RIPPLE-FREE DEADBEAT CONTROL OF SAMPLED-DATA SYSTEMS

A THESIS
SUBMITTED TO THE DEPARTMENT OF ELECTRICAL AND ELECTRONICS ENGINEERING
AND THE INSTITUTE OF ENGINEERING AND SCIENCES
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
Erkan Ünal Mumcuoglu
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I certify that I have read this thesis and that, in my opinion, it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. M. Erol Sezer (Principal Advisor)

I certify that I have read this thesis and that, in my opinion, it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. A. Bülent Özgüler

I certify that I have read this thesis and that, in my opinion, it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

RIPPLE-FREE DEADBEAT CONTROL PROBLEM

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Supervisor: Prof. Dr. M. Erol Sezer
February, 1990

In this thesis, we consider the ripple-free deadbeat control problem for linear, multivariable sampled-data systems represented by state-space models. Existing results concerning the deadbeat/ripple-free deadbeat regulation and tracking problems are based on controller configurations of either constant state-feedback or discrete dynamic output feedback. In the thesis, the problem is analyzed for two new sampled-data controllers, namely, generalized sampled-data hold functions and multirate-output controllers. Some necessary and sufficient solvability conditions for the problem are stated by theorems in time-domain and frequency domain in terms of the open-loop system parameters. Several special cases are also considered as corollaries.

Key words: Multivariable Systems, Sampled-Data Systems, Ripple-Free Deadbeat Control, Tracking, Regulation, Generalized Sampled-Data Hold Functions, Multirate-Output Controllers.

ÖZET

DALGACIKSIZ SIFIRA DÖNÜMLÜ DENETİM KURALI

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Elektrik ve Elektronik Mühendisliği Bölümü Yüksek Lisans
Tez Yöneticisi: Profesör Dr. M. Erol Sezer
Mayıs, 1990

Bu tezde, durum uzayında tanımlanmış, doğrusal, çok degişkenli örneklenmiş sistemlerin dalgacıksız sıfıra dönümlü denetim problemi araştırılmıştır. Sıfıra dönümlü ve dalgacıksız sıfıra dönümlü izleme ve düzenleme amaçlı denetleyicilere ilişkin varolan sonuçlar sadece değişmez durum geribeslemeli ya da zamanda ayrık dinamik çıkıştan geribeslemeli denetim yapıları için elde edilmiştir. Bu tezde ise problem iki yeni bilgi örnekleme denetleyici sistemi kullanılarak analiz edilmiştir. Bu yöntemler, genelleştirilmiş bilgi örnekleme-tutma fonksiyonları ve farklı sıklıkta çıkış örnekleyici denetleyicilerdir. Teoremlerde verilen gerek ve yeter çözüm koşulları hem zaman hem de frekans tanım bölgelerinde açık döngü sistem değişmezleri türünden ifade edilmiştir. Özel durumlar ise teoremlerin sonuçlarında incelenmiştir.

Anahtar sözcükler: Çok Değişkenli Sistemler, Örnekleme-Tutma Sistemleri, Dalgacıksız Sıfıra Dönümlü Denetim, İzleme Problemi, Durağanlaştırma problemi, Genelleştirilmiş Bilgi Örnekleme-Tutma Fonksiyonları, Farklı Sıklıkta Çıkış Denetleyicileri.

ACKNOWLEDGEMENT

I am grateful to Prof. Dr. M. Erol Sezer for his invaluable guidance, suggestions and helps during the development and writing of this thesis.

It is my pleasent duty to express my thanks to all individuals who assisted in the typing and gave suggestions. In particular, I would like to thank Assoc. Prof. A. Bülent Özgüler for his kindly helps and suggestions.

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Chapter 1

INTRODUCTION

Since World War II, the digital computers have experienced a period of remarkable growth, because their application to scientific computation provide high accuracy, computational speed as well as flexibility and versatility. As advantages over analog techniques became apparent, it seemed quite natural that control system engineers should also consider the application of digital techniques in control system design. By the use of sampling, a continuous-time system can be converted into a discrete-time system upon which digital control techniques can be applied easily to change both the continuous- and discrete-time behavior of the system in a desired way.

One of the fundamental problems associated with discrete-time control of linear (either discrete- or continuous-time) systems is that of driving some signals to zero in finite time and holding it there for all discrete (sampling) times thereafter. This problem is called the deadbeat control problem, since the signals are beaten to a dead stop.

If it is the system's state which is to be driven to zero in finite time, then the problem is deadbeat state regulation [1],[2], and the very definition of controllability can be applied to solve this problem. The state deadbeat controller is independent of the system's initial state and results in a nilpotent state transition matrix.

Deadbeat regulation problem arises if it is the system's output that is to be driven to zero in finite time. This problem was solved by Leden [3] using state feedback, who pointed out that the closed-loop sometimes loses stability and the control input diverges exponentially to keep the output zero. In many applications, such a situation is not acceptable, therefore, controllers should be designed such that the control input converges to zero as time goes to infinity. Leden proposed in [3] a procedure for designing such a controller for a rather restricted class of systems.

Akashi & Imai [4] extended this result to the case of output feedback. They derived an elegant geometric characterization of the settling time, inspired by the geometric approach developed by Wonham [5]. Kimura & Tanaka [6] considered the problem with internal stability constraint in its full generality.

A more difficult problem is deadbeat tracking, when one wishes the output of the given system to track a reference signal in a deadbeat fashion. Such a deadbeat controller is a dynamic system which depends on the initial states of both the given plant and the reference generator. The earlier works of Tuo [7] and Kuçera [8] do not insist on the independence of the controller upon the initial states. The most complete results were given by Kuçera & Šebek [9], whose approach was in the transfer function setting, with dynamic output feedback configuration. They also showed that the same problem is solvable with internal stability constraint if and only if no unstable poles of the reference generator occur as a zero of the plant.

However, since ordinary deadbeat control requires the deadbeat response only at sampling times, there may be non-decaying ripples in the steady-state response between the sampling instants, even if the deadbeat control system is internally stable. The ripples, in the most general case, appear due to the modes which can not be made unobservable in the error system. This problem, namely the ripple-free deadbeat control problem(RFDB) was analyzed recently by Urikura & Nagata [10], who considered the constant state feedback approach as a control scheme. They stated a geometric solvability condition, which requires plant to include the continuous-time signal model of the given reference. This is a somewhat obvious solvability condition and the ripples are not eliminated by the digital control alone unless the plant is precompensated by a suitable order compensator. The resulting closed-loop system is relatively strong, since the system is essentially of feedback form and is internally stable, but robustness of the system is weak as far as the deadbeat property is concerned, since the deadbeat control is sensitive to the variation of the system parameters. Also assumed in this paper was that states of both plant and the reference model are directly detectable. However, in an actual construction of the control system, suitable order estimators are required if this assumption is not satisfied.

In the design of controllers based on the state-space method, observers are often used to estimate inaccessible elements of the state vector. The advantage of using an observer exists in the fact that the observer design is separated from the controller design, and therefore the whole design procedure is simplified. Nonetheless, we can point out two clear disadvantages which accompany the introduction of an observer, namely increase in the order of the system and the possibility of producing an unstable controller. Therefore, we desire to apply a new type of sampled-data output feedback controller which internally stabilizes the closed-loop system, and at the same time, is capable of satisfying deadbeat and ripple-free deadbeat response, independent of the initial states.

Among the sampled-data controllers that exist in the literature, we can think of three different control structures. Chammas & Leondes [11],[12] proposed to use a certain type of periodically time-varying gain controllers, namely multirate-input controllers (MRIC) which detects all the plant outputs once in a frame period T_0 and change the i-th plant input N_i times in T_0 with uniform sampling periods. Hagiwara & Araki [13] proposed another kind of sampled-data controllers which detect the i-th plant output N_i times during a frame period T_0 and change the plant inputs once during T_0, i.e. multirate-output controllers (MROC). They have shown, in particular, that an arbitrary state feedback can be realized by such a controller, with arbitrary degree of controller stability. The most general form of sampled-data controllers is the generalized sampled-data hold functions (GSHF) considered by Kabamba [14]. The idea of GSHF is to periodically sample the outputs of the system, and let the control be a linear periodic time-varying weighting of the output sequence. The freedom inherent to this method allows the system designer to achieve simultaneous objectives.

With these new sampled-data controllers at hand, we analyze in this thesis deadbeat and ripple-free deadbeat control problems deeply, considering the stability criterion as well.

In Chapter 2, the ripple-free deadbeat control problem is considered using generalized sampled-data hold functions. Initially, we investigate the deadbeat control problem. Together with the main theorems regarding the algebraic and geometric solvability conditions, a necessary condition relating the state space orders of the reference model and the plant is provided. it is also shown in the corollaries that the above necessary condition is, at the same time, a sufficient condition for the single output case and for the nonsingular output matrix case of the reference model. Furthermore, we present some computational methods for realization of generalized sampled-data hold functions by continuous and piecewise constant functions of time. In the second part of the chapter, algebraic solvability conditions of the ripple-free deadbeat problem are given in time domain. In addition to that, a frequency domain solvability condition is presented by a corollary. Finally, the special case of zero reference input, namely deadbeat regulation and ripple-free deadbeat regulation problems are investigated. We show that they are equivalent problems and are always solvable. We note throughout the analysis that the

... stability property holds within the closed-loop structure.

In Chapter 3, we deal with the ripple-free deadbeat control problem with MROC, which is the dual form of MRIC. We formulate the problem and provide a solvability condition, which internally stabilize, and at the same time, strongly stabilize the closed-loop structure. The motivation behind this method is by Araki & Nagata [10] and [13], who have shown that an arbitrary state-feedback can be realized by MROC. In the final subsection, the deadbeat and ripple-free deadbeat regulation problems are discussed and a method for the solution is provided by a theorem.

Finally, Chapter 4 contains conclusions and comments on further research of the problem.

1.1 Notation

Throughout the thesis, matrices and vectors are denoted by upper and lower case italic letters, abstract objects such as a subspace, a system etc. by script letters. A bar over a symbol indicates that this symbol is related with a continuous-time system, while a symbol without a bar indicates that this symbol is related with a discrete -time system. A hat over a symbol indicates that the symbol is related with the closed-loop system. Subscripts p,r and e indicate the plant, reference and error systems respectively. A super superscript d indicates that symbols are related with sampled-data systems in small-time intervals.

Chapter 2

RIPPLE-FREE DEADBEAT CONTROL USING GENERALIZED SAMPLED-DATA HOLD FUNCTIONS

In this chapter, we consider the ripple-free deadbeat control problem using output feedback and generalized sampled-data hold functions. In the first section, we introduce the control configuration, and state the problem. In Section 2, we investigate the deadbeat control problem in detail, and present some necessary and/or sufficient solvability conditions. The last section is devoted to ripple-free deadbeat control problem, where some partial results are reported.

2.1 Formulation of the Problem

Consider a linear time-invariant plant $\bar{\mathcal{S}}_p$ described by continuous-time state equations

$$\bar{\mathcal{S}}_p : \begin{array}{rcl} \dot{\bar{x}}_p(t) & = & \bar{A}_p \bar{x}_p(t) + \bar{B}_p \bar{u}(t) \\ \bar{y}_p(t) & = & C_p \bar{x}_p(t), \end{array} \tag{2.1}$$

where $\bar{x}_p(t) \in \mathcal{R}^{n_p}$, $\bar{u}(t) \in \mathcal{R}^m$, and $\bar{y}_p(t) \in \mathcal{R}^l$ are the state, input, and output of $\bar{\mathcal{S}}_p$, respectively.

Suppose that a reference input to $\bar{\mathcal{S}}_p$ is generated by a reference model $\bar{\mathcal{S}}_r$

$$\bar{\mathcal{S}}_r : \begin{array}{rcl} \dot{\bar{x}}_r(t) & = & A_r\bar{x}_r(t) \\ \bar{y}_r(t) & = & C_r\bar{x}_r(t), \end{array} \tag{2.2}$$

where $\bar{x}_r(t)\,\epsilon\,\mathcal{R}^{n_r}$, and $\bar{y}_r(t)\,\epsilon\,\mathcal{R}^l$ are the state and output of $\bar{\mathcal{S}}_r$. $\bar{y}_r(t)$ describes the desired output of the plant. To avoid trivialities, we assume that $rank\ C_p = rank\ C_r = l$.

The augmented system $\bar{\mathcal{S}}$ consisting of the plant $\bar{\mathcal{S}}_p$ and the reference system $\bar{\mathcal{S}}_r$ is described by

$$\bar{\mathcal{S}} : \begin{array}{rcl} \dot{\bar{x}}(t) & = & \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t) \\ \bar{y}(t) & = & C\bar{x}(t), \end{array} \tag{2.3}$$

where $\bar{x}(t) = \begin{bmatrix} \bar{x}_p^T(t) & \bar{x}_r^T(t) \end{bmatrix}^T \epsilon\,\mathcal{R}^n$, $n = n_p + n_r$, and

$$\bar{A} = \begin{bmatrix} \bar{A}_p & 0 \\ 0 & \bar{A}_r \end{bmatrix}, \ \bar{B} = \begin{bmatrix} \bar{B}_p \\ 0 \end{bmatrix}, C = \begin{bmatrix} C_p & 0 \\ 0 & C_r \end{bmatrix}. \tag{2.4}$$

Similarly, we define the error system $\bar{\mathcal{S}}_e$ as

$$\bar{\mathcal{S}}_e : \begin{array}{rcl} \dot{\bar{x}}(t) & = & \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t) \\ \bar{e}(t) & = & D\bar{x}(t), \end{array} \tag{2.5}$$

where $\bar{e}(t)\,\epsilon\,\mathcal{R}^l$ is the error defined by the difference between $\bar{y}_r(t)$ and $\bar{y}_p(t)$, and

$$D = \begin{bmatrix} -C_p & C_r \end{bmatrix}. \tag{2.6}$$

We assume that the plant $\bar{\mathcal{S}}_p$ is controllable and observable, and the system $\bar{\mathcal{S}}_r$ is observable. Thus the augmented system $\bar{\mathcal{S}}$ is observable. Note, however, that the error system $\bar{\mathcal{S}}_e$ may be unobservable.

The control structure for the error system $\bar{\mathcal{S}}_e$ is defined as

$$\bar{u}(t) = \bar{F}_p(t)y_p(k) + \bar{F}_r(t)y_r(k), \ \ kT \leq t < (k+1)T, \tag{2.7}$$

where T is the sampling period; $y_p(k)$ and $y_r(k)$ are discrete-time signals obtained by sampling $\bar{y}_p(t)$ and $\bar{y}_r(t)$, i.e., $y_i(k) = \bar{y}_i(kT)$, $k\,\epsilon\,\mathcal{Z}$, $i = p, r$; and $\bar{F}_p(t)$ and $\bar{F}_r(t)$ are T-periodic generalized sampled-data hold functions (GSHF); i.e.,

$$\bar{F}_i(t+T) = \bar{F}_i(t), \ i = p, r, \ \ t\,\epsilon\,\mathcal{R}. \tag{2.8}$$

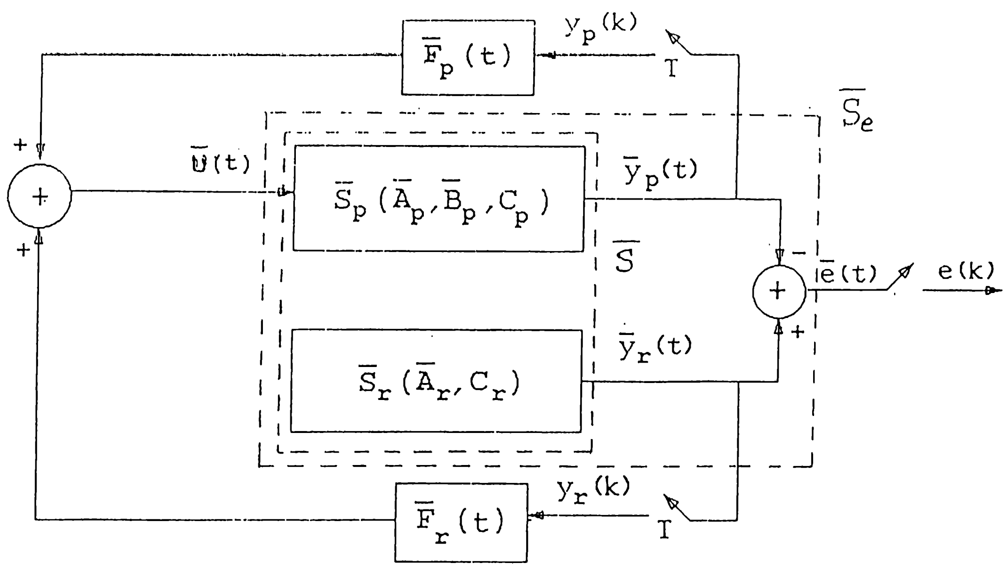

Figure 2.1. Control Scheme with GSHF Controller

The closed-loop sampled-data system has the configuration shown in Fig. 2.1, where $\bar{S}$ and $\bar{S}_e$ are also indicated. To obtain a discrete-time description of the sampled-data system in Fig. 2.1, let us define

$$\begin{aligned}
x_i(k) &= \bar{x}_i(kT),\ i = p, r, \\
u(k) &= \bar{u}(kT), \\
x(k) &= \left[\ x_p^T(k) \quad x_r^T(k)\ \right]^T, \\
y(k) &= \left[\ y_p^T(k) \quad y_r^T(k)\ \right]^T, \\
\Phi_i &= e^{\bar{A}_i T},\ i = p, r, \\
\Phi &= e^{\bar{A}T} = diag.\{\Phi_p\ ,\ \Phi_r\}, \\
\Gamma_p &= \int_0^T e^{\bar{A}_p(T-\tau)}\bar{B}_p d\tau, \\
\Gamma &= \int_0^T e^{\bar{A}(T-\tau)}\bar{B} d\tau = \left[\ \Gamma_p{}^T \quad 0\ \right]^T.
\end{aligned} \tag{2.9}$$

Then, the discrete-time augmented system $\mathcal{S}$ and the error system $\mathcal{S}_e$ are described respectively as

$$\mathcal{S}: \begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k), \end{aligned} \tag{2.10}$$

$$\mathcal{S}_e: \begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ e(k) &= Dx(k). \end{aligned} \tag{2.11}$$

We assume that the sampling process does not introduce any unobservable modes into S_p and S_r, i.e., (Φ_p, C_p) and (Φ_r, C_r) are observable pairs.

Next, we obtain descriptions of the closed-loop sampled-data and the corresponding discrete-time systems as follows. First, we rewrite (2.7) as

$$\bar{u}(t) = \bar{F}(t)y(k), \ kT \leq t < (k+1)T, \tag{2.12}$$

where y(k) is defined in (2.9), and

$$\bar{F}(t) = \begin{bmatrix} \bar{F}_p(t) & \bar{F}_r(t) \end{bmatrix}. \tag{2.13}$$

Substituting (2.12) in the expression

$$\bar{x}(kT+\delta) = e^{\bar{A}\delta}\bar{x}(kT) + \int_{kT}^{kT+\delta} e^{\bar{A}(kT+\delta-\tau)}\bar{B}\bar{u}(\tau)d\tau, \ 0 \leq \delta \leq T \tag{2.14}$$

which describes the evolution of the state of $\bar{S}$, we obtain

$$\begin{aligned} \bar{x}(kT+\delta) &= e^{\bar{A}\delta}\bar{x}(kT) + \int_{kT}^{kT+\delta} e^{\bar{A}(kT+\delta-\tau)}\bar{B}\bar{F}(\tau)y(k)d\tau \\ &= [\, e^{\bar{A}\delta} + \int_0^{\delta} e^{\bar{A}(\delta-\tau)}\bar{B}\bar{F}(\tau)Cd\tau \,]\, \bar{x}(kT) \\ &= \hat{\bar{\Phi}}(\delta)\bar{x}(kT), \end{aligned} \tag{2.15}$$

where

$$\begin{aligned} \hat{\bar{\Phi}}(\delta) &= e^{\bar{A}\delta} + \int_0^{\delta} e^{\bar{A}(\delta-\tau)}\bar{B}\bar{F}(\tau)Cd\tau, \ 0 \leq \delta \leq T \\ &= \begin{bmatrix} e^{\bar{A}_p\delta} + G_p(\delta)C_p & G_r(\delta)C_r \\ 0 & e^{\bar{A}_r\delta} \end{bmatrix} \end{aligned} \tag{2.16}$$

with

$$G_i(\delta) = \int_0^{\delta} e^{\bar{A}_p(\delta-\tau)}\bar{B}_p\bar{F}_i(\tau)d\tau, \ i = p, r. \tag{2.17}$$

Thus the closed-loop sampled-data error system is described by

$$\hat{\bar{S}}_e : \begin{array}{rcl} \bar{x}(kT+\delta) &=& \hat{\bar{\Phi}}(\delta)\bar{x}(kT) \\ \bar{e}(kT+\delta) &=& D\bar{x}(kT+\delta) \end{array}, \ 0 \leq \delta \leq T. \tag{2.18}$$

The description of the closed-loop discrete-time error system is obtained from (2.18) with $\delta = T$ as

$$\hat{S}_e : \begin{array}{rcl} x(k+1) &=& \hat{\Phi}x(k) \\ e(k) &=& Dx(k) \end{array} \tag{2.19}$$

where

$$\hat{\Phi} = \hat{\hat{\Phi}}(T) = \begin{bmatrix} \Phi_p + G_p C_p & G_r C_r \\ 0 & \Phi_r \end{bmatrix} \tag{2.20}$$

with

$$G_i = G_i(T) = \int_0^T e^{\bar{A}_p(T-\tau)} \bar{B}_p \bar{F}_i(\tau) d\tau, \;\; i = p, r. \tag{2.21}$$

Having obtained the descriptions for $\hat{\hat{\mathcal{S}}}_e$ and $\hat{\mathcal{S}}_e$, we now formulate the deadbeat and ripple-free deadbeat control problems as follows.

Deadbeat Control Problem :Find T-periodic generalized sampled-data hold functions $\bar{F}_p(t)$ and $\bar{F}_r(t)$ such that for all $\bar{x}(0) \epsilon \mathcal{R}^n$

$$e(k) = 0, \;\; for\ all\ \ k \geq N \tag{2.22}$$

for some $N \epsilon \mathcal{Z}_+$.

Ripple-Free Deadbeat Control Problem :Find T-periodic generalized sampled-data hold functions $\bar{F}_p(t)$ and $\bar{F}_r(t)$ such that for all $\bar{x}(0) \epsilon \mathcal{R}^n$

$$\bar{e}(t) = 0, \;\; for\ all\ \ t \geq NT \tag{2.23}$$

for some $N \epsilon \mathcal{Z}_+$.

2.2 Deadbeat Control Problem

In this section, the deadbeat control problem is investigated in detail. In the first subsection, the main theorem is stated. The second subsection is devoted to some implications of the main theorem. In the third subsection, the realization of generalized sampled-data hold functions is considered. In the final subsection, geometric solvability conditions are given.

2.2.1 General Solvability Condition

The output of the closed-loop discrete-time error system $\hat{\mathcal{S}}_e$ in (2.19) is given by

$$e(k) = D\hat{\Phi}^k x_0, \quad x_0 = x(0) = \bar{x}(0). \tag{2.24}$$

Thus deadbeat control problem is equivalent to finding $\bar{F}_p(t)$ and $\bar{F}_r(t)$ such that

$$Im\ \hat{\Phi}^k \ \subset \ Ker\ D, \quad for\ all\ \ k \geq N, \quad for\ some\ \ N \in Z_+. \tag{2.25}$$

Noting that $Im\ \hat{\Phi}^k \ \subset \ Im\ \hat{\Phi}^N$, for all $k \geq N$, (2.25) is equivalent to

$$Im\ \hat{\Phi}^N \ \subset \ Ker\ D. \tag{2.26}$$

The following theorem provides a necessary and sufficient condition for solvability of the deadbeat control problem.

Theorem 2.1 *Deadbeat control problem is solvable if and only if there exists* $X \in \mathcal{R}^{n_p \times n_r}$ *and* $Y \in \mathcal{R}^{n_p \times l}$ *such that the following equalities are satisfied simultaneously:*

$$(\Phi_p + YC_p)X = X\Phi_r, \tag{2.27}$$

$$C_p X = C_r. \tag{2.28}$$

Proof: **[Necessity]** Using (2.6) and (2.20), (2.26) can be rewritten as

$$\begin{bmatrix} -C_p & C_r \end{bmatrix} \begin{bmatrix} \Phi_p + G_p C_p & G_r C_r \\ 0 & \Phi_r \end{bmatrix}^N = 0. \tag{2.29}$$

Defining,

$$\begin{aligned} \hat{\Phi}_p &= \Phi_p + G_p C_p \\ \Omega_N &= \hat{\Phi}_p^{N-1} G_r C_r + \hat{\Phi}_p^{N-2} G_r C_r \Phi_r + \ldots + G_r C_r {\Phi_r}^{N-1}, \end{aligned} \tag{2.30}$$

(2.29) becomes

$$\begin{bmatrix} -C_p & C_r \end{bmatrix} \begin{bmatrix} \hat{\Phi}_p^N & \Omega_N \\ 0 & {\Phi_r}^N \end{bmatrix} = 0, \tag{2.31}$$

which in turn implies

$$C_p \hat{\Phi}_p^N = 0, \tag{2.32}$$

and

$$C_p \Omega_N = C_r {\Phi_r}^N. \tag{2.33}$$

From (2.32), we obtain

$$\begin{bmatrix} C_p \\ C_p \hat{\Phi}_p \\ \vdots \\ C_p \hat{\Phi}_p^{n_p - 1} \end{bmatrix} \hat{\Phi}_p^N = \hat{Q}_p \, \hat{\Phi}_p^N = 0, \tag{2.34}$$

where $\hat{Q}_p$ is the observability matrix of the pair $(C_p, \hat{\Phi}_p)$. Since (C_p, Φ_p) is observable by assumption, so is $(C_p, \hat{\Phi}_p)$ [15], so that $\hat{Q}_p$ has full column rank. Then, (2.34) implies that

$$\hat{\Phi}_p^N = 0. \tag{2.35}$$

Noting that Φ_r is nonsingular, we now define

$$X = \Omega_N {\Phi_r}^{-N}, \quad Y = G_p + G_r. \tag{2.36}$$

Then, (2.33) implies (2.28). On the other hand, using (2.35) we obtain

$$\begin{aligned} (\Phi_p + Y C_p) X &= (\Phi_p + G_p C_p) X + G_r C_p X \\ &= \hat{\Phi}_p \Omega_N \Phi_r^{-N} + G_r C_r \\ &= (\hat{\Phi}_p^N G_r C_r + \hat{\Phi}_p^{N-1} G_r C_r \Phi_r + \ldots + \hat{\Phi}_p G_r C_r {\Phi_r}^{N-1}) {\Phi_r}^{-N} \\ &\quad + G_r C_r \\ &= (\hat{\Phi}_p^{N-1} G_r C_r + \ldots + \hat{\Phi}_p G_r C_r {\Phi_r}^{N-2} + G_r C_r \Phi_r^{N-1}) \, \Phi_r^{1-N} \\ &= \Omega_N \Phi_r^{1-N} \\ &= X \, \Phi_r \, , \end{aligned} \tag{2.37}$$

i.e., (2.27) is also satisfied.

[Sufficiency] Since (C_p, Φ_p) is observable, G_p can be chosen to make $\hat{\Phi}_p = \Phi_p + G_p C_p$ nilpotent; i.e.

$$\hat{\Phi}_p^N = 0 \tag{2.38}$$

for some N. Let

$$G_r = Y - G_p. \tag{2.39}$$

Then, (2.27) and (2.28) imply that

$$\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{\Phi}_p & G_r C_r \\ 0 & \Phi_r \end{bmatrix} = \begin{bmatrix} \hat{\Phi}_p & 0 \\ 0 & \Phi_r \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}, \tag{2.40}$$

and, therefore

$$\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{\Phi}_p & G_r C_r \\ 0 & \Phi_r \end{bmatrix}^N = \begin{bmatrix} \hat{\Phi}_p^N & 0 \\ 0 & \Phi_r^N \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}. \tag{2.41}$$

Hence, from (2.38), we obtain

$$\begin{aligned} D\hat{\Phi}^N &= \begin{bmatrix} -C_p & C_r \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{\Phi}_p & G_r C_r \\ 0 & \Phi_r \end{bmatrix}^N \\ &= \begin{bmatrix} -C_p & 0 \end{bmatrix} \begin{bmatrix} \hat{\Phi}_p^N & 0 \\ 0 & \Phi_r^N \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \\ &= 0. \end{aligned} \tag{2.42}$$

Thus when (2.27) and (2.28) are satisfied, G_p and G_r can be constructed to satisfy the deadbeat condition (2.26). It remains to show that given G_p and G_r, $\bar{F}_p(t)$ and $\bar{F}_r(t)$ can be solved from (2.21). This is done in section 2.2.3.

2.2.2 Implications of the Main Theorem

Following the necessity part of the Theorem 2.1, (2.35) implies that the closed-loop system is internally stable. Hence, the deadbeat control problem inherently includes the internal stability constraint.

We now elaborate on conditions (2.27) and (2.28) of Theorem 2.1. They together imply that

$$C_p(\Phi_p + YC_p)^i X = C_r \Phi_r^i, \quad i = 0, 1, \ldots. \tag{2.43}$$

In particular, we have

$$\begin{bmatrix} C_p \\ C_p(\Phi_p + YC_p) \\ \vdots \\ C_p(\Phi_p + YC_p)^{\eta-1} \end{bmatrix} X = \begin{bmatrix} C_r \\ C_r \Phi_r \\ \vdots \\ C_r \Phi_r^{\eta-1} \end{bmatrix}, \tag{2.44}$$

where $\eta = max\ \{n_p,\ n_r\}$. Since the pair (Φ_p, C_p) is observable, then so is $[(\Phi_p + YC_p), C_p]$, so that the coefficient matrix on the left-hand side of (2.44) has rank n_p. Similarly, the matrix on the right-hand side has rank n_r. This observation leads to the following corollary of Theorem 2.1.

Corollary 2.1 *A necessary condition for the solvability of the deadbeat control problem is that* $n_p \geq n_r$.

We now turn our attention to the special case of single-output system and reference plant, i.e., when $l = 1$.

Without loss of generality let us assume that the pair (Φ_p, C_p) is in observable canonical form, that is

$$\Phi_p = \begin{bmatrix} 0 & & & \phi_{n_p} \\ 1 & & & \vdots \\ & & 0 & \phi_2 \\ & & 1 & \phi_1 \end{bmatrix}, \tag{2.45}$$

$$C_p = \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix}. \tag{2.46}$$

Also, assume without loss of generality that Φ_r is in jordan form with

$$\Phi_r = diag.\ \{J_1,\ J_2, \ldots.\ J_q\}, \tag{2.47}$$

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_i & 1 \\ & & & \lambda_i \end{bmatrix} \epsilon\ \mathcal{R}^{n_i \times n_i}, \tag{2.48}$$

where $\sum_{i=1}^{q} n_i = n_r$, and $\lambda_i \neq \lambda_j\ \ for\ \ i \neq j$. Let C_r be partitioned conformably as

$$C_r = \begin{bmatrix} C_1 & C_2 & & C_q \end{bmatrix} \tag{2.49}$$

where

$$C_i = \begin{bmatrix} c_{i1} & c_{i2} & \ldots & c_{in_i} \end{bmatrix}, i = 1, 2, \ldots, q. \tag{2.50}$$

Note that since (Φ_r, C_r) is observable, we have $c_{i1} \neq 0,\ \ i = 1, 2, \ldots, q$.

Finally, let

$$G_p = \begin{bmatrix} g_{pn_p} \\ \vdots \\ g_{p2} \\ g_{p1} \end{bmatrix}, \quad G_r = \begin{bmatrix} g_{rn_p} \\ \vdots \\ g_{r2} \\ g_{r1} \end{bmatrix}. \tag{2.51}$$

We are now ready to prove the following.

Corollary 2.2 *For the single-output case, the deadbeat control problem is solvable if and only if* $n_p \geq n_r$.

Proof : [Necessity] The necessity of the condition $n_p \geq n_r$ has already been stated in Corollary 1.1. Below we provide an alternative proof.

Consider (2.35), which has been shown to be a necessary condition. From the structure of Φ_p, C_p and G_p, it follows that (2.35) is satisfied if and only if $N \geq n_p$ and

$$g_{pj} + \phi_j = 0, \quad j = 1, 2, \ldots, n_p, \tag{2.52}$$

that is

$$\hat{\Phi}_p = \begin{bmatrix} 0 & & & 0 \\ 1 & & & \vdots \\ & & 0 & 0 \\ & & 1 & 0 \end{bmatrix}. \tag{2.53}$$

Substituting C_p, C_r, $\hat{\Phi}_p$, Φ_r and G_r in (2.29), we get

$$\begin{bmatrix} 0 & & 0 & -1 & | & C_1 & & C_q \end{bmatrix} \left[\begin{array}{cccc|ccc} 0 & & & 0 & g_{rn_p}C_1 & & g_{rn_p}C_q \\ 1 & & & \vdots & \vdots & & \vdots \\ & & 0 & 0 & g_{r2}C_1 & & g_{r2}C_q \\ & & 1 & 0 & g_{r1}C_1 & \ldots & g_{r1}C_q \\ \hline & & & & J_1 & & \\ & & & & & & \\ & & & & & & J_q \end{array}\right]^N = 0. \tag{2.54}$$

Carrying out multiplications, we obtain the following equivalent equations

$$C_i(J_i^{n_p} - g_{r1}J_i^{n_r-1} - \ldots - g_{rn_p}I)J_i^{N-n_p} = 0, \quad i = 1, 2, \ldots, q. \tag{2.55}$$

Since J_i's are nonsingular, (2.55) is equivalent to

$$C_r(\Phi_r^{n_p} - g_{r1}\Phi_r^{n_p-1} - \ldots - g_{rn_p}I) = 0. \tag{2.56}$$

Now, (2.56) implies

$$Q_r(\Phi_r^{n_p} - g_{r1}\Phi_r^{n_p-1} - \ldots - g_{rn_p}I) = 0, \tag{2.57}$$

where

$$Q_r = \begin{bmatrix} C_r \\ C_r\Phi_r \\ \vdots \\ C_r{\Phi_r}^{n_r-1} \end{bmatrix}$$

is the observability matrix of the pair (Φ_r, C_r). Since (Φ_r, C_r) is observable, (2.57) is satisfied only if

$$\Phi_r^{n_p} - g_{r1}\Phi_r^{n_p-1} - \ldots - g_{rn_p}I = 0. \tag{2.58}$$

Finally, from the structure of Φ_r, it follows that degree of the minimal polynomial of Φ_r is n_r, so that a necessary condition for (2.58) to be satisfied by some g_{r_i}, $i = 1, 2, \ldots, n_p$ is obtained as $n_p \geq n_r$.

[Sufficiency] Assuming that $n_p \geq n_r$, let us choose the elements of G_p to satisfy (2.52), and those of G_r to satisfy (2.56). Then $\hat{\Phi}_p$ is of the form given in (2.53) , and (2.54) is satisfied for any $N \geq n_p$. This completes the proof.

Another special case that deserves attention is when C_r is square, i.e., when $l = n_r$.

Corollary 2.3 *For the case $l = n_r$, the deadbeat control problem is solvable if and only if $n_p \geq n_r$.*

Proof : Necessity follows from Corollary 2.1. To prove sufficiency, first note that since rank $C_r = l = n_r$, C_r is invertible . Now , choose

$$X = {C_p}^{\dagger}C_r, \tag{2.59}$$

and

$$Y = X\Phi_r C_r^{-1} - \Phi_p X C_r^{-1}, \tag{2.60}$$

where $C_p^{\natural}$ is any right inverse of C_p satisfying $C_p C_p^{\natural} = I_l$. Then,

$$C_p X = C_p C_p^{\natural} C_r = C_r, \tag{2.61}$$

and

$$\begin{aligned}(\Phi_p + YC_p)X &= \Phi_p X + YC_r \\ &= \Phi_p X + X\Phi_r - \Phi_p X = X\Phi_r\end{aligned} \tag{2.62}$$

so that both (2.27) and (2.28) are satisfied.

2.2.3 Realizations of Generalized Sampled-Data Hold Functions

To complete the proof of Theorem 2.1 we need to show the existence of GSHF's $\bar{F}_i(t)$, $i = p, r$, which satisfy

$$\int_0^T e^{\bar{A}_p(T-\tau)} \bar{B}_p \bar{F}_i(\tau) d\tau = G_i \tag{2.63}$$

for any given G_i. This is the well-known controllability problem [16]. Since the pair $(\bar{A}_p, \bar{B}_p)$ is controllable, the controllability grammian

$$\bar{W}(T) = \int_0^T e^{\bar{A}_p(T-\tau)} \bar{B}_p \bar{B}_p^T e^{\bar{A}_p^T(T-\tau)} d\tau \tag{2.64}$$

is nonsingular [16]. It is then a trivial matter to show that

$$\bar{F}_i(t) = \bar{B}_p^T e^{\bar{A}_p^T(T-t)} \bar{W}(T)^{-1} G_i, \tag{2.65}$$

satisfies (2.63).

In the rest of this subsection, we review a method by Araki and Hagiwara[17] to construct piecewise constant GSHF's $\bar{F}_i(t)$ which satisfy (2.63). For this, we first recall the following definition.[17]

Definition 2.1 *Let $(\bar{A}_p, \bar{B}_p)$ be a controllable pair, where $\bar{A}_p \in \mathcal{R}^{n_p \times n_p}$ and $\bar{B}_p = [\ \bar{b}_1 \ldots \bar{b}_m\ ] \in \mathcal{R}^{n_p \times m}$. A set of integers $\{N_1, N_2, \ldots, N_m\}$ with $N_k \geq 0$ and $\sum N_k = n_p$ is said to be the locally minimum controllability indices (LMCI) if*

$$rank\ [\ \bar{b}_1 \ldots \bar{A}_p^{N_1-1}\bar{b}_1 \ldots \bar{b}_m \ldots \bar{A}_p^{N_m-1}\bar{b}_m\ ] = n_p. \tag{2.66}$$

Let $\{N_1, N_2, \ldots, N_m\}$ be a set of LMCI for $(\bar{A}_p, \bar{B}_p)$. Define,

$$N = l.c.m.\{N_1, N_2, \ldots, N_m\}, \tag{2.67}$$

$$T_0 = T/N, \quad T_k = T/N_k, \quad k = 1, 2, \ldots, m. \tag{2.68}$$

Let $\bar{F}_i(t) = [\ \bar{f}_{kj}(t)\ ]$, $k = 1, 2, \ldots, m$; $j = 1, 2, \ldots, l$, where

$$\bar{f}_{kj}(t) = f_\mu^{kj}, \quad \mu T_k \leq t < (\mu+1)T_k; \quad \mu = 0, 1, \ldots, N_k - 1, \tag{2.69}$$

with f_μ^{kj} being real constants. Then, from (2.63), the j^{th} column of G_i is expressed as

$$\begin{aligned} g_j &= \sum_{k=1}^{m} \sum_{\mu=0}^{N_k-1} \int_{\mu T_k}^{(\mu+1)T_k} e^{\bar{A}_p(T-\tau)}\ \bar{b}_k\ f_\mu^{kj}\ d\tau \\ &= \sum_{k=1}^{m} \sum_{\mu=0}^{N_k-1} e^{\bar{A}_p(N_k-\mu-1)T_k}\ (\int_0^{T_k} e^{\bar{A}_p(T_k-\tau)}\ \bar{b}_k\ d\tau\ ) f_\mu^{kj} \\ &= \sum_{k=1}^{m} \sum_{\mu=0}^{N_k-1} \Phi_p^{N_k-\mu-1}(T_k)\, b_k(T_k)\ f_\mu^{kj}, \quad j = 1, 2, \ldots, l \end{aligned} \tag{2.70}$$

where,

$$\Phi_p(T_k) = e^{\bar{A}_p T_k} \tag{2.71}$$

and

$$b_k(T_k) = \int_0^{T_k} e^{\bar{A}_p(T_k-\tau)}\ \bar{b}_k\ d\tau = \int_0^{T_k} e^{\bar{A}_p \tau}\ \bar{b}_k\ d\tau. \tag{2.72}$$

Equations (2.70) can be written in compact form as

$$G_i = E\ F \tag{2.73}$$

where

$$E = \left[\begin{array}{cccccc} b_1(T_1) & \ldots & \Phi_p^{N_1-1}(T_1)b_1(T_1) & \ldots & b_m(T_m) & \ldots & \Phi_p^{N_m-1}(T_m)b_m(T_m) \end{array} \right] \quad (2.74)$$

and $F = [\ \hat{f}_{kj}\ ]$, $k = 1, 2, \ldots, m$; $j = 1, 2, \ldots, l$, with

$$\hat{f}_{kj} = \begin{bmatrix} f^{kj}_{N_k-1} \\ \vdots \\ f^{kj}_0 \end{bmatrix}. \quad (2.75)$$

From (2.73), a necessary and sufficient condition for the existence of a solution F for any given G_i is obtained as

$$rank\ E = n_p. \quad (2.76)$$

However, as shown in [17], (2.66) implies (2.76) for almost all T.

2.2.4 Geometric Solvability Conditions

In this subsection, we provide a geometric interpretation(in the sense of Wonham [5]) of the necessary and sufficient conditions (2.27) and (2.28) for solvability of the deadbeat control problem . Let us define the controllability subspace by

$$\mathcal{B} = Im\ B =< \Phi \mid \Gamma > . \quad (2.77)$$

Assuming that sampling does not introduce any uncontrollable modes into the system, we obtain

$$B = \begin{bmatrix} I_{n_p} \\ 0_{n_r \times n_p} \end{bmatrix}, \quad (2.78)$$

and state the following.

Lemma 2.1 *Conditions (2.27) and (2.28) are simultaneously satisfied by a pair (X, Y) if and only if there exists a matrix* $U \in \mathcal{R}^{n_p \times 2l}$ *and a subspace* $\mathcal{V} \subset \mathcal{R}^n$ *such that*

$$(\Phi + BUC)\mathcal{V} \subset \mathcal{V} \quad (2.79)$$

$$\mathcal{V} \subset Ker\, D \tag{2.80}$$

$$\mathcal{V} \oplus \mathcal{B} = \mathcal{R}^n \tag{2.81}$$

Proof : **[Necessity]** If (2.27) and (2.28) are satisfied by some pair (X,Y), let

$$V = \begin{bmatrix} X \\ I_{n_r} \end{bmatrix}, \quad U = \begin{bmatrix} Y & 0 \end{bmatrix}. \tag{2.82}$$

Then, (2.27) and (2.28) imply

$$(\Phi + BUC)V = V\Phi_r \tag{2.83}$$

and

$$DV = 0. \tag{2.84}$$

Letting $\mathcal{V} = Im\ V$, the proof follows.

[Sufficiency] Let

$$V = \begin{bmatrix} V_p \\ V_r \end{bmatrix} \tag{2.85}$$

be a basis for $\mathcal{V}$, where $V_p \in \mathcal{R}^{n_p \times n_r}$ and $V_r \in \mathcal{R}^{n_r \times n_r}$; and let U be partitioned as

$$U = \begin{bmatrix} U_p & U_r \end{bmatrix}, \tag{2.86}$$

where $U_p,\ U_r \in \mathcal{R}^{n_p \times l}$. (2.81) implies that V_r is nonsingular. Let $X = V_p V_r^{-1}$ and $Y = U_p + U_r$. Then, it is a simple matter to show that (2.79) and (2.80) imply (2.27) and (2.28).

It has been shown [18] that a subspace $\mathcal{V} \subset \mathcal{R}^n$ satisfies (2.79) for some matrix U if and only if it is both Φ(mod $\mathcal{B}$)- and $\Phi|Ker(C)$- invariant; i.e., there exists $U_1 \in \mathcal{R}^{n_p \times l}$ and $U_2 \in \mathcal{R}^{n \times n}$ such that

$$(\Phi + BU_1)\mathcal{V} \subset \mathcal{V} \tag{2.87}$$

and

$$(\Phi + U_2 C)\mathcal{V} \subset \mathcal{V}. \tag{2.88}$$

Using this result and Lemma 2.1, we reach the following.

Theorem 2.2 *The deadbeat control problem is solvable if and only if there exists a subspace $\mathcal{V} \subset \mathcal{R}^n$ such that*

$$\mathcal{V} \text{ is } \Phi|Ker\,(C) - invariant \tag{2.89}$$

$$\mathcal{V} \subset Ker\,D \tag{2.90}$$

$$\mathcal{V} \oplus <\Phi \mid \Gamma> = \mathcal{R}^n \tag{2.91}$$

Proof : [Necessity] (2.79) implies (2.88), which is equivalent to (2.89). The proof then follows from Lemma 2.1.

[Sufficiency] Let V in (2.85) be a basis for $\mathcal{V}$ with V_r nonsingular. Then,

$$\begin{bmatrix} \Phi_p & \\ & \Phi_r \end{bmatrix} \begin{bmatrix} V_p \\ V_r \end{bmatrix} = \begin{bmatrix} V_p \\ V_r \end{bmatrix} (V_r{}^{-1}\Phi_r V_r) + \begin{bmatrix} I \\ 0 \end{bmatrix} (\Phi_p V_p - V_p V_r{}^{-1}\Phi_r V_r), \tag{2.92}$$

that is

$$\Phi\mathcal{V} \subset \mathcal{V} + \mathcal{B}. \tag{2.93}$$

(2.93) is equivalent to (2.87), and therefore, together with (2.89) implies (2.79). Lemma 2.1 completes the proof.

The geometric conditions of Theorem 2.2 may be useful in developing an algorithm to construct $\mathcal{V}$ if it exists, from which X and Y can easily be generated as in Lemma 2.1. Our attempts so far have failed to come up with such an algorithm. However, it is interesting to compare the above conditions with the solvability conditions of output regulation problem stated by Theorem 8.1 of Wonham[5]. (2.90) and (2.91) are exactly the same in both cases. The only difference is in (2.89), which is replaced by (Φ,Γ)-invariance of $\mathcal{V}$. Remembering that Φ|Ker (C)-invariance and $(\Phi|\Gamma)$-invariance concepts are dual, it might be useful to apply the methods and results of geometric approach of Wonham.

2.3 Ripple-Free Deadbeat Control Problem

In this section, we investigate the ripple-free deadbeat control problem using output feedback and generalized sampled-data hold functions. In the first subsection, we provide a general solvability result and alternative necessity and/or sufficiency theorems. Some examples verifying the results are also given. A special case is the subject of the second subsection.

2.3.1 General Solvability Conditions

From (2.18) and (2.19), the output of the closed-loop sampled-data error system is obtained as

$$\bar{e}(kT+\delta) = D\hat{\bar{\Phi}}(\delta)\hat{\Phi}^k x_0, \;\; 0 \le \delta \le T. \tag{2.94}$$

Hence, the ripple-free deadbeat control problem is equivalent to satisfying

$$D\hat{\Phi}^N = 0 \tag{2.95}$$

and

$$D\hat{\bar{\Phi}}(\delta)\hat{\Phi}^N = 0, \;\; \delta \,\epsilon\, (0,T) \tag{2.96}$$

for some integer N, where $\hat{\bar{\Phi}}(\delta)$ and $\hat{\Phi}^N$ are defined in (2.16) and (2.31).

Let us define

$$\bar{Y}(\delta) = \int_0^\delta e^{\bar{A}_p(\delta-\tau)}\bar{B}_p\bar{F}_s(\tau)d\tau, \;\; \delta \,\epsilon\, [0,T] \tag{2.97}$$

for some GSHF $\bar{F}_s(t) \,\epsilon\, \mathcal{R}^{m\times l}$. The following theorem provides necessary and sufficient conditions for solvability of the ripple-free deadbeat control problem.

Theorem 2.3 *Ripple-free deadbeat control problem is solvable if and only if there exists* $X \,\epsilon\, \mathcal{R}^{n_p\times n_r}$ *and a GSHF* $\bar{F}_s(.) \,\epsilon\, \mathcal{R}^{m\times l}$ *such that*

$$C_p X = C_r \tag{2.98}$$

$$(\Phi_p + \bar{Y}(T)C_p\,)X = X\Phi_r \tag{2.99}$$

$$C_p[\,e^{\bar{A}_p(\delta)} + \bar{Y}(\delta)C_p\,]X = C_r e^{\bar{A}_r(\delta)},\ \delta\,\epsilon\,(0,T). \tag{2.100}$$

Proof [**Necessity**]: Following the proof of the necessity part of Theorem 2.1, (2.95) implies (2.33) and (2.35), which in turn, reduces (2.96) to

$$C_p(e^{\bar{A}_p(\delta)} + \int_0^\delta e^{\bar{A}_p(\delta-\tau)}\bar{B}_p[\,\bar{F}_p(\tau) + \bar{F}_r(\tau)\,]C_p d\tau\,)\Omega_N\Phi_r{}^{-N} = C_r e^{\bar{A}_r(\delta)}. \tag{2.101}$$

Letting $X = \Omega_N\Phi_r{}^{-N}$, $\bar{F}_s(t) := \bar{F}_p(t) + \bar{F}_r(t)$, and noting that

$$\bar{Y}(T) = G_p + G_r, \tag{2.102}$$

the proof follows from Theorem 2.1.

[**Sufficiency**]: Choose G_p to make $\hat{\Phi}_p = \Phi_p + G_pC_p$ nilpotent, compute $\bar{F}_p(t)$ from G_p using the procedure of Section 2.2.3, and let $\bar{F}_r(t) = \bar{F}_s(t) - \bar{F}_p(t)$. Then, $G_r = \bar{Y}(T) - G_p$, and (2.98) and (2.99) imply (2.41) and (2.42), which is the same as (2.95). Finally, using (2.41), left-hand side of (2.96) can be evaluated as

$$\begin{aligned} D\bar{\hat{\Phi}}(\delta)\hat{\Phi}^N &= D\bar{\hat{\Phi}}(\delta)\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}\hat{\Phi}^N \\ &= \begin{bmatrix} * & 0 \end{bmatrix}\begin{bmatrix} \hat{\Phi}_p^N & 0 \\ 0 & \Phi_r{}^N \end{bmatrix}\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \\ &= 0, \end{aligned} \tag{2.103}$$

where * denotes some matrix, which is irrelevant to the result. This completes the proof.

We now look at the ripple-free deadbeat control problem from another perspective. Let

$$\bar{Z}(\delta) = \begin{bmatrix} \bar{Z}_p^T(\delta) & \bar{Z}_r^T(\delta) \end{bmatrix}^T, \tag{2.104}$$

where

$$\bar{Z}_p(\delta) = e^{\bar{A}_p(\delta)}X + \bar{Y}(\delta)C_r,\ \ \delta \geq 0 \tag{2.105}$$

for some $X \in \mathcal{R}^{n_p \times n_r}$ and

$$\bar{Z}_r(\delta) = e^{\bar{A}_r(\delta)}, \quad \delta \geq 0. \tag{2.106}$$

Note that

$$\dot{\bar{Z}}_p(\delta) = \bar{A}_p \bar{Z}_p(\delta) + \bar{B}_p \bar{F}_s(\delta) C_r, \quad \bar{Z}_p(0) = X \tag{2.107}$$

and

$$\dot{\bar{Z}}_r(\delta) = \bar{A}_r \bar{Z}_r(\delta), \quad \bar{Z}_r(0) = I_{n_r} \tag{2.108}$$

so that

$$\dot{\bar{Z}}(\delta) = \bar{A}\bar{Z}(\delta) + \bar{B}\bar{F}_s(\delta)C_r, \quad \bar{Z}(0) = \begin{bmatrix} X^T & I \end{bmatrix}^T. \tag{2.109}$$

The following result is obvious from Theorem 2.3.

Corollary 2.4 *Ripple-free deadbeat control problem is solvable if and only if there exists* $X \in \mathcal{R}^{n_p \times n_r}$ *and a GSHF* $\bar{F}_s(.)$ *such that the solution of (2.109) satisfies*

$$\begin{array}{rcl} D\bar{Z}(\delta) & = & 0, \quad \delta \in [0, T), \\ Im\ \bar{Z}(T) & \subset & Im\ \bar{Z}(0). \end{array} \tag{2.110}$$

Let $\mathcal{V} = Im\ \bar{Z}(0)$. Then the solution space of (2.109) is exactly the space of solutions of the closed-loop sampled-data system $\bar{S}$ starting in $\mathcal{V}$. referring to the geometric conditions of Section 2.2.4., it follows that $\mathcal{V}$ is exactly the space of states which results in a deadbeat response. Once the state of the overall system is driven into $\mathcal{V}$, then (2.110) guarantees a ripple-free deadbeat response from then on.

Corollary 2.5 *Ripple-free deadbeat control problem is solvable if there exists* $X \in \mathcal{R}^{n_p \times n_r}$ *and* $\bar{F}_s(s) \in \mathcal{R}(s)^{m \times l}$ *with a bounded inverse laplace transform such that*

$$C_p X = C_r \tag{2.111}$$

$$(sI - \bar{A}_p)^{-1} \begin{bmatrix} I + \bar{B}_p \bar{F}_s(s) C_p \end{bmatrix} X = X(sI - \bar{A}_r)^{-1} \tag{2.112}$$

Proof : Follows immediately from Theorem 2.3 by taking inverse laplace transform of (2.112).

Example 2.1 *To illustrate the result of Theorem 2.3 and Corollary 2.5, consider a scalar plant whose reference input is generated also by a scalar system; that is* $\bar{S}_p = (a_p, b_p, c_p)$, $\bar{S}_r = (a_r, c_r)$. *Condition (2.111) gives* $x = c_r/c_p$, *and substituting into (2.112), we obtain*

$$\bar{f}_s(s) = \frac{a_r - a_p}{b_p c_p}\frac{1}{s - a_r}. \tag{2.113}$$

Thus

$$\bar{f}_s(t) = \frac{a_r - a_p}{b_p c_p} e^{a_r t}, \quad t \in [0, T]. \tag{2.114}$$

Now, the deadbeat condition requires

$$g_p = \int_0^T e^{a_p(T-\tau)} b_p \bar{f}_p(\tau) d\tau = -\frac{1}{c_p}\Phi_p = -\frac{1}{c_p} e^{a_p T}, \tag{2.115}$$

and after choosing $\bar{f}_p(t)$ *to satisfy (2.115),* $\bar{f}_r(t)$ *is computed as* $\bar{f}_r(t) = \bar{f}_s(t) - \bar{f}_p(t)$.

Example 2.2 *Consider a single-output plant with*

$$\begin{aligned} \bar{A}_p &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \ \bar{B}_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C_p &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \end{aligned} \tag{2.116}$$

and a single-output reference system with

$$\begin{aligned} \bar{A}_r &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ C_r &= \begin{bmatrix} 1 & 0 \end{bmatrix}. \end{aligned} \tag{2.117}$$

(2.98) requires that

$$X = \begin{bmatrix} 1 & 0 \\ \alpha & \beta \end{bmatrix}, \quad \alpha, \beta \in \mathcal{R}. \tag{2.118}$$

With X as above and $\bar{Y}(\delta) = \begin{bmatrix} \bar{y}_1(\delta) & \bar{y}_2(\delta) \end{bmatrix}^T$, *(2.100) becomes*

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 + \bar{y}_1(\delta) & 1 - e^{-\delta} \\ \bar{y}_2(\delta) & e^{-\delta} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \alpha & \beta \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix} \tag{2.119}$$

which is never satisfied. Hence, the ripple-free deadbeat control is not possible.

However, it is easy to check that the deadbeat conditions (2.27) and (2.28) are satisfied with

$$\alpha = -1,\ \beta = \frac{T}{1-e^{-T}},\ Y = (1-e^{-T})\begin{bmatrix} 1 & -1 \end{bmatrix}^T. \tag{2.120}$$

Example 2.3 *In this example, we show the existence of a system for which the set of sufficient conditions in corollary 2.5 are not satisfied, while the ripple-free deadbeat control problem is solvable. Consider a second order single output plant represented by*

$$\begin{aligned} \bar{A}_p &= \begin{bmatrix} m & 0 \\ 0 & 3m \end{bmatrix},\ \bar{B}_p = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ C_p &= \begin{bmatrix} 1 & 1 \end{bmatrix},\ m \neq 0 \end{aligned} \tag{2.121}$$

and a scalar reference system with

$$\bar{A}_r = 2m, \quad C_r = 1. \tag{2.122}$$

Checking (2.111) and (2.112), we see that sufficient conditions are not satisfied. Next, we can easily show that the GSHF $\bar{f}_s(t)$ and $X = \begin{bmatrix} x_1 & 1-x_1 \end{bmatrix}^T$ satisfies (2.98), (2.99) and (2.100), where

$$\begin{aligned} \bar{f}_s(t) &= [\tfrac{m(2x_1-1)}{2}]e^{2mt} - (\tfrac{m^2}{2})te^{2mt},\ t \in [0,T] \\ x_1 &= [(2+3mT)e^{3mT}-2]/[18((e^{mT}-e^{2mT})+6(e^{3mT}-1)], \end{aligned} \tag{2.123}$$

and T is the solution of

$$\begin{aligned} &[(2+3mT)e^{3mT}-2][50(e^{2mT}-e^{3mT})+10[(e^{5mT}-1)] = \\ &[18((e^{mT}-e^{2mT})+6(e^{3mT}-1)][50(e^{2mT}-e^{3mT})+(4+5mT)e^{5mT}-4] \end{aligned} \tag{2.124}$$

which is approximately $(\frac{-2.947}{m})$ for negative m, and $(\frac{4.824}{m})$ for positive m values.

2.3.2 Ripple-Free Deadbeat Regulation

In this subsection, we consider the deadbeat and ripple-free deadbeat control problems for the special case of zero reference input.

In the absence of a reference input, the ripple-free deadbeat conditions (2.95) and (2.96) reduce to

$$C_p \hat{\Phi}_p^N = 0 \tag{2.125}$$

and

$$C_p \hat{\bar{\Phi}}_p(\delta) \hat{\Phi}_p^N = 0, \quad \delta \in (0, T) \tag{2.126}$$

respectively, where

$$\hat{\bar{\Phi}}_p(\delta) = e^{\bar{A}_p \delta} + \bar{G}_p(\delta) C_p. \tag{2.127}$$

Solvability conditions are provided by the following theorem.

Theorem 2.4 *Deadbeat and ripple-free deadbeat regulation problems with internal stability are equivalent, and are always solvable.*

Proof: Following the same argument as in the proof of Theorem 2.1, it follows that the deadbeat condition (2.125) is equivalent to

$$\hat{\Phi}_p^N = 0, \tag{2.128}$$

which also satisfies the ripple-free deadbeat regulation condition (2.126). (2.128), however, implies internal stability, and can always be satisfied by a suitable G_p, which can be realized using a GSHF $\bar{F}_p(t)$ as discussed in Section 2.2.3.

Chapter 3

RIPPLE-FREE DEADBEAT CONTROL USING MULTIRATE-OUTPUT CONTROLLERS

In this chapter, we investigate the ripple-free deadbeat control problem using dynamic feedback from multirate-sampled output values. The first section is devoted to problem formulation. In the second section, a solvability condition is presented, and several examples verifying the results are provided. A special case is a subject of the third section.

3.1 Formulation of the Problem

Consider the error system $\bar{\mathcal{S}}_e$ of (2.5), which consists of the plant $\bar{\mathcal{S}}_p$ of (2.1) and the reference model of $\bar{\mathcal{S}}_r$ of (2.2). For the purpose of ripple-free deadbeat control of $\bar{\mathcal{S}}_p$, we consider a MROC operating on sampled values of the plant output and the reference signal as shown in Fig. 3.1.

In the control configuration of Fig. 3.1, the samplers at the outputs of $\bar{\mathcal{S}}_p$ and $\bar{\mathcal{S}}_r$ operate at rates T_p and T_r respectively, where

$$N_p T_p = N_r T_r = T, \tag{3.1}$$

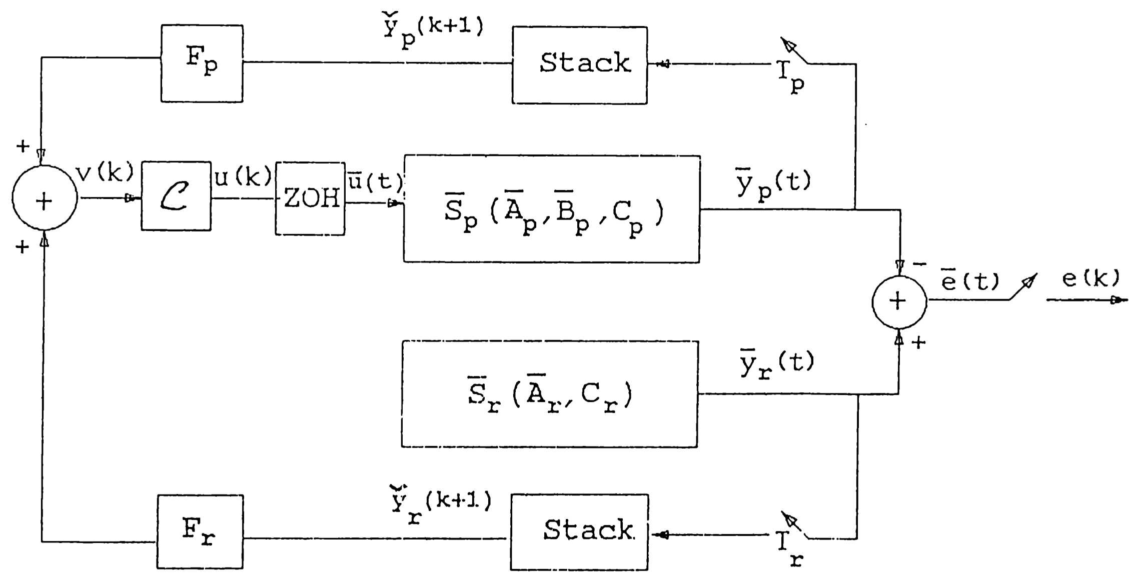

Figure 3.1. Control Scheme with Multirate-Output Controller

for some integers N_p and N_r, and T is the basic sampling period. We define doubly-indexed discrete sequences $x_i^d(k,j)$ and $y_i^d(k,j)$ as

$$\begin{aligned} x_i^d(k,j) &= \bar{x}_i(kT + jT_i) \\ y_i^d(k,j) &= \bar{y}_i(kT + jT_i) \end{aligned} \tag{3.2}$$

for $k \in \mathcal{Z}$, $j = 0, 1, \ldots, N_i - 1$, $i = p, r$; and let

$$\begin{aligned} x_i(k) &= x_i^d(k,0) \\ y_i(k) &= y_i^d(k,0). \end{aligned} \tag{3.3}$$

For convenience, we also let

$$\begin{aligned} x_i^d(k,N_i) &= x_i^d(k+1,0) &= x_i(k+1) \\ y_i^d(k,N_i) &= y_i^d(k+1,0) &= y_i(k+1). \end{aligned} \tag{3.4}$$

The output samples $y_i^d(k,j)$ are stacked into a vector $\check{y}_i(k+1)$ over each basic sampling period, that is,

$$\check{y}_i(k+1) = \begin{bmatrix} y_i^d(k,0) \\ \vdots \\ y_i^d(k, N_i - 1) \end{bmatrix}, \quad k \in \mathcal{Z}, \quad i = p, r. \tag{3.5}$$

The discrete controller $\mathcal{C}$ which operates at a rate compatible with the basic sampling period T is defined as

$$\mathcal{C}: \quad u(k+1) = Hu(k) + v(k), \tag{3.6}$$

where

$$v(k) = F_p \check{y}_p(k+1) + F_r \check{y}_r(k+1). \tag{3.7}$$

The control input $\bar{u}(t)$ to the plant $\bar{\mathcal{S}}_p$ is generated by holding the outputs of $\mathcal{C}$ over each basic sampling period, i.e.,

$$\bar{u}(t) = u(k), \quad kT \leq t < (k+1)T. \tag{3.8}$$

To obtain a description of the closed-loop sampled-data system, we first note that the discrete-time models for $\bar{\mathcal{S}}_p$ and $\bar{\mathcal{S}}_r$ at their own sampling rates are described as

$$\bar{\mathcal{S}}_p^d: \begin{array}{rcl} x_p^d(k,j+1) & = & \Phi_p^d x_p^d(k,j) + \Gamma_p^d u(k), \quad k \in \mathcal{Z}, \quad j = 0,1,\ldots,N_p-1 \\ y_p^d(k,j+1) & = & C_p x_p^d(k,j), \end{array} \tag{3.9}$$

and

$$\bar{\mathcal{S}}_r^d: \begin{array}{rcl} x_r^d(k,j+1) & = & \Phi_r^d x_r^d(k,j), \quad k \in \mathcal{Z}, \quad j = 0,1,\ldots,N_r-1 \\ y_r^d(k,j+1) & = & C_r x_r^d(k,j), \end{array} \tag{3.10}$$

where

$$\Phi_i^d = e^{\bar{A}_i T_i}, \quad i = p, r, \tag{3.11}$$

and

$$\Gamma_p^d = \int_0^{T_p} e^{\bar{A}_p \tau} \bar{B}_p d\tau. \tag{3.12}$$

From (3.9), it follows that

$$x_p^d(k,j) = (\Phi_p^d)^j x_p^d(k,0) + \sum_{i=0}^{j-1} (\Phi_p^d)^i \Gamma_p^d u(k), \tag{3.13}$$

and similarly from (3.10)

$$x_r^d(k,j) = (\Phi_r^d)^j x_r^d(k,0). \tag{3.14}$$

Hence, the stacked samples are obtained as

$$\check{y}_p(k+1) = Q_p^d x_p(k) + R_p^d u(k), \tag{3.15}$$

$$y_r(k+1) = Q_r^d x_r(k), \tag{3.16}$$

$$Q_i^d = \begin{bmatrix} C_i \\ C_i \Phi_i^d \\ \vdots \\ C_i (\Phi_i^d)^{N_i - 1} \end{bmatrix}, \quad i = p, r, \tag{3.17}$$

and

$$R_p^d = \begin{bmatrix} 0 \\ C_p \Gamma_p^d \\ \vdots \\ C_p \sum_{i=0}^{N_p - 2} (\Phi_i^d)^i (\Gamma_p^d) \end{bmatrix}. \tag{3.18}$$

On the other hand, again from (3.9) and (3.10), the discrete models over the basic sampling period are obtained as

$$\mathcal{S}_p : \begin{array}{rcl} x_p(k+1) & = & \Phi_p x_p(k) + \Gamma_p u(k) \\ y_p(k) & = & C_p x_p(k), \end{array} \tag{3.19}$$

$$\mathcal{S}_r : \begin{array}{rcl} x_r(k+1) & = & \Phi_r x_r(k) \\ y_r(k) & = & C_r x_r(k), \end{array} \tag{3.20}$$

where

$$\Phi_i = (\Phi_i^d)^{N_i} = e^{\bar{A}_i T}, \quad i = p, r, \tag{3.21}$$

and

$$\Gamma_p = \sum_{i=0}^{N_p - 1} (\Phi_p^d)^i (\Gamma_p^d). \tag{3.22}$$

Here, we assume that the sampling process does not introduce any unobservable and uncontrollable modes into $\mathcal{S}_p$ and $\mathcal{S}_r$, i.e., (Φ_p, C_p) and (Φ_r, C_r) are observable pairs and (Φ_p, Γ_p) is a controllable pair.

Now, combining (3.6), (3.7), (3.15), (3.16), (3.19) and (3.20), a discrete-time description of the closed-loop error system over the basic sampling period T is obtained as

$$\mathcal{S}_a : \begin{array}{rcl} x_a(k+1) & = & \Phi_a x_a(k) \\ e(k) & = & D_a x_a(k), \end{array} \tag{3.23}$$

where

$$\begin{aligned} x_a(k) &= \begin{bmatrix} x_p^T(k) & x_r^T(k) & u^T(k) \end{bmatrix}^T \\ \Phi_a &= \begin{bmatrix} \Phi_p & 0 & \Gamma_p \\ 0 & \Phi_r & 0 \\ F_p Q_p^d & F_r Q_r^d & H + F_p R_p^d \end{bmatrix}, \\ D_a &= \begin{bmatrix} -C_p & C_r & 0 \end{bmatrix}. \end{aligned} \tag{3.24}$$

Finally, to obtain an expression for the continuous-time error signal $\bar{e}(t)$, we note that

$$\begin{aligned} \bar{x}(kT+\delta) &= e^{\bar{A}\delta}\bar{x}(kT) + \int_{kT}^{kT+\delta} e^{\bar{A}(kT+\delta-\tau)}\bar{B}u(k)d\tau \\ &= e^{\bar{A}\delta}x(k) + \int_0^{\delta} e^{\bar{A}(\tau)}\bar{B}u(k)d\tau \\ &= x(k) + \int_0^{\delta} e^{\bar{A}(\tau)}\left[\,\bar{A}x(k) + \bar{B}u(k)\,\right]d\tau, \quad 0 \le \delta \le T, \end{aligned} \tag{3.25}$$

where $\bar{A}$, $\bar{B}$ and $x(k)$ are as defined in (2.4) and (2.9), and the last equality follows from the fact that

$$e^{\bar{A}\delta} + \int_0^{\delta} e^{\bar{A}(\tau)}d\tau = I_n + \int_0^{\delta} e^{\bar{A}(\tau)}\bar{A}d\tau. \tag{3.26}$$

Noting that

$$x(k) = \begin{bmatrix} I_n & 0 \end{bmatrix} x_a(k), \tag{3.27}$$

and

$$\bar{A}x(k) + \bar{B}u(k) = \begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} x_a(k), \tag{3.28}$$

(3.25) can be rewritten as

$$\bar{x}(kT+\delta) = \left(\begin{bmatrix} I_n & 0 \end{bmatrix} + \int_0^{\delta} e^{\bar{A}(\tau)} \begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} d\tau \right) x_a(k). \tag{3.29}$$

Substituting $x_a(k) = {\Phi_a}^k x_a(0)$ into (3.29) and using $\bar{e}(t) = D\bar{x}(t)$, we obtain

$$\bar{e}(kT+\delta) = D\left(\begin{bmatrix} I_n & 0 \end{bmatrix} + \int_0^{\delta} e^{\bar{A}(\tau)} \begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} d\tau \right) {\Phi_a}^k x_a(0), \quad 0 \le \delta \le T. \tag{3.30}$$

We now formulate the ripple-free deadbeat control problem as follows:

Ripple-Free Deadbeat Control Problem: Choose the integers N_p and N_r, and the matrices F_p, F_r and H such that for all $x_a(0) \in \mathcal{R}^{n+m}$

$$\bar{e}(t) = 0, \quad for\ all \ \ t \geq NT, \tag{3.31}$$

for some $N \in \mathcal{Z}_+$.

3.2 A Solvability Condition

From (3.30), necessary and sufficient conditions for ripple-free deadbeat response are obtained as

$$D \begin{bmatrix} I_n & 0 \end{bmatrix} {\Phi_a}^N = 0, \tag{3.32}$$

and

$$D \left(\int_0^\delta e^{\bar{A}(\tau)} d\tau \right) \begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} {\Phi_a}^N = 0, \quad \delta \in (0, T),$$

or equivalently,

$$\bar{Q}_e \begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} {\Phi_a}^N = 0, \tag{3.33}$$

where

$$\bar{Q}_e = \begin{bmatrix} D \\ D\bar{A} \\ \vdots \\ D\bar{A}^{n-1} \end{bmatrix} \tag{3.34}$$

is the observability matrix of $\bar{S}_e$. We note that (3.32) is alone the deadbeat condition.

Since the conditions (3.32) and (3.33) involve the controller parameters H, F_p and F_r, they are not practical to use in design. The following theorem provides a sufficient condition in terms of the open-loop system parameters.

Theorem 3.5 *Ripple-free deadbeat control problem is solvable in N steps if there exists $G \in \mathcal{R}^{m \times n}$ such that*

$$D \, (\Phi + \Gamma G)^{N-1} = 0, \tag{3.35}$$

$$\bar{Q}_e(\bar{A} + \bar{B}G)(\Phi + \Gamma G)^{N-1} = 0, \tag{3.36}$$

where $\Gamma = \begin{bmatrix} \Gamma_p^T & 0 \end{bmatrix}^T$, *with* Γ_p *as in (3.22).*

Proof: For $G = \begin{bmatrix} G_p & G_r \end{bmatrix}$,

$$(\Phi + \Gamma G)^{N-1} = \begin{bmatrix} \hat{\Phi}_p^{N-1} & \Omega_{N-1} \\ 0 & \Phi_r^{N-1} \end{bmatrix}, \tag{3.37}$$

where $\hat{\Phi}_p = \Phi_p + \Gamma_p G_p$ and

$$\Omega_{N-1} = \hat{\Phi}_p^{N-2}\Gamma_p G_r + \hat{\Phi}_p^{N-3}\Gamma_p G_r \Phi_r + \ldots + \Gamma_p G_r \Phi_r^{N-2}. \tag{3.38}$$

Thus (3.35) and (3.36) are equivalent to

$$D \begin{bmatrix} \hat{\Phi}_p^{N-1} & \Omega_{N-1} \\ 0 & \Phi_r^{N-1} \end{bmatrix} = 0, \tag{3.39}$$

$$\bar{Q}_e \begin{bmatrix} \hat{\bar{A}}_p \hat{\Phi}_p^{N-1} & \hat{\bar{A}}_p \Omega_{N-1} + \bar{B}_p G_r \Phi_r^{N-1} \\ 0 & \bar{A}_r \Phi_r^{N-1} \end{bmatrix} = 0. \tag{3.40}$$

where $\hat{\bar{A}}_p = \bar{A}_p + \bar{B}_p G_p$.

We now choose F_i, $i = p, r$, such that

$$G_i = F_i Q_i^d \Phi_i^{-1}, \quad i = p, r, \tag{3.41}$$

and let

$$H = F_p(Q_p^d \Phi_i^{-1}\Gamma_p - R_p^d). \tag{3.42}$$

(3.41) requires that Q_i^d have full column rank, which is true provided $N_i \geq \eta_i$, $i = p, r$, where η_i are the observability indices of (Φ_p^d, C_p) and (Φ_r^d, C_r) respectively. With

$$T_a = \begin{bmatrix} I_{n_p} & 0 & 0 \\ 0 & I_{n_r} & 0 \\ G_p & 0 & I_m \end{bmatrix},$$

it follows from (3.41) and (3.42) that

$$T_a^{-1}\Phi_a T_a = \begin{bmatrix} \hat{\Phi}_p & 0 & \Gamma_p \\ 0 & \Phi_r & 0 \\ 0 & G_r \Phi_r & 0 \end{bmatrix} = \tilde{\Phi}_a. \tag{3.43}$$

Consequently, (3.42) and (3.43) can be rewritten as

$$D \left[\begin{array}{cc} I_n & 0 \end{array} \right] \tilde{\Phi}_a^N = 0 \tag{3.44}$$

$$\bar{Q}_e \left[\begin{array}{cc} \hat{\bar{A}} & \bar{B} \end{array} \right] \tilde{\Phi}_a^N = 0, \tag{3.45}$$

where

$$\hat{\bar{A}} = \left[\begin{array}{cc} \hat{\bar{A}}_p & 0 \\ 0 & \bar{A}_r \end{array} \right], \tag{3.46}$$

with $\hat{\bar{A}}_p$ as defined in (3.40). Due to the special structure of $\tilde{\Phi}_a$, $\tilde{\Phi}_a^N$ can easily be constructed as

$$\tilde{\Phi}_a^N = \left[\begin{array}{ccc} \hat{\Phi}_p^N & \Omega_{N-1}\Phi_r & \hat{\Phi}_p^{N-1}\Gamma_p \\ 0 & {\Phi_r}^N & 0 \\ 0 & G_r{\Phi_r}^N & 0 \end{array} \right]. \tag{3.47}$$

It is then a simple job of substitution to show that (3.39) and (3.40) imply (3.44) and (3.45) respectively. This completes the proof.

We note as a side-remark that since $(\Phi + \Gamma G)$ has only n_p-free eigenvalues and other eigenvalues are not zero, it is unnecessary to check for existence of G for $N \geq n_p + 1$.

It is interesting to compare the solvability conditions of Theorem 3.5 with the necessary and sufficient conditions of Urikura & Nagata [10], who considered ripple-free deadbeat control using constant state feedback under the assumptions that $m = l$ and

$$\left[\begin{array}{cc} \Phi_p - \mu I & \Gamma_p \\ C_p & 0 \end{array} \right] \tag{3.48}$$

is nonsingular for all eigenvalues μ of Φ_r. Urikura & Nagata [10] showed that, under the stated assumptions, there exists unique matrices $X \in \mathcal{R}^{n_p \times n_r}$ and $Y \in \mathcal{R}^{m \times n_r}$ such that

$$\left[\begin{array}{cc} \Phi_p & \Gamma_p \\ C_p & 0 \end{array} \right] \left[\begin{array}{c} X \\ Y \end{array} \right] = \left[\begin{array}{c} X\Phi_r \\ C_r \end{array} \right]. \tag{3.49}$$

Defining

$$V = \left[\begin{array}{c} X \\ I_{n_r} \end{array} \right], \quad \mathcal{V} = Im\, V, \tag{3.50}$$

they proved that ripple-free deadbeat control with constant state feedback is possible if and only if

$$\bar{A}\mathcal{V} \subset Im\ \bar{B} + Ker\ \bar{Q}_e. \tag{3.51}$$

In the proof of the sufficiency part of the result they constructed a state feedback matrix G which satisfies (3.35) and (3.36).

On the other hand, with H and G chosen as in the proof of Theorem 3.5 we have from (3.6), (3.7), (3.15), (3.16), (3.41), (3.42), (3.19) and (3.20)

$$\begin{aligned} u(k+1) &= Hu(k) + F_p\breve{y}_p(k+1) + F_r\breve{y}_r(k+1) \\ &= (H + F_pR_p^d)u(k) + F_pQ_p^dx_p(k) + F_rQ_r^dx_r(k) \\ &= G_p\left[\ \Phi_px_p(k) + \Gamma_pu(k)\ \right] + G_r\Phi_rx_r(k) \\ &= G_px_p(k+1) + G_rx_r(k+1) \\ &= Gx(k+1), \end{aligned} \tag{3.52}$$

so that dynamic feedback from multirate sampled outputs is equivalent to constant state feedback after the first basic sampling period.

In conclusion, under the assumptions of Urikura & Nagata[10] and H restricted to the form in (3.42), solvability conditions (3.35) and (3.36) are equivalent to the condition (3.51) of Urikura & Nagata. However, while (3.51) is also necessary for state feedback control, (3.35) and (3.36) are not for dynamic multirate output feedback control. Obviously, the reason is the freedom in the choice of H.

As a final remark, we are going to show that MROC can also strongly stabilize the closed-loop structure under some mild assumptions. For that purpose, let us define

$$\tilde{\bar{A}}_p = \begin{bmatrix} \bar{A}_p & \bar{B}_p \\ 0 & 0 \end{bmatrix},\ \tilde{C}_p = \begin{bmatrix} C_p & 0 \end{bmatrix}. \tag{3.53}$$

Corollary 3.6 *Assuming that*

$$\begin{bmatrix} \bar{A}_p & \bar{B}_p \\ C_p & 0 \end{bmatrix} \tag{3.54}$$

is full-column rank and $N_p \geq \bar{\eta}_p$*, where* $\bar{\eta}_p$ *is the observability index of the augmented system* $(\tilde{\bar{A}}_p, \tilde{C}_p)$*, the RFDB control problem is solvable with strong stability property.*

Proof: By (3.41) and (3.42), H and F_p satisfy

$$F_p \left[\begin{array}{c|c} Q_p^d \Phi_p^{-1} & R_p - Q_p^d \Phi_p^{-1} \Gamma_p \end{array} \right] = \left[\begin{array}{c|c} G_p & -H \end{array} \right]. \tag{3.55}$$

Noting that the appropriate matrices for the augmented system are

$$\begin{aligned} \tilde{\Phi}_p^d &= \begin{bmatrix} \Phi_p^d & \Gamma_p^d \\ 0 & I \end{bmatrix} \\ \tilde{Q}_p^d &= \begin{bmatrix} Q_p^d & R_p^d \end{bmatrix}, \end{aligned} \tag{3.56}$$

we have

$$\tilde{Q}_p^d \tilde{\Phi}_p^{-1} = \left[\begin{array}{c|c} Q_p^d \Phi_p^{-1} & R_p - Q_p^d \Phi_p^{-1} \Gamma_p \end{array} \right]. \tag{3.57}$$

Two assumptions above imply that (3.57) is full-column rank , and hence, (3.55) has a solution for F_p, given any stable H. The proof is complete by Theorem 3.5.

3.3 Ripple-Free Deadbeat Regulation

In this section, deadbeat and ripple-free deadbeat regulation problems are considered for the special case of zero reference input.

In the absence of a reference input and together with the choice of $H = F_p \ (\ Q_p^d \Phi_p{}^{-1} \Gamma_p - R_p^d \ )$, the sufficient conditions (3.35) and (3.36) of the ripple-free deadbeat control problem reduce to

$$C_p(\Phi_p + \Gamma_p G_p)^{N-1} = 0, \tag{3.58}$$

and

$$(\bar{A}_p + \bar{B}_p G_p)(\Phi_p + \Gamma_p G_p)^{N-1} = 0. \tag{3.59}$$

where (3.58) alone is the deadbeat condition. Solvability conditions are provided by the following theorem.

Theorem 3.6 *Deadbeat and ripple-free deadbeat regulation problems are always solvable with internal stability constraint.*

Proof: Sufficient conditions (3.58) and (3.59) can be satisfied by choosing G_p so as to make

$$(\Phi_p + \Gamma_p G_p)^{N-1} = 0, \tag{3.60}$$

which is always possible since (Φ_p, Γ_p) is a controllable pair, where G_p can be realized by F_p as

$$G_p = F_p Q_p^d {\Phi_p}^{-1}, \tag{3.61}$$

provided $N_p \geq n_p + 1$. Internal stability follows from (3.60).

Chapter 4

CONCLUSIONS

In this thesis, the ripple-free deadbeat regulation and tracking problems are considered for linear, time-invariant systems. The problem is formulated in state-space setting, and is analyzed with two new sampled-data controllers, namely generalized sampled-data hold functions and multirate-output controllers. The methods provide simplicity in implementation since they are in output feedback form. The necessary and sufficient solvability conditions are stated by theorems in terms of open-loop system parameters.

The main contributions of this thesis are Theorems 2.1, 2.3 and 3.5. Partial results related with the regulation problem already exist in the literature. Theorems 2.4 and 3.6 provide complete results. The solvability conditions are in terms of simultaneous linear/nonlinear matrix equations involving system transition, input, and output matrices of the reference model and the plant. The solvability conditions of Theorem 2.1 are restated in the geometric setting by Theorem 2.2. In other theorems and corollaries, various special cases of the reference model are considered, which help deeply in understanding the solvability conditions. Internal stability and strong stability properties are also investigated.

It is clear that, there is work to be done towards obtaining system theoretic interpretations and geometric counterparts of the solvability conditions. This line of research was deliberately avoided in this thesis since its development requires a different algebraic background.

A problem which is left open in this thesis is the minimum-time constraint. Our approach is not aimed for obtaining the ripple-free deadbeat response in minimum-time, but rather expressing the solvability conditions in its simplest form.

Another open question is the robustness analysis of the deadbeat and ripple-free deadbeat controllers, which are believed to be quite weak since the controller is highly sensitive to the variations of system parameters.

As a final remark, we note that the almost/approximate ripple-free deadbeat control problems are challenging concepts for further research since their results would be very helpful in industrial applications.

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