	6.74819	-.94199	-.27071	.87181
1.20	1.27027	1.19488	1.52113	6.23571	-.90950	-.25894	.83690
1.25	1.33375	1.24223	1.42955	5.80215	-.87419	-.24744	.80500
1.30	1.39580	1.28913	1.34593	5.43226	-.83731	-.23624	.77572
1.35	1.45651	1.33560	1.26920	5.11432	-.79975	-.22539	.74873
1.40	1.51597	1.38169	1.19844	4.83923	-.76215	-.21490	.72375
1.45	1.57427	1.42742	1.13291	4.59983	-.72495	-.20477	.70056
1.50	1.63149	1.47282	1.07199	4.39040	-.68848	-.19530	.67897
1.55	1.68771	1.51792	1.01515	4.20636	-.65296	-.18551	.65880
1.60	1.74300	1.56273	.96196	4.04396	-.61852	-.17657	.63991
1.65	1.79743	1.60728	.91202	3.90015	-.58527	-.16738	.62217
1.70	1.85104	1.65[illegible]	.86502	3.77238	-.55324	-.15953	.60548
1.75	1.90391	1.69566	.82058	3.65855	-.52245	-.15151	.58974
1.80	1.95608	1.73952	.77874	3.55688	-.49291	-.14380	.57487
1.85	2.00760	1.78317	.73899	3.46588	-.46459	-.13639	.56080
1.90	2.05850	1.82664	.70124	3.38428	-.43747	-.12927	.54745
1.95	2.10885	1.86993	.66533	3.31100	-.41150	-.12243	.53478
2.00	2.15866	1.91306	.63111	3.24509	-.38666	-.11586	.52272
2.25	2.40084	2.12650	.48121	3.00148	-.27762	-.08655	.47026
2.50	2.63389	2.33696	.35863	2.85678	-.18983	-.06224	.42791
2.75	2.86013	2.54511	.25589	2.77332	-.11846	-.04186	.39291
3.00	3.08119	2.75144	.16810	2.72946	-.05977	-.02460	.36345
3.25	3.29822	2.95630	.09196	2.71207	-.01093	-.00983	.33826
3.50	3.51206	3.15997	.02511	2.71273	.03018	.00292	.31646
3.75	3.72336	3.36265	-.03419	2.72591	.06515	.01402	.29738
4.00	3.93253	3.56450	-.03724	2.74783	.09518	.02376	.28054
4.25	4.14008	3.76564	-.13504	2.77585	.12119	.03237	.26556
4.50	4.34616	3.96619	-.17838	2.80811	.14390	.04002	.25213
5.00	4.75490	4.36580	-.25411	2.88029	.18156	.05302	.22905
6.00	5.56274	5.16066	-.37326	3.03546	.23559	.07245	.19377
7.00	6.36237	5.95160	-.46319	3.18718	.27219	.08621	.16802
8.00	7.15690	6.73996	-.53373	3.32768	.29847	.09645	.14837

Table 5.4: Estimated values of parameters

	Data Sets					
	1		2		3	
	Weibull	Gamma	Weibull	Gamma	Weibull	Gamma
α	2.44	5.22	1.55	2.29	1.75	2.9
λ	0.00193	0.0113	0.0013	0.0033	0.000115	0.000374
γ	295	295	750	750	1051	1051

For Gamma distribution, with truncation parameter γ,

$$f(x;\lambda,\alpha,\gamma) = \frac{\lambda^{\alpha}(x-\gamma)^{\alpha-1}e^{-\lambda(x-\gamma)}}{\Gamma(\alpha)} \quad \gamma < x < \infty, \alpha > 0, \lambda > 0.$$

As explained before, $\hat{\gamma} = x_1$. The first two moments are:

$$E(X - x_1) = \frac{\alpha}{\lambda} \quad V(X - x_1) = \frac{\alpha}{\lambda^2}$$

On equating the first two sample moments to corresponding distribution moments, the moment estimators become

$$\hat{\lambda} = \frac{\bar{x} - x_1}{s^2} \quad and \quad \hat{\alpha} = \hat{\lambda}(\bar{x} - x_1) = \frac{(\bar{x} - x_1)^2}{s^2}$$

The results are given in table 5.4. As it can be observed from this table, the shape parameters are strictly greater than 1, so the items are aging in time.

First, to simplify the calculations, the shape parameter for Gamma distribution is rounded to the nearest integer and the results obtained in Chapter 2 are applied, for finding the mean residual lifetime function. The reliability and MRL functions are plotted in figures 5.1 to 5.2, 5.3 to 5.4, and 5.5 to 5.6 for the data sets 1,2 and 3, respectively. The empirical reliability function is found from the following equation:

$$\hat{R}(t) = \frac{S}{n}$$

where S and n are as defined before.

Moreover, in the graphs of $e(t)$ versus t, the upper bound (UB) and lower bound (LB) are found to be the approximate 95 % confidence limits as follows:

$$\begin{aligned} UB &= \hat{e}(t) + 2\frac{s(t)}{\sqrt{S}} \\ LB &= \hat{e}(t) - 2\frac{s(t)}{\sqrt{S}} \end{aligned} ,$$

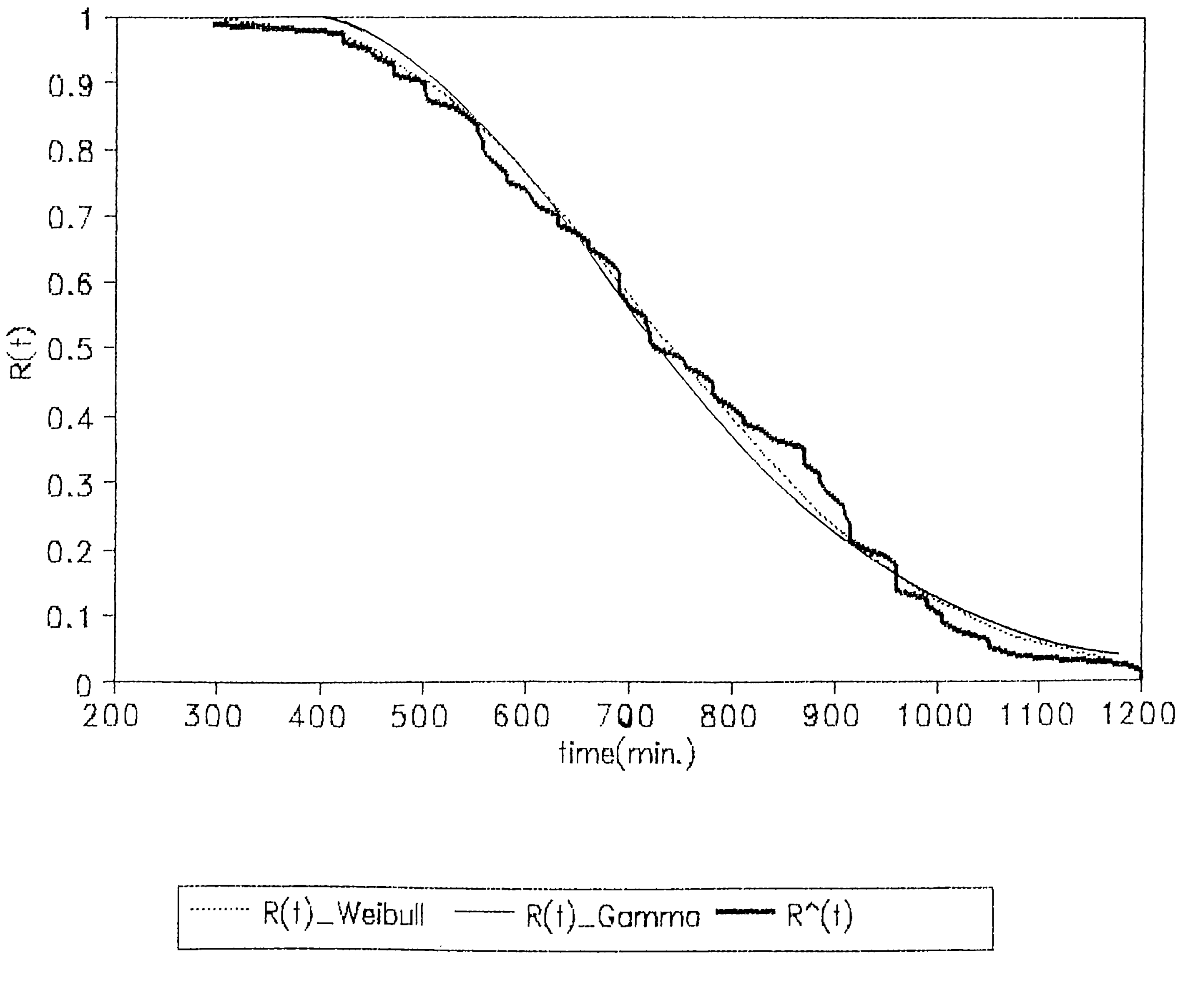

Figure 5.1: The fitted and empirical ($\hat{R}(t)$) reliability functions for Data Set 1

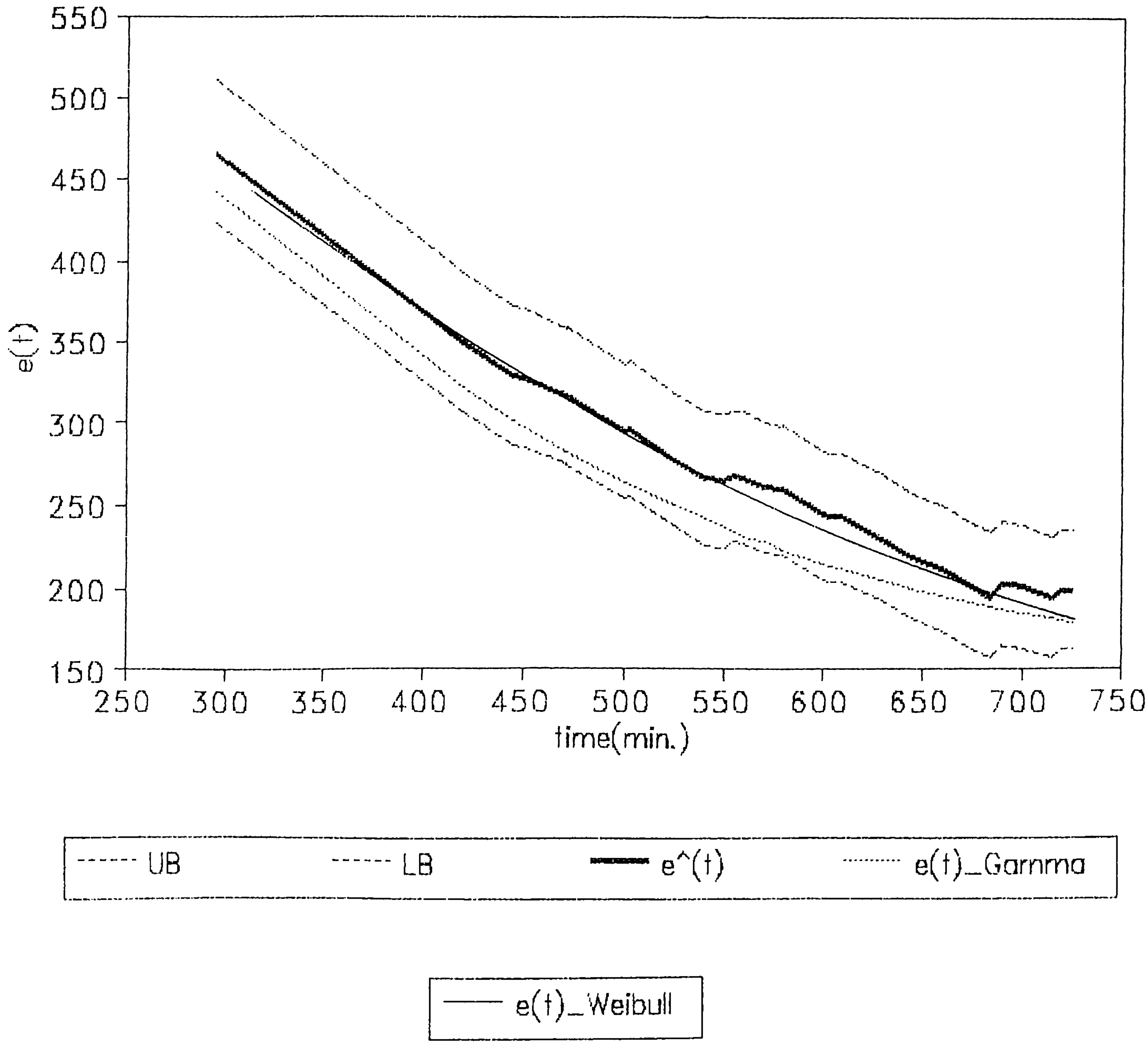

Figure 5.2: The fitted Gamma and Weibull $e(t)$ and the empirical $\hat{e}(t)$ for Data Set 1

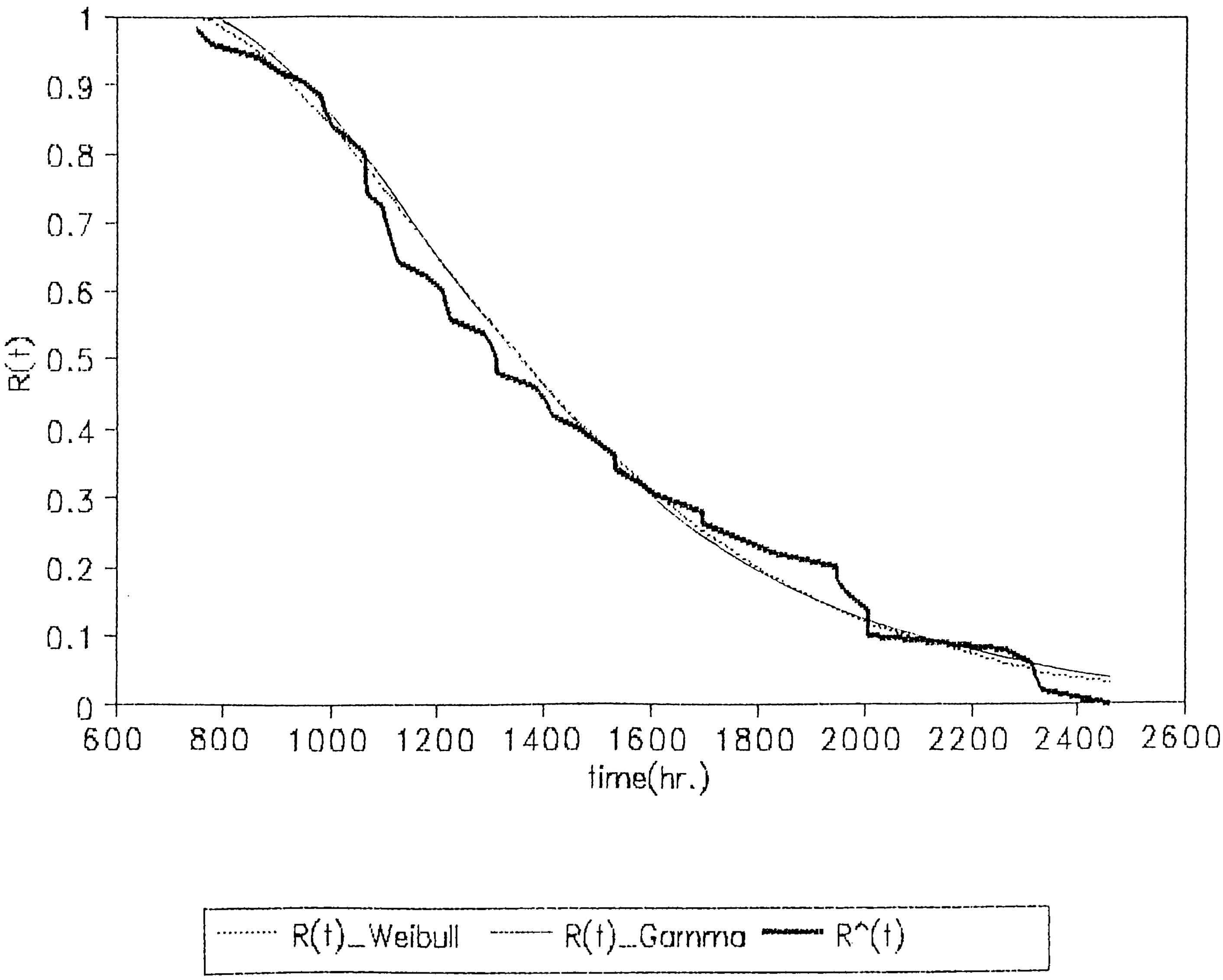

Figure 5.3: The fitted and empirical ($\hat{R}(t)$) reliability functions for Data Set 2

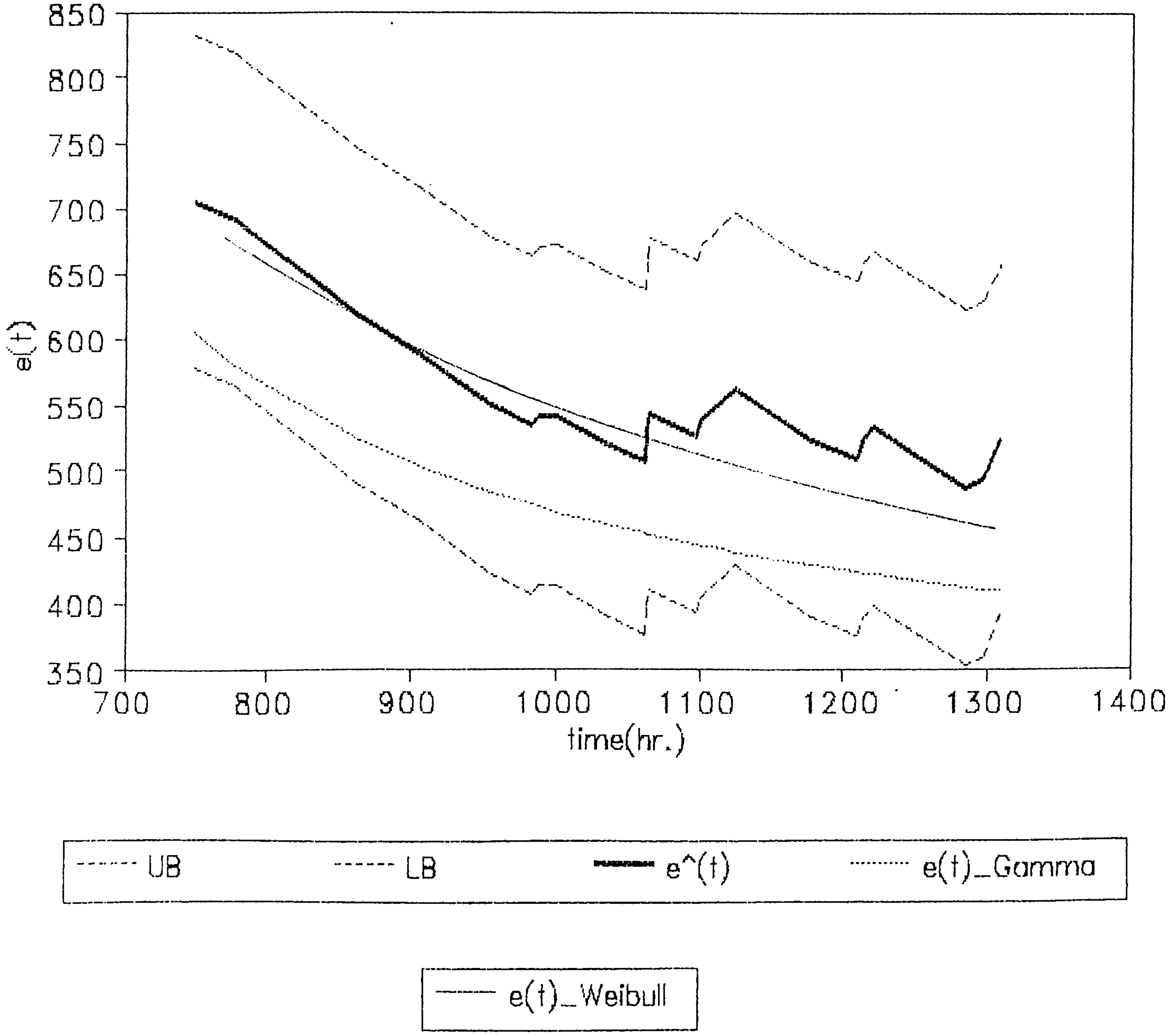

Figure 5.4: The fitted Gamma and Weibull $e(t)$ and the empirical $\hat{e}(t)$for Data Set 2

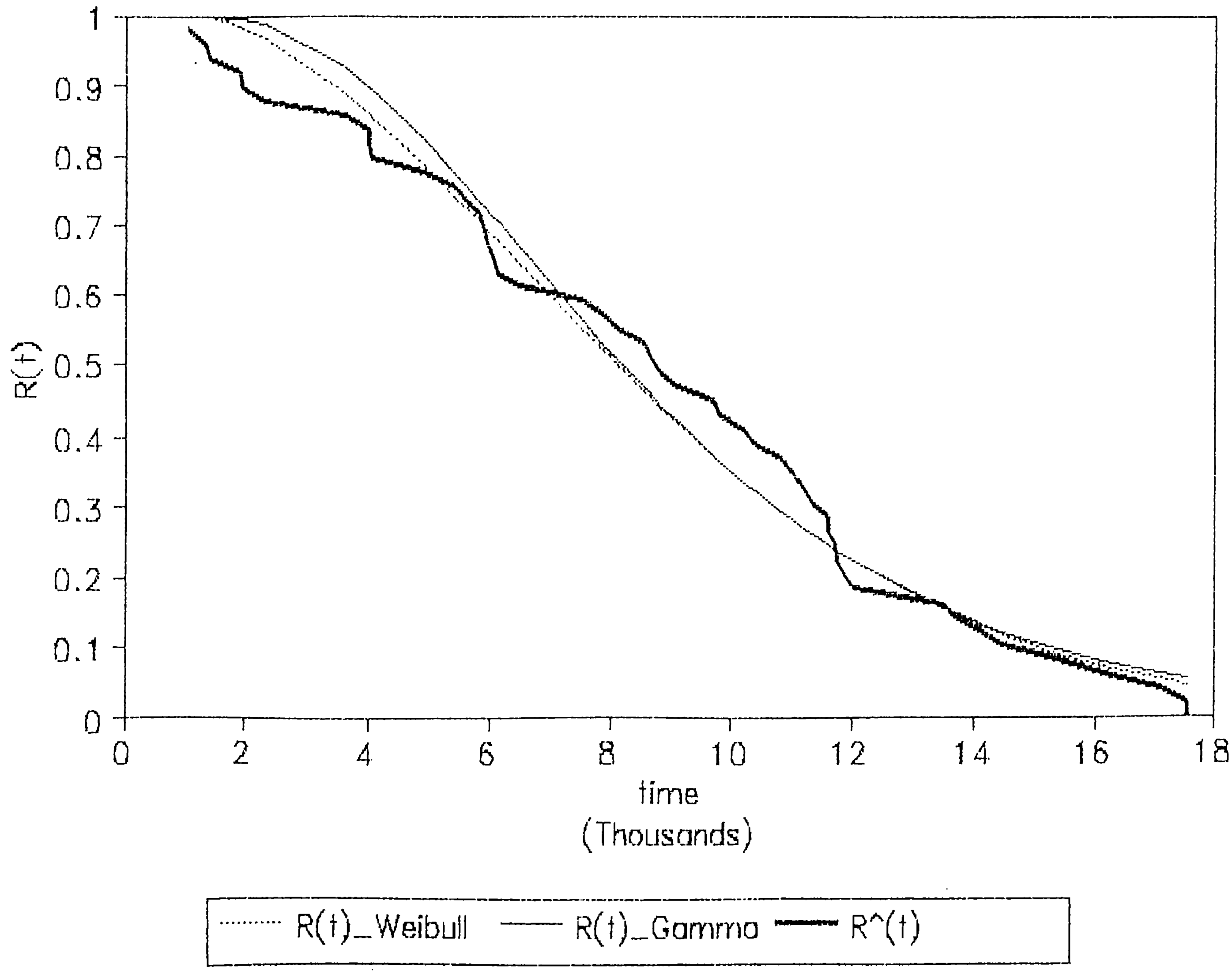

Figure 5.5: The fitted and empirical ($\hat{R}(t)$) reliability functions for Data Set 3

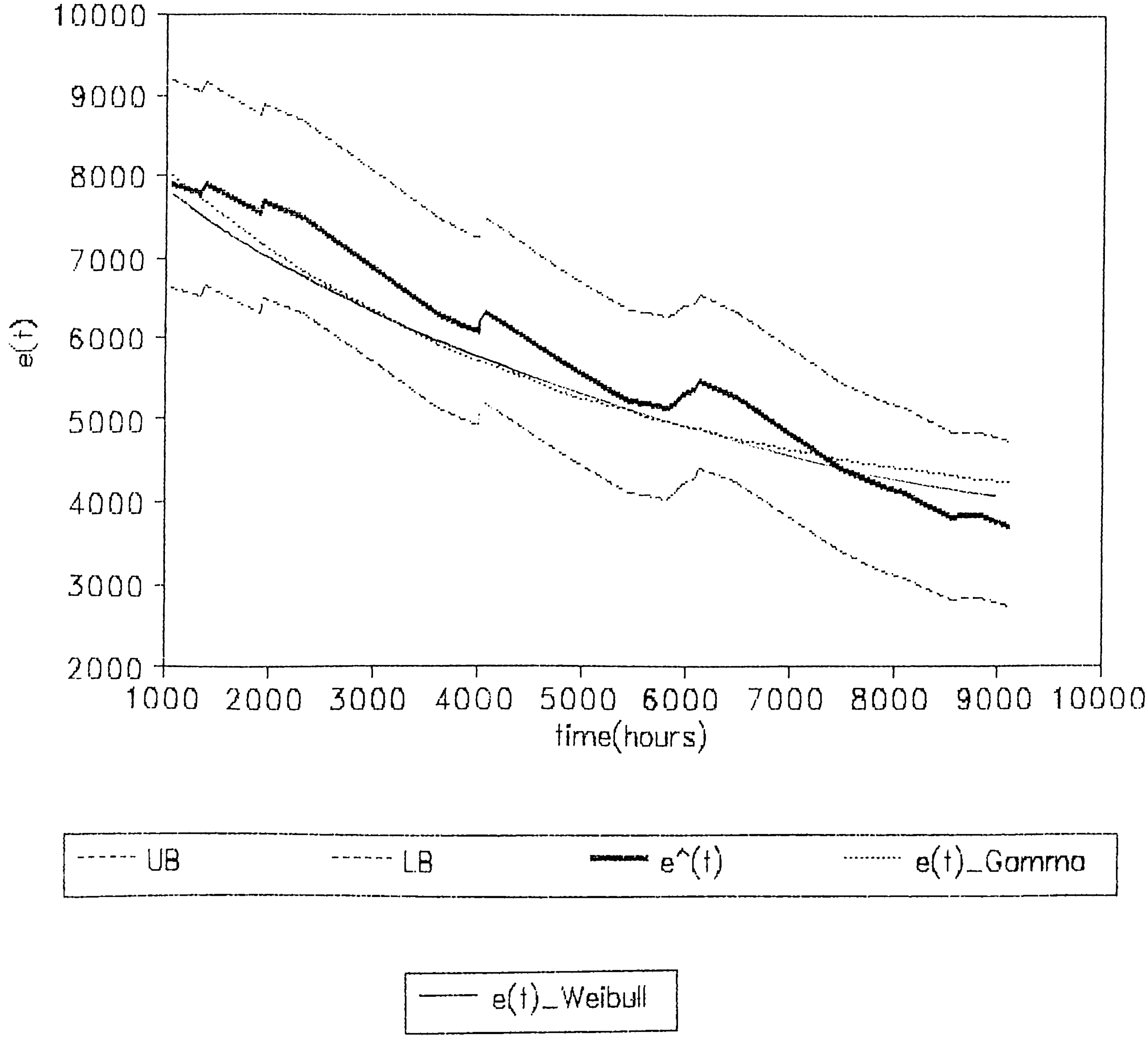

Figure 5.6: The fitted Gamma and Weibull $e(t)$ and the empirical $\hat{e}(t)$ for Data Set 3

where $s(t)$ is the estimated standard deviation of the residual lifetime at t, i.e.

$$s(t) = (S-1)^{-1} \sum_{j=1}^{S} [t_j - e(t)]^2$$

In figures 5.2 and 5.4, it can be seen that the theoretical values for Gamma distribution underestimate the empirical mean residual lifetime (MRL) function because we underestimated the shape parameter for the data sets 1 and 2 by rounding them off to the nearest integer. The corrected graphs of $e(t)$ versus t for Gamma distribution are shown in figures 5.7 and 5.8 for data sets 1 and 2, respectively. In those graphs, theoretical and emprical values fit better.

The last data set was fitted a Gamma Distribution using MLE's in [8] without assuming a truncation parameter, and $\hat{\alpha}$ and $\hat{\lambda}$ were found to be 1.06 and 0.0156, respectively. So we can consider this set as a sample from an exponential distribution with parameter 0.0156. The reliability and MRL functions are plotted in figures 5.9 and 5.10. The reliability plot fits well and this is supported by MRL plot which remains quite constant in time.

In order to test the goodness of fit, Kolmogorov-Smirnov statistic D_n is used. It is defined as [12]

$$D_n = \sup_t |R_n(t) - R(t)| \quad ,$$

where n is the sample size. The distribution of D_n is independent of the cdf $F(t)$ $(= 1 - R(t))$ that defines H_0. Accordingly, the acceptance limits for the test of Goodness of Fit are tabulated in [12, page 580]. The critical region $D_n > D_{n,\alpha}$ is used to test $F(t)$ against the alternative that the cdf is *not* $F(t)$, where $D_{n,\alpha}$ is the constant corresponding to the significance level α. The values of test statistic are given in table 5.5 for all the data sets, for $\alpha = .05$. With those values both Weibull and Gamma distributions are accepted for the first three sets, and the one whose D_n value is smaller, is regarded as the better fitting distribution and marked with a star. With this criterion, in all of them Weibull distribution came out to be better. But we can comment that for the second data set, considering the MRL plot (figure 5.8), Gamma distribution might be preferred, because $\hat{e}(t)$ does not decrease, but rather fluctuates after some time, and this obeys to our results for Gamma distribution in Chapter 2. In fact, for this set the Kolmogorov-Smirnov test statistics for the two distributions are very close to eachother. Finally, for the last set Exponential distribution is also accepted.

Some data analysis is performed and the effect of truncating the range of random

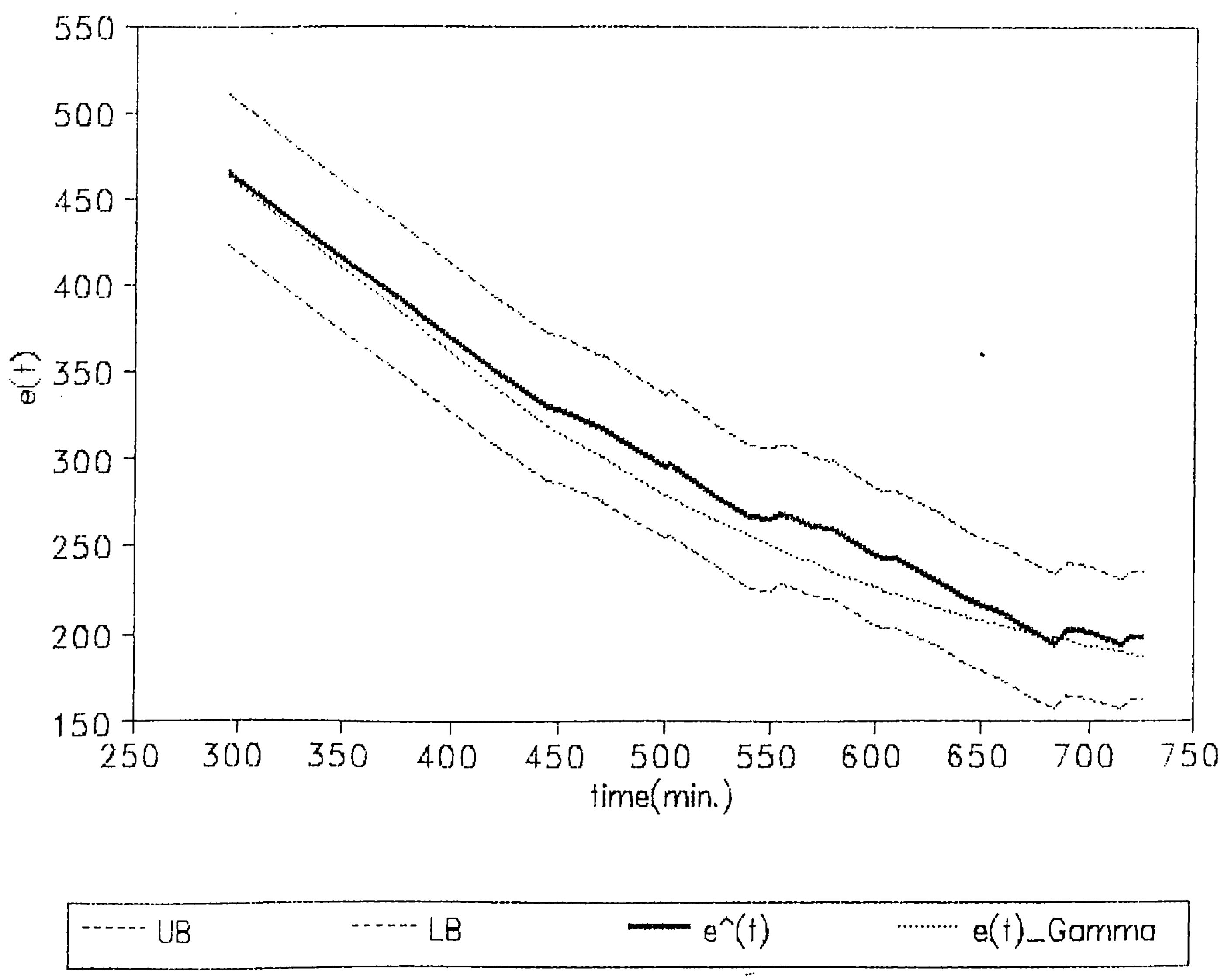

Figure 5.7: The fitted Gamma $e(t)$ and the empirical $\hat{e}(t)$ for Data Set 1 (real shape parameter)

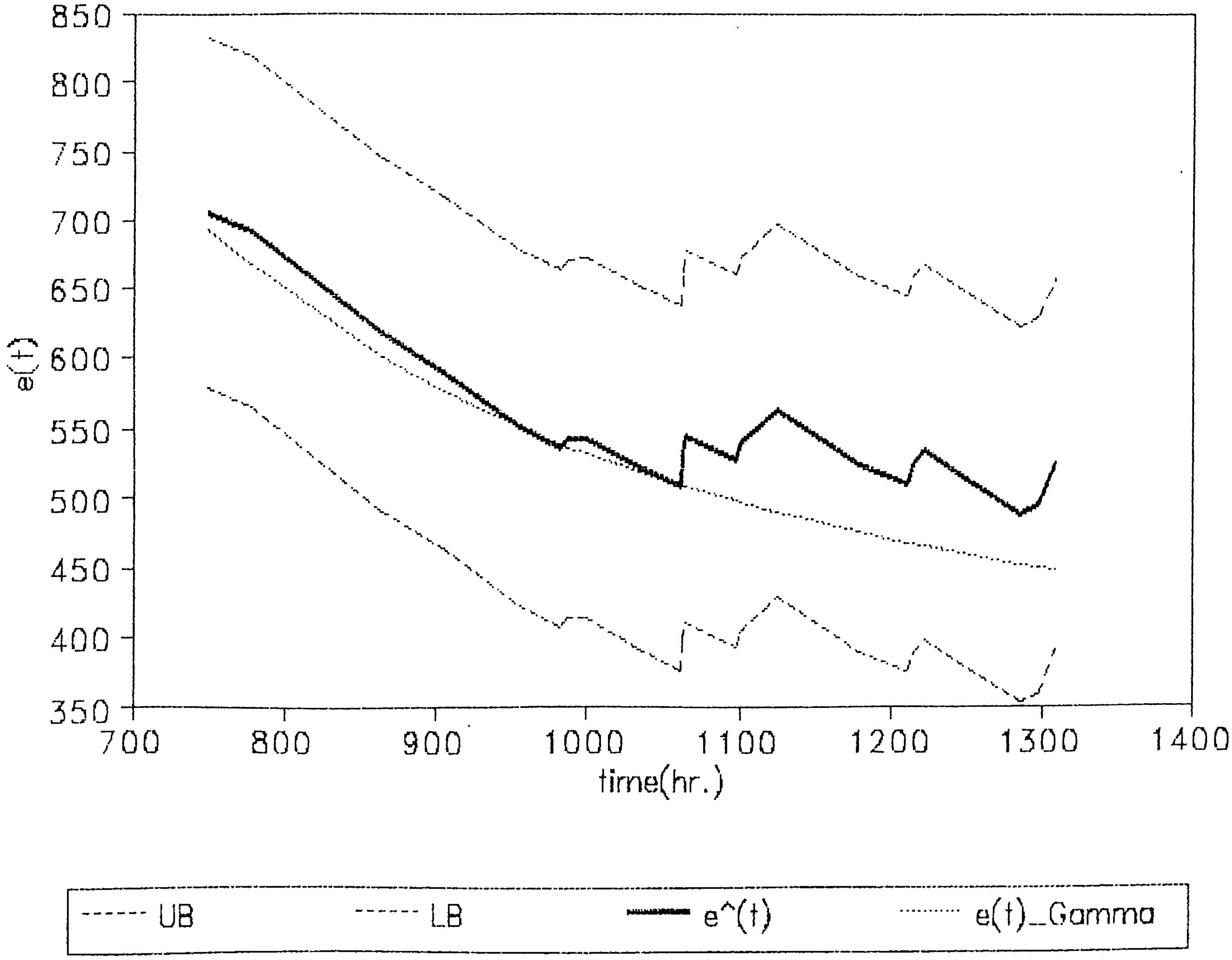

Figure 5.8: The fitted Gamma $e(t)$ and the empirical $\hat{e}(t)$ for Data Set 2 (real shape parameter)

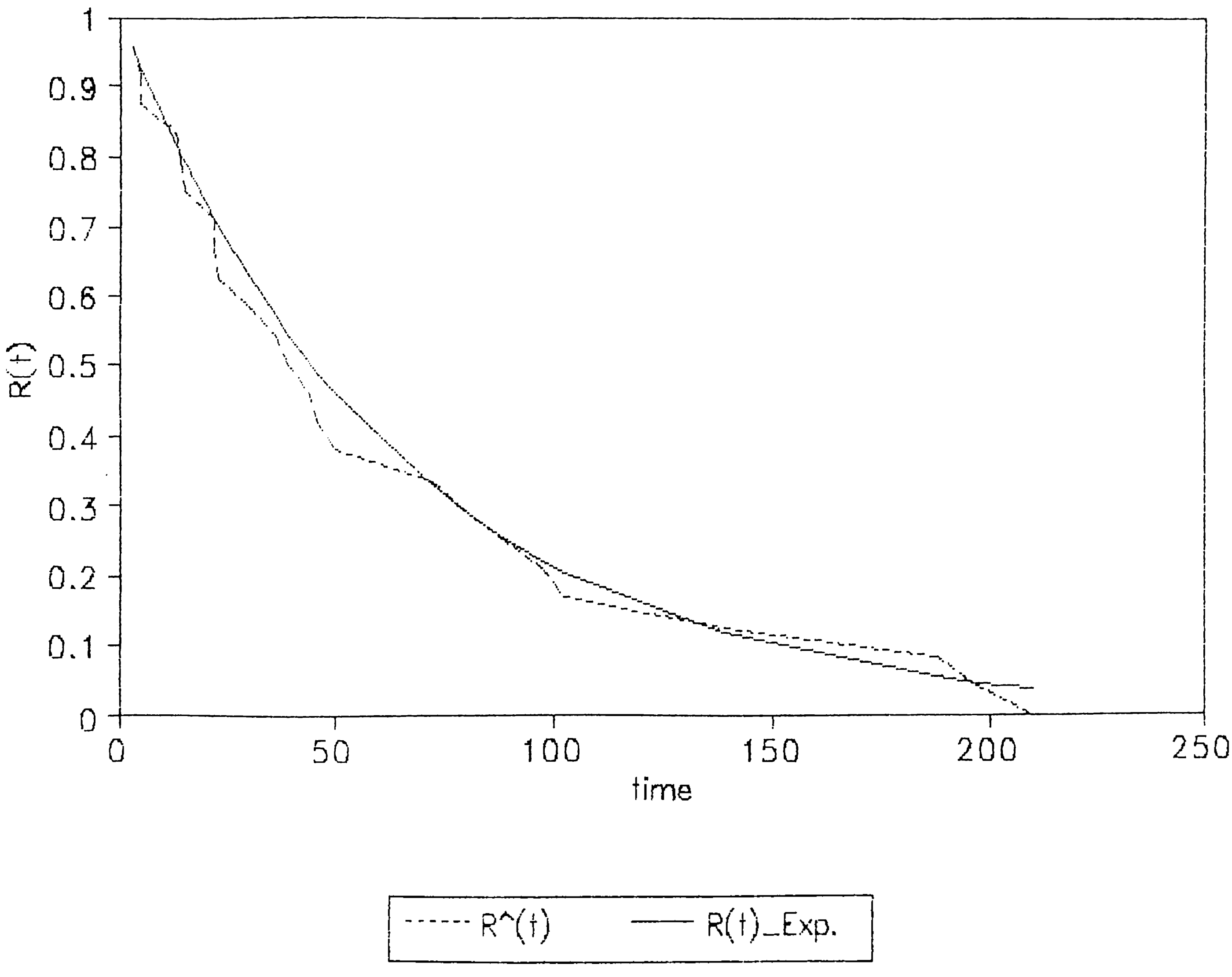

Figure 5.9: The fitted and empirical ($\hat{R}(t)$) reliability functions for Data Set 4

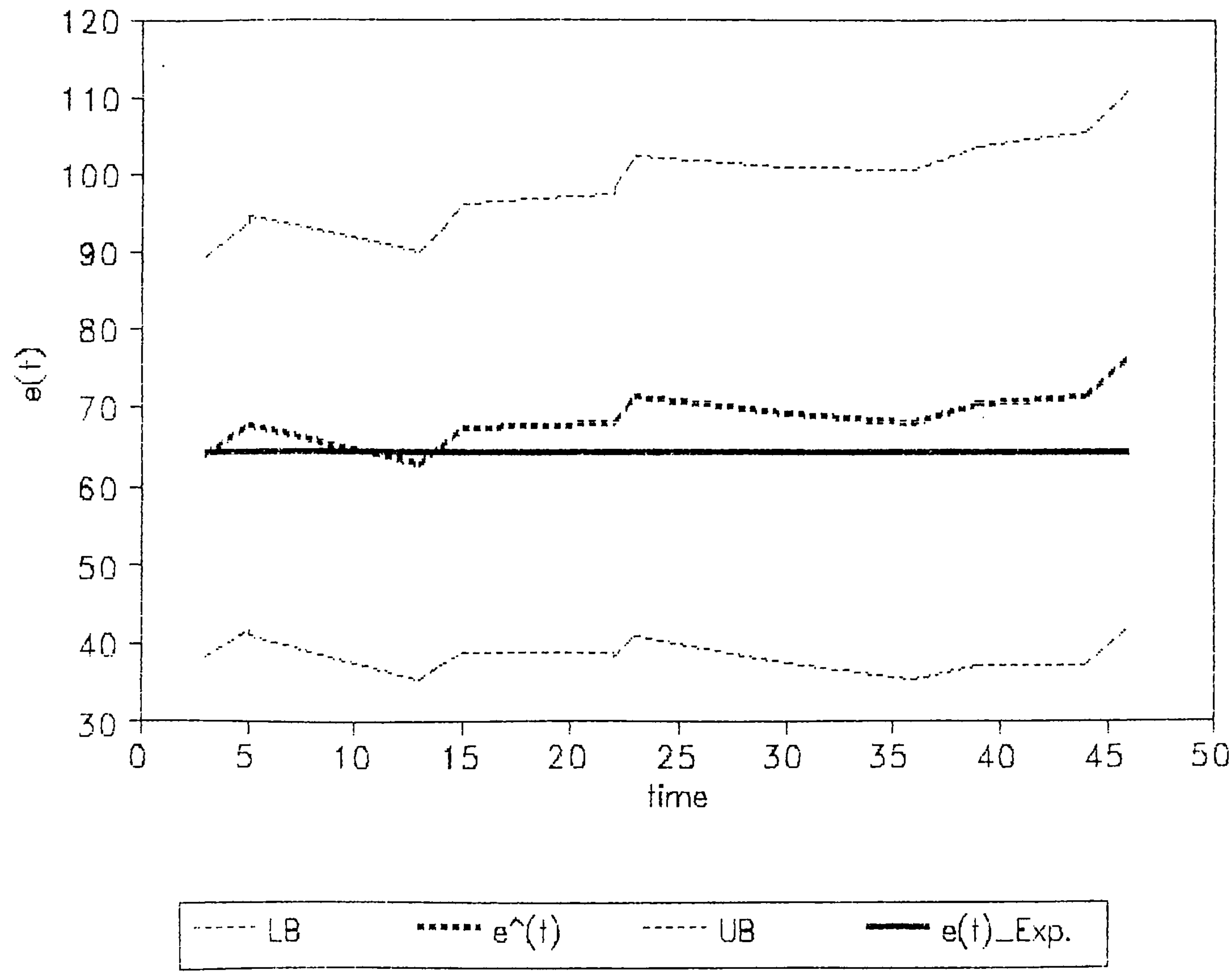

Figure 5.10: The fitted Exponential $e(t)$ and the empirical $\hat{e}(t)$ for Data Set 4

Table 5.5: Kolmogorov-Smirnov Test Statistic Values

	Data Sets						
	1		2		3		4
	Weibull*	Gamma	Weibull*	Gamma	Weibull*	Gamma	Exponential*
D_n	0.0723	0.0973	0.081	0.091	0.0884	0.1098	0.0835
$D_{n,.05}$	0.15		0.19		0.192		0.27

variables on $e(t)$ is demonstrated in [15].

We can then, conclude that for the first three data sets, items are aging in time, and exponential distribution cannot explain the behavior of the bulbs' lifetimes. From the graphs of $e(t)$ versus t, it can be seen that the theoretical curves fit the emprical results. quite well. In relevant situations such as this, MRL function being one of the distribution identities, can also be used as a diagnostic procedure to make a distinction among different distributions, together with the reliability function.

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